

# The Maxwell–Dirac Equations, Some Non-Perturbative Results

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In this talk I will review some recent work on the Maxwell–Dirac equations. This system of equations can be thought of as the classical equations for electronic matter, the quantisation of which yields that most successful of physical theories, QED. The talk will focus on qualitative, non-perturbative properties of this highly non-linear system of equations. We will be particularly interested in properties which might be used to describe a single isolated electron.

## 1 Introduction

The Maxwell–Dirac system consists of the Dirac equation

$$\gamma^\alpha(\partial_\alpha - ieA_\alpha)\psi + im\psi = 0, \quad (1)$$

with electromagnetic interaction given by the potential  $A_\alpha$ ; and the Maxwell equations (sourced by the Dirac current,  $j^\alpha$ ),

$$\begin{aligned} F_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha, \\ \partial^\alpha F_{\alpha\beta} &= -4\pi e j_\beta = -4\pi e \bar{\psi} \gamma_\beta \psi. \end{aligned} \quad (2)$$

Most studies of the Dirac equation treat the electromagnetic field as given and ignore the Dirac current as a source for the Maxwell equations, i.e. these treatments ignore the electron “self-field”. A comprehensive survey of these results can be found in the book by Thaller [1]. This is not surprising, inclusion of the electron self-field via the Dirac current leads to a very difficult, highly non-linear set of partial differential equations. So difficult in fact that the existence theory and solution of the Cauchy problem for small initial data was only solved in 1997 (Gross [2], Chadam [3], Georgiev [4], Esteban *et al* [5], Bournaveas [6], and Flato, Simon and Taffin [7]) – seventy years after Dirac first wrote down his equation!

There are no known non-trivial, exact solutions to the Maxwell–Dirac equations in  $1 + 3$  dimensions – all known solutions involve some numerical work. These solutions do, however, exhibit interesting non-linear behaviour which would not have been apparent through perturbation expansions. The particular solutions found in [8] and [9] exhibit just this sort of behaviour – localisation and charge screening. See also Das [10] and the recent work of Finster, Smoller and Yau [11].

Finster, Smoller and Yau also point out in [12] that solving the system (Einstein–Maxwell–Dirac system in their case) gives, in effect, all the Feynman diagrams of the quantum field theory, with the exception of the fermionic loop diagrams. Study of the Maxwell–Dirac system should provide an interesting insight into non-perturbative QED.

In the discussion which follows we will focus on two broad reductions of the equations, the static case (including the spherically symmetric sub-case) and the stationary case. Precise statements of theorems will be given, however only brief indications as to the methods of proof are supplied – details can be found in the original papers cited in the bibliography.

## 2 The Maxwell–Dirac equations

In [8] the 2-spinor form of the Dirac equations was employed to solve (1) for the electromagnetic potential, under the non-degeneracy condition  $j^\alpha j_\alpha \neq 0$ . Requiring  $A^\alpha$  to be a real four-vector then gave a set of partial differential equations in the Dirac field alone, *the reality conditions*.

For 2-spinors  $u_A$  and  $v^B$  (see [13] for an exposition of the 2-spinor formalism) we have

$$\psi = \begin{pmatrix} u_A \\ \bar{v}^{\dot{B}} \end{pmatrix}, \quad \text{with} \quad u_C v^C \neq 0 \quad (\text{non-degeneracy}),$$

where  $A, B = 0, 1$ ,  $\dot{A}, \dot{B} = \dot{0}, \dot{1}$  are two-spinor indices. The *Dirac equations* are

$$\begin{aligned} (\partial^{A\dot{A}} - i e A^{A\dot{A}}) u_A + \frac{im}{\sqrt{2}} \bar{v}^{\dot{A}} &= 0, \\ (\partial^{A\dot{A}} + i e A^{A\dot{A}}) v_A + \frac{im}{\sqrt{2}} \bar{u}^{\dot{A}} &= 0, \end{aligned} \quad (3)$$

where  $\partial^{A\dot{A}} \equiv \sigma^{\alpha A\dot{A}} \partial_\alpha$ ,  $A^{A\dot{A}} = \sigma^{\alpha A\dot{A}} A_\alpha$ ; here  $\sigma^{\alpha A\dot{A}} A_\alpha$  are the Infeld-van der Waerden symbols. The *electromagnetic potential* is (see [8] for details),

$$A^{A\dot{A}} = \frac{i}{e(u^c v_c)} \left\{ v^A \partial^{B\dot{A}} u_B + u^A \partial^{B\dot{A}} v_B + \frac{im}{\sqrt{2}} (u^A \bar{u}^{\dot{A}} + v^A \bar{v}^{\dot{A}}) \right\}. \quad (4)$$

