Determinations of $\alpha(M_Z)$: Comparison and Prospects

Jens Erler*
Department of Physics and Astronomy, University of Pennsylvania, Philadelphia

I review and compare various techniques to obtain the value of the QED coupling, α, at the Z pole. GigaZ precisions would require a much more accurate determination than available today. A combination of the virtues of current methods may help to achieve this goal.

The value of the QED coupling constant at Z pole energies,

$$\alpha(M_Z) = \frac{\alpha}{1 - \Delta\alpha(M_Z)}, \tag{1}$$

continues to induce the dominant theoretical uncertainty in the interpretation of the observables from LEP 1 and the SLC. A future linear e^+e^- collider with GigaZ [1, 2] capability would be able to greatly improve the current measurements, with $\alpha(M_Z)$ ultimately dominating the overall uncertainty. Therefore, it would be essential to improve the present error, $\delta\alpha(M_Z)/\alpha(M_Z) \approx \pm 2\times 10^{-4}$, by at least a factor of two or three. While the discussion will be limited to $\alpha(M_Z)$, in its essence it also applies to the two-loop hadronic uncertainty in the muon anomalous magnetic moment.

All methods and renormalization schemes to determine $\alpha(M_Z)$ utilize experimental data up to some cut-off, $s_{\rm cut}$, beyond which perturbative QCD (PQCD) is evoked. They differ due to differences in the data sets and treatments; the choice of $s_{\rm cut}$; the reference renormalization scheme; the option to add experimental information from τ-decays [3] (assuming isospin invariance); space-like vs. time-like integration; the treatment of heavy quarks; the use of a QCD sum rule [4] and/or a resummation optimization; and so on.

In terms of the photon polarization function, $\Pi_\gamma(s)$, $\Delta\alpha$ is given by,

$$\Delta\alpha(s) = -4\pi\alpha\left[\Pi'_\gamma(s) - \Pi'_\gamma(0)\right] \quad \text{(on-shell scheme)}, \qquad \Delta\hat\alpha(\mu) = 4\pi\alpha\hat\Pi_\gamma(0) \quad (\overline{\rm MS}\text{ scheme}). \tag{2}$$

At the one-loop level in perturbation theory one finds for a fermion of charge Q_f (N_c^f is the color factor),

$$\Delta\alpha = \frac{\alpha}{3\pi}\sum_f Q_f^2 N_c^f\left[\ln\frac{s}{m_f^2} - \frac{5}{3}\right] \quad \text{(on-shell)}, \qquad \Delta\hat\alpha(\mu) = \frac{\alpha}{3\pi}\sum_f Q_f^2 N_c^f \ln\frac{\mu^2}{m_f^2} \quad (\overline{\rm MS}). \tag{3}$$

Numerically, $\alpha^{-1}(M_Z) \sim 129$ and $\hat\alpha^{-1}(M_Z) \sim 128$. Alternatively, the on-shell quantity can be represented by a once subtracted dispersion relation (SDR),

$$\Delta\alpha(s) = -4\alpha s \int_{4m_\pi^2}^{\infty} ds' \frac{{\rm Im}\Pi'_\gamma(s')}{s'(s'-s-i\epsilon)}. \tag{4}$$

In the case of hadrons, $R(s) = 12\pi{\rm Im}\Pi'_\gamma(s)$, and in the standard SDR approach one has,

$$\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = -\frac{\alpha M_Z^2}{3\pi}\left[\underbrace{\int_{4m_\pi^2}^{s_{\rm cut}} \frac{R(s)ds}{s(s-M_Z^2-i\epsilon)}}_{\rm DATA} + \underbrace{\int_{s_{\rm cut}}^{\infty} \frac{R(s)ds}{s(s-M_Z^2-i\epsilon)}}_{\rm PQCD}\right], \tag{5}$$

*erler@ginger.hep.upenn.edu

Table I Comparison of QCD analyses. The values and uncertainties quoted in the original papers are adjusted to $\alpha_s(M_Z) = 0.120$ (fixed). The quark mass uncertainty in the BF-MOM scheme is from the pole masses which cannot be improved. The UDR approach uses $\overline{\text{MS}}$ masses; their error can be expected to decrease significantly in the future. The theory errors in the SDR and UDR approaches include the uncertainty introduced by assuming quark-hadron duality near the cut-off of the dispersion integrals. The PQCD error in the BF-MOM approach has not been estimated, yet.

