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# On the Center of $q$-Deformed Algebra $U'_q(\mathrm{so}_3)$ Related to Quantum Gravity at $q$ a Root of 1

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It is known that Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$ appears as algebra of observables in quantum gravity in $(2+1)$-dimensional de Sitter space with space being torus. In this paper, we study the center of this algebra at $q$ a root of 1. It turns out that Casimir elements in this case are algebraically dependent. Using realization of the algebra $U'_q(\mathrm{so}_3)$ in terms of quantized lengths of geodesics on torus with one hole, we find this dependence in an explicit form. It is expressed in terms of Chebyshev polynomials of the first kind. The properties of Casimir elements in the cyclic type representations are studied.

## 1 Introduction

It is shown by Nelson, Regge and Zertuche [1] that the algebra of observables in quantum gravity in (2+1)-dimensional de Sitter space with space being torus is related to Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$ [2, 3], where $q$ is related to the Plank constant and the curvature of the de Sitter space. Thus it is important, from point of view of physics, to study the structure (in particular, the center) of this algebra. The center of the algebra $U'_q(\mathrm{so}_3)$ in the case of $q$ being not a root of 1 is generated by the element $C$, which is deformation of the Casimir element of Lie algebra $\mathrm{so}_3$. The center of this algebra at $q$ a root of 1 contains three more elements $C_1$, $C_2$, $C_3$ [2, 4]. It turns out that all four Casimir elements are algebraically dependent. The main goal of this paper is to describe this dependence in an explicit form. To find it we use the realization of algebra $U'_q(\mathrm{so}_3)$ in terms of quantum geodesics on torus $\mathcal{T}$ with one hole proposed by Chekhov and Fock [5]. Namely, generators $I_1$ and $I_2$ (resp. Casimir element $C$) of algebra $U'_q(\mathrm{so}_3)$ are related to quantized lengths of geodesics corresponding to two basis cycles (resp. cycle around the hole) on $\mathcal{T}$. They are expressed in terms of $z_1$, $z_2$, $z_3$, which are "coordinates" on quantized Teichmüller space $\mathcal{A}_q$ of $\mathcal{T}$. In this realization, the fact that elements $C$, $C_1$, $C_2$, $C_3$ belong to the center of $U'_q(\mathrm{so}_3)$ is almost obvious. We note, that the same algebra $U'_q(\mathrm{so}_3)$ appeared also in the paper [6] as Kauffman bracket skein algebra of $\mathcal{T}$.

It is known that algebra $U'_q(\mathrm{so}_3)$, at $q$ a root of 1, possesses cyclic type irreducible representations [7, 8]. The action formulas for Casimir operators on the spaces of these representations are presented in explicit form. It is shown that $C_1$, $C_2$, $C_3$ do not separate this type of representations. To separate them we also need to include $C$.

## 2 Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$

The Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$ [2, 3] is an associative unital algebra with generating elements $I_1$, $I_2$, $I_3$ and defining relations

$$q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \qquad q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \qquad q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2,$$

where $q \neq 0, \pm 1$, is a complex number called deformation parameter. In the limit $q \to 1$, the algebra $U'_q(\mathrm{so}_3)$ reduces to the Lie algebra $\mathrm{so}_3$. Algebra $U'_q(\mathrm{so}_3)$ has a linear basis $I_1^{k_1}I_2^{k_2}I_3^{k_3}$,

$k_1, k_2, k_3 \geq 0$ (Poincaré–Birkhoff–Witt basis) [7]. At arbitrary $q$, the algebra $U'_q(\mathrm{so}_3)$ has central element

$$C = -q^{1/2}\left(q - q^{-1}\right) I_1 I_2 I_3 + q I_1^2 + q^{-1} I_2^2 + q I_3^2. \tag{1}$$

It generates the center of $U'_q(\mathrm{so}_3)$ when $q$ is not a root of 1 (see [6]).