The *reality conditions* are,

$$\begin{aligned} \partial^{A\dot{A}} (u_A \bar{u}_{\dot{A}}) &= -\frac{im}{\sqrt{2}} (u^C v_C - \bar{u}^{\dot{C}} \bar{v}_{\dot{C}}), \\ \partial^{A\dot{A}} (v_A \bar{v}_{\dot{A}}) &= \frac{im}{\sqrt{2}} (u^C v_C - \bar{u}^{\dot{C}} \bar{v}_{\dot{C}}), \\ u_A \partial^{A\dot{A}} \bar{v}_{\dot{A}} - \bar{v}_{\dot{A}} \partial^{A\dot{A}} u_A &= 0. \end{aligned} \quad (5)$$

The Maxwell equations are,

$$\partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta = -4\pi e \sqrt{2} \sigma_\beta^{A\dot{A}} (u_A \bar{u}_{\dot{A}} + v_A \bar{v}_{\dot{A}}). \quad (6)$$

The equations (4), (5) and (6) are entirely equivalent to the original Maxwell–Dirac equations, (1) and (2).

## 3 The static Maxwell–Dirac equations

A Maxwell–Dirac system is said to be *static* if there exists a Lorentz frame in which the Dirac current vector is purely timelike, i.e.  $j^\alpha = j^0 \delta_0^\alpha$ , in this Lorentz frame there is no current flow.

As noted in [8] this definition implies,

$$v^A = e^{i\chi} \sqrt{2} \sigma^{0A\dot{A}} \bar{u}_{\dot{A}}, \quad \text{with } \chi \text{ a real function.}$$

The gauge may be fixed (see [8]) by the choice,

$$u^0 = X e^{\frac{i}{2}(\chi+\eta)}, \quad u^1 = Y e^{\frac{i}{2}(\chi-\eta)},$$

with  $X$ ,  $Y$ , and  $\eta$  real functions on  $\mathbb{R}^4$ .

Defining the null vector  $L$ ,

$$L = \left( \sigma_{AA'} u^A \bar{u}^{\dot{A}} \right) = \left( L^0, \frac{1}{\sqrt{2}} \mathbf{V} \right), \quad \text{with} \quad L^0 = \frac{1}{\sqrt{2}} (X^2 + Y^2) \quad \text{and}$$

$$\mathbf{V} = (2XY \cos \eta, 2XY \sin \eta, X^2 - Y^2),$$

our equations become,

$$\begin{aligned} \frac{\partial}{\partial t} (X^2 + Y^2) &= 0, \\ \nabla \cdot \mathbf{V} &= -2m (X^2 + Y^2) \sin \chi, \\ \frac{\partial \mathbf{V}}{\partial t} + (\nabla \chi) \times \mathbf{V} &= \mathbf{0}. \end{aligned} \tag{7}$$

With electromagnetic potential

$$A^0 = \frac{m}{e} \cos \chi + \frac{(X^2 - Y^2)}{2e(X^2 + Y^2)} \frac{\partial \eta}{\partial t} + \frac{(\nabla \chi) \cdot \mathbf{V}}{2e(X^2 + Y^2)},$$

$$\mathbf{A} = \frac{1}{2e(X^2 + Y^2)} \left[ \frac{\partial \chi}{\partial t} \mathbf{V} + (X^2 - Y^2) \nabla \eta - \nabla \times \mathbf{V} \right], \quad \text{where} \quad \mathbf{A} = (A^1, A^2, A^3). \tag{8}$$

The full system is given by the above two sets of equations and the Maxwell equations.

Further simplification can be made to the system by imposing the stationary condition: A Maxwell–Dirac system is said to be *stationary* if there is a gauge in which  $\psi = e^{i\omega t} \phi$ , with the bi-spinor  $\phi$  independent of  $t$ . Such a gauge will be referred to as a stationary gauge. We will be examining isolated, stationary, static systems in Section 3.2. A stationary gauge is not unique.

### 3.1 Spherical symmetry

Spherical symmetry of the stationary and static Maxwell–Dirac system is imposed (in a gauge independent way) by demanding that the null vector  $L$ , defined above, is spherically symmetric. This has the following consequences, in terms of spherical polar coordinates,

$$X = \sqrt{R} \cos(\theta/2), \quad Y = \sqrt{R} \sin(\theta/2), \quad \text{and} \quad \eta = \phi.$$

The equations are

$$\begin{aligned} \mathbf{A} &= \frac{1}{2e} \frac{\cot \theta}{r} \hat{\phi}, \quad A^0 = \frac{m}{e} \cos \chi + \frac{1}{2e} \frac{d\chi}{dr}, \\ \frac{d}{dr} (r^2 R) &= -2mr^2 R \sin \chi, \quad \frac{d}{dr} \left( r^2 \frac{dA_0}{dr} \right) = 8\pi e r^2 R, \end{aligned}$$

with  $\chi$  and  $R$  functions of  $r$  only. The Dirac field is

$$\psi = \sqrt{R} \begin{pmatrix} -e^{\frac{i}{2}(\chi-\phi)} \sin\left(\frac{\theta}{2}\right) \\ e^{\frac{i}{2}(\chi+\phi)} \cos\left(\frac{\theta}{2}\right) \\ -e^{\frac{-i}{2}(\chi+\phi)} \sin\left(\frac{\theta}{2}\right) \\ e^{\frac{-i}{2}(\chi-\phi)} \cos\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

