Beam Rounders for Circular Colliders

A. Burov*, S. Nagaitsev, FNAL, Batavia, IL
Ya. Derbenev, Jefferson Laboratory, Newport News, VA

Abstract

By means of linear optics, an arbitrary uncoupled beam can be locally transformed into a round (rotation-invariant) state and then back. This provides an efficient way to round beams in the interaction region of circular colliders.

1 ROUND BEAMS AND ROTATION-INVARIANT MAPS

Round beams in the interaction region of a circular collider are widely believed to be an effective way to increase the luminosity (see e. g. [1] and the references therein).

Canonical angular momentum (CAM) preservation by the IP revolution mapping might play a crucial role in the luminosity upgrade of circular colliders. The CAM is preserved when 2 conditions are satisfied:

- The lattice IP revolution map is CAM-preserving;
- The beams are round in the IR.

General form of the CAM-preserving matrices was found by E. Pozdeev and E. Perevedentsev [2]:

$$\mathcal{T}=\begin{pmatrix} \mathsf{T}\cdot\cos\theta & \mathsf{T}\cdot\sin\theta \\ -\mathsf{T}\cdot\sin\theta & \mathsf{T}\cdot\cos\theta \end{pmatrix} \equiv \mathcal{R}(\theta)\begin{pmatrix} \mathsf{T} & 0 \\ 0 & \mathsf{T} \end{pmatrix} \tag{1}$$

The CAM-preserving group is identical to the symplectic rotation-invariant transformations.

Parameterization of the 2×2 unimodular matrix T can be taken in the conventional Courant-Snyder form, in terms of its input $\alpha_1,\ \beta_1$ and output $\alpha_2,\ \beta_2$ parameters and a phase advance μ : (see e. g. [3]):

$$\mathsf{T}=\begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\mathsf{c}+\alpha_1\mathsf{s}) & \sqrt{\beta_1\beta_2}\,\mathsf{s} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\mathsf{s}+\frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\mathsf{c} & \sqrt{\frac{\beta_1}{\beta_2}}(\mathsf{c}-\alpha_2\mathsf{s}) \end{pmatrix}. \tag{2}$$

where $\mathsf{s}=\sin\mu$ $\mathsf{c}=\cos\mu$, the subscript 1 of the Courant-Snyder parameters relates to an initial and 2 to a final states.

2 CIRCULAR BASIS

The simplectic basis which form is preserved by the rotation-invariant transformations:

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\beta}\mathsf{c}_+ & \sqrt{\beta}\mathsf{s}_+ & -\sqrt{\beta}\mathsf{c}_- & -\sqrt{\beta}\mathsf{s}_- \\ \frac{-\mathsf{s}_+-\alpha\mathsf{c}_+}{\sqrt{\beta}} & \frac{\mathsf{c}_+-\alpha\mathsf{s}_+}{\sqrt{\beta}} & \frac{\mathsf{s}_-+\alpha\mathsf{c}_-}{\sqrt{\beta}} & \frac{-\mathsf{c}_-+\alpha\mathsf{s}_-}{\sqrt{\beta}} \\ \sqrt{\beta}\mathsf{s}_+ & -\sqrt{\beta}\mathsf{c}_+ & \sqrt{\beta}\mathsf{s}_- & -\sqrt{\beta}\mathsf{c}_- \\ \frac{\mathsf{c}_+-\alpha\mathsf{s}_+}{\sqrt{\beta}} & \frac{\mathsf{s}_++\alpha\mathsf{c}_+}{\sqrt{\beta}} & \frac{\mathsf{c}_--\alpha\mathsf{s}_-}{\sqrt{\beta}} & \frac{\mathsf{s}_-+\alpha\mathsf{c}_-}{\sqrt{\beta}} \end{pmatrix}. \tag{3}$$

where $\mathsf{c}_\pm=\cos\phi_\pm$ $\mathsf{s}_\pm=\sin\phi_\pm$ with arbitrary phases $\phi_\pm$. Similar, but not exactly same presentation of the circular modes was used by V. Lebedev and S. Bogacz [4]. A great feature of this parameterization:
Under the rotation-invariant transformations (1) the circular set (3) is transformed similar to how the linear basis does under the uncoupled mappings:

$$\tilde{U}\equiv\mathcal{T}\cdot U(\alpha,\beta,\phi_+,\phi_-)=U(\alpha_2,\beta_2,\phi_++\mu-\theta,\phi_-+\mu+\theta)\ . \tag{4}$$

