Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 685–691

Supplementary Solutions for Quasi-Exactly Solvable One-Dimensional Equations

Nathalie DEBERGH † *and Bruno VAN DEN BOSSCHE* ‡

† *Techniques du Son et de l'Image, Montéfiore Institute (B28), University of Liège, B4000 Liège, Belgique*
E-mail: *Nathalie.Debergh@ulg.ac.be*

‡ *B4000 Liège, Belgique*

Quasi-exactly solvable one-dimensional Schrödinger equations can be specified in order to exhibit supplementary analytic eigenstates. While the usual solutions are preserved by the $sl(2,\mathbb{R})$ generators, the additional ones are stabilized at the level of the universal enveloping algebra of this Lie structure. We discuss the square-integrability, the orthogonality of these supplementary solutions as well as the reality of the corresponding energies.

1 Introduction

Since Schrödinger wrote his famous equation in 1926, there has been a constant effort to find different methods for solving it. The present stage of the problem is that one-dimensional time-independent Schrödinger equations can be divided into three categories according to their degree of solvability. There are the *exactly solvable* ones for which the whole set of eigenvalues and eigenstates can be analytically determined, the *non solvable* ones which require a numerical or a perturbative treatment and the intermediate category, the *quasi-exactly solvable* (Q.E.S.) ones [1], for which a finite number of analytic solutions are known. The first explicit example of Q.E.S. potentials was found by Razavy [2].

Exactly solvable as well as most of the Q.E.S. Schrödinger equations are built up from linear and quadratic combinations of the $sl(2,\mathbb{R})$ generators [3]. For example, the Razavy potential illustrates this statement. What we want to emphasize here is that we can constrain in a specific way these combinations [4, 5] so that supplementary analytic eigenstates (and energies) arise. More precisely, the usual solutions are preserved by each of the $sl(2,\mathbb{R})$ generators while the supplementary ones are stabilized by adequate elements of the universal enveloping algebra of $sl(2,\mathbb{R})$. Moreover, the whole set of eigenstates exhibit interesting physical properties and call in question the self-adjointness of the corresponding Hamiltonians.

In Section 2, we prove how to construct a Q.E.S. operator such that it admits supplementary eigenstates with respect to the standard ones. We illustrate these developments on a Q.E.S. interaction generalizing the Razavy potential in Section 3. Finally, we discuss in Section 4 the square-integrability, the orthogonality of these supplementary solutions, as well as the reality of the corresponding energies.

2 Supplementary solutions to Q.E.S. equations

It is well known [3] that the $(n+1)$-dimensional space of monomials

$$V_1 = \{1, y, \dots, y^n\}, \tag{1}$$

where n is a positive integer, is preserved by the differential operators

$$j_+ = -y^2\frac{d}{dy} + ny, \tag{2}$$

$$j_0 = y\frac{d}{dy} - \frac{n}{2}, \tag{3}$$

$$j_- = \frac{d}{dy}, \tag{4}$$

in the sense that

$$j_+ y^k = (n-k)\, y^{k+1}, \qquad j_0 y^k = \left(k - \frac{n}{2}\right) y^k, \qquad j_- y^k = k\, y^{k-1}, \qquad k = 0, 1, \ldots, n.$$

The operators (2)–(4) generate the $sl(2,\mathbb{R})$ Lie algebra characterized by the commutation relations

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0,$$

as it can be checked directly.

Moreover, any function of the $sl(2,\mathbb{R})$ generators will still keep the space V_1 invariant. In particular, being interested in the Schrödinger second-order differential operator, we consider the most general operator preserving V_1 i.e.

$$T = C_{+0} j_+ j_0 + C_{00} j_0^2 + C_{0-} j_0 j_- + C_+ j_+ + C_0 j_0 + C_- j_- + C_*. \tag{5}$$

In this expression, we have suppressed, without losing generality, the term $C_{+-} j_+ j_-$ due to the fact that the $sl(2,\mathbb{R})$ Casimir operator is given by

$$C = j_+ j_- + j_0^2 - j_0,$$

as well as the two terms $C_{++} j_+^2$ and $C_{--} j_-^2$, since it has been proved in [6] that these terms are redundant with respect to the canonical forms.

