# X-ray Scatter-to-Primary Ratio versus Thickness, Two analytic Models Evaluated Against Monte Carlo Calculations

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# 1 Introduction

An exact calculation of scatter can be expressed formally by a transport equation; [1] however, the exact solution is not generally known even for the simplest geometries. Two analytic approaches to scatter modeling, one by Swank [2] and another by Smith and Kruger [3] are tested against the results of Monte Carlo calculations using GEANT4. By treating parameters in the analytic equations as fitting parameters, one obtains a convenient way to parameterize measured data. The comparison to Monte Carlo results allows a match of parameters appearing in the analytic expressions to the physical parameters in the transport theory. Specifically, mono-energetic photons having single absorption and scatter cross-sections are studied. The two analytic approaches, although of very different derivation, yield similar expressions that capture the overall magnitude, the thickness dependence and photon energy dependence of the scatter-to-primary ratio.

# 2 Smith and Kruger 1D Model

Consider first the forward scatter trajectories labeled as process F1 and F2 in Figure 1. For the purposes of the following analysis, one may generalize to a class of trajectories **F(x)**={$F_i$} to include all cases of multiple scatter following the initial scatter event at point s from the top surface. Trajectories of type F are to be distinguished from B where the first event is scattered away from the output surface. Together, F and B represent all possible scattering trajectories. One may expect that the distinction between F and B is not important for cases where s is far from either the top or bottom surface. For thick slabs, each makes an identical contribution to the total scatter. However, for s close to the top surface (i.e. within a scattering length), the trajectories of class B are attenuated because of loss out the top. A complementary situation exists at the bottom surface where

1st backscattered photons are likely to be rescattered and then exit out the bottom surface.

As described by Smith and Kruger (SK), one makes an approximation that all forward scatter processes can be described by a single one-dimensional forward scatter cross-section $\beta\mu_C$ where $0<\beta<1$ is a constant and $\mu_C$ is the total scatter cross-section (including Compton and coherent scattering). Furthermore the attenuation of the scatter from the 1st generation site to the

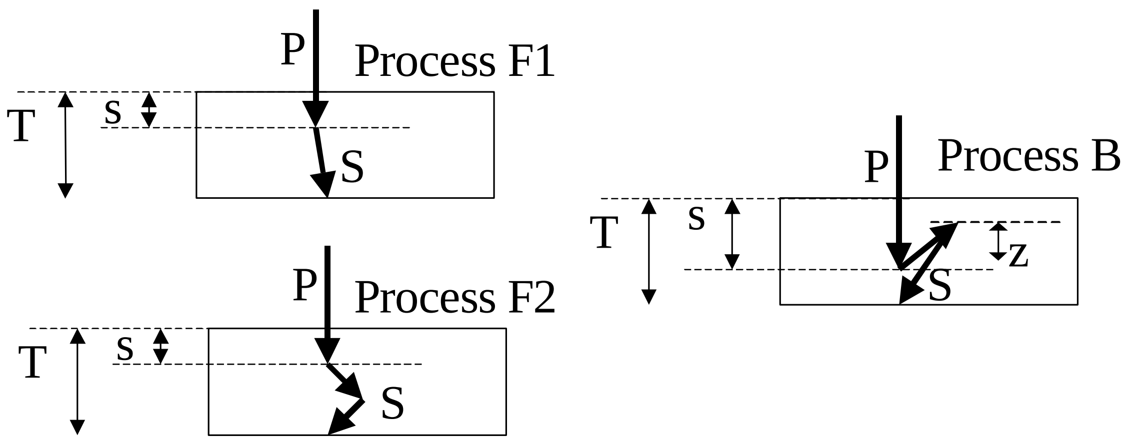


Figure 1: Forward and backscatter processes considered in the 1D model.

output surface, irrespective of the actual path length, is given by $\exp(-\eta\mu_P(T-x))$ where $0<\eta<1$ is a constant and $\mu_P$ is the attenuation constant of the primary. Within the context of the SK model, $\beta$~0.5 corresponding to the part of the total (isotropic) scatter cross-section which is forward scattered. Similarly, one may expect that $\eta\mu_P \sim \mu_P-\beta\mu_C$. That is, the attenuation of the scatter is less than that of the primary by the amount equal to the forward scatter cross-section.

