

Normal Form Methods and Optimization for Nonlinear Properties of Cooling Channels—Part I

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The normal form method provides an excellent mechanism to diagnose the nonlinear behavior of repetitive systems. The proposed cooling channels for muon accelerators are long chains of cells consisting of accelerating cavities, energy absorbers and guiding magnets as solenoids, and the Differential Algebraic (DA) normal form algorithm is applicable to optimize them. This paper together with a sister paper [1] will explain the normal form method.

1. Introduction

The study of an ensemble of particles in beamlines is analogous to that of astronomical systems. There is a long history of the perturbative treatment of such motion, beginning with the work of Poincare[2]. The first step of the treatment is finding the flow of the 2ν equations of motion $d/dt \vec{z} = \vec{f}(\vec{z}, t)$, i.e. the function representing the transformation of initial conditions $\vec{z}_i$ into final solutions $\vec{z}_f$ if the equations have unique solutions. In the case the motion is autonomous or has a periodicity Δt , it is common to study the motion along discretized time steps Δt , resulting in the so-called Poincare map that transports coordinates by the period Δt :

$$\vec{z}_f = \mathcal{M}(\vec{z}_i).$$

The Differential Algebraic (DA) technique provides an efficient and elegant way to compute such maps with Taylor coefficients to arbitrary order [3].

Currently designed beamlines for muon beam cooling are long channels of cells, each cell consisting of beam guiding magnets like solenoids, accelerating cavities and energy absorbers. The whole channel can be viewed as a repetitive system.

The normal form method provides an excellent mechanism to analyze repetitive systems, and the DA technique allows an efficient algorithm for the method. The algorithm consists of a series of coordinate transformations. The idea is to perform a nonlinear change of variables in such a way that depending on the eigenvalue spectrum of the linear part of the Poincare map, approximate invariants can be read off, so the motion in the new variables becomes highly regular. Indeed, in the simplest non-resonant case, without damping it just follows an approximately circular motion. When any damping is present, which is the case for beam cooling systems, it leads to a spiral-like structure. Since a mere change of coordinates does not affect the general topological properties, this approach usually allows for a much more detailed analysis of the motion. Thus the method can assist the analysis and optimization of muon cooling channels in a systematic and efficient manner.

In this paper, we will show the mechanism of the DA normal form coordinate transformation. In a sister paper [1], we will discuss about various special cases including damped systems.

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2. The Differential Algebraic Normal Form Algorithm

The normal form algorithm consists of a sequence of coordinate transformations $\mathcal{A}$ of the n th order Taylor transfer map $\mathcal{M}_n$:

$$\mathcal{A} \circ \mathcal{M}_n \circ \mathcal{A}^{-1}. \quad (1)$$

2.1. Linear Diagonalization and Eigenvalues

The first such coordinate transformation is a linear coordinate transformation that diagonalizes the linear part of the map. For this process, we have to assume that there are 2ν pairwise distinct eigenvalues. This together with the fact that no eigenvalue should be unity and that their product is positive are the only requirements we have to demand for the map. The eigenvalues are grouped in such a way that complex conjugate pairs are together; the remaining real eigenvalues are grouped in pairs of equal sign, which is possible because of the positive product of all eigenvalues. After diagonalization, the linear map assumes the form:

$$\begin{pmatrix} r_1 e^{+i\mu_1} & & & & \\ & r_1 e^{-i\mu_1} & & & \\ & & 0 & \ddots & 0 \\ & & & \ddots & r_\nu e^{+i\mu_\nu} \\ & & & & r_\nu e^{-i\mu_\nu} \end{pmatrix}, \quad (2)$$

here the phases μ_j are either purely real or purely imaginary. For stable systems, none of the $r_j e^{\pm i\mu_j}$ must exceed unity in modulus. For area preserving systems the determinant is unity, so the product of the r_j must be unity. This implies that for such systems, for any $r_j < 1$ there is another with $r_j > 1$. Thus stable area preserving systems have $r_j = 1$ for all j , because otherwise there would be one j for which r_j exceeds unity, and thus at least one of $r_j e^{\pm i\mu_j}$ would have modulus larger than unity. This would also happen if a μ_j were imaginary. So all μ_j are real, and they are even nonzero because we demanded distinct eigenvalues.

The eigenvectors $s_j^\pm$ belonging to the eigenvalue $r_j e^{\pm i\mu_j}$ provide a basis. We consider another set of vectors $t_j^\pm$ associated to the $s_j^\pm$ as follows:

$$t_j^+ = (s_j^+ + s_j^-)/2, \quad t_j^- = (s_j^+ - s_j^-)/2i. \quad (3)$$

In case of complex $s_j^\pm$, the $t_j^\pm$ are just the real and imaginary parts and thus are real. In the unstable case, t_j^+ is real and t_j^- is imaginary. Obviously the $s_j^\pm$ can be expressed in terms of the $t_j^\pm$ as

$$s_j^+ = t_j^+ + i t_j^-, \quad s_j^- = t_j^+ - i t_j^-.$$

We will perform the manipulations in the $s_j^\pm$, whereas the results are most easily interpreted in the $t_j^\pm$.

2.2. Order-by-Order Construction of Transformation

We now show that a map in the $s_j^\pm$ can be subjected to nonlinear coordinate transformations that simplify the nonlinear terms. The transformation to the new coordinates is carried out iteratively order-by-order; the first was the diagonalization of the linear part. All further steps are purely nonlinear and do not affect the linear part anymore. The m th step transforms only the m th order of the map and leaves the lower orders unaffected.

