ENVELOPE MODES OF BEAMS WITH ANGULAR MOMENTUM*

John J. Barnard and Bojan Losic, Virtual National Laboratory for Heavy Ion Fusion, LLNL
Livermore, CA, and LBNL, Berkeley, CA

Abstract

For a particle beam propagating in an alternating gradient focusing system, envelope equations are often employed to describe the evolution of the beam radii in the two directions transverse to the direction of propagation, and aligned with the principle axes of the alternating gradient system. When the beams have zero net angular momentum and when the alternating gradient focusing is approximated by a continuous focusing system, there are two normal modes to the envelope equations: the 'breathing' mode and a 'quadrupole' mode. In the former, the two radii oscillate in phase, and in the latter the radii oscillate 180 degrees out of phase. In this paper, we extend the analysis to include beams that have a finite angular momentum. We perturb the moment equations of ref. [1], wherein it was assumed that space charge is a distributed in a uniform density ellipse. Two additional modes are obtained. The breathing mode remains, but the quadrupole mode is split into two modes, and a new low frequency mode appears. We calculate the frequencies and eigenmodes of these four modes as a function of tune depression and a dimensionless net angular momentum. These modes can be excited by rotational errors of the quadrupoles in an alternating gradient focusing channel..

1 INTRODUCTION

When a beam has angular momentum about the axis parallel to the propagation axis, or when the principle axes of a beam with elliptical cross-section do not align with the principle axes of a quadrupole, the *x* and *y* momentum equations are coupled and hence, the *x* and *y* normalized emittances are not conserved, even for a beam with an initial Kapchinskij-Vladimirskij (K-V) distribution with a linear space charge force profile propagating under linear external forces. However, if the equations of motion result from linear forces and are derivable from a Hamiltonian system, constants of the motion may be obtained analogous to the normalized *x* and *y* emittances [2]. Further, the K-V distribution has been generalized [3] to distributions in which the principal axes do not align with the *x* and *y* axes, and moment equations have been derived [1, 5] that assume the space-charge profile remains linear, consistent with the assumption of the KV-like distribution of ref. [3]. In ref. [1], a drifting, non-relativistic beam was assumed, and a conserved emittance was derived that is equivalent to the first of the conservation constraints in ref. [2]. The second invariant was independently derived in ref. [5]. In ref. [4], the equations were generalized to include acceleration, and the two normalized generalized emittances were evaluated using the methodology of ref. [2]. In ref. [5], moment equations were derived for the case of quadrupoles in which the principle axes continuously rotate along the longitudinal (z) direction, and the stability of the system was examined.

2 MOMENT EQUATIONS

In this section, we use the moment equations of ref. [1], to describe the evolution of an elliptical beam with arbitrary rotation angle. As in ref. [1], for simplicity we consider non-relativistic beams. We assume the space charge force can be calculated from that of a beam with elliptical symmetry but that is rotated with respect to the *z* axis. The transverse (*x* and *y*) equations of motion of a single particle are then:

$$d^2x/dz^2 = K_{qxx}x + K_{qxy}y + K_{sxx}(x-\langle x\rangle) + K_{sxy}(y-\langle y\rangle)$$
$$d^2y/dz^2 = K_{qyy}y + K_{qxy}x + K_{syy}(y-\langle y\rangle) + K_{sxy}(x-\langle x\rangle) \qquad (1)$$

Here $K_{qxx} = K_{q0}\cos 2\theta$, $K_{qxy} = K_{q0}\sin 2\theta$, (where θ is the rotation angle of the quadrupole about the *z*-axis), and $K_{q0} = (B'/[B\rho]$ for magnetic quadrupole focusing, $E'/(2V)$ for electric quadrupole focusing, and $-k_{\beta 0}^2$ for uniform focusing). Here B' (or E') is the magnetic (or electric) field gradient, $[B\rho]$ is the ion rigidity, and qV is ion kinetic energy, where q is the ion charge.

