Discrete Ambiguities and Sensitivity Estimates in Measurements of γ and $2\beta + \gamma$

Abner Soffer*
Colorado State University
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Discrete ambiguities in CKM phase measurements have a significant statistical impact on measurement sensitivities, which must be taken into account when assessing the luminosity needs of CP violation experiments. We discuss both trigonometric and accidental ambiguities in measurements of γ and $2\beta+\gamma$, and quantify their effect on the measurement sensitivity. In the presence of ambiguities, we calculate sensitivity estimates for current- and next-generation B factories.

1. Ambiguities in CKM Phase Measurement

Measurements of CKM phases utilize the interference of decay amplitudes, which depend on the CKM Phase of interest and one or more CP-conserving phase. Because this dependence is trigonometric, these amplitudes are invariant under several symmetry operations, resulting in discrete ambiguities.

For example, measurements of the CKM phase γ typically involve decay rates of the form $\Gamma_\pm = |A_1|^2 + |A_2|^2 + 2|A_1 A_2|\cos(\delta_B \pm \gamma)$, where $|A_1|$ and $|A_2|$ are the magnitudes of the interfering amplitudes, and δ_B is a CP-conserving phase. Thus measurement of γ boils down to comparing $\cos(\delta_B + \gamma)$ and $\cos(\delta_B - \gamma)$, both of which are invariant under three symmetry operations [1]:

$$\begin{aligned} S_{\rm exchange} &= \gamma \leftrightarrow \delta_B \\ S_{\rm sign} &= \gamma \to -\gamma, \quad \delta_B \to -\delta_B \\ S_\pi &= \gamma \to \gamma + \pi, \quad \delta_B \to \delta_B + \pi. \end{aligned} \tag{1}$$

Together with the trivial operation $\theta \to \theta \pm 2\pi$ ($\theta = \gamma,\ \delta_B$), these operations form a symmetry group which results in an 8-fold ambiguity in the measurement of γ. Thus, lacking a-priori knowledge of δ_B, an otherwise accurate measurement of γ will be unable to distinguish between the following 8 values:

$$\begin{array}{cccc} \gamma, & -\gamma \ , & \gamma + \pi, & -(\gamma + \pi), \\ \delta_B, & -\delta_B \ , & \delta_B + \pi, & -(\delta_B + \pi). \end{array} \tag{2}$$

Similarly, measurements of $\sin(\delta_B \pm \phi)$, where ϕ is the CKM phase[1], are invariant under

$$\begin{aligned} S'_{\rm exchange} &= \phi \to \delta_B - \pi/2, \quad \delta_B \to \phi + \pi/2 \\ S'_{\rm sign} &= \phi \to -(\phi + \pi), \quad \delta_B \to -\delta_B \\ S_\pi &= \phi \to \phi + \pi, \quad \delta_B \to \delta_B + \pi. \end{aligned} \tag{3}$$

The resulting 8-fold ambiguity is

$$\begin{array}{cccc} \phi, & -\phi \ , & \phi + \pi, & -(\phi + \pi), \\ \delta_B - \pi/2, & -(\delta_B - \pi/2) \ , & \delta_B + \pi/2, & -(\delta_B + \pi/2). \end{array} \tag{4}$$

A-priori knowledge of δ_B can only come from theoretical understanding of final state interaction phases. In some cases, theorists claim to be able to calculate these phases [2], but several

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*abi@slac.stanford.edu

[1]For example, $\phi = 2\beta + \gamma$.

measurements of CKM phases will probably be required before these calculations can be deemed reliable. Very high accuracy measurements are therefore crucial. In principle, measurements involving different values of CP-conserving phases help resolve the ambiguities. In practice, however, this requires large statistics, and even resolved ambiguities significantly reduce the sensitivity of the measurement, as shown in Sec. 2. It should also be realized that in most CKM phase measurements the ratio r between the interfering amplitudes is not known a-priori. r must therefore be measured as well, resulting in accidental ambiguities. These occur when a solution exists for which both r and γ are mis-measured, and statistics are not large enough to differentiate between this faulty result and the correct one. Some examples of trigonometric and accidental ambiguities are shown in Sec. 2.