With this approximation, SK obtain the forward scatter by integrating over the thickness as follows:

$$S_F = I_o \int_{s=0}^{T} ds\, \beta\, \mu_C\, e^{-\mu_P s} e^{-\eta\,\mu_P (T-s)} = I_o \left[ \frac{\beta W}{(1-\eta)} \right] e^{-\eta\,\mu_P T} \left( 1 - e^{-(1-\eta)\mu_P T} \right) \quad \textbf{(1)}$$

where $I_o$ is the input intensity at the top surface and $W=\mu_C/\mu_P$ is called generally called the albedo. Within the integrand, the first and second exponential terms account for the attenuation of the primary and scatter, with attenuation constants $\mu_P$ and $\eta\mu_P$ respectively. Note that at small values of $\mu_P T$~0, this

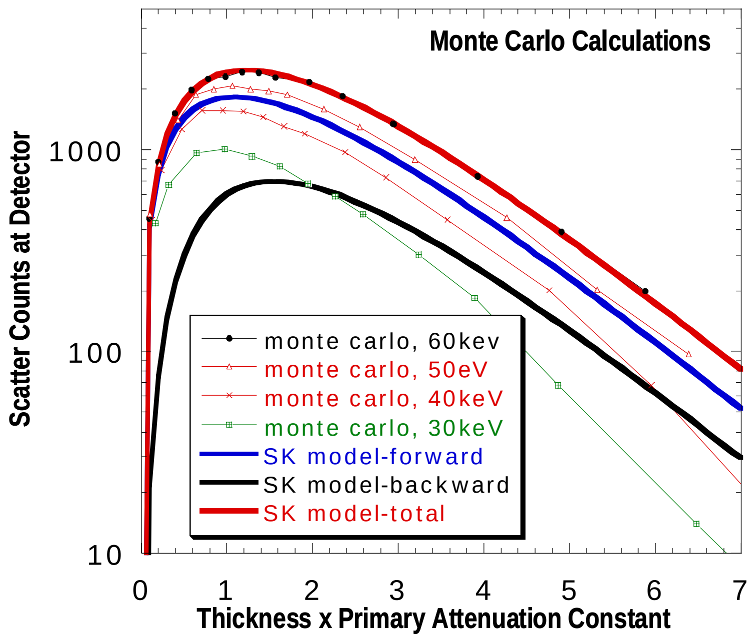


Figure 2 The monte carlo results at 60kev are fit to a modified Smith-Kruger 1d scatter model which includes backscatter, fit parameters are $\eta$=0.80, and c=1.0. The contribution from backscattering at large thickness is 44% of forward contribution

expression makes a linear contribution $S_F = I_o\beta\mu_C T$, which is the expected result given the definition of $\beta\mu_C$ as the forward scatter cross-section for the single scatter case.

Note that in the SK paper the bracketed prefactor $\beta\mu_C /(1-\eta)\mu_P$ is set equal to 1. This is equivalent to making the following identity $\eta\mu_P = \mu_P - \beta\mu_C$, that is the effective scatter attenuation constant is the primary attenuation constant minus the forward scatter contribution. This is an approximation used in the SK paper but causes of the model to underestimate the measured value of scatter by a factor of order 2 or 3. In fact, the nature of the fits to the data at large thicknesses force $\eta$~0.7 to 0.9. Given that $\beta$~0.5 and $\mu_C /\mu_P$ ~0.7 to 0.9, we find the bracketed factor is of the order of 2 or 3 because of the small value of 1-$\eta$.