	SDR	BF-MOM	UDR
quantity	$R(s)$	$D(-s)$	$\beta(\mu)$
$\Delta\alpha_{\text{had}}^{(5)}$	0.02770	0.02773	0.02779
quoted uncertainty	0.00015	0.00018	0.00020
α_s-dependence	linear approximation	not available	fully analytic
contribution from J/Ψ resonances	65%	15%	0%
error from quark masses	0	0.00010	0.00015
error from data	0.00015	0.00015	0.00011
theory error	0.00002	(0)	0.00007
s_{cut}	1.8 GeV	2.5 GeV	1.8 GeV
reference	[10]	[12]	[5]

where the superscript indicates application to all quarks except the top. In the $\overline{\text{MS}}$ scheme it is more natural to work with an unsubtracted dispersion relation (UDR) [5],

$$\Delta\hat{\alpha}_{\text{had}}^{(3)}(s_{\text{cut}}) = \frac{\alpha}{3\pi} \underbrace{\int_{4m_\pi^2}^{s_{\text{cut}}} \frac{R(s)ds}{s - i\varepsilon}}_{\text{DATA}} + 2\alpha \underbrace{\int_0^{2\pi} d\theta\, \hat{\Pi}_\gamma^{(3)}(\theta)}_{\text{PQCD}}, \tag{6}$$

where the second integral is along a circle with radius $s = s_{\text{cut}}$. Since typical values for $\sqrt{s_{\text{cut}}}$ are 1.8 GeV [6] and 2.5 GeV [7], one only needs to include the three light quarks in Eq. (6). One then uses an analytical solution [5] to the order $\alpha\alpha_s^3$ and α^2 renormalization group evolution (RGE) to *decrease* $\mu^2 = s_{\text{cut}}$ to $\mu^2 = \hat{m}_c^2(\hat{m}_c)$ (the $\overline{\text{MS}}$ charm mass) where one matches the effective field theories with three and four effective quark flavors. The matching is performed at order $\alpha\alpha_s^2$ at which subtle effects from internal charm quark loops have to be taken into account. However, well below the charmonium threshold these are small and strongly decoupling despite $\hat{m}_c < s_{\text{cut}}$. Then one evolves up in energy and includes the τ-lepton and the b-quark. However, this is successful *only* when a short-distance quark mass definition (such as $\overline{\text{MS}}$) is used. Transition to the on-shell mass definition would introduce large π^2 terms, rendering application to bottom (charm) quarks questionable (impossible). Thus, in the UDR approach bottom and charm effects can be described *entirely* within PQCD, avoiding complications at heavy quark resonances. On the other hand, in the SDR approach one has to abandon PQCD in the vicinity of resonance regions.

Focussing on only one quark flavor at a time, one could relate the integral expression of the SDR approach to the analytical expressions [5] of the UDR approach. The resulting equation has the form of a specific type of QCD sum rule which could be used to *determine* $\hat{m}_c$ and $\hat{m}_b$. However, this is only the first entry in an infinite series of sum rules [8] — and not the one which uses the data most efficiently. This implies that combining the UDR approach with an appropriate QCD sum rule is a recipe to minimize the uncertainties from the b and c quark sector [9]. Another advantage of the UDR approach is that all theoretical contributions are available as explicit analytical expressions, with no need for a numerical integration. In particular, the α_s and quark mass dependences are all taken into account. This is important for global analyses in which these parameters enter in many different places, causing non-trivial and non-linear correlations. In the SDR approach only a crude linear approximation [10] is available.

Another way to reduce the impact of the resonance region is to use the analytic structure of Π_γ and to work in the Euclidean (space-like) region [7],

$$\Delta\alpha^{(5)}(M_Z^2) = \underbrace{\left[\Delta\alpha^{(5)}(M_Z^2) - \Delta\alpha^{(5)}(-M_Z^2)\right]}_{\text{PQCD}} + \underbrace{\left[\Delta\alpha^{(5)}(-M_Z^2) - \Delta\alpha^{(5)}(-s_{\text{cut}})\right]}_{\text{PQCD}} + \underbrace{\Delta\alpha^{(5)}(-s_{\text{cut}})}_{\text{DATA}}. \tag{7}$$

The first term is the analytical continuation from the Minkowski (time-like) to the Euklidean region. The second term represents the RGE in the perturbative Euklidean domain in which $R(s)$ is replaced by the Adler D function. It is computed in the *gauge dependent* background field momentum subtraction (BF-MOM) scheme up to three-loop order [11]. Unlike the $\overline{\rm MS}$ scheme, the BF-MOM scheme is a mass-dependent renormalization scheme. Thus both, the UDR and BF-MOM approaches, depend explicitly on the quark masses. In the latter, the quark pole masses are used. This is disadvantageous since a long-distance mass definition such as the pole mass has an intrinsic renormalon ambiguity of order $\Lambda_{\rm QCD}$. The heavy quark sector also contributes via the last term in Eq. (7) where, like in the SDR approach, the resonance region complicates the analysis. However, in the BF-MOM approach the resonance contribution is suppressed by about a factor of four [12]. Note, that in this approach there may be a subtle correlation between the uncertainty from the theory (quark masses) and data (resonance region) parts.