Any phase space vector x can be expanded over this rotating basis:

$$x=U\cdot a\ . \tag{5}$$

$$a=(\sqrt{2J_+}\sin\chi_+\,,\sqrt{2J_+}\cos\chi_+\,,\sqrt{2J_-}\sin\chi_-\,,\sqrt{2J_-}\cos\chi_-) \tag{6}$$

Taking the amplitudes from their definition (5), the actions can be expressed in terms of 2D vectors of the offset and transverse momentum $\vec{r}=(x,y)\,,\vec{p}=(p_x,p_y)$:

$$J_\pm=\gamma\vec{r}^2/4+\alpha\vec{r}\vec{p}/2+\beta\vec{p}^2/4\pm M/2 \tag{7}$$

where $\gamma\equiv(1+\alpha^2)/\beta$ and $M=xp_y-yp_x$ is the CAM. Note a similarity of this expression to the corresponding formula in the uncoupled case.

Preservation of the circular actions $J_\pm$ under the invariant mappings means that both their sum and difference are preserved as well:

$$J_+-J_-=M=\text{const}\,;$$

$$J_++J_-=\gamma\vec{r}^2/2+\alpha\vec{r}\vec{p}+\beta\vec{p}^2/2=\text{const}\,. \tag{8}$$

Inverse expressions are found as

$$\begin{aligned} \vec{r}^2/\beta= &\quad J_++J_-+2\sqrt{J_+J_-}\cos\psi \\ \vec{p}^2\beta= &\quad (J_++J_-)(1+\alpha^2)+ \\ &\quad 2\sqrt{J_+J_-}(-1+\alpha^2)\cos\psi+4\sqrt{J_+J_-}\alpha\sin\psi \\ \vec{r}\vec{p}= &\quad -\alpha(J_++J_-)-2\sqrt{J_+J_-}\alpha\cos\psi-2\sqrt{J_+J_-}\sin\psi \end{aligned} \tag{9}$$

where $\psi=\phi_++\chi_++\phi_-+\chi_-$. When only one of the two circular modes is excited (either J_+ or J_- is zero), then

$$\vec{r}^2=\beta J\,,\ \vec{p}^2=\gamma J\,,\ \vec{r}\vec{p}=-\alpha J\,,\ M=\pm J\,. \tag{10}$$

3 ADAPTERS

Both uncoupled V and circular U (3) basic sets are symplectic; therefore, they can be mapped on each other. Symplectic transformations

$$\mathcal{C}=U\cdot V^{-1}\quad\text{and}\quad\tilde{\mathcal{C}}=V\cdot U^{-1} \tag{11}$$

* burov@fnal.gov

map the uncoupled basis V on the circular basis U, and back, respectfully. Note that the uncoupled-to-circular transformation $\mathcal{C}$ maps the horizontal and vertical phase spaces on the modes of opposite helicities. So the corresponding uncoupled and circular Courant-Snyder invariants are equal:

$$J_x = J_+ \, ; J_y = J_- \, . \tag{12}$$

Adaptive transformations are illustrated schematically by Fig. 1.

4 IMPLEMENTATION OF ADAPTERS

A particular solution for the adaptive transformation [Ya. Derbenev]:

$$\mathcal{C} = \mathcal{R}(\pi/4)\langle \mathsf{M}, \mathsf{N}\rangle \mathcal{R}(-\pi/4) \tag{13}$$

where $\langle \mathsf{M}, \mathsf{N}\rangle$ stands for a block-diagonal 4×4 matrix with M and N as its 2×2 diagonal blocks:

$$\mathsf{M} = \begin{pmatrix} \sqrt{\dfrac{\beta}{\beta_0}}(\cos\phi_0 - \alpha_0 \sin\phi_0) & -\sqrt{\beta\beta_0}\sin\phi_0 \\ \dfrac{\alpha_0\cos\phi_0 + \sin\phi_0}{\sqrt{\beta\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta}}\cos\phi_0 \end{pmatrix} \tag{14}$$

and

$$\mathsf{N} = \begin{pmatrix} -\sqrt{\dfrac{\beta}{\beta_0}}(\alpha_0\cos\phi_0 + \sin\phi_0) & -\sqrt{\beta\beta_0}\cos\phi_0 \\ \dfrac{\cos\phi_0 - \alpha_0\sin\phi_0}{\sqrt{\beta\beta_0}} & -\sqrt{\dfrac{\beta_0}{\beta}}\sin\phi_0 \end{pmatrix} . \tag{15}$$

The matrices M , N are related as

$$\mathsf{N} = \mathsf{F}\cdot\mathsf{M} \quad \mathsf{F} = \begin{pmatrix} 0 & -\beta \\ 1/\beta & 0 \end{pmatrix} . \tag{16}$$

this particular adapter transforms initial uncoupled basis (subscript 0) into a circular basis at its waist point ($\alpha = 0$).