The operator (5) is constructed from three operators of different gradings

$$J_+ \equiv j_+ (C_{+0} j_0 + C_+), \qquad J_- \equiv (C_{0-} j_0 + C_-) j_-$$

of respective gradings $+1$ and -1 and the diagonal (i.e. of grading 0)

$$J_0 \equiv C_{00} j_0^2 + C_0 j_0 + C_*.$$

Let us now ask for supplementary elements, say $\lambda_0(y), \lambda_1(y), \ldots, \lambda_{N-n-1}(y)$ with N a positive integer such that $N \geq n+1$, to be preserved by T. In other words, we do not limit ourselves to the standard Q.E.S. space V_1 but we add a $(N-n)$-dimensional space which we require in direct sum with V_1 without loss of generality (the dimension being adaptable). Following the gradings in (5), we thus ask for

$$\begin{aligned} &J_+ \lambda_r(y) \sim \lambda_{r+1}(y), \qquad r = 0, 1, \ldots, N-n-2; \qquad J_+ \lambda_{N-n-1}(y) = 0, \\ &J_- \lambda_r(y) \sim \lambda_{r-1}(y), \qquad r = 1, 2, \ldots, N-n-1; \qquad J_- \lambda_0(y) = 0. \end{aligned}$$

These relations constrain the reals C_+ and C_- as well as the elements $\lambda_r(y)$ $(r = 0, 1, \ldots, N-n-1)$ in the following way

$$\begin{aligned} &C_+ = \left(1 + \frac{3n}{2} - N - a\right) C_{+0}, \qquad C_- = \left(1 + \frac{n}{2} - a\right) C_{0-}, \\ &\lambda_r(y) = y^{a+r}, \qquad r = 0, 1, \ldots, N-n-1, \end{aligned}$$

where a is an arbitrary real number. Consequently, the operator

$$T = C_{+0} j_+ \left(j_0 + 1 + \frac{3n}{2} - N - a\right) + C_{0-}(j_0 + 1 + \frac{n}{2} - a) j_- + C_{00} j_0^2 + C_0 j_0 + C_* \tag{6}$$

does preserve the $(N+1)$-dimensional space

$$V = V_1 \oplus V_2,$$

where V_1 is given in equation (1) while V_2 is

$$V_2 = \{y^a, y^{a+1}, \ldots, y^{a+N-n-1}\}.$$

With account of equations (2)–(4), the operator (6) simply reads

$$T = F(y)\frac{d^2}{dy^2} + G(y)\frac{d}{dy} + H(y) \tag{7}$$

with

$$\begin{aligned} F(y) &= -C_{+0}y^3 + C_{00}y^2 + C_{0-}y, \\ G(y) &= C_{+0}(N+a-2)y^2 + (C_{00}(1-n) + C_0)y + C_{0-}(1-a), \\ H(y) &= C_{+0}(1+n-N-a)n\,y + \frac{n^2}{4}C_{00} - \frac{n}{2}C_0 + C_*. \end{aligned} \tag{8}$$

The usual way [3] for transforming the operator (7) into a Schrödinger one

$$T \to H = -\frac{d^2}{dx^2} + V(x)$$

is to perform the change of variables

$$x \equiv f(y) \equiv \int \sqrt{\frac{-1}{F(y)}}\, dy \tag{9}$$

as well as the "gauge" transformation

$$\Lambda(x) \equiv \exp\left\{\frac{1}{2}\int^{y(x)} f'(y)\left[F(y)f''(y) + G(y)f'(y)\right] dy\right\}.$$

The potential $V(x)$ is then given by

$$V(x) = \left\{H(y) + \frac{1}{\Lambda(x)}\frac{d\Lambda}{dx}\left[F(y)f''(y) + G(y)f'(y)\right] - \frac{1}{\Lambda(x)}\frac{d^2\Lambda}{dx^2}\right\}_{y\to y(x)}.$$

3 An example

The change of variables (9) together with (8) leads to an elliptic function technically difficult to handle. We avoid this difficulty by choosing

$$C_{+0} = -1, \qquad C_{00} = 2, \qquad C_{0-} = 1$$

which implies that

$$x = -\left(\tanh\frac{y}{2}\right)^2.$$

We also introduce the new parameter a_1

$$a_1 \equiv 2(N-n) + C_0$$

and fix C_* as

$$C_* = \frac{1}{2}(n-N)(a_1+n-N)$$

for esthetical reasons. In this case, we have

$$\Lambda(x) = \exp\left(\frac{1}{2}(2a+a_1-1-N)\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^{-N}\left(\tanh\frac{x}{2}\right)^{a-\frac{1}{2}}$$

