Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 780–787

Intertwiners of Pseudo-Hermitian 2×2-Block-Operator Matrices and a No-Go Theorem for Isospectral MHD Dynamo Operators

Uwe GÜNTHER

Research Center Rossendorf, P.O. Box 510119, D-01314 Dresden, Germany
E-mail: *u.guenther@fz-rossendorf.de*

Pseudo-Hermiticity as a generalization of usual Hermiticity is a rather common feature of (differential) operators emerging in various physical setups. Examples are Hamiltonians of PT- and CPT-symmetric quantum mechanical systems [1] as well as the operator of the spherically symmetric α^2-dynamo [2] in magnetohydrodynamics (MHD). In order to solve the inverse spectral problem for these operators, appropriate uniqueness theorems should be obtained and possibly existing isospectral configurations should be found and classified. As a step toward clarifying the isospectrality problem of dynamo operators, we discuss an intertwining technique for η-pseudo-Hermitian 2×2-block-operator matrices with second-order differential operators as matrix elements. The intertwiners are assumed as first-order matrix differential operators with coefficients which are highly constrained by a system of nonlinear matrix differential equations. We analyze the (hidden) symmetries of this equation system, transforming it into a set of constrained and interlinked matrix Riccati equations. Finally, we test the structure of the spherically symmetric MHD α^2-dynamo operator on its compatibility with the considered intertwining ansatz and derive a no-go theorem.

1 Introduction

This article is based on a talk given at the Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" which was held in Kiev, June 23–29, 2003. A more detailed presentation of the material is contained in Ref. [2].

Operator intertwining techniques are one of the basic ingredients of supersymmetric (SUSY) quantum mechanics (QM) [3, 4]. With their help classes of isospectral operators have been constructed for Hermitian Hamiltonians as well as recently for pseudo-Hermitian ones [5, 6]. The success of these techniques in obtaining new and wider classes of isospectral operators raised the natural question whether a suitable extension of them can be developed for analyzing the isospectrality problem [7] of the MHD α^2-dynamo operator [8]

$$\hat{H}_l[\alpha] \equiv \begin{pmatrix} -p^2 - \dfrac{l(l+1)}{r^2} & \alpha(r) \\ p\alpha(r)p + \alpha(r)\dfrac{l(l+1)}{r^2} & -p^2 - \dfrac{l(l+1)}{r^2} \end{pmatrix} \tag{1}$$

which acts on the domain[1]

$$\mathcal{D}(\hat{H}_l[\alpha]) := \left\{ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \ \psi \in \tilde{\mathcal{H}} \equiv \mathcal{H} \oplus \mathcal{H}, \ \mathcal{H} = L_2\big(\Omega, r^2 dr\big), \right.$$
$$\left. \Omega = [0,1], \ \psi(1) = 0, \ r\psi(r)|_{r\to 0} \to 0 \right\}$$

[1]It should be noted that this domain $\mathcal{D}(\hat{H}_l[\alpha])$ corresponds to highly idealized and physically non-realistic boundary conditions (see the corresponding comments in Ref. [2] and the setup of the model in Ref. [8]).

in the Hilbert space $\tilde{\mathcal{H}}$ and where $p = -i(\partial_r + 1/r)$ denotes the radial momentum operator. The scalar function $\alpha(r)$ is the helical turbulence function (α-profile) of the α^2-dynamo [2, 8] and plays a similar role like the potential $V(r)$ in QM models.

Subsequently, it is shown in a sketchy outline that a no-go theorem exists which forbids an extension of the operator intertwining formalism from pseudo-Hermitian QM Hamiltonians to the 2×2-operator matrix (1) of the spherically symmetric α^2-dynamo in its simplest ansatz. The obstruction for such an extension consists in the presence of the centrifugal terms $l(l+1)/r^2$.