5 CIRCULAR EIGENMODES FOR A SOLENOID

Circular eigenmodes of an extended solenoid: CS parameters remain constant, and only the phases run. The solenoidal transformation:

$$\mathcal{T}_s = \mathcal{R}(-\theta_s)\cdot\langle \mathsf{T}_s, \mathsf{T}_s\rangle \tag{17}$$

with

$$\mathsf{T}_s = \begin{pmatrix} \cos\theta_s & \beta_s\sin\theta_s \\ -\beta_s^{-1}\sin\theta_s & \cos\theta_s \end{pmatrix} . \tag{18}$$

Here $\theta_s = eBz/(2p_0c) \equiv z/(2\rho)$ is the Larmor phase advance and

$$\beta_s = 2c/(eB) \tag{19}$$

can be referred to as the Larmor β-function. The Courant-Snyder parameters of the circular basis with $\beta = \beta_s$ and $\alpha = 0$ are preserved inside the solenoid: the first pair of the basis vectors turns by an angle $\Delta\phi_+ = \theta_s + \theta_s = 2\theta_s$ and the second pair by $\Delta\phi_- = -\theta_s + \theta_s = 0$, i. e. remains constant.

The canonical variables $\tilde{a}$ associated with these circular modes describe the kinetic momenta

$$k_y = p_y + x/\beta_s \quad k_x = p_x - y/\beta_s \tag{20}$$

and coordinates of the Larmor center

$$d_x = x/2 - \beta_s p_y/2 \quad d_y = y/2 + \beta_s p_x/2 \, ; \tag{21}$$

namely,

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = \sqrt{\frac{\beta_s}{2}}\begin{pmatrix} k_y \\ k_x \end{pmatrix} , \quad \begin{pmatrix} \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} = -\sqrt{\frac{2}{\beta_s}}\begin{pmatrix} d_x \\ d_y \end{pmatrix} . \tag{22}$$

When the adapter $\mathcal{C}$ is matched with an adjacent downstream solenoid, i. e. $\alpha = 0, \beta = \beta_s$, the horizontal degree of freedom of the incoming uncoupled beam transforms into the cyclotron mode inside the solenoid, while the vertical one transforms into the drift mode, and the emittances are preserved:

$$\begin{aligned} \varepsilon_x^2 \; &\equiv \langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2 = \varepsilon_c^2 \equiv \langle \tilde{a}_1^2\rangle\langle \tilde{a}_2^2\rangle - \langle \tilde{a}_1\tilde{a}_2\rangle^2 = \\ &= (4/\beta^2)\left(\langle k_x^2\rangle\langle k_y^2\rangle - \langle k_x k_y\rangle^2\right) \\ \varepsilon_y^2 \; &\equiv \langle y^2\rangle\langle p_y^2\rangle - \langle y p_y\rangle^2 = \varepsilon_d^2 \equiv \langle \tilde{a}_3^2\rangle\langle \tilde{a}_4^2\rangle - \langle \tilde{a}_3\tilde{a}_4\rangle^2 = \\ &= (\beta^2/4)\left(\langle d_x^2\rangle\langle d_y^2\rangle - \langle d_x d_y\rangle^2\right) \end{aligned} \tag{23}$$

with the brackets $\langle ...\rangle$ standing for an ensemble averaging. For a particular case of the round beam inside the solenoid, when $\langle d_x^2\rangle = \langle d_y^2\rangle \equiv d^2$, $\langle d_x d_y\rangle = 0$ and similar momentum relations, it yields

$$\varepsilon_x = \beta k^2/2 \, , \quad \varepsilon_y = 2d^2/\beta \, . \tag{24}$$

The solenoid with an opposite field switches mapping: the horizontal degree of freedom is mapped onto the drift mode and the vertical plane is mapped onto the cyclotron mode.

Similar relations take place for the reverse, circular-to-uncoupled transformations $\tilde{\mathcal{C}}$.

