Observed Smooth Energy Fitted by Parametrized Quintessence

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This paper proposes two phenomenological quintessence potentials with parameters fitted to the presently observed ratio $\mathcal{R}_0$ of smooth energy to clustered mass and limits on the equation of state parameter $w_Q(a) = P/\rho_Q$, for which the quintessence potential is now slow- or fast-rolling. These two potentials are intended to illustrate rather different quintessence phenomenology since tracking and into the near future. Neither can be a fundamental potential derivable from present-day string theory.

The condition for a tracker solution to exist is [1] that the logarithmic derivative of the scalar field potential, $-V'/V$, $V' \equiv dV/d\phi$, be a slowly decreasing function of scalar field ϕ or that

$$0 < \Gamma(a) - 1 \equiv d(-V/V')/d\phi \tag{1}$$

be nearly constant as function of cosmological scale a. It is convenient to measure the steepness of the potential by the logarithmic derivative $\beta(\phi) \equiv -d\ln V/d\ln\phi$, so that

$$\Gamma(a) - 1 = d(\phi/\beta)/d\phi = 1/\beta(\phi) \cdot (1 - d\ln\beta/d\ln\phi). \tag{2}$$

We will consider below, an inverse-power potential, for which β is strictly constant, and an "isothermal" potential, for which β slowly decreases with ϕ so as to keep the equation of state w_Q = constant [2].

The ordinary inflationary parameters are then

$$\eta(\phi) \equiv V''/V, \quad 2\varepsilon \equiv (V'/V)^2, \quad \Gamma = V\ddot{V}/\dot{V}^2 = \eta(\phi)/2\varepsilon(\phi). \tag{3}$$

When η, $2\varepsilon \ll 1$, a tracking potential will be *slow-rolling*, meaning that $\ddot{\phi}$ in the scalar field equation of motion and $\dot{\phi}^2$ in the quintessence energy density are both negligible. This slow-roll approximation is usually satisfied in ordinary inflation. In the early e-folds of tracking quintessence, however, $-V'/V = \beta(\phi)/\phi$ is slowly changing, but need not itself be small, if $\beta \neq 0$. This establishes the important distinction between static or quasi-static quintessence, for which the EOS parameter and scale exponent $n_Q(a) = 3(1 + w_Q)$ have present values $n_Q(1) \sim 0$, $w_{Q0} \sim -1$, and sensibly dynamical quintessence, for which $n_Q > 1$, $w_{Q0} > -2/3$. At the observationally allowed upper limit $w_{Q0} = -0.5$, the first few scale e-folds would still be *fast-rolling*. Such dynamical quintessence will become slow-roll only in the very far future, after many e-folds of the quintessence field slowly driving Δ from 1 to zero and w_Q from its present value to zero. This means that while the slow-roll approximation is applicable to ordinary inflation, dynamical quintessence generally requires exact solution of the equations of motion.

For tracking potentials derived from physical principles, it is reasonable to assume that $\beta \sim$ constant, at least since tracking began. We therefore parametrize quintessence so that the SUSY-inspired potential

$$V(\phi) = V_0(\phi_0/\phi)^\beta, \quad \beta = 3.5 \tag{4}$$

is inverse power law, and $\Gamma - 1 = 1/\beta$ = constant.

For comparison, we also consider the isothermal equation of state

$$V(\phi) = V_0[\sinh(\alpha\phi_0)/\sinh(\alpha\phi)]^\beta, \quad \beta = 2, \quad \alpha \equiv \sqrt{3/(2\beta + \beta^2)} = 0.612, \tag{5}$$

for which $\Gamma - 1 = \Omega_B/\beta$ is not constant, but decreases from $1/\beta$ when tracking begins, to zero in the far future, as the clustered mass fraction $\Omega_B - > 0$. This potential is called "isothermal" because

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*University of Pennsylvania, Philadelphia for it, the quintessence pressure/density ratio $w_Q = -2/(2+\beta) = -1/2$ and $n_Q = 3\beta/(2+\beta) = 3/2$ are constant once matter dominates over radiation. This makes the tracker approximation exact for the isothermal potential: $\Delta = 1$ and $-V'/V = n_Q \cdot da/a$. This potential allows a scaling solution $V(a)$, $\rho_Q(a) \sim a^{-n_Q}$ [3] and has the inverse power potential $V = V_0(\phi_0/\phi)^\beta$ and the exponential potential $V = V_0 \exp(-\sqrt{n_Q}\phi)$ as limits for $\alpha\phi \ll 1$ and $\gg 1$, respectively. It therefore interpolates between an inverse power potential at early times and an exponential potential at late times.