2. Measuring γ in $B \to DK$

Gronau and Wyler [3] proposed a method to measure γ by constructing the triangle corresponding to the decay amplitudes $A = \mathrm{Amp}(B^+ \to D^0 K^+)$, $\overline{A} = \mathrm{Amp}(B^+ \to \overline{D}^0 K^+)$, and their interference, $A_{\mathrm{CP}} = \mathrm{Amp}(B^+ \to \frac{1}{\sqrt{2}}(D^0 + \overline{D}^0)K^+)$. γ is then extracted from this triangle and the triangle corresponding to B^- decays. B mesons decaying via the A ($\overline{A}$) amplitude are identified through D final states containing a K^- (K^+), and A_{CP} is identified through CP-eigenstate D final states, such as $\pi^+\pi^-$. This method has the advantage that it does not require a decay rate asymmetry in order to measure γ. However, the amplitude ratio r is predicted by factorization to be only 10%. Several variations of the method have proposed [4].

Atwood, Dunietz and Soni [5] noted that the $B^+ \to D^0 K^+$ amplitude cannot be cleanly identified in the $D^0 \to K^- (n\pi)^+$ final state, due to interference from the doubly-Cabibbo suppressed (DCS) decay $D^0 \to K^+ (n\pi)^-$. Since the interfering amplitudes are expected to have very similar magnitudes and a large CP-conserving phase difference δ_D may be introduced by the D decays, a large decay rate asymmetry may result.

This problem with the Gronau-Wyler method is solved by a measurement of $\cos\delta_D$, which may be accurately carried out at a charm factory [6]. It should be pointed out, that measuring $\cos\delta_D$ improves the statistical sensitivity of the Atwood-Dunietz-Soni method, but does not resolve the ambiguity. Whether with or without the $\cos\delta_D$ measurement, the effect of the DCS D decay can be incorporated into the Gronau-Wyler triangle construction. Thus, both the $K^\pm (n\pi)^\mp$ and CP-eigenstate final state of the D meson may be used, improving the statistical sensitivity of the γ measurement and preserving the advantages of both the Atwood-Dunietz-Soni and the Gronau-Wyler methods [1]. Using this approach, a sensitivity study has been conducted [1] for an integrated luminosity of 600 fb^{-1} collected at an e^+e^- machine running at the $\Upsilon(4S)$ resonance. Full GEANT Monte Carlo was used to estimate the signal efficiency and background rates. No measurement of $\cos\delta_D$ or a-priori information about δ_D or δ_B was assumed. For given values of γ, δ_B, and δ_D, the signal and background yields of the average experiment were calculated, and the measured values of r, γ, δ_B, and δ_D were calculated by minimizing a χ^2 function. Depending on the values of the phases, the error of γ was in the range $\sigma_\gamma \sim 5^\circ - 10^\circ$. However, ambiguities severely limit the physical significance of these results.

Plots of the χ^2 vs. trial values of γ are shown in Figure 1 (left). The input values of the phases were chosen so as to illustrate the effect of ambiguities. In Figure 1a, the 8-fold ambiguity of Eq. (2) is evident from the 8 points for which $\chi^2 = 0$. The curvature of the χ^2 at the true value of γ yields the error $\sigma_\gamma \approx 5^\circ$. However, if one uses $\chi^2 > 10$ as the criterion for determining that a particular value of γ is inconsistent with this measurement, it is clear that almost any value of γ fails this criterion. Thus, not being able to exclude any value of γ, the measurement has almost no physical significance.

In Figure 1b, a slightly larger value of δ_D was used. This resolves the S_{exchange} ambiguity in principle, but in practice, the ambiguity is not resolved at the $\chi^2 > 10$ level. In Figure 1c, the true value of γ is around 90°, causing the S_π and S_{sign} ambiguities to overlap. The S_{exchange} ambiguity is in principle resolved, but the dip in χ^2 around the true value of δ_B demonstrates that this ambiguity resolution costs statistics, and the region of values of γ excluded at the $\chi^2 > 10$ level is still extremely small.