2 Operator intertwining ansatz

In SUSY QM the operator intertwining technique heavily relies on the Hermiticity or pseudo-Hermiticity of the corresponding Hamiltonians

$$H = H^\dagger \qquad \text{or} \qquad H = H^\sharp \equiv \eta H^\dagger \eta.$$

The operator η is Hermititian, involutory and unitary

$$\eta = \eta^\dagger, \qquad \eta^2 = I, \qquad \eta^{-1} = \eta^\dagger$$

and, depending on the concrete model, it can be, e.g., the operator of a parity-transformation P [5, 9] or of some other symmetry [10]. For the operator matrix of the α^2-dynamo one obtains the similar relation

$$\hat{H}_l[\alpha] = \hat{H}_l^\sharp[\alpha] \equiv \eta \hat{H}_l^\dagger[\alpha]\eta, \qquad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The structure of the α^2-dynamo operator itself cannot be reduced by appropriate complexification to the simpler setup of one-component pseudo-Hermitian QM Hamiltonians [11].

Due to the matrix structure of the dynamo operator and the α-coupling in its highest (second) order differential expressions we generalize the intertwining technique of QM by an ansatz

$$\hat{H}_{l_0}[\alpha_0] - EI = -\hat{A}\hat{A}^\sharp, \qquad \hat{H}_{l_1}[\alpha_1] - EI = -\hat{A}^\sharp\hat{A} \tag{2}$$

which is based on first-order differential operators

$$\hat{A} := iR(r)p + Q(r), \qquad \hat{A}^\sharp := -ipR^\sharp(r) + Q^\sharp(r). \tag{3}$$

Dynamos with different α-profiles $\alpha_0(r) \neq \alpha_1(r)$ will be isospectral, if appropriate 2×2-matrix functions $R(r)$ and $Q(r)$ can be constructed so that the relations (2), (3) are fulfilled simultaneously. Otherwise, intrinsic contradictions should be found which could be interpreted as a no-go theorem.

For the subsequent analysis it is convenient to introduce the auxiliary matrices $K_{0,1}$, $M_{0,1}$

$$K_{0,1} := I - \alpha_{0,1}\sigma_-,$$
$$M_{0,1} := K_{0,1}\frac{l_{0,1}(l_{0,1}+1)}{r^2} + EI - \alpha_{0,1}\sigma_+$$

with the nilpotent matrices $\sigma_\pm$ defined as $\sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The shifted α^2-dynamo operator matrices in (2) take then the short form

$$\hat{H}_{l_{0,1}}[\alpha_{0,1}] - EI = -pK_{0,1}p - M_{0,1}. \tag{4}$$

As next step, equations (3), (4) can be substituted into the intertwining ansatz (2) and terms of the same type of differential operators p^2, p, $p^0 = I$ can be equated (after appropriate commutations like $[p, R(r)] = -iR'(r)$). As a result, one obtains the following six consistency conditions

$$\begin{aligned}
\hat{H}_{l_0}: \quad & p^2: \quad && RR^\sharp = K_0, && (5)\\
& p: && RQ^\sharp - QR^\sharp - R(R^\sharp)' + R'R^\sharp = 0, && (6)\\
& I: && QQ^\sharp - R(R^\sharp)'' + R(Q^\sharp)' - Q(R^\sharp)' = M_0, && (7)\\
\hat{H}_{l_1}: \quad & p^2: && R^\sharp R = K_1, && (8)\\
& p: && -R^\sharp Q + Q^\sharp R = 0, && (9)\\
& I: && Q^\sharp Q - \left(R^\sharp Q\right)' = M_1. && (10)
\end{aligned}$$

For a successful intertwining construction these matrix equations should be fulfilled simultaneously. So, the main task consists in finding explicit solution sets for (5)–(10). Alternatively, intrinsic contradictions within this equation system should be obtained which could be interpreted as a no-go theorem forbidding this construction for α^2-dynamo operator matrices.