# 3 Extension of Smith and Kruger model

Referring to the diagram for process B in Figure 1, the contribution from the1st backscattered photons is estimated as follows.

$$S_B = I_o \int_{s=0}^{T} ds\,(1-\beta)\,\mu_C \int_{z=0}^{s} dz\, c\mu_S\; e^{-\mu_p s} e^{-\eta\,\mu_p (s-z)} e^{-\eta\,\mu_p (T-s)}$$
$$= I_o \frac{\beta W}{(1-\eta)} e^{-\eta\,\mu_p T} \left(\frac{1-\beta}{\beta}\right)\left(\frac{c\eta}{1+\eta}\right)\left[1 - \frac{1+\eta}{2\eta} e^{-(1-\eta)\mu_p T} + \frac{1-\eta}{2\eta} e^{-(1+\eta)\mu_p T}\right] \quad \textbf{(2)}$$

A new parameter, c, is introduced to parameterized the effective cross-section for backscattered photons. One can place a constraint on the constant c by restricting the backscattered contribution at large thickness to be equal to or less than the forward one. Consider the expression above at large values of T. $S_B$ makes a contribution similar to the forward one but weighted by the ratio of the forward and backward cross-sections $(1-\beta)/\beta$ and the factor $c\eta/(1+\eta)$. Since for Compton scattering, $\beta \geq 0.5$ (i.e. forward directed at high energy), the former factor is equal to or less then 1. Restricting the backscattered contribution at large thickness to be equal to or less than the forward one suggests that $0 < c \leq (1+\eta)/\eta$. For $c=(1+\eta)/\eta$ the forward and backscattered contributions are equal at large thicknesses. At small values of $\mu_P T < 1$, $S_B$ is of order $(\mu_P T)^2 << 1$ and make little contribution compared to the forward scattering component.

Assuming that $\beta=(1-\beta)=0.5$, the total scatter is obtained by summing the forward and backward contributions calculated above to obtain the formula shown in Equation 3. This is the extension of the SK model and it will be compared to the diffusion model. A fit of this two parameter model to the Monte Carlo results at 60kev is shown in Figure 2 along with the individual forward and backward contributions. The fit parameters are $\eta=0.80$, and $c=1.0$. At large thicknesses the backscattered contribution is 44% of the forward scatter contribution. Fits of similar quality are obtained at other energies with fit parameters shown in table I. Note that for 30 and 40kev there is no backscatter contribution.

The trends with decreasing kev, both the decrease in magnitude and shift in the peak to lower thickness, are reproduced by this two parameter model in a very satisfactory way. This may be expected because the photoelectric cross section is a larger percentage of the total cross section. There is no detailed accounting in this 1d model of the photoelectric versus Compton cross-sections except for the presence of the albedo (i.e. W) as a prefactor in Equation 3. However, this W prefactor does is not sufficient to account for observed change in scatter magnitude The values of the albedo are listed in Table II for different keVs. Rather, within this 1d model it is the change in the backscattered contribution (i.e. the c parameter) which effectively parameterizes the decrease in scatter with decreasing keV.

**Equation 3 The 1d scatter model of Smith and Kruger extended to include backscatter**

$$S_T = I_o \frac{\beta W}{(1-\eta)} e^{-\eta \mu_p T} \left( \frac{1+(1+c)\eta}{1+\eta} \right) \times$$
$$\left[ 1 - \frac{(1+c/2)(1+\eta)}{1+(1+c)\eta} e^{-(1-\eta)\mu_p T} + \frac{c}{2} \left( \frac{1-\eta}{1+\eta} \right) \frac{1+\eta}{1+(1+c)\eta} e^{-(1+\eta)\mu_p T} \right]$$