In Figure 1d, the phases are such that the S_{exchange} ambiguity is in principle resolved. However, an accidental ambiguity appears at $\gamma \approx 28^\circ$, with the value of r found to be $\sim 4/3$ its true value.

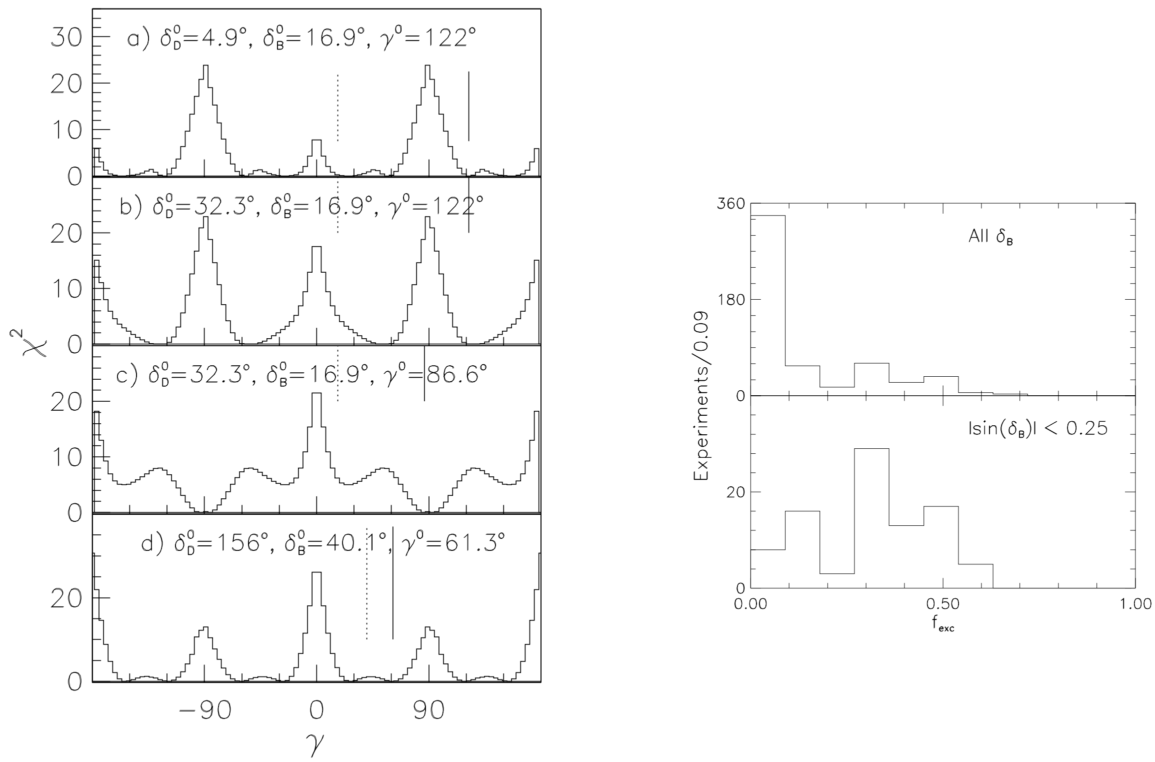

Figure 1: Left: χ^2 as a function of trial values of γ, minimized with respect to δ_B, δ_D, and r at each point. The phase values with which the events were generated are listed, and the integrated luminosity is 600 fb^{-1}. The vertical solid (dashed) lines indicates the generated value of γ (δ_B). Right: The $f_{\rm exc}$ distribution of all Monte Carlo experiments conducted, and experiments with $|\sin(\delta_B)| < 0.25$, for 600 fb^{-1}.

Such a small mistake in the value of r is highly unlikely to be forbidden by theoretical calculations, thus demonstrating the danger of accidental ambiguities.