6 LOCAL ROTATION INVARIANCE

When the rotation invariance is local (continuous):

$$\frac{d\beta}{ds} = -\frac{2\alpha}{p_0} \, , \quad \frac{d\phi_\pm}{ds} = \frac{1}{p_0}\left(\frac{1}{\beta} \pm \frac{1}{\beta_s}\right) . \tag{25}$$

and

$$\beta'' - \frac{\beta'^2}{2\beta} + \frac{(\gamma_0'\beta)'}{\beta_0^2\gamma_0} + \frac{2\beta}{p_0^2}\left(\frac{1}{\beta_s^2} - \frac{1}{\beta^2}\right) - \frac{2K}{|M_m|} = 0 \, . \tag{26}$$

Here β_0 and γ_0 are the relativistic factors, $p_0 = mc\beta_0\gamma_0$ is the total (longitudinal) momentum, M_m is the CAM of the boundary particle with the offset r_m and $K = \dfrac{2Ie}{mc^3\beta_0^3\gamma_0^3}$ is the so-called generalized perveance, which takes into account the space charge.

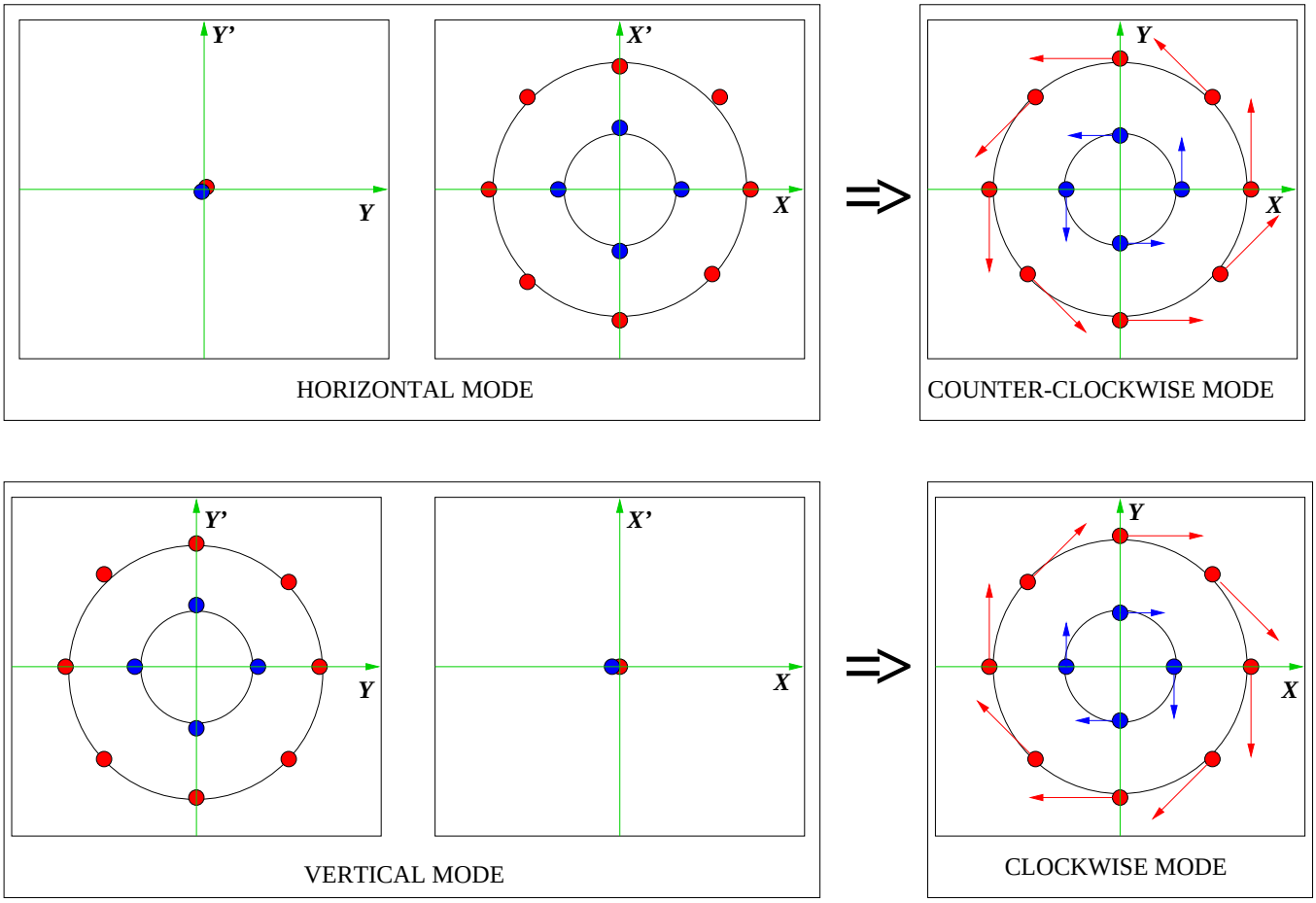

Figure 1: Schematic illustration of the uncoupled-to-circular beam adapter: horizontally and vertically polarized modes are transformed into circular modes of opposite helicities. Blue and red dots represent particles with smaller or r actions. Arrows on the circular mode portrets show particle momenta, proportional to the offsets. For simplicity, all the phase portrets are depicted as circles; generally, tilted ellipses are mapped onto each other. Direction of external arrows => specify the direction of transformation. Reverse direction of both upper and lower arrows (<=) would correspond to the reverse, circular-to-uncoupled adapter.