The first thing one notices is that there is a central magnetic monopole, with Dirac magnetic charge  $\frac{1}{2e}$ . In fact, we can obtain a reasonably complete characterisation of these solutions [8]. Briefly, under quite weak (physically reasonable) assumptions, we find that the solutions can be thought of as a central magnetically and electrically charged point source (external to the Dirac

field) surrounded by an electrically (oppositely) charged Dirac field. Near  $\infty$  the electrostatic potential behaves as  $A^0 \sim -\frac{m}{e} + \frac{1/(me)}{r^2}$  and near  $r = 0$  the potential behaves as  $A^0 \sim -\frac{m}{e} + \frac{\gamma/e}{r}$  (for some constant  $\gamma$ ).

The object is highly compact, with a radius of about  $1/m$  a (reduced) Compton wavelength. Inside this radius it has an onion like structure consisting of an infinite series of spherical shells. The system is electrically neutral, with the central Coulomb point source effectively screened by the Dirac field for  $r > 1/m$ .

### 3.2 Isolated systems

In most physical processes that we would wish to model using the Maxwell–Dirac system we would be interested in isolated systems – systems where the fields and sources are largely confined to a compact region of  $\mathbb{R}^3$ . This requires that the fields ‘die-off’ sufficiently quickly as  $|x| \rightarrow \infty$ .

The best language for the discussion of such decay conditions and other regularity issues is the language of weighted function spaces; specifically weighted classical and Sobolev spaces. In [14] the weighted Sobolev spaces  $W_\delta^{k,p}$  are used following the definitions of [15]. These definitions have the advantage that the decay rate is explicit: under appropriate circumstances a function in  $W_\delta^{k,p}$  behaves as  $|x|^\delta$  with  $|x| \rightarrow \infty$ . An element  $f$  of  $W_\delta^{k,p}$  has  $\sigma^{-\delta+|\alpha|-\frac{3}{p}}\partial^{|\alpha|}f$  in  $L^p$  for each multi-index  $\alpha$  for which  $0 \leq |\alpha| \leq k$ ; here  $\sigma = \sqrt{1 + |x|^2}$  and we are working on  $\mathbb{R}^3$  (or some appropriate subset thereof) – see [15] or [16] and [17] (the later papers use a different indexing of the Sobolev spaces).

We will be interested in the asymptotic region (spatially) of the Maxwell–Dirac system, which we denote by  $E_\rho = \mathbb{R}^3 \setminus B_\rho$ , where  $B_\rho$  is the ball of radius  $\rho$ . A minimal condition that one may impose on the Dirac field is that it have finite total charge in the region  $E_\rho$ , this amounts to

$$\int_{E_\rho} j^0 dx = \int_{E_\rho} (|u_0|^2 + |u_1|^2 + |v^0|^2 + |v^1|^2) dx < \infty.$$

This, of course, simply means that  $u_A$  and  $v^A$  are in  $L^2$ .

Suppose we have a stationary system and we are in a stationary gauge for which  $A^\alpha \rightarrow 0$  as  $|x| \rightarrow \infty$ . Write,  $u_A = e^{-iEt}U_A$  and  $\bar{v}^A = e^{-iEt}\bar{V}^A$  with  $U_A, V_A$  and  $A^\alpha$  all independent of time  $t$ . Then  $U_A$  and  $V_A$  must be in  $L^2(E_\rho)$  if the total charge due to the Dirac field is finite. So  $U$  and  $V$  must have  $L^2$  decay as  $|x| \rightarrow \infty$ , roughly  $U$  and  $V$  must decay faster than  $|x|^{-\frac{3}{2}}$ . We also note that  $A^\alpha$  is given by equation (4) in terms of  $U$  and  $V$  and their first derivatives. If we substitute this expression for the electromagnetic potential into the Maxwell equations then we have equations that are of third order for  $U$  and  $V$ . For these equations to make sense we require that  $U$  and  $V$  are three times differentiable (in the weak sense at least). This suggests that  $U$  and  $V$  should be in  $W_{-\tau}^{3,2}(E_\rho)$ , where  $\tau > \frac{3}{2}$ .