Let us fix $q$ to be a primitive root of 1 of order $p > 2$, that is $q^p = 1$, $q^{p'} \neq 1$, $1 \leq p' < p$. Then elements

$$C_k = 2\ T_p\left(I_k\left(q - q^{-1}\right)/2\right), \qquad k = 1, 2, 3, \tag{2}$$

where $T_p(x)$ is Chebyshev polynomial of the first kind, are also central in $U'_q(\mathrm{so}_3)$. The Chebyshev polynomial $T_p(x)$ is uniquely defined through $T_p(\cos\theta) = \cos(p\theta)$. It explicit form is

$$T_p(x) = \frac{p}{2}\sum_{k=0}^{\lfloor p/2\rfloor} \frac{(-1)^k (p-k-1)!}{k!(p-2k)!}(2x)^{p-2k}, \tag{3}$$

where $\lfloor p/2\rfloor$ is integral part of $p/2$. Some examples of Chebyshev polynomials of the first kind:

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x, \qquad T_4(x) = 8x^4 - 8x^2 + 1, \qquad T_5(x) = 16x^5 - 20x^3 + 5x, \quad \ldots .$$

The central elements $C_1$, $C_2$ and $C_3$ were first described in implicit form (and without proof) in [2]. In the paper [4], these elements were given in explicit form as sum of type (3). It was pointed out to me by V. Fock the coincidence of this sum with Chebyshev polynomial of the first kind. The elements $C$, $C_1$, $C_2$ and $C_3$ are algebraically dependent. Our main goal is to describe this dependence in an explicit form.

## 3 Algebra $U'_q(\mathrm{so}_3)$ as algebra of quantum geodesics on torus with one hole

Now we describe the algebra $\mathcal{A}_q$ of quantized Teichmüller space of torus with one hole [5]. It is an associative unital algebra with generating elements $z_1$, $z_1^{-1}$, $z_2$, $z_2^{-1}$, $z_3$, $z_3^{-1}$ and defining relations

$$z_k z_k^{-1} = z_k^{-1} z_k = 1, \qquad z_1 z_2 = q z_2 z_1, \qquad z_2 z_3 = q z_3 z_2, \qquad z_3 z_1 = q z_1 z_3. \tag{4}$$

It is easy to realize that $z_1^{k_1} z_2^{k_2} z_3^{k_3}$, $k_1, k_2, k_3 \in \mathbb{Z}$, constitute a linear basis in $\mathcal{A}_q$. Geodesic functions $G_1$, $G_2$ and $G_3$, which are related to lengths $L_1$, $L_2$ and $L_3$ of geodesics $(1, 0)$, $(0, 1)$ (corresponding to two basis cycles) and $(1, 1)$ (corresponding to sum of these cycles) on torus with one hole as $G_k = 2\cosh(L_k/2)$, after quantization take the form [5]:

$$G_1 = q^{-1/2} z_3^{-1} z_1^{-1} + q^{1/2} z_3^{-1} z_1 + q^{-1/2} z_3 z_1, \tag{5}$$
$$G_2 = q^{-1/2} z_2^{-1} z_3^{-1} + q^{1/2} z_2^{-1} z_3 + q^{-1/2} z_2 z_3, \tag{6}$$
$$G_3 = q^{-1/2} z_1^{-1} z_2^{-1} + q^{1/2} z_1^{-1} z_2 + q^{-1/2} z_1 z_2. \tag{7}$$

**Proposition 1 ([5]).** *The map $\phi$ given by*

$$\phi: \ I_k \mapsto G_k/\left(q - q^{-1}\right), \qquad k = 1, 2, 3,$$

*defines an injective homomorphism $\phi : U'_q(\mathrm{so}_3) \to \mathcal{A}_q$.*

**Proof.** It is easy to show by straightforward calculation that

$$q^{1/2}G_1G_2 - q^{-1/2}G_2G_1 = \left(q - q^{-1}\right) G_3,$$
$$q^{1/2}G_2G_3 - q^{-1/2}G_3G_2 = \left(q - q^{-1}\right) G_1,$$
$$q^{1/2}G_3G_1 - q^{-1/2}G_1G_3 = \left(q - q^{-1}\right) G_2.$$