Also, $K_{qyy} = -K_{qxx}$ for electric or magnetic quadrupole focusing, and $K_{qyy}=K_{qxx}$ for uniform focusing. $K_{sxx}=K_{sxb}\cos 2\alpha + K_{syb}\sin 2\alpha$, and $K_{syy}=K_{syb}\cos 2\alpha + K_{sxb}\sin 2\alpha$, where $K_{sxb}=Q/(2[\Delta x_b^2 + [\Delta x_b^2\Delta y_b^2]^{1/2}])$, $K_{syb}=Q/(2[\Delta y_b^2 + [\Delta x_b^2\Delta y_b^2]^{1/2}])$. Here the beam widths along the principle axes are given by $\Delta x_b^2=\Delta x^2\cos^2\alpha + \Delta y^2\sin^2\alpha + 2\Delta xy \sin\alpha\cos\alpha$ and $\Delta y_b^2=\Delta y^2\cos^2\alpha + \Delta x^2\sin^2\alpha - 2\Delta xy \sin\alpha\cos\alpha$. The quantity α is the rotation angle of the beam given by $\tan 2\alpha = 2\Delta xy/(\Delta x^2-\Delta y^2)$; and $Q = \lambda/(4\pi\varepsilon_0 V)$ is the generalized perveance for a non-relativistic beam, with line charge density λ and where ε_0 is the permittivity of free space. We have used the notation $\Delta ab = \langle ab\rangle - \langle a\rangle\langle b\rangle$, where $\langle\ \rangle$ denotes average over the distribution function.

The set of ten first order equations for the quadratic moments of the distribution, obtained by averaging eq. (1) over the distribution function, was found in ref. [1] to be:

$$d\Delta x^2/dz = 2\Delta xx'$$
$$d\Delta xx'/dz = \Delta x'^2 + K_{xx}\Delta x^2 + K_{xy}\Delta xy$$
$$d\Delta x'^2/dz = 2K_{xx}\Delta xx' + 2K_{xy}\Delta x'y$$
$$d\Delta y^2/dz = 2\Delta yy'$$
$$d\Delta yy'/dz = \Delta y'^2 + K_{xx}\Delta y^2 + K_{xy}\Delta xy$$
$$d\Delta y'^2/dz = 2K_{yy}\Delta yy' + 2K_{xy}\Delta xy'$$
$$d\Delta xy/dz = \Delta xy' + \Delta x'y$$
$$d\Delta x'y/dz = \Delta x'y' + K_{xx}\Delta xy + K_{xy}\Delta y^2$$
$$d\Delta xy'/dz = \Delta x'y' + K_{yy}\Delta xy + K_{xy}\Delta x^2$$
$$d\Delta x'y'/dz = K_{xx}\Delta xy' + K_{xy}\Delta yy' + K_{yy}\Delta x'y + K_{xy}\Delta xx' \qquad (2)$$

Here, $K_{xx} = K_{qxx} + K_{sxx}$; $K_{xy} = K_{qxy} + K_{sxy}$; and $K_{yy} = K_{qyy} + K_{syy}$.

3 EQUILIBRIUM

We first examine the case of uniform focusing, which represents the focusing force averaged over a lattice period, (or the force arising from a background of uniform density space charge with sign opposite to the beam charge.) In this case, $K_{qxx} = K_{qyy} \equiv -k_{\beta 0}{}^2$; $K_{sxx0}=K_{syy0} \equiv k_{sc0}{}^2=Q/(4\Delta x_0{}^2)$; and $K_{qxy} = 0$. Each of the 10 moments in eq. 2, have the equilibrium values (subscript 0) given by: $\Delta x_0^2 = \Delta y_0^2$, $\Delta x_0'^2 = \Delta y_0'^2 = (k_{\beta 0}^2 - k_{sc0}^2)\Delta x_0^2$, $\Delta xy_0' = -\Delta x' y_0 \equiv l_0 / 2$; $\Delta xx_0' = \Delta yy_0' = \Delta xy_0 = \Delta x'y_0' = K_{xy0} = 0$, where l_0 is proportional to the angular momentum of the beam. With these values, the right hand sides of eq. 2 are zero, and the beam moments are stationary.