3 Consistency analysis

The easiest way for starting the analysis is from equations (5) and (8). From the tautologies $RR^\sharp R = RR^\sharp R$ and $R^\sharp RR^\sharp = R^\sharp RR^\sharp$ follows

$$RK_1 = K_0 R, \qquad K_1 R^\sharp = R^\sharp K_0$$

which with

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \qquad R^\sharp = \begin{pmatrix} r_{22}^* & r_{12}^* \\ r_{21}^* & r_{11}^* \end{pmatrix}$$

yields

$$r_{12} = 0, \qquad \frac{\alpha_1}{\alpha_0} = \frac{r_{11}}{r_{22}} = \frac{r_{11}^*}{r_{22}^*}. \tag{11}$$

Hence, one can set

$$r_{11} = |r_{11}|e^{i\gamma}, \qquad r_{22} = |r_{22}|e^{i\gamma}, \qquad r_{21} = |r_{21}|e^{i(\gamma+\varepsilon)}.$$

Using this and (11) in

$$RR^\sharp = K_0 = \begin{pmatrix} 1 & 0 \\ -\alpha_0 & 1 \end{pmatrix}, \qquad R^\sharp R = K_1 = \begin{pmatrix} 1 & 0 \\ -\alpha_1 & 1 \end{pmatrix}$$

one finds

$$R = e^{i\gamma} \begin{pmatrix} \sqrt{\dfrac{\alpha_1}{\alpha_0}} & 0 \\ -\dfrac{1}{2}\sqrt{\alpha_0\alpha_1}\,(1 + i\tan\varepsilon) & \sqrt{\dfrac{\alpha_0}{\alpha_1}} \end{pmatrix}, \tag{12}$$

where the phases γ and ε are still undefined.

As next step, equations (6) and (9) will be analyzed. It is easily seen that for the matrices

$$U := R\big[Q^\sharp - (R^\sharp)'\big], \qquad B := R^\sharp Q \tag{13}$$

these equations are equivalent to the J-symmetry relations

$$U = U^\sharp, \qquad B = B^\sharp.$$

Due to the different symmetry content of B and Q it is natural to consider B as primary structural element of the intertwining construction, and Q as a secondary one. So, the subsequent investigation can be performed in terms of B and R. Explicitly, the η-symmetry is realized by the matrix structure

$$B = \begin{pmatrix} b_1 + ib_4 & b_2 \\ b_3 & b_1 - ib_4 \end{pmatrix}, \qquad \Im b_k = 0, \qquad k = 1, \ldots, 4. \tag{14}$$

Furthermore, Q can be excluded from (13) to obtain the interlinking constraint

$$U = RBR^{-1} - R(R^\sharp)'. \tag{15}$$

Introducing the notation $N := R^{-1}R'$ and substituting (15) into the symmetry relation $U = U^\sharp$ yields the additional constraint

$$\big[B, K_1^{-1}\big] = N^\sharp - N. \tag{16}$$

From equation (12) one finds

$$N = i\gamma' I + \begin{pmatrix} -q & 0 \\ f & q \end{pmatrix},$$

$$q = \frac{1}{2}\left(\frac{\alpha_0'}{\alpha_0} - \frac{\alpha_1'}{\alpha_1}\right), \tag{17}$$

$$f = -\frac{\alpha_1}{2}\left[\frac{\alpha_0'}{\alpha_0}(1 + i\tan\varepsilon) + i\frac{\varepsilon'}{\cos^2\varepsilon}\right] \tag{18}$$

so that (16) transforms to

$$\alpha_1 \begin{pmatrix} b_2 & 0 \\ -2ib_4 & -b_2 \end{pmatrix} = -2i\gamma' I + \begin{pmatrix} 2q & 0 \\ f^* - f & -2q \end{pmatrix}.$$

Finally, one arrives at the following restrictions on the phase γ and the components b_2 and b_4 of the matrix B:

$$\gamma' = 0, \qquad b_2 = \frac{2q}{\alpha_1}, \qquad b_4 = \frac{\Im f}{\alpha_1} = -\frac{1}{2}\left(\frac{\alpha_0'}{\alpha_0}\tan\varepsilon + \frac{\varepsilon'}{\cos^2\varepsilon}\right). \tag{19}$$