Due to ambiguities, the error σ_γ is not meaningful when quantifying the sensitivity of an experiment. A better metric is the fraction $f_{\rm exc}$ of the allowed range of γ, taken to be between 40° and 100° in this study, for which $\chi^2 > 10$. The distribution of $f_{\rm exc}$ for 540 random values of the phases is shown in Figure 1 (right). If is seen that for most cases, $f_{\rm exc}$ is very small, indicating a physically weak measurement. Also shown is the distribution of $f_{\rm exc}$ for small values of $\sin\delta_B$. In this case, $f_{\rm exc}$ tends to be larger, since the $S_{\rm exchange}$ ambiguity is pushed away from the true value of γ.

The situation changes dramatically for an integrated luminosity of 10 ab^{-1}. In this case, the χ^2 scale is expanded by a factor of almost 16 with respect to Fig. 1. Most values of γ are thus excluded at the $\chi^2 > 10$ level, despite the ambiguities. The measurement becomes accurate and physically meaningful, with no theoretical assumptions regarding CP-conserving phases, and with small statistical errors in the range $1^\circ - 2.5^\circ$. In addition, the sensitivity is large enough that ambiguities are quite likely to be removed if different $B \to DK$ modes have somewhat different values of δ_B.

3. Measuring $\sin(2\beta + \gamma)$

The CKM parameter $\sin\phi$ ($\phi \equiv 2\beta + \gamma$) may be measured in decays of the type $B \to D^{(*)\mp}h^\pm$, where $h^\pm$ is a $\pi^\pm$, $\rho^\pm$ or $a_1^\pm$ [7]. The CKM phase difference arises due to interference between the direct decay of the B^0 ($\overline{B}^0$) with its mixing into $\overline{B}^0$ (B^0), followed by a doubly-Cabibbo suppressed decay into the same final state. The analysis is time-dependent, similar to the $\sin 2\beta$ analysis. However, all four possible final-state flavor and tag B flavor combinations are used to extract $\sin(\delta_B \pm \phi)$.

In this method, $\sin\phi$ is obtained from $r\sin\phi$ observables, where the amplitude ratio r ($r \sim 0.01 - 0.04$ from factorization) is extracted from measurements of $1 \pm r^2$ terms. The resulting statistical error in $\sin\phi$ is proportional to $1/r^2$. However, London, Sinha and Sinha [8] demonstrated that by performing a time-dependent angular analysis, one does not rely on r^2 terms. The statistical error is thus proportional to $1/\sqrt{r}$. Thus, a large statistical advantage may be obtained at the cost of a more complicated analysis.

In the BaBar Physics Book [10] we estimated that the statistical error in the measurement of $\sin\phi$ is about twice as large as the error in $\sin 2\beta$ for a given luminosity. This estimate was based on partial reconstruction using the $D^{*-}\pi^+$ mode only, and without using the method of [8]. Scaling from the current BaBar $\sin 2\beta$ error, it is expected that with 10 ab^{-1}, it will be possible to obtain the statistical error $\sigma_{\sin\phi} \sim 0.007$. A toy Monte Carlo study using fully reconstructed $D^{*-}\pi^+$ and $D^-\pi^+$ events found a somewhat smaller error [9]. Since partially and fully reconstructed samples are almost independent, we estimate the total statistical error to be $\sigma_{\sin\phi} \sim 0.005$. This small error should help resolve the ambiguities very efficiently, and may be further reduced if the method of [8] is proven to work.

Diehl and Hiller [11] attempted to overcome the small r problem in the $B \to D^{(*)\mp}h^\pm$ modes by using light hadrons $h^\pm$ which have a suppressed decay constant, and thus couple very weakly to the $W^\pm$. For the lightest such hadron, a_0^+, they estimate the branching fraction $Br(B^0 \to D^{(*)-}a_0^+) \sim (1-4)\times 10^{-6}$. Since a_0^+ decays to $\pi^+\eta$, it is possible to estimate the signal and background rates from the BaBar measurement of $B \to D^{(*)\mp}\rho^\pm$. We thus find that in 10 ab^{-1} one may expect 30 – 140 signal events, with no less than 3500 background events. The figure of merit $S/\sqrt{B}$ is then of order 0.5 – 2.5. We conclude that this mode is not competitive with other measurements of $\sin\phi$, although it could provide a useful cross-check if one uses all the decay constant-suppressed mesons listed in [11].

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