and

$$\begin{aligned}V(x) = \frac{1}{32}\Bigg\{ & 7+28a^2-a_1(6+a_1)+32n(n+1)-2N-2N(3a_1+16n)+15N^2 \\ & + 4a(-7+3a_1-8n+N)-8(N+1)(2a+a_1-1-N)\cosh x \\ & + (2a+a_1-1-N)^2\cosh 2x-(8(N-2n+a-2)(N-2n+a)+6)\frac{1}{(\cosh\frac{x}{2})^2} \\ & + 2(4a^2-1)\frac{1}{(\sinh\frac{x}{2})^2}\Bigg\}. \end{aligned} \tag{10}$$

Let us first point out some characteristics of this Q.E.S. potential.

The behavior near the origin is

$$V(x\to 0) \sim \left(a^2-\frac{1}{4}\right)\frac{1}{x^2}. \tag{11}$$

The coefficient in front of $\frac{1}{x^2}$ implies that the potential is attractive and unbounded at the origin if $a \in]-\frac{1}{2},\frac{1}{2}[$ and repulsive otherwise. The cases $a=\pm\frac{1}{2}$ are particular since the potential is not singular anymore. We recover here the Razavy interaction [2]. Up to these values, the problem is thus restricted to a half-line. Moreover equation (11) is such that the particle does not fall on the center [7]. Let us also notice that

$$\lim_{x\to\pm\infty} V(x) = +\infty,$$

a physical eigenstate will thus have to vanish at infinity.

Second we turn to the eigenstates which are given by

$$\begin{aligned}\psi_n(x) &= \exp\left(-\frac{1}{2}(2a+a_1-1-N)\left(\cosh\frac{x}{2}\right)^2\right) \\ &\quad \times\left(\cosh\frac{x}{2}\right)^N\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}-a}P_n\left(\left(\tanh\frac{x}{2}\right)^2\right), \\ \tilde{\psi}_{N-n-1}(x) &= \exp\left(-\frac{1}{2}(2a+a_1-1-N)\left(\cosh\frac{x}{2}\right)^2\right) \\ &\quad \times\left(\cosh\frac{x}{2}\right)^N\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}+a}\tilde{P}_{N-n-1}\left(\left(\tanh\frac{x}{2}\right)^2\right),\end{aligned}$$

where $P_n(u)$ and $\tilde{P}_{N-n-1}(u)$ are polynomials of respective orders n and $(N-n-1)$ of the real variable u. The untilded solutions are those belonging to the standard Q.E.S. space V_1 while the tilded ones refer to the additional space V_2.

In what concerns the energies, they can be obtained as solutions of algebraic equations of degree $N+1$. For example, if $N=3$ and $n=1$, they are

$$E_\pm = \frac{5}{2}-a_1\pm\frac{1}{2}\sqrt{a_1^2-8a_1-4a^2+20} \tag{12}$$

and

$$\tilde{E}_{\pm} = \frac{5}{2} - a_1 - 4a + 2a^2 + aa_1 \pm \frac{1}{2}\sqrt{a_1^2 - 8a_1 + 8aa_1 + 12a^2 - 32a + 20} \tag{13}$$

corresponding to the respective eigenstates

$$\psi_{\pm}(x) \sim \exp\left(-\frac{1}{2}(2a + a_1 - 4)\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}-a} \times\left[(a-1) + \left(E_{\pm} + \frac{3}{2}(a_1 - 3)\right)\left(\tanh\frac{x}{2}\right)^2\right], \tag{14}$$

and

$$\tilde{\psi}_{\pm}(x) \sim \exp\left(-\frac{1}{2}(2a + a_1 - 4)\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}+a} \times\left[(a+1) - \left(\tilde{E}_{\pm} + \frac{1}{2}(3 - 2a)(2a + a_1 - 3)\right)\left(\tanh\frac{x}{2}\right)^2\right]. \tag{15}$$

4 Real energies and self-adjointness

A look at equations (12), (13) shows that a particular choice of a and a_1 can lead to complex energies despite of the fact that the potential (10) is always real. This situation is the opposite of the one encountered in [8] where complex potentials lead to real spectra.

We thus have to analyze the self-adjointness of the kinetic term. This point is usually left aside because the radial eigenstates traditionally vanish at the origin. This is no longer true here (see equations (14), (15) for an arbitrary a) and we thus have to come back to the behavior of the functions (14), (15) at the origin.