In summary, one finds that the first four consistency conditions are free of intrinsic contradictions. From the initially eight arbitrary complex-valued functions contained in the matrices R and Q only the three real-valued functions (b_1, b_3, ε) are still undefined. Together with the helical turbulence functions (α_0, α_1) and the constants $(\gamma, E, l_0, l_1) \in \mathbb{R}^2 \times \mathbb{Z}_+^2$ they can be expected to be highly fine-tuned by the remaining two consistency conditions (7) and (10).

These conditions can be strongly simplified with the help of the definitions of U and B in (13), their implications

$$Q^\sharp - (R^\sharp)' = R^{-1}U, \tag{20}$$

$$(Q^\sharp)' - (R^\sharp)'' = -R^{-1}R'R^{-1}U + R^{-1}U',$$

$$Q = (R^\sharp)^{-1}B \tag{21}$$

and the relations $RR^\sharp = K_0$, $R^\sharp R = K_1$. According to equations (5), (8), one finds that (7) and (10) transform to the matrix Riccati equations (MREs)

$$U' = M_0 - UK_0^{-1}U, \tag{22}$$
$$B' = -M_1 + BK_1^{-1}B. \tag{23}$$

These MREs can be linearized by an ansatz [12, 13]

$$U = VW^{-1}, \qquad V, W \in \mathbb{C}^{2\times 2}, \qquad \det(W) \neq 0,$$
$$B = XY^{-1}, \qquad X, Y \in \mathbb{C}^{2\times 2}, \qquad \det(Y) \neq 0$$

to give the equation systems

$$\begin{pmatrix} V' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & M_0 \\ K_0^{-1} & 0 \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}, \qquad \begin{pmatrix} X' \\ Y' \end{pmatrix} = -\begin{pmatrix} 0 & M_1 \\ K_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{24}$$

The 4×2 matrices $\begin{pmatrix} V \\ W \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{C}^{4\times 2}$ are defined up to $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$-transformations

$$\begin{pmatrix} \tilde V \\ \tilde W \end{pmatrix} = \begin{pmatrix} VG_0 \\ WG_0 \end{pmatrix}, \qquad \begin{pmatrix} \tilde X \\ \tilde Y \end{pmatrix} = \begin{pmatrix} XG_1 \\ YG_1 \end{pmatrix}, \qquad G_0, G_1 \in GL(2,\mathbb{C})$$

and can be interpreted as homogeneous coordinates of two points on a complex Grassmann manifold $G_2(\mathbb{C}^4)$ which consists of 2-dimensional complex subspaces in $\mathbb{C}^4$ (see, e.g. [12, 13]). The matrices $U = VW^{-1}$ and $B = XY^{-1}$ are the corresponding affine coordinates of these points.

By differentiating (24) and substituting $V = K_0 W'$, $X = -K_1 Y'$ it is easily seen that the equation systems (24) are equivalent to the second-order matrix differential equations

$$(\partial_r K_0 \partial_r - M_0)\, W = 0,$$
$$(\partial_r K_1 \partial_r - M_1)\, Y = 0. \tag{25}$$

This implies that the matrices $\tilde W = r^{-1} W$, $\tilde Y = r^{-1} Y$ should be formal (non-normalized) solutions of the eigenvalue equations for the dynamo operator matrices $\hat H_{l_0}[\alpha_0]$, $\hat H_{l_1}[\alpha_1]$, respectively

$$\hat H_{l_0}[\alpha_0]\tilde W = E\tilde W, \qquad \hat H_{l_1}[\alpha_1]\tilde Y = E\tilde Y.$$

Similar like in QM models, the intertwining operator matrix $\hat A$ can be expressed in terms of W or Y. With the help of (20), (21) and (24) one finds

$$\hat A = R\left(ip - Y'Y^{-1}\right) = \left(ip + K_0 W' W^{-1} K_0^{-1}\right) R.$$