The advantage of splitting the data and theory parts as in Eq. (7) is that no reference to global or local quark hadron duality is needed. In contrast, the SDR and UDR approaches both have an explicit momentum cut-off where the transition from data (hadrons) to theory (quarks) occurs. In principle, this could give rise to a significant cut-off dependence, especially when non-perturbative (NP) effects produce a strongly oscillating form of the hadronic cross section, *i.e.* $R(s)$. Such oscillations arise neither in PQCD nor in the operator product expansion (OPE) which accounts for a certain class of NP effects. Clearly, the cut-off dependence should be kept small. However, it is not necessarily optimal to demand that it vanishes. Indeed, the importance of duality violating effects has been overemphasized in the past, and there are good reasons to believe that they are small [13]. While unidentified sources of (OPE breaking) NP effects are hazardous, they are not the only source of uncertainty. For example, a poorly converging perturbative expansion can be even more perilous.

Table I summarizes the comparison of the three methods defined by Eqs. (5), (6), and (7). Table II gives a breakdown of the theory error in $\alpha^{-1}(M_Z)$ in the UDR approach. The corresponding error in $\Delta\alpha$ is obtained by multiplying by α.

Table II Error breakdown in the UDR approach. The quoted non-perturbative QCD uncertainty is due to OPE breaking effects typically of the form e^{-C/α_s}, where C is in general complex leading to an oscillating $R(s)$. Davier and Höcker [10] fit a variety of oscillating curves to the experimental $R(s)$ around $s_{\rm cut}$ and conclude $\Delta\alpha^{-1} = \pm 0.002$. Here I use a more conservative estimate [5]. There are other non-perturbative (higher twist) effects *within* the OPE. These are of $\mathcal{O}(\alpha_s^2/\pi^2 \Lambda^4_{\rm QCD}/s^2_{\rm cut}; \alpha_s^2/\pi^2 m_K^2 f_\pi^2/s^2_{\rm cut}) \sim 2\times 10^{-7}$ and of $\mathcal{O}(\Lambda^4_{\rm QCD}/m_c^4; \Lambda^4_{\rm QCD}/m_b^4) \sim -3\ldots -7\times 10^{-6}$, respectively, and can be neglected. The parametric error due to the imperfect knowledge or α_s is excluded here; the α_s dependence is fully included in electroweak fits.

	sector	uncertainty	comment
perturbative QCD	u, d, s	0.005	missing $\mathcal{O}(\alpha\alpha_s^3)$ corrections
perturbative QCD	c, b	0.004	missing $\mathcal{O}(\alpha\alpha_s^3)$ corrections
perturbative QCD	RGE	0.003	missing $\mathcal{O}(\alpha\alpha_s^4)$ corrections
non-perturbative QCD	u, d, s	0.006	quark-hadron duality
total QCD	u, d, s, c, b	0.009	theory
$\overline{\rm MS}$ quark mass	c	0.019	$\hat{m}_c = 1.31 \pm 0.07$ GeV
$\overline{\rm MS}$ quark mass	b	0.002	$\hat{m}_b = 4.24 \pm 0.11$ GeV
total quark masses	c, b	0.021	parametric
$R(s)$ [10]	u, d, s	0.015	data
grand total	u, d, s, c, b	0.027	data + theory + parametric

The available data in the region $m_c \lesssim \sqrt{s} \lesssim 2m_c$ are rather poor, and one would like to replace them by a robust theoretical description as far as possible. It is therefore desirable to improve the UDR approach by lowering the cut-off dependence and to decrease ones exposure to duality violating effects. Conversely, the BF-MOM approach would benefit by utilizing a short-distance mass definition. Furthermore, as a matter of practice, it may be unwieldy to properly correlate the uncertainties from resonances and quark masses; this may eventually result in a somewhat larger uncertainty compared to the UDR approach. Nevertheless, it appears that within both, the UDR and BF-MOM methods, one has the potential to obtain a solid theory driven evaluation of $\alpha(M_Z)$. In contrast, the more traditional SDR approach with its strong reliance on the complicated function $R(s)$ seems inadequate for the high demands of GigaZ precisions.

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