Let us start with

$$H\psi = E_\psi \psi, \qquad H\phi = E_\phi \phi.$$

Multiplying both sides of these equations by $\phi^\dagger$ and $\psi^\dagger$, respectively, taking the conjugate of the resulting second equation and subtracting them, we obtain after integration and for a real potential, the standard relationship

$$\left(\psi\frac{d}{dx}\phi^\dagger - \phi^\dagger\frac{d}{dx}\psi\right)\Big|_{x_1}^{x_2} = (E_\psi - \bar{E}_\phi)\int_{x_1}^{x_2}\phi^\dagger\psi\, dx. \tag{16}$$

The behavior of the functions (14), (15) at the origin is

$$\psi(x \to 0) \sim A_\psi x^{-\delta} + B_\psi x^{2-\delta} + \cdots, \tag{17}$$

where A_ψ, B_ψ, ... are expansion coefficients while

$$\delta = -\frac{1}{2} + a \qquad \text{or} \qquad \delta = -\frac{1}{2} - a$$

if the spaces V_1 or V_2 are under study, respectively. Inserting (17) in (16), we obtain

$$2(A_\psi \bar{B}_\phi - B_\psi \bar{A}_\phi)\left[x^{1-2\delta} + O(x^{3-2\delta})\right]|_{x\to 0} = (E_\psi - \bar{E}_\phi)\int_0^{+\infty}\phi^\dagger\psi\, dx. \tag{18}$$

If the functions ψ and ϕ belong to the same space, either V_1 or V_2, equation (18) proves that these functions are orthogonal if

$$1 - 2\delta > 0,$$

i.e.

$$a < 1 \quad \text{for } V_1 \quad \text{and} \quad a > -1 \quad \text{for } V_2. \tag{19}$$

Moreover, taking $\psi = \phi$, we deduce from equation (18) that the energies are real. On the other hand, if $1 - 2\delta \leq 0$, neither the orthogonality nor the reality of the energies can be guaranteed. Note that the condition $1-2\delta > 0$ is also the one for the eigenstates (14), (15) (see equation (17)) to be square-integrable at the origin. Besides these eigenstates are square-integrable at infinity if

$$a_1 + 2a - 4 > 0. \tag{20}$$

The constraints (19), (20) thus fix the domains of a and a_1 in order to ensure the reality of the energies as well as the physical relevance of the eigenstates. These constraints are for example met with $a = \frac{1}{4}$, $a_1 = 4$. In this case, we have

$$E_\pm = -\frac{3}{2} \pm \frac{1}{4}\sqrt{15}, \qquad \tilde{E}_\pm = -\frac{11}{8} \pm \frac{1}{4}\sqrt{19}$$

while

$$\psi_\pm = \exp\left(-\frac{1}{4}\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{\frac{1}{4}}\left(3 \pm \sqrt{15}\left(\tanh\frac{x}{2}\right)^2\right), \tag{21}$$

$$\tilde{\psi}_\pm = \exp\left(-\frac{1}{4}\left(\cosh\frac{x}{2}\right)^2\right)\cosh\frac{x}{2}\left(\tanh\frac{x}{2}\right)^{\frac{3}{4}}\left(7 \pm \sqrt{19} + \left(3 \mp \sqrt{19}\right)\cosh x\right). \tag{22}$$

Among these eigenstates, all vanishing at the origin, two (ψ_+ and $\tilde{\psi}_+$) have one node while the two others do not vanish on R_0^+. Another particular case is $a = \frac{1}{2}$, $a_1 = 4$. This implies

$$E_\pm = -\frac{3}{2} \pm \frac{1}{2}\sqrt{3}, \qquad \tilde{E}_\pm = -1 \pm \frac{1}{2}\sqrt{7}$$

and

$$\psi_\pm = \exp\left(-\frac{1}{2}\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(1 \pm \sqrt{3}\left(\tanh\frac{x}{2}\right)^2\right), \tag{23}$$

$$\tilde{\psi}_\pm = \exp\left(-\frac{1}{2}\left(\cosh\frac{x}{2}\right)^2\right)\sinh\frac{x}{2}\left(5 \pm \sqrt{7} + \left(1 \mp \sqrt{7}\right)\cosh x\right). \tag{24}$$