4 LINEARIZED EQUATIONS

We now examine perturbations about this equilibrium of the form, $\Delta x^2 = \Delta x_0^2 + \Delta x_1^2 \exp(ikz)$, and $\Delta y^2 = \Delta y_0^2 + \Delta y_1^2 \exp(ikz)$, etc. for each of the 10 moments. Here subscript 1, indicates the amplitude of the perturbation. As in ref. [5], we linearize equation 2, obtaining a set of equations for the eigenfrequency *k:*

$$ik\Delta x_1^2 = 2\Delta xx_1'$$
$$ik\Delta xx_1' = \Delta x_1'^2 - k_{\beta 0}^2 \Delta x_1^2 + k_{sc0}^2 (\Delta x_1^2 - \Delta y_1^2)/4$$
$$ik\Delta x_1'^2 = -2(k_{\beta 0}^2 - k_{sc0}^2)\Delta xx_1' + k_{sc0}^2 (\Delta xy_0' / \Delta x_0^2)\Delta xy_1$$
$$ik\Delta y_1^2 = 2\Delta yy_1'$$
$$ik\Delta yy_1' = \Delta y_1'^2 - k_{\beta 0}^2 \Delta y_1^2 + k_{sc0}^2 (\Delta y_1^2 - \Delta x_1^2)/4$$
$$ik\Delta y_1'^2 = -2(k_{\beta 0}^2 - k_{sc0}^2)\Delta yy_1' - k_{sc0}^2 (\Delta xy_0' / \Delta x_0^2)\Delta xy_1$$
$$ik\Delta xy_1 = \Delta xy_1' + \Delta x' y_1$$
$$ik\Delta x' y_1 = \Delta x' y_1' - (k_{\beta 0}^2 - k_{sc0}^2/2)\Delta xy_1 \qquad (3)$$
$$ik\Delta xy_1' = \Delta x' y_1' - (k_{\beta 0}^2 - k_{sc0}^2/2)\Delta xy_1$$
$$ik\Delta x' y_1' = -(k_{\beta 0}^2 - k_{sc0}^2)(\Delta xy_1' + \Delta x' y_1) - k_{sc0}^2 (\Delta xy_0' / 2\Delta x_0^2)(\Delta x_1^2 - \Delta y_1^2)$$

Here $k_{sc0}^2 \equiv Q / 4\Delta x_0^2$, and we have found and used the relations:

$K_{sxx1} = -k_{sc0}^2 (3\Delta x_1^2 + \Delta y_1^2)/(4\Delta x_0^2)$,

$K_{syy1} = -k_{sc0}^2 (3\Delta y_1^2 + \Delta x_1^2)/(4\Delta x_0^2)$, and

$K_{sxy1} = -k_{sc0}^2 \Delta xy_1 /(2\Delta x_0^2)$.

5 EIGENVALUES AND EIGENMODES

Equation (3) can be expressed as matrix equation of the form M.x=0, where x is the column vector of the 10 quadratic moments and M is a 10 by 10 matrix. The determinant of M yields an eigenvalue equation with 4 distinct non-zero frequencies given by:

$$k_B / k_{\beta 0} = \sqrt{2(1+\mu^2)}$$

$$k_{Q\pm} / k_{\beta 0} = \frac{(1-\mu^2)^{1/3}\delta_+}{3(2^{1/3})} + \frac{2^{1/3}(1+3\mu^2)}{(1-\mu^2)^{1/3}\delta_\pm}$$

$$k_L / k_{\beta 0} = \frac{(1-\mu^2)^{1/3}\delta_+(-1-\sqrt{3}i)}{6(2^{1/3})} - \frac{(1+3\mu^2)(1-\sqrt{3}i)}{2^{2/3}(1-\mu^2)^{1/3}\delta_\pm} \qquad (4)$$

Here, $\delta_\pm \equiv 3(2\Gamma)^{1/3}\left(\pm 1 + i\left[\frac{1}{27\alpha} - 1\right]^{1/2}\right)^{1/3}$

$\Gamma \equiv \Delta xy_0' /(k_{\beta 0}\Delta x_0^2)$; $\alpha=\Gamma^2(1-\mu^2)^2/(1+3\mu^2)^3$; and $\mu^2 \equiv 1 - k_{sc0}^2 / k_{\beta 0}^2$.