In each of these potentials, the two parameters have been chosen to fit the present values $\mathcal{R}_0 = 2$ and $\rho_{Q0} = 2\rho_{cr}/3 = 2.7E - 47\,\mathrm{GeV}^4$, so that for $w_{Q0} = -0.5$, $n_{Q0} = 3/2$ the present value of the potential is $V_0 = V(\phi_0) = 1.013E - 47\,\mathrm{GeV}^4$. This requires tracking to begin after matter dominance, reaching present values $\phi_0 = 2.47$ for the inverse-power and 1.87 for the isothermal EOS, respectively. Tracking continues indefinitely for the isothermal EOS, but is of relatively short duration for the inverse-power potential. We believe these parametrizations of $V(\phi)$ to be representative of reasonable smooth potentials, over the red-shift range $z < 5$ that is observationally accessible. If we considered lower values of β for the present ratio $\mathcal{R}_0 = 2$, w_{Q0} would then decrease from $-1/2$ to -1 as the potential became faster rolling.

For the inverse-power potential, $-V'/V = \beta/\phi$, and for the isothermal potential $-V'/V = \beta\alpha/\tanh(\alpha\phi)$ respectively, so that, for $w_{Q0} = -0.5$, η and perforce ε are not now small. Thus both the dynamical quintessence potentials we are considering are now still fast-rolling. The isothermal potential, with $\beta = 2$, asymptotically approaches $\exp(-1.22\phi)$ and will never be slow-rolling. The inverse-power potential, with $\beta = 3.5$, will become slow-rolling once $\phi > \beta$ and will asymptotically approach a de Sitter solution in the distant future. In the observable recent past, its w_Q increases with z approximately as $w_Q(z) \approx -0.5 + 0.016z$.

After transients depending on initial conditions, and a frozen epoch ($w_Q \approx -1$), the scalar field overshoots and then converges onto a solution

$$n_Q(a) = n_B(a)\beta/(2+\beta) = n_B(a)/2 \tag{6}$$

which tracks the background $n_B(a)$. During the tracking regime, until quintessence dominates, $\Delta \approx 1$ and $\Omega_Q \approx n_Q\phi^2/\beta(\phi)$ increases quadratically with field strength. Driven by the background EOS which is changing around $z_{cr} = 3880$, the quintessence equation of state parameter $w_Q(a)$ slowly decreases. The inverse-power potential reaches its tracker at $\log a \approx -3.2$, but remains there only until $\log a \approx -1.8$, when the growth of the quintessence field slowly drives w_Q down from the tracker value -0.364 towards -1 in the very far future (solid curve in Figure 1). For the inverse-power potential, the tracker approximation holds only briefly, and thereafter seriously overestimates w_Q. The exact isothermal solution reaches its tracker later, at $\log a > -1$, but remains exactly on tracker thereafter (dashed curve in Figure 1).

Once reaching the tracker, the isothermal equation of state stays constant at $w_Q(\phi) = -2/(2+\beta) = -1/2$. The evolution with cosmological scale is fixed by the scaling $V/V_0 = \rho_Q/\rho_{Q0} = a^{-3/2}$, which makes

$$\mathcal{R}(a) = \sinh^2(\alpha\phi), \quad \Omega_Q = 1 - \Omega_B = \tanh^2(\alpha\phi) = \mathcal{R}_0 a^{3/2}/(1 + \mathcal{R}_0 a^{3/2}). \tag{7}$$

For the present ratio $\mathcal{R}_0 \sim 2$ one has $\alpha\phi_0 \leq 1.146$ and $\phi_0 \leq 1.87$. The observations that the Universe is now accelerating fix the bounds

$$-2/(2+\beta) \leq -1/2, \quad \beta \leq 2, \quad \alpha \geq 0.612, \quad \phi_0 \leq 1.87. \tag{8}$$

For $\beta = 0$, the potential is static: Λ-dominance started at redshift $R_0^{1/3} - 1 = 0.260$, the universe was then already accelerating $-q = 0.333$, and is now accelerating faster $-q_0 = 0.5$. For larger β, the scalar field is more dynamic. At the upper limit, $\beta = 2$ the isothermal potential rolls only as fast as the observed acceleration ($-q_0 \geq 0$) allows, tracking starts only after matter domination, and acceleration starts only now. Nevertheless, $\rho_Q(a)$ was still subdominant to the radiation density all the way back into the era of Big Bang nucleosynthesis [5].

As mentioned above, the tracking approximation is exact for the isothermal potential, but only briefly valid for the inverse-power potential. For $\alpha\phi \ll 1$, the background dominates and the isothermal equation of state reduces to the inverse power potential $V \sim \phi^{-\beta}$, for which, along the tracker, ϕ, $\sqrt{\mathcal{R}} \sim a^{3/(2+\beta)}$. We shall, however, also be interested in the quintessence dominated era $\alpha\phi > 1$, when different potentials give different present and future behavior.