It proves that $\phi$ defines a homomorphism. Let us show that $\ker(\phi) = 0$. We assume that there exists an element $a = \sum a_{k_1,k_2,k_3} I_1^{k_1} I_2^{k_2} I_3^{k_3}$, where only finite number of coefficients $a_{k_1,k_2,k_3}$ are non-zero, such that $\phi(a) = 0$. Let $a_{l_1,l_2,l_3} \neq 0$ for some $l_1, l_2, l_3 \geq 0$ and $a_{k_1,k_2,k_3} = 0$ for all $k_1$, $k_2$, $k_3$ such that $k_1 + k_2 + k_3 > l_1 + l_2 + l_3$. Then $\phi(I_1^{l_1} I_2^{l_2} I_3^{l_3})$ contains summand $\alpha z_1^{l_1+l_3} z_2^{l_2+l_3} z_3^{l_1+l_2}$, $\alpha \neq 0$. It is unique summand with maximal sum of powers of $z_1$, $z_2$ and $z_3$. Only $\phi(I_1^{k_1} I_2^{k_2} I_3^{k_3})$ with $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$ and $a_{k_1,k_2,k_3} \neq 0$ contain summands with the same sum of powers of $z_1$, $z_2$, $z_3$. But the very monomials in $z_1$, $z_2$, $z_3$ are not coinciding, because from $z_1^{k_1+k_3} z_2^{k_2+k_3} z_3^{k_1+k_2} = z_1^{l_1+l_3} z_2^{l_2+l_3} z_3^{l_1+l_2}$ it follows that $k_i = l_i$. Thus coefficient at $z_1^{l_1+l_3} z_2^{l_2+l_3} z_3^{l_1+l_2}$ in $\phi(a)$ is non-zero. It contradicts the assumption that $\phi(a) = 0$. ■

The injectivity of homomorphism $\phi$ follows from construction given in [5]. We proved the injectivity in purely algebraic way.

Now our strategy is following. We find the images of $C$, $C_1$, $C_2$, $C_3$ in $\mathcal{A}_q$. Then, due to Proposition 1, the relations between the obtained images will imply the relations between of $C$, $C_1$, $C_2$ and $C_3$. Instead of $C$, we will use

$$\partial = \left(q + q^{-1}\right) \mathbf{1} - \left(q - q^{-1}\right)^2 C. \tag{8}$$

Straightforward calculation shows that (see (1))

$$\phi(\partial) = q^{-2} \left(z_1^{-2} z_2^{-2} z_3^{-2} + z_1^2 z_2^2 z_3^2\right). \tag{9}$$

It is easy to see that $\phi(\partial)$ commutes with $z_1$, $z_2$, $z_3$ and, therefore, with $\phi(I_1)$, $\phi(I_2)$, $\phi(I_3)$. Hence, due to Proposition 1, $\partial$ is central in $U_q'(\mathrm{so}_3)$. The images of $C_k$, $k = 1, 2, 3$, in $\mathcal{A}_q$ are (see (2))

$$\phi(C_k) = 2T_p(G_k/2), \qquad k = 1, 2, 3. \tag{10}$$

Let us define an associative algebra $L_q$ with generating elements $\Lambda$, $\Lambda^{-1}$, $\Lambda_0$ which satisfy the relations

$$\Lambda\Lambda^{-1} = \Lambda^{-1}\Lambda = 1, \qquad \Lambda\Lambda_0 = q^2 \Lambda_0 \Lambda,$$

where $q$ is a non-zero complex number. In order to formulate an important lemma, we remind the standard notations for $q$-numbers:

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \tag{11}$$

$q$-factorials and $q$-binomial coefficients:

$$[m]! = [m][m-1]\cdots[1], \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!} = \frac{[n][n-1]\cdots[n-m+1]}{[1][2]\cdots[m]}.$$

**Lemma 1.** *In algebra $L_q$ at non-zero complex number $q$, we have*

$$2T_p\left(\frac{\Lambda + \Lambda_0 + \Lambda^{-1}}{2}\right) = \Lambda^p + \Lambda^{-p} + \sum R_{p,k,l} \Lambda^l \Lambda_0^k,$$