7 DIAGONALIZATION OF BEAM MATRIX

The beam matrix

$$\Sigma_{i,j} = \langle x_i x_j \rangle$$

describes the beam distribution. If $\mathcal{M}$ is a transfer matrix, then the new Σ-matrix is $\mathcal{M}\Sigma\mathcal{M}^T$. The uncoupled state is described by the block-diagonal Σ-matrix in the original Cartesian coordinates; its 4D emittance is a product of the 2D emittances. Normally the phase distributions are homogeneous, in this case the Σ-matrix is diagonal in the matched uncoupled basis (the transfer matrix in this case $\mathcal{M} = V^{-1}$):

$$\Sigma = \mathrm{Diag}(\varepsilon_x, \varepsilon_x, \varepsilon_y, \varepsilon_y), \qquad (27)$$

where Diag(...) is a diagonal matrix with elements listed as the arguments. In the same way, the Σ-matrix of a round beam is diagonal in the matched circular basis.

The Σ-matrix of a round beam can be expressed in rotation-invariant terms:

$$\Sigma = \frac{1}{2}\begin{pmatrix} \Sigma & \langle M\rangle \mathsf{J} \\ -\langle M\rangle \mathsf{J} & \Sigma \end{pmatrix}; \quad \Sigma = \begin{pmatrix} \langle r^2\rangle & \langle \vec{r}\vec{p}\rangle \\ \langle \vec{r}\vec{p}\rangle & \langle p^2\rangle \end{pmatrix}; \qquad (28)$$

$$\mathsf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This beam matrix is diagonalized by the circular basis with

$$\beta = \frac{\langle r^2\rangle}{\sqrt{\langle r^2\rangle\langle p^2\rangle - \langle \vec{r}\vec{p}\rangle^2}}, \quad \alpha = -\frac{\langle \vec{r}\vec{p}\rangle}{\sqrt{\langle r^2\rangle\langle p^2\rangle - \langle \vec{r}\vec{p}\rangle^2}} \qquad (29)$$

leading to

$$\Sigma = \mathrm{Diag}(\varepsilon_1, \varepsilon_1, \varepsilon_2, \varepsilon_2) \qquad (30)$$

with the emittances

$$2\varepsilon_{1,2} = \pm\langle M\rangle + \sqrt{\langle r^2\rangle\langle p^2\rangle - \langle \vec{r}\vec{p}\rangle^2} \geq 0\,. \qquad (31)$$

These partial emittances are preserved by any symplectic transformation.

The total 4D emittance is a product of these partial emittances :

$$4\epsilon \equiv 4\varepsilon_1\varepsilon_2 = \langle r^2\rangle\langle p^2\rangle - \langle \vec{r}\vec{p}\rangle^2 - \langle M\rangle^2 \qquad (32)$$

[S. Nagaitsev, A Shemyakin]. The 4D emittance in terms of the canonical and kinetic momenta are absolutely identical: a transfer from one to another is equivalent to rotation imposed on the beam as a whole, which does not change the total emittance.

8 ROUND BEAMS FOR CIRCULAR COLLIDERS

For circular colliders, round beams in the interaction region (IR) are known to be beneficial: angular momentum preservation allows to increase the beam-beam tune shift and so the luminosity. Conventional round-beams schemes require $\varepsilon_x = \varepsilon_y$ and $\nu_x = \nu_y$ Another approach to get the beams round, the Möbius accelerator [5], based on beam rotator optics [6], is studied experimentally at CESR [7]. This scheme also leads to emittance identity and effective tune degeneration:x the resulting normal tunes are inevitably separated by $1/2$.

Matched adapters bounding the IR opens a way that is free from all these limitations.

Figure 2: Beam Rounder

This beam rounder allows to have:

- round beam inside it;
- the same uncoupled beam outside it, as it was without the rounder;
- rotation-invariant revolution matrix;
- all these features are kept for any tunes, emittances and the solenoidal field inside.

9 REFERENCES

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