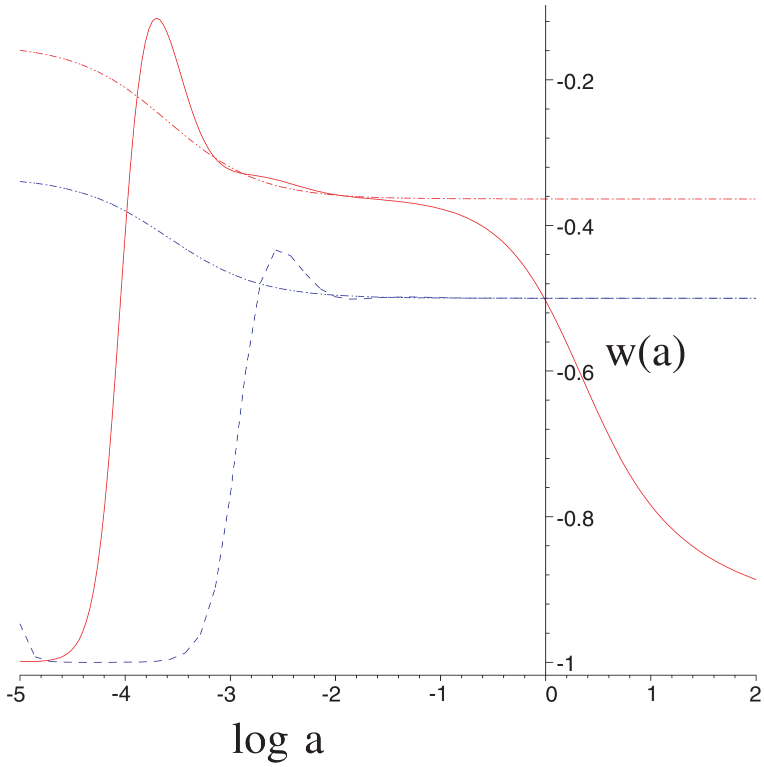

Figure 1: Exact and tracker approximation solutions w_Q for inverse power and isothermal potentials chosen to give present values $\mathcal{R}_0 = 2$, $w_{Q0} = -0.5$. The exact w_Q values for the inverse-power and the isothermal quintessence potentials (4) and (5) are shown by the solid and dashed curves. The tracker approximation (dash-dotted curves) is exact for the isothermal potential, but holds only briefly for the inverse-power potential and considerably overestimates w_Q once quintessence is appreciable. Asymptotically, $w_Q - > -0.5, 1$, so that, if unchanged, both potentials would show an event horizon.

If w_Q is not constant, we could still reconstruct the quintessence potential from $w_Q(z)$. Indeed $dw_Q(z)/dz$ is positive generically for quintessence and generally negative for k-essence [6], an alternative in which the scalar field has a non-linear kinetic energy instead of a potential. In principle, $w_Q(z)$ is observable in high red-shift supernovae [13], in cluster evolution [7, 8] and in gravitational lensing [9]. In practice, $w_Q(z)$ is poorly constrained observationally and theoretically [10, 11, 12]. Over the small redshift range for which smooth energy dominates and for which sensitive measurements are possible, the effects of varying $w_Q(z)$ are much smaller than the present uncertainties in measurement and in cosmological model. Theoretically, the luminosity distance to be measured in high-redshift supernova and in comoving volume number density measurements depends on integrals which smooth out the sensitivity to $w_Q(z)$ [11]. Until Ω_{m0} is determined to (1–2)% accuracy, SNAP [13] and other future experiments will only be able to determine some w_Q effective over the observable redshift range $z < 2$, to tell whether the smooth energy is static or dynamic. In the far future, however, the differences among different quintessence potentials will asymptotically become substantial.

The difference between the two potentials (4,5) appears only in the evolution of w_Q now and in the future, when $\alpha\phi > 1$. For the inverse power potential, the growth of quintessence drives $\Delta(a)$ below unity and $w_Q(a)$ decreases slowly towards -1. The quintessence energy density ρ_Q and the expansion rate $H \to$ constant, drive ultimately towards a de Sitter universe, which inflates exponentially. The isothermal EOS will approach the exponential potential, w_Q stays constant, ρ_Q and H continue to decrease, and inflation is power-law. For both these potentials, acceleration continues indefinitely, so that two observers separated by fixed coordinate distance, ultimately have relative speed $\geq c$. Because an event horizon exists in both cases, a local observer cannot construct an S-matrix. This shows that neither of our phenomenological potentials is derivable from field theory or string theory. To be string-inspired and lead to an event horizon, the quintessence EOS would have to ultimately change back from accelerating to decellerating

($w_Q > -1/3$) [14].

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