**Equation 4 The solution to the diffusion equation for slab geometry with perpendicular incidence.**

$$S_T = I_o \frac{W/2}{(1-\eta)} e^{-\eta \mu_p T} \left( \frac{4\xi(1+\xi\eta)}{(1+\xi)^2 (1+\eta)} \right) \times$$
$$\left[ 1 - \frac{(1+\xi)(1+\eta)}{2(1+\xi\eta)} e^{-(1-\eta)\mu_p T} + \frac{1}{2} \left( \frac{1-\eta}{1+\eta} \right) (1-\xi) e^{-(1+\eta)\mu_p T} \right]$$

| | Monte Carlo Params | | | | 1D model Fits | | Diffusion Model Fits | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| kev | μ–photo | μ–comp | μ–total | albedo | η | c | η | ξ | η–expected | ξ–expected |
| 30 | 0.1410 | 0.183 | 0.324 | 0.56 | 1.1 | 0 | 0.9 | 0.2 | 1.14 | 0.847222 |
| 40 | 0.0543 | 0.184 | 0.238 | 0.77 | 0.9 | 0.0 | 0.85 | 0.4 | 0.83 | 1.157773 |
| 50 | 0.0265 | 0.187 | 0.213 | 0.88 | 0.85 | 0.5 | 0.85 | 1.0 | 0.61 | 1.31338 |
| 60 | 0.0151 | 0.181 | 0.196 | 0.92 | 0.8 | 1.0 | 0.85 | 3.0 | 0.48 | 1.384439 |

Table I Parameters used in the Monte Carlo simulation and the fit parameters of analytic (but approximate) scatter models.

# 4 Diffusion Model

The solution of the diffusion equation is obtained by Swank (his equation 23) , and by Ishimaru (his section 9-4)and shown in Equation 4 and involves two parameters, $\eta = \mu_S / \mu_P$ and $\xi = \mu_{tr} / \mu_P$. Note the similarity to Equation 3 derived above from the 1d approximation. In fact, in the limit of $\xi$=1, the expression for forward scatter (Equation 1) is obtained exactly.

The derivation of Equation 4 is involved and will not be repeated here. However, some insight as to these two parameters can be gained from inspection of how they enter into the derivation. The parameter η is present in the attenuation term in the diffusion equation. It represents the effective attenuation length scale of the scatter field in the bulk and parallels equivalent $\eta$ parameter in the 1D model. The second parameter in Equation 4, $\xi$, is the ratio of the "transport cross-section" to the primary cross-section. It appears in only in the boundary condition and characterizes the reciprocal length scale of relaxation of the scatter intensity away from surfaces or sources.

Within the context of the diffusion the two parameters $\eta$, $\xi$ are related to the physical attenuation coefficients for photoelectric and Compton processes.[4]. However, because the diffusion equation is only approximate near surfaces, this relationship is not found to hold for the fits. We treat these as fitting parameters in the model. The functional dependence of these parameters on keV ( or equivalently the albedo) is obtained by fits to the Monte Carlo data as listed in Table II. The quality of the fits is similar to Figure 2. Operationally, the $\eta$ parameter controls the magnitude and shape of the scatter as a function of thickness. The parameter $\xi$ largely scales the magnitude of the curve without much influence on the shape.

As a function of keV, the $\eta$ parameter is close to 0.85, a value similar to that found in the 1D model. This suggests a consistent picture for this parameter, that the attenuation of the scatter field is slightly less than that of the primary field.

The functional dependence on object thickness is strongly controlled by this parameter.

The $\xi$ parameter (and c parameter in the 1D model) increases with increasing kvp. This effectively parameterizes the shrinking role of the top boundary on the scatter field. At low kvP, the stronger absorptive attenuation of the primary means that much of the scatter is generated near the top surface of the object and much of this escapes.

[1] A. Ishimaru, Wave Propagation and Scattering in Random Media, Academic Press, Inc. N.Y. 1978; S. Chandrasekhar, Radiative Transfer, Dover Pub. N.Y. 1960.

[2] R.K. Swank, Appl. Optics, 12, 1865 (1973).

[3] S.W. Smith and R.A. Kruger, Med. Phys. 13, 831 (1986).

[4] The expected relationships are $\eta$=sqrt(3 $\mu_a$ /$\mu_P$) and $\xi$=3/2, where $\mu_a$ is the cross-section for photoelectric absorption and $\mu_P$ is the total attenuation coefficient of the primary, that is the sum of absorption and scattering.