Here k_B is the frequency of the breathing mode, unaltered in form by the beam; k_{Q+} and k_{Q-} are the quadrupole modes, now split into two modes as a result of the rotation; and k_L is the frequency of the low frequency mode, a new mode present only with rotation. Eq. (4) lists positive roots; the negatives of these roots are also solutions (see fig. 1).

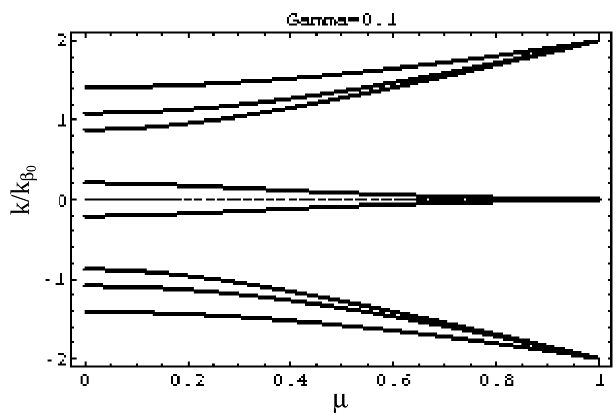

Figure 1. Mode frequency vs. tune depression for Γ=0.1. For $\alpha < 1/27 \cong 0.037$ all modes are real.

One can express eq. (4) such that the real parts of the quadrupole and low frequency modes are explicit, and are expressed relative to the quadrupole frequency in the absence of rotation $k_{Q0} \equiv k_{\beta 0}\sqrt{1+3\mu^2}$:

$$k_L / k_{Q0} = \sqrt{(2/3)(1-\cos\theta)} \text{ and}$$

$$k_{Q\pm} / k_{Q0} = \sqrt{(2/3)(1+(1/2)\cos\theta \pm (\sqrt{3}/2)\sin\theta)}$$

where $\theta = \frac{1}{3}\tan^{-1}\left[\frac{6\sqrt{3\alpha(1-27\alpha)}}{1-54\alpha}\right]$.

Thus k_L / k_{Q0} and $k_{Q\pm} / k_{Q0}$ are functions of the single parameter α which we plot in figure 2.

The breathing eigenmode is unaltered by rotation with the perturbation in Δx^2 in phase with the motion in Δy^2, and with the four perturbations to the cross moments zero. Remarkably, all of the other modes, maintain a constant ellipticity during a complete cycle of the perturbation. (It was found by numerically integrating eqs. (2) that the values of Δx_b^2 and Δy_b^2 were constant over z for an initial condition which initiated the beam in a pure k_{Q+}, k_{Q-}, or k_L mode.) The ellipse rotates at the frequency of the mode. This would seem to contradict the behavior of the known quadrupolar mode with zero rotation in which the ellipticity changes over the cycle of the perturbation. However, the contradiction is resolved when it is noted that at zero angular momentum the k_{Q+} mode and k_{Q-} mode have the same frequency k_{Q0}, with the result that a normal quadrupole mode can be formed by the summation of a clockwise and counterclockwise propagating perturbation.

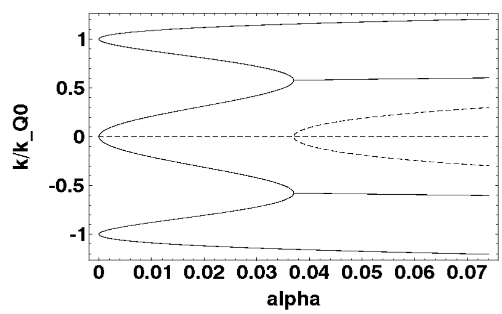

Figure 2. Frequency (in units of k_{Q0}) of quadrupole and low frequency modes vs. parameter α (see text). (Solid is real, dashed is imaginary). Note for α > 1/27 the low frequency quadrupole and low frequency merge, and an unstable mode appears.