As already noticed, there is no more singularity so that here x belongs to the whole line. The number of nodes is 0, 1, 2 and 3 corresponding to the eigenstates ψ_-, $\tilde{\psi}_-$, ψ_+ and $\tilde{\psi}_+$, respectively. Finally, we choose $a = \frac{3}{4}$, $a_1 = 4$. We obtain

$$E_\pm = -\frac{3}{2} \pm \frac{1}{4}\sqrt{7}, \qquad \tilde{E}_\pm = -\frac{3}{8} \pm \frac{1}{4}\sqrt{43}$$

and

$$\psi_\pm = \exp\left(-\frac{3}{4}\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{-\frac{1}{4}}\left(1 \mp \sqrt{7}\left(\tanh\frac{x}{2}\right)^2\right), \tag{25}$$

$$\tilde{\psi}_\pm = \exp\left(-\frac{3}{4}\left(\cosh\frac{x}{2}\right)^2\right)\cosh\frac{x}{2}\left(\tanh\frac{x}{2}\right)^{\frac{5}{4}}\left(13 \pm \sqrt{43} + \left(1 \mp \sqrt{43}\right)\cosh x\right). \tag{26}$$

The number of respective nodes is kept with respect to the context $a = \frac{1}{4}$, while the eigenstates have a different behavior at the origin: If the tilded functions still vanish when $x \to 0$, $\psi_\pm$ goes to $\mp\infty$ at the origin.

In each of these three cases, beside the reality of the energies, we confirm the square-integrability of each eigenstate (on R_0^+ or R) as well as their orthogonality within the same subspace ($\psi_\pm$ are orthogonal, $\tilde{\psi}_\pm$, too).

Let us now take a look at equation (18) when ψ and ϕ belong to different subspaces, say ψ to V_1 and ϕ to V_2. We have

$$2aA_\psi\bar{A}_\phi\left[1+O\left(x^2\right)\right]|_{x\to 0}=\left(E_\psi-\bar{E}_\phi\right)\int_0^{+\infty}\phi^\dagger\psi\, dx \tag{27}$$

It is clear that the left-hand side of the relations (27) is not vanishing. Hence the eigenstates of two different subspaces are not orthogonal in general. This can be checked directly for the states (21), (22) as well as for the ones in equations (25), (26). However, the situation is special when $a=\pm\frac{1}{2}$. We have $\delta=0$ either for the eigenstates lying in V_1 or for the ones of V_2. This means that the eigenstate at the origin goes to a constant in one case and is vanishing in the other case. One of the subspaces contains the even eigenstates and the second subspace contains the odd ones. These subspaces are now orthogonal to each other. All these statements are confirmed through equations (23), (24).

Acknowledgements

We would like to warmly thank the Organizers for giving us the opportunity to present these results.

[1] Ushveridze A., Quasi-exactly solvable models in quantum mechanics, Bristol, IOP Publishing Ltd, 1994;
Turbiner A.V., Quasi-exactly solvable differential equations, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3, New Trends in Theoretical Developments and Computational Methods, Editor N.H. Ibraginov, 1995;
Shifman M.A., ITEP lectures on particle physics and field theory, Vol. 62, Singapore, World Scientific Publishing Ltd, 1999.

[2] Razavy M., An exactly solvable Schrödinger equation with a bistable potential, *Am. J. Phys.*, 1980, V.48, 285–288;
Razavy M., A potential model for torsional vibrations of molecules, *Phys. Lett. A*, 1981, V.82, 7–9.

[3] Turbiner A.V., Quasi-exactly solvable problems and $sl(2)$ algebra, *Comm. Math. Phys.*, 1988, V.118, 467–474.

[4] Debergh N. and Van den Bossche B., Differential realizations of polynomial algebras in finite-dimensional spaces of monomials, *Ann. Phys.*, 2003, V.308, 605–638.

[5] Debergh N. and Van den Bossche B., Quasi-exactly solvable one-dimensional equations: self-adjointness and supplementary solutions, Preprint ULg, 2003.

[6] Finkel F., Gonzalez-Lopez A. and Rodriguez M.A., Quasi-exactly solvable potentials on the line and orthogonal polynomials, *J. Math. Phys.*, V.37, 1996, 3954–3972.

[7] Landau L. and Lifchitz E., Quantum mechanics, London, Pergamon Press, 1977.

[8] Bender C.M. and Boettcher S., Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry, *Phys. Rev. Lett.*, V.80, 1998, 5243–5246.