To make this all a little more precise we introduce some more notation. Note that  $u_C v^C = U_C V^C$  is a gauge and Lorentz invariant complex scalar function, this means we can introduce a (unique up to sign) ‘‘spinor dyad’’  $\{o_A, \iota_B\}$ , with  $\iota^A o_A = 1$ . The dyad is defined as follows, let  $U_C V^C = R e^{i\chi}$  – where  $R$  and  $\chi$  are real functions – then write,

$$U_A = \sqrt{R} e^{i\frac{\chi}{2}} o_A \quad \text{and} \quad V_A = \sqrt{R} e^{i\frac{\chi}{2}} \iota_A.$$

**Definition 1.** A stationary Maxwell–Dirac system will be said to be isolated if, in some stationary gauge, we have

$$\psi = e^{-iEt} \sqrt{R} \begin{pmatrix} e^{i\frac{\chi}{2}} o_A \\ e^{-i\frac{\chi}{2}} \iota_{\dot{A}} \end{pmatrix},$$

with  $E$  constant and  $\sqrt{R}e^{i\frac{\chi}{2}} \in W_{-\tau}^{3,2}(E_\rho)$ ;  $o_A, \iota_A \in W_\epsilon^{3,2}(E_\rho)$  and  $A^\alpha \in W_{-1+\epsilon}^{2,2}(E_\rho)$ , for some  $\tau > \frac{3}{2}$  and some  $\rho > 0$  and any  $\epsilon > 0$ .

**Remark 1.** This definition ensures, after use of the Sobolev inequality and the multiplication lemma, that  $\psi = o(r^{-\tau})$  and  $A^\alpha = o(r^{-1+\epsilon})$ .

**Remark 2.** Notice our condition places regularity restrictions on the fields in the region  $E_\rho$  only. In the “interior”  $B_\rho$  there are no regularity assumptions.

The spherically symmetric solution in fact provides an excellent example of an *isolated, stationary* and *static* Maxwell–Dirac system.

The main theorem proved in [14] shows that the electric neutrality of the spherically symmetric solution is generic for these isolated, static systems.

**Theorem 1.** *An isolated, stationary, static Maxwell–Dirac system is electrically neutral.*

The theorem is remarkable in that it depends only on asymptotic regularity and decay – almost anything can happen in  $B_\rho$ ! Another theorem of [14] shows that the association of a magnetic monopole with the central, external Coulomb field, in the spherically symmetric case, is also generic (at least for axial symmetry). That is, associated to each external Coulomb point charge in a stationary, static Maxwell–Dirac system there is a magnetic monopole with magnetic charge of Dirac value  $\frac{1}{2e}$ .

## 4 Stationary isolated systems

To close this brief overview of the Maxwell–Dirac system we will take a quick look at some very recent results [18].

The first observation one makes is that under the regularity and decay conditions assumed (those of an isolated system) we can always perform a gauge transformation to the Lorenz gauge. The Maxwell equation for  $A^\alpha$  now becomes an elliptic equation (remember the system is stationary)

$$\Delta A^\alpha = 4\pi e \sqrt{2} R \sigma_\beta^{A\dot{A}} (o_A \bar{o}_{\dot{A}} + \iota_A \bar{\iota}_{\dot{A}}). \tag{9}$$

Writing  $a^\alpha = E\delta_0^\alpha + A^\alpha$  the Dirac equations (3) are,

$$\begin{aligned} \frac{o_A}{2} \left( \frac{\partial^{A\dot{A}} R}{R} + i\partial^{A\dot{A}} \chi \right) + \partial^{A\dot{A}} o_A - iea^{A\dot{A}} o_A + \frac{im}{\sqrt{2}} \bar{t}^{\dot{A}} e^{-i\chi} &= 0, \\ \frac{\iota_A}{2} \left( \frac{\partial^{A\dot{A}} R}{R} + i\partial^{A\dot{A}} \chi \right) + \partial^{A\dot{A}} \iota_A + iea^{A\dot{A}} \iota_A + \frac{im}{\sqrt{2}} \bar{t}^{\dot{A}} e^{-i\chi} &= 0. \end{aligned} \tag{10}$$

A straightforward “bootstrap” argument (based on elliptic regularity) can be made to show that  $A$  and  $U$  and  $V$  must in fact be  $C^\infty$  if  $U$  and  $V$  are taken to be in  $L^2(E_\rho)$  and  $A$  is in  $L^1_{loc}$ .

One can show (using an argument based on Thaller, [1]) that the essential spectrum of the Dirac operator in this case is the same as that for the free Dirac operator, i.e.  $(-\infty, -m] \cup [m, \infty)$ . So we would expect to get bound states for  $E \in (-m, m)$  – cf. [5]. In fact under very weak assumptions we can show that there are *no embedded eigenvalues*, i.e.  $E \in [-m, m]$  – cf. [1].

Under more restrictive assumptions there is also a version of the “electric neutrality” theorem. The interested reader may find details of these and other results in the forthcoming paper.

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