Figure 2 indicates that for large Γ the modes become unstable. In the model in this paper, $\Delta xy_0'$ is an independent quantity. However, when the distribution function which underlies the model is considered, limits can be placed on $\Delta xy_0'$ relative to Δx_0^2. In particular, for a rigidly rotating equilibrium, a particle transverse velocity (angle) satisfies $x' = -(\omega / v_z) y + \delta x'$, and $y' = (\omega / v_z) x + \delta y'$, where ω is the angular frequency of the beam, v_z is the longitudinal velocity, and $\delta x'$ and $\delta y'$ are the particular transverse angles of the particle. With this ansatz, the quantity $\Gamma = \omega/(k_{\beta 0}\, v_z) - \langle x \delta y' \rangle = \omega/(k_{\beta 0}\, v_z)$, if $\langle x \delta y' \rangle = 0$, i.e. Γ is the ratio of the rigid body rotation frequency to the betatron frequency. The equilibrium value of $\Delta x_0'^2 = \mu^2 k_{\beta 0}^2 \Delta x_0^2$, but for rigid rotation $\Delta x_0'^2 = \Gamma^2 k_{\beta 0}^2 \Delta x_0^2 + \langle \delta x'^2 \rangle$, (again assuming that the correlations $\langle y \delta x' \rangle = \langle x \delta y' \rangle = 0$), and hence $\Gamma \le \mu$, for a self-consistent equilibrium undergoing rigid rotation. For $\Gamma \le \mu$, we find that all modes are stable. The model equation (eq. 2) may be integrated for arbitrary values of Γ, however, including those with unstable modes. The question of whether equilibria exist with $\Gamma > \mu$ (for example with non-uniform rotation) is still open.

6 ALTERNATING GRADIENT MODES

When a beam that is matched to an alternating gradient focusing system is given an arbitrary perturbation including a finite l_0 it will oscillate under the normal modes of that system. We have given a general perturabation to such a system by numerically integrating eq. 2, and examined the Fourier components of the resulting perturbation. We plot the Fourier spectrum if figure 3.

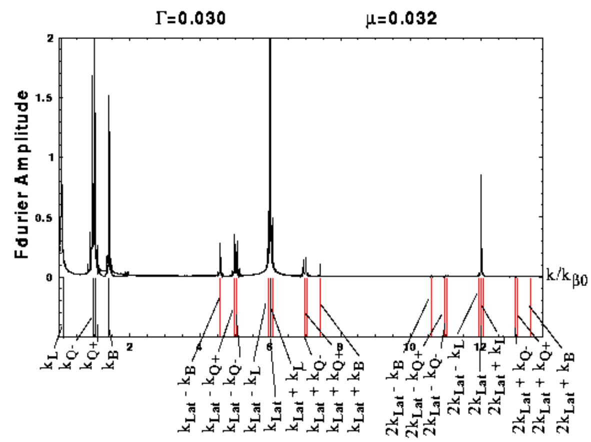

Figure 3. Fourier decomposition of modes for an initially mismatched beam in an alternating gradient lattice (with μ=0.030, and Γ=0.032) Frequencies shown relative to $k_{\beta 0}$.

As can be seen, spectral lines exist at the mode frequencies for a uniform beam and also approximately at the fundamental lattice frequency $k_{Lat} = \pi/L$ (where L is the half-lattice period of the alternating gradient lattice) and integer multiples of this frequency, plus and minus the frequency of the four modes discussed in the uniform focusing case. Here $\Delta xy_0'$ and Δx_0^2 are averaged over a lattice period to calculate Γ. We thus expect that the rotational modes will appear in an "average" sense in less idealized systems.

7 CONCLUSIONS

This work has shown the existence of additional low order "envelope" modes in beams that acquire finite angular momentum through, for example, the presence of skew quadrupole errors or longitudinal magnetic fields. The presence of these beam modes provide additional possibilities for particle/envelope resonances and possible halo formation, and should be considered in the context of general beam mismatches.

8 ACKNOWLEDGEMENTS

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9 REFERENCES

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