*where sum runs over integral $k$ and $l$ such that $k > 0$, $k \pm l \leq p$, $k + l \equiv p \pmod 2$, and*

$$R_{p,k,l} = q^{-kl} \frac{[p]}{[k]} \begin{bmatrix} \frac{p+k+l}{2} - 1 \\ k - 1 \end{bmatrix} \begin{bmatrix} \frac{p+k-l}{2} - 1 \\ k - 1 \end{bmatrix}.$$

**Proof.** We will prove the lemma by induction. It is easy to see that the lemma is correct at $p = 0$ and $p = 1$. For $p = 2$, we have $2T_2(x/2) = x^2 - 2$, and validity of lemma follows from direct calculation. The left-hand sides of the relations given in lemma satisfy the recurrent relation which follows from recurrent relation for Chebyshev polynomials: $T_p(x) = 2xT_{p-1}(x) - T_{p-2}(x)$. Hence, the right-hand sides also must satisfy the same relation. In terms of $R_{p,k,l}$ it looks like

$$R_{p,k,l} = R_{p-1,k,l-1} + R_{p-1,k,l+1} + R_{p-1,k,l+1} + q^{-2l} R_{p-1,k-1,l} - R_{p-2,k,l}.$$

Substituting explicit expressions for $R_{p,k,l}$ and cancelling common multiplier we obtain the relation

$$\begin{aligned}&[p]\left[\frac{p+k+l}{2}-1\right]\left[\frac{p+k-l}{2}-1\right] = q^{-k}[p-1]\left[\frac{p-k-l}{2}\right]\left[\frac{p+k+l}{2}-1\right]\\ &\quad + q^{k}[p-1]\left[\frac{p-k+l}{2}\right]\left[\frac{p+k-l}{2}-1\right] + q^{-l}[p-1][k][k-1]\\ &\quad - [p-2]\left[\frac{p-k+l}{2}\right]\left[\frac{p-k-l}{2}\right],\end{aligned}$$

which can be verified in direct way using definition (11) of $q$-numbers. ■

**Corollary 1.** *In algebra $L_q$, when $q$ is a primitive root of 1 of order $p > 2$ , we have*

$$2T_p\left(\frac{\Lambda + \Lambda_0 + \Lambda^{-1}}{2}\right) = \Lambda^p + \Lambda^{-p} + \Lambda_0^p$$

*if $p$ is odd, and*

$$2T_p\left(\frac{\Lambda + \Lambda_0 + \Lambda^{-1}}{2}\right) = \Lambda^p + \Lambda^{-p} + \Lambda_0^p + 2\Lambda^{p/2}\Lambda_0^{p/2} + 2\Lambda^{-p/2}\Lambda_0^{p/2}$$

*if $p$ is even.*

**Proof.** Let us make some remarks on the values of $q$-numbers at $q$ a root of 1. If $p$ is odd, then $[p] = 0$ and $[s] \neq 0$, if $s = 1, 2, \ldots, p-1$. If $p$ is even, then $[p/2] = [p] = 0$ and $[s] \neq 0$, if $s = 1, 2, \ldots, p/2-1, p/2+1, \ldots, p-1$. Let $p$ be an odd number. Then numerators and denominators in the both $q$-binomial coefficients included in $R_{p,k,l}$ are non-zero. Thus all the $R_{p,k,l} = 0$ (due to $[p] = 0$), unless $k = p$. In the case $k = p$, we have $l = 0$ and $R_{p,p,0} = 1$. Now we consider the case of even $p$. Simple analysis shows that if the denominator of a $q$-binomial coefficient included in $R_{p,k,l}$ contains $[p/2]$ then the corresponding numerator also contains this $q$-number. Cancelling it, we obtain non-zero $q$-binomial coefficient. All the $R_{p,k,l} = 0$ (due to $[p] = 0$), unless $k = p$ or $k = p/2$. If $k = p$, we obtain $l = 0$ and $R_{p,p,0} = 1$ in full analogy with odd $p$ case. Analyzing numerators and denominators in $q$-binomial coefficients in the case of $k = p/2$, we find that the $q$-binomial coefficients are non-zero only if $l = \pm p/2$. Since $q^{p/2} = -1$ (not $+1$ because $q$ is a primitive root of 1), we have $[p]/[p/2] \equiv q^{p/2} + q^{-p/2} = -2$ and $(-1)^{\mp p^2/4} = (-1)^{p/2}$. Using the relation $[p-r] = -[r]$, we obtain $R_{p,p/2,\pm p/2} = 2$. Thus we have found all the non-zero coefficients $R_{p,k,l}$. ■