We begin the m th step by splitting the momentary map $\mathcal{M}$ into its linear and nonlinear parts $\mathcal{R}$ and S_m , i.e. $\mathcal{M} = \mathcal{R} + S_m$. The linear part $\mathcal{R}$ has the form of (2). Then we perform a transformation using $\mathcal{A}_m$ that has the form

$$\mathcal{A}_m = \mathcal{I} + \mathcal{T}_m, \quad (4)$$

where $\mathcal{T}_m$ vanishes to order $m - 1$. Then, up to order m , we have

$$\mathcal{A}_m^{-1} =_m \mathcal{I} - \mathcal{T}_m, \quad (5)$$

where “ $=_m$ ” is an equivalence relation: For f and g in C^m , $f =_m g$ if and only if $f(0) = g(0)$ and all the partial derivatives of f and g agree at 0 up to order m . In fact,

$$(\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{I} - \mathcal{T}_m) =_m (\mathcal{I} - \mathcal{T}_m) + \mathcal{T}_m \circ (\mathcal{I} - \mathcal{T}_m) =_m (\mathcal{I} - \mathcal{T}_m) + \mathcal{T}_m = \mathcal{I},$$

where in the second step we used the fact that $\mathcal{T}_m$ has no parts of order lower than m , hence only the linear part of anything that is inserted into it contributes to order m . We study the effect of the transformation up to order m .

$$\begin{aligned} \mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1} &= (\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{R} + S_m) \circ (\mathcal{I} - \mathcal{T}_m) = (\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{R} + S_m - \mathcal{R} \circ \mathcal{T}_m) \\ &= \mathcal{R} + S_m + (\mathcal{T}_m \circ \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m) \end{aligned}$$

For the first step, we used $S_m \circ (\mathcal{I} - \mathcal{T}_m) =_m S_m$, which holds because S_m is nonlinear and $\mathcal{T}_m$ is of order m . In the second step we used $\mathcal{T}_m \circ (\mathcal{R} + S_m - \mathcal{R} \circ \mathcal{T}_m) =_m \mathcal{T}_m \circ \mathcal{R}$, which holds because $\mathcal{T}_m$ is of exact order m and everything in the second term is nonlinear except $\mathcal{R}$. The last line indicates that S_m can be simplified by choosing the commutator

$$C_m \stackrel{\text{def}}{=} \{\mathcal{T}_m, \mathcal{R}\} = \mathcal{T}_m \circ \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m$$

appropriately. Recall that we now have the form

$$\mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1} =_m \mathcal{R} + S_m + C_m.$$

Indeed, if the range of C_m is the full space, then S_m can be removed entirely. However, as we shall see, most of the time this is not the case.

2.3. Removal of m -th Order Nonlinear Terms

Let $(\mathcal{T}_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-)$ be the Taylor expansion coefficient of $\mathcal{T}_{mj}^\pm$ in the j -th component pair of $\mathcal{T}_m$ with respect to $(s_1^+)^{k_1^+} (s_1^-)^{k_1^-} \dots (s_v^+)^{k_v^+} (s_v^-)^{k_v^-}$. So $\mathcal{T}_{mj}^\pm$ is written as

$$\mathcal{T}_{mj}^\pm = \sum_{k_1^+, \dots, k_v^-} (\mathcal{T}_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-) \cdot (s_1^+)^{k_1^+} (s_1^-)^{k_1^-} \dots (s_v^+)^{k_v^+} (s_v^-)^{k_v^-}.$$

Similarly we denote the coefficients of C_m by $(C_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-)$. Because $\mathcal{R}$ is diagonal, it is possible to express the coefficients of C_m in terms of the coefficients of $\mathcal{T}_m$.

$$\begin{aligned} (C_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-) &= \left(\left(\prod_{l=1}^v r_l^{(k_l^+ + k_l^-)} \right) \cdot e^{i\vec{\mu} \cdot (\vec{k}^+ - \vec{k}^-)} - r_j \cdot e^{\pm i\mu_j} \right) \cdot (\mathcal{T}_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-) \\ &= C_{mj}^\pm(\vec{k}^+, \vec{k}^-) \cdot (\mathcal{T}_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-). \end{aligned} \quad (6)$$

A term in $S_{mj}^\pm$ can be removed if and only if the factor $C_{mj}^\pm(\vec{k}^+, \vec{k}^-)$ is nonzero; if it is nonzero, we demand

$$(S_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-) + (C_{mj}^\pm | k_1^+, k_1^-, \dots, k_v^+, k_v^-) = 0,$$

so the required term in $\mathcal{T}_{mj}^\pm$ is just the negative of the respective term in $S_{mj}^\pm$ divided by $C_{mj}^\pm(\vec{k}^+, \vec{k}^-)$.

The outcome of the whole normal form transformation depends upon the conditions under which $C_{mj}^\pm(\vec{k}^+, \vec{k}^-)$ vanishes. This is the case if and only if the moduli and the arguments of $r_j \cdot e^{\pm i\mu_j}$ and $(\prod_{l=1}^v r_l^{(k_l^+ + k_l^-)}) \cdot e^{i\vec{\mu} \cdot (\vec{k}^+ - \vec{k}^-)}$ are identical.

3. Outlook

The m th step of the normal form method shows how to choose the terms of exact order m of $\mathcal{T}_m$. Once $\mathcal{T}_m$ is obtained, the transformation map to m th order $\mathcal{A}_m$ and its inverse transformation map $\mathcal{A}_m^{-1}$ are known through (4) and (5), and we can perform the nonlinear coordinate transformation (1), which simplifies the system.

The discussion about the conditions under which the term $C_{m,j}^{\pm}(\vec{k}^+, \vec{k}^-)$ vanishes was not covered yet. We will show various special cases and draw conclusions in the sister paper [1].

Acknowledgments

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References

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