Additionally, the interlinking constraint (15) induces a product invariant for the matrices W and Y. The latter can be obtained by substitution of

$$U = RR^\sharp W' W^{-1}, \qquad B^\sharp = -(Y^\sharp)^{-1}(Y^\sharp)' R^\sharp R$$

into the slightly modified version of the interlinking constraint (15)

$$U = RB^\sharp R^{-1} - R(R^\sharp)'.$$

As intermediate relation one obtains

$$W'W^{-1} = -(R^\sharp)^{-1}(Y^\sharp)^{-1}(Y^\sharp)' R^\sharp - (R^\sharp)^{-1}(R^\sharp)'$$

which is of Lie algebra conjugation type $g = g_1 n$, $(\partial_r g)g^{-1} = (\partial_r g_1)g_1^{-1} + g_1(\partial_r n)n^{-1}g_1^{-1}$ and which can be integrated to yield the product invariant

$$Y^\sharp R^\sharp W = C, \qquad \det(C) \neq 0,$$

with C a constant non-singular matrix.

So far, the intertwining technique of pseudo-Hermitian QM is generalized to the η-symmetric dynamo operator model. It remains to test whether the MREs of this model are consistent.

4 No-go theorem

In order to test the pair of MREs (22), (23) for consistency, one can use equations (15), (16) as well as the relation

$$N + K_1^{-1} N^\sharp K_1 = K_1^{-1} K_1' = K_1'$$

and transform the MRE for U (equation (22)) into an equivalent MRE for B. As result, one arrives at the following pair of MREs

$$B' = R^{-1} M_0 R - K_1^{-1} BB + BK_1' + \left[NN^\sharp + (N^\sharp)'\right] K_1, \tag{26}$$

$$B' = -M_1 + BK_1^{-1}B, \tag{27}$$

which should be satisfied simultaneously. The corresponding consistency test will be performed in two steps:

1. From the limiting behavior at $r \to 0$ a relation between l_0 and l_1 will be derived.
2. From equations (26), (27) a system of non-linear ODEs will be extracted for the helical turbulence functions α_0, α_1 and for the components $b_1, \ldots, b_4$ of the matrix B. By mutual substitutions of these ODEs an inconsistency will be found which can be interpreted as a no-go theorem.

4.1 Limiting behavior at $r \to 0$

From the assumed non-singular behavior of the helical turbulence functions at $r \to 0$ it follows that they can be approximated as

$$\alpha_{0,1}(r \to 0) \approx c_{0,1} + a_{0,1} r + \mathcal{O}\left(r^2\right), \qquad c_{0,1} \neq 0.$$

Substituting this approximation in a slightly rewritten version of the defining equation (25) for the matrix Y

$$\left[I\partial_r^2 - \alpha_1' \sigma_- \partial_r - \frac{l_1(l_1+1)}{r^2} I - \begin{pmatrix} E & -\alpha_1 \\ \alpha_1 & E - \alpha_1^2 \end{pmatrix}\right] Y = 0 \tag{28}$$

one obtains the estimate

$$Y(r \to 0) \approx r^{-l_1} \left(I + \frac{a_1}{2}\sigma_- r + \mathcal{O}\left(r^2\right)\right) \left(r^{2l_1+1} C_+ + C_-\right),$$

where C_+, C_- are arbitrary non-singular constant matrices $\det(C_\pm) \neq 0$. Correspondingly, it holds

$$Z := Y' Y^{-1} \approx -l_1 r^{-1} I + \frac{a_1}{2}\sigma_- + \mathcal{O}(r), \tag{29}$$

$$B = -K_1 Y' Y^{-1} \approx l_1 r^{-1} I - \left[c_1 l_1 r^{-1} + (l_1 + 1/2) a_1\right] \sigma_- + \mathcal{O}(r). \tag{30}$$

Comparison of (30) with (14) shows that the components b_2 and b_4 of the matrix B vanish at least as

$$b_2, b_4 \approx \mathcal{O}(r) \qquad \text{for} \quad r \to 0.$$

Furthermore, one finds with the help of equations (17), (18) and (19) that $q \approx \mathcal{O}(r)$ and, hence, $a_0/c_0 = a_1/c_1$, as well as $q', f, f' \approx \mathcal{O}(1)$ what implies $N, N^\sharp, (N^\sharp)' \approx \mathcal{O}(1)$.