**Corollary 2.** *The map $\phi$ on $C_1$, $C_2$ and $C_3$, when $q$ is a primitive root of 1 of order $p > 2$, is*

$$\left.\begin{aligned}C_1 &\mapsto q^{p/2}\left(z_3^{-p}z_1^{-p} + z_3^{-p}z_1^{p} + z_3^{p}z_1^{p}\right),\\ C_2 &\mapsto q^{p/2}\left(z_2^{-p}z_3^{-p} + z_2^{-p}z_3^{p} + z_2^{p}z_3^{p}\right),\\ C_3 &\mapsto q^{p/2}\left(z_1^{-p}z_2^{-p} + z_1^{-p}z_2^{p} + z_1^{p}z_2^{p}\right)\end{aligned}\right\}\qquad \text{at odd } p,$$

$$\left.\begin{aligned} &C_1 \mapsto z_3^{-p}z_1^{-p} + z_3^{-p}z_1^p + z_3^p z_1^p + (-1)^{p/2}2\left(z_1^p + z_3^{-p}\right),\\ &C_2 \mapsto z_2^{-p}z_3^{-p} + z_2^{-p}z_3^p + z_2^p z_3^p + (-1)^{p/2}2\left(z_3^p + z_2^{-p}\right),\\ &C_3 \mapsto z_1^{-p}z_2^{-p} + z_1^{-p}z_2^p + z_1^p z_2^p + (-1)^{p/2}2\left(z_2^p + z_1^{-p}\right)\end{aligned}\right\} \quad \text{at even } p.$$

**Proof.** Let us find $\phi(C_1)$ (see (10)). We denote three summands in $G_1 = q^{-1/2}z_3^{-1}z_1^{-1} + q^{1/2}z_3^{-1}z_1 + q^{-1/2}z_3z_1$ by $\Lambda^{-1}$, $\Lambda_0$ and $\Lambda$, respectively. It is easy to verify that these three objects give realization of algebra $L_q$ in $\mathcal{A}_q$. Then this corollary can be obtained using Corollary 1 and commutation relations (4) for $\mathcal{A}_q$. The cases of the elements $C_2$ and $C_3$ can be analyzed in full analogy with the case of element $C_1$. ■

It is obvious that images of $C_1$, $C_2$, $C_3$, at $q$ a root of 1, commute with $z_k$, and therefore with $\phi(I_1)$, $\phi(I_2)$, $\phi(I_3)$. It gives one more proof of the fact that $C_1$, $C_2$, $C_3$ are central in $U_q'(\mathrm{so}_3)$.

**Proposition 2.** *The algebraic dependence of the central elements $\partial$, $C_1$, $C_2$, $C_3$ of $U_q'(\mathrm{so}_3)$ at $q$ a primitive root of 1 of order $p > 2$ has the form*

$$\begin{aligned} p = 2k+1: \quad & -q^{p/2}C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) - 2 = 0,\\ p = 4k: \quad & -C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) + 16T_{p/2}(\partial/2) + 10\\ & \quad + 4(T_{p/2}(\partial/2)+1)(C_1+C_2+C_3) = 0,\\ p = 4k+2: \quad & -C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) - 16T_{p/2}(\partial/2) + 10\\ & \quad - 4(T_{p/2}(\partial/2)-1)(C_1+C_2+C_3) = 0. \end{aligned} \tag{12}$$

*(The relation between $C$ and $\partial$ is given by (8)).*

**Proof.** To prove this proposition we map by $\phi$ left-hand sides of these relations to $\mathcal{A}_q$. It is easy to verify (using (9) and $2T_k((t+t^{-1})/2) = t^k + t^{-k}$) the relations

$$2T_p(\phi(\partial)/2) = z_1^{-2p}z_2^{-2p}z_3^{-2p} + z_1^{2p}z_2^{2p}z_3^{2p}, \qquad 2T_{p/2}(\phi(\partial)/2) = z_1^{-p}z_2^{-p}z_3^{-p} + z_1^p z_2^p z_3^p,$$

where second relation is given only for even $p$. Thus the images of left-hand sides of relations (12) can be rewritten in terms of commuting variables $x_k = z_k^p$, $k = 1, 2, 3$. We obtain three relations (with respect to cases $p = 2k+1$, $p = 4k$ and $p = 4k+2$) each of them not depending on $p$ of commuting variables $x_1$, $x_2$ and $x_3$. They can be verified directly. ■

In private communication, V. Fock informed me about the form of algebraic dependence of central elements, in the case of odd $p$. Independently, V. Levandovskyy found this dependence when $p = 3, 4$ by using Computer Algebra System PLURAL for Non-commuting Polynomial Computation. This information was very important for me to formulate Proposition 2. Note, that algebraic dependence of central elements of Drinfeld–Jimbo algebra $U_q(\mathrm{sl}_2)$ at $q$ a root of unity is also expressed in terms of Chebyshev polynomials [9].

**Conjecture 1.** *The elements $C$ (or, equivalently, $\partial$), $C_1$, $C_2$, $C_3$ of $U_q'(\mathrm{so}_3)$ at $q$ a root of 1 generate the center of this algebra. All the algebraic relations among them follow from the relations described in Proposition 2.*

## 4 Cyclic type representations of $U_q'(\mathrm{so}_3)$ at $q$ a root of 1

Let $q^p = 1$. Then all the irreducible representations of $U_q'(\mathrm{so}_3)$ are finite-dimensional [8]. We describe one class of such representations, namely, cyclic type representations $T \equiv T_{l,h,M}$, where $h$, $l$ and $M$ are complex numbers, $h, h+l, h-l \notin \frac{1}{2}\mathbb{Z}$. These representations are given on

$p$-dimensional vector space $\mathcal{V}_{l,h,M}$ with basis $|h\rangle$, $|h+1\rangle, \ldots,$ $|h+p-1\rangle$. It useful to identify $|h+p\rangle \equiv |h\rangle$, $|h-1\rangle \equiv |h+p-1\rangle$. The action formulas are

$$T(I_1)|m\rangle = \mathrm{i}[m]|m\rangle,$$
$$T(I_2)|m\rangle = \frac{[m]}{[2m]}\left(M[l-m]|m+1\rangle - M^{-1}[l+m]|m-1\rangle\right),$$
$$T(I_1)|m\rangle = \mathrm{i}q^{1/2}\frac{[m]}{[2m]}\left(Mq^{m}[l-m]|m+1\rangle + M^{-1}q^{-m}[l+m]|m-1\rangle\right),$$

where $m = h, h+1, \ldots, h+p-1$, and definition of $q$-numbers (11) is used.

**Proposition 3 ([8]).** *Any of irreducible representations $T_{l',h',M'}$ has unique equivalent representation among $T_{l,h,M}$ with $|\mathrm{Re}\, h| < 1/4$, $0 < \mathrm{Re}\, l < p/4$, $0 \leq \arg M < 2\pi/p$.*

**Proposition 4.** *The action of $T(C)$, $T(C_1)$, $T(C_2)$ and $T(C_3)$ is given by the formulas:*

$$T(C)|m\rangle = -[l][l+1]|m\rangle,$$

*if $p = 2k+1$:*

$$T(C_1)|m\rangle = \mathrm{i}^p\left(q^{ph} - q^{-ph}\right)|m\rangle,$$
$$T(C_2)|m\rangle = \left(M^pA_+ - M^{-p}A_-\right)|m\rangle,$$
$$T(C_3)|m\rangle = \mathrm{i}^pq^{p/2}\left(M^pq^{ph}A_+ + M^{-p}q^{-ph}A_-\right)|m\rangle,$$
$$A_\pm = \frac{q^{p(l\mp h)} - q^{-p(l\mp h)}}{q^{ph} + q^{-ph}};$$