Now, a partial consistency test of (26) and (27) can be performed by comparing the singular terms of these equations in the vicinity of the origin $r = 0$. From the MREs (26) and (27) one finds

$$-K_1^{-1}K_1'Z - Z' = \frac{l_0(l_0+1)}{r^2}I - K_1^{-1}ZK_1Z - ZK_1' + \mathcal{O}(1), \tag{31}$$

$$-K_1^{-1}K_1'Z - Z' = -\frac{l_1(l_1+1)}{r^2}I + ZZ + \mathcal{O}(1), \tag{32}$$

respectively. Substituting Z from (29) and equating the coefficients of the r^{-2}-, r^{-1}-terms one obtains from equation (31)

$$l_1 = l_0 + 1, \qquad a_1 = 0$$

and, hence, also $a_0 = 0$. Equation (32) is automatically satisfied, because Y is defined by the corresponding linearized equation (28). The incremental relation $l_1 = l_0 + 1$ is well known from ladder operator constructions for spherically symmetric Hamiltonians in QM [4]. This is not surprising, because this ladder operator construction can be recovered from the intertwining construction (2) for the α^2-dynamo operator matrices by the two-step transition: 1. $\alpha_0 = \alpha_1 = \alpha$, 2. $\alpha \to 0$.

4.2 Systems of coupled non-linear ODEs and their inconsistency

The system of eight coupled non-linear ODEs for the components $b_1, \ldots, b_4$ of the matrix B is easily obtained from the MREs (26), (27), e.g. with the help of the matrix multiplication package of MATHEMATICA©. For the analysis it is sufficient to consider only the simplest four equations of this system, i.e. the σ_+ and I projections of (26) and (27):

$$b_2' = 2b_1b_2 + \alpha_1(1 + b_2^2) \tag{33}$$

$$= -2b_1b_2 - \frac{\alpha_0^2}{\alpha_1}, \tag{34}$$

$$b_1' = b_1^2 + b_2b_3 - b_4^2 - E - \frac{l_1(l_1+1)}{r^2} + \alpha_1b_1b_2, \tag{35}$$

$$= -b_1^2 - b_2b_3 + b_4^2 + E + \frac{l_0(l_0+1)}{r^2} - \alpha_1'b_2 + \frac{\alpha_0^2}{2} + q' - q^2. \tag{36}$$

Equating the right-hand sides of (33), (34) and using $b_2 = 2q/\alpha_1$ from (19) one expresses b_1 as

$$b_1 = -\frac{4q^2 + \alpha_0^2 + \alpha_1^2}{8q}. \tag{37}$$

Taking into account that $q = \partial_r \ln(\alpha_0/\alpha_1)/2$ according to equation (17) and that the helical turbulence functions α_0 and α_1 do not depend on l_0 or l_1 one concludes from equation (37) that b_1 should not depend on l_0 or l_1 too. On the other hand, addition of (35) and (36) together with the relation $l_0 = l_1 - 1$ gives

$$2b_1' = -\frac{2l_1}{r^2} + 2q\left(b_1 - \frac{\alpha_1'}{\alpha_1}\right) + \frac{\alpha_0^2}{2} + q' - q^2$$

what by integration leads to a function b_1 which depends on l_1. I.e. the term depending on l_1 cannot be compensated by a combination of l_1-independent terms. This is an obvious contradiction to (37) and one has to conclude that the consistency conditions (5)–(10) cannot be fulfilled simultaneously. This means that it holds following