*if $p = 4k$:*

$$T(C_1)|m\rangle = \left(q^{ph} + q^{-ph}\right)|m\rangle,$$
$$T(C_2)|m\rangle = \left(M^p\tilde{A}_+ + \tilde{A}_0 + M^{-p}\tilde{A}_-\right)|m\rangle,$$
$$T(C_3)|m\rangle = \left(M^pq^{ph}\tilde{A}_+ + \tilde{A}_0 + M^{-p}q^{-ph}\tilde{A}_-\right)|m\rangle,$$
$$\tilde{A}_\pm = \frac{\left(q^{\frac{p}{2}(l\mp h)} - q^{-\frac{p}{2}(l\mp h)}\right)^2}{\left(q^{\frac{p}{2}h} - q^{-\frac{p}{2}h}\right)^2}, \qquad \tilde{A}_0 = -2\frac{\left(q^{\frac{p}{2}l} - q^{-\frac{p}{2}l}\right)^2}{\left(q^{\frac{p}{2}h} - q^{-\frac{p}{2}h}\right)^2};$$

*if $p = 4k+2$:*

$$T(C_1)|m\rangle = -\left(q^{ph} + q^{-ph}\right)|m\rangle,$$
$$T(C_2)|m\rangle = \left(M^p\breve{A}_+ + \breve{A}_0 + M^{-p}\breve{A}_-\right)|m\rangle,$$
$$T(C_3)|m\rangle = \left(-M^pq^{ph}\breve{A}_+ + \breve{A}_0 - M^{-p}q^{-ph}\breve{A}_-\right)|m\rangle,$$
$$\breve{A}_\pm = \frac{\left(q^{\frac{p}{2}(l\mp h)} - q^{-\frac{p}{2}(l\mp h)}\right)^2}{\left(q^{\frac{p}{2}h} + q^{-\frac{p}{2}h}\right)^2}, \qquad \breve{A}_0 = -2\frac{\left(q^{\frac{p}{2}l} + q^{-\frac{p}{2}l}\right)^2}{\left(q^{\frac{p}{2}h} + q^{-\frac{p}{2}h}\right)^2}.$$

**Proof.** From Schur lemma, it follows that $T(C)$, $T(C_1)$, $T(C_2)$ and $T(C_3)$ are proportional to unit matrix. That is the vectors $|m\rangle$ are eigenvectors with eigenvalues not depending on $m$.

The action of $T(C)$ and $T(C_1)$ can be found directly using the definition of $q$-numbers (11). From the action formulas for $T(I_2)$ and $T(I_3)$, we can see that matrix elements of diagonal action of $T(C_2)$ and $T(C_3)$ may include only summands which are proportional to $M^{\pm p}$ or summands which have no dependence on $M$. To find the coefficients $\tilde{A}_\pm$, $\breve{A}_\pm$ at $M^{\pm p}$ in action formulas for $T(C_2)$ (resp. $T(C_3)$), we observe that only highest order summand in the expression of $C_2$ (resp. $C_3$) in terms of Chebyshev polynomial of $I_2$ (resp. $I_3$) gives contribution to these coefficients. It is easy to calculate them. Now we use the relations of Proposition 2 to find the coefficients $\tilde{A}_0$ and $\breve{A}_0$ in the case of even $p$. The coefficients at $M^{\pm 2p}$ after substitution of $\tilde{A}_\pm$ and $\breve{A}_\pm$ are zero. The condition on the coefficients at $M^{\pm p}$ to be zero gives $\tilde{A}_0$ and $\breve{A}_0$. Of course, the found matrix elements also identically satisfy relations of Proposition 2 constructed from terms not depending on $M$. ■

It follows from Proposition 3 that representations $T_{l,h,M}$ and $T_{l+1,h,M}$ with $l, h, M$ as in that proposition are not equivalent. But, it is easy to see, $T_{l,h,M}(C_k) = T_{l+1,h,M}(C_k)$, $k = 1, 2, 3$. Thus central elements $C_1$, $C_2$ and $C_3$ do not separate non-equivalent cyclic type representations. In fact, they separate almost all of them, namely, there exists at least one of central elements $C_k$ such that $T_{l,h,M}(C_k) \neq T_{l',h',M'}(C_k)$ if $(l - l') \notin \mathbb{Z}$. To separate all of them we also need to include $C$.

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