No-go theorem. *The structure of the MHD α^2-dynamo operator matrix is incompatible with an operator intertwining technique which is based on first-order differential intertwining operators.*

A similar situation occurs also for three-dimensional spherically symmetric models in QM [4]. There the l-dependent centrifugal term sets so strong restrictions on the form of the allowed potential that an intertwining construction built on first-order differential intertwining operators is only possible for the following three cases: the constant potential $V(r) = \text{const}$, the Coulomb potential $V(r) \propto 1/r$, and the potential of the three-dimensional isotropic harmonic oscillator with $V(r) \propto r^2$. Richer classes of allowed potentials are only found for models in their s states, when $l = 0$. Such states are *a priori* excluded for the α^2-dynamo operator matrix due to its construction [2].

Acknowledgements

The author thanks F. Stefani for fruitful collaboration, and G. Gerbeth, C. Tretter and M. Znojil for useful discussions. The project was supported by the German Research Foundation, DFG, under grant GE-682/12-1.

[1] Bender C.M. and Boettcher S., Real spectra in non-Hermitian Hamiltonians having PT symmetry, *Phys. Rev. Lett.*, 1998, V.80, N 24, 5243–5246; physics/9712001;
Znojil M., PT-symmetric harmonic oscillators, *Phys. Lett. A*, 1999, V.259, 220–223; quant-ph/9905020;
Bender C.M., Brody D.C. and Jones H.F., Complex extension of quantum mechanics, *Phys. Rev. Lett.*, 2002, V.89, N 27, 270401; quant-ph/0208076.

[2] Günther U. and Stefani F., Isospectrality of spherical MHD dynamo operators: pseudo-Hermiticity and a no-go theorem, *J. Math. Phys.*, 2003, V.44, N 6, 3097–3111; math-ph/0208012.

[3] Witten E., Constraints on sypersymmetry breaking, *Nucl. Phys. B*, 1982, V.202, 253–316;
Gendenshtein L.E. and Krive I.V., Supersymmetry in quantum mechanics, *Usp. Fiz. Nauk*, 1985, V.146, 553–590 (*Sov. Phys. Usp.*, 1985, V.28, 645–666);
Cooper F., Khare A. and Sukhatme U., Supersymmetry and quantum mechanics, *Phys. Rep.*, 1995, V.251, 267–385; hep-th/9405029.

[4] De Lange O.L. and Raab R.E., Operator methods in quantum mechanics, Oxford, Clarendon Press, 1991.

[5] Mostafazadeh A., Pseudo-Hermiticity versus PT symmetry: the necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian, *J. Math. Phys.*, 2002, V.43, 205–214; math-ph/0107001.

[6] Mostafazadeh A., Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians, *Nucl. Phys. B*, 2002, V.640, 419–434; math-ph/0203041.

[7] Stefani F. and Gerbeth G., A toy model for inverse dynamo theory, *Phys. Earth Planet. Inter.*, 2001, V.128, 109–124.

[8] Krause F. and Rädler K.-H., Mean-field magnetohydrodynamics and dynamo theory, Oxford, Pergamon Press, 1980.

[9] Mostafazadeh A., Generalized PT-, C-, and CPT-symmetries, position operators, and localized states of Klein–Gordon fields, quant-ph/0307059.

[10] Langer H. and Tretter C., Diagonalization of certain block operator matrices and applications to Dirac operators, *Operator Theory: Adv. Appl.*, 2001, V.122, 331–358.

[11] Günther U., Jakubsky V. and Znojil M., PT-symmetric quantum mechanics and MHD dynamo operators: a unified approach, in preparation.

[12] Zelikin M.I., Control theory and optimization I. Homogeneous spaces and the Riccati equation in the calculus of variations, Berlin, Springer, 2000.

[13] Anderson R.L., Harnad J. and Winternitz P., Superposition principles for matrix Riccati equations, *J. Math. Phys.*, 1983, V.24, 1062–1072.