Modeling High-Energy Gamma-Rays from the Fermi Bubbles

Megan Splettstoesser*

August 2015

Abstract

In 2010, the Fermi Bubbles were discovered at the galactic center of the Milky Way. These giant gamma-ray structures, extending 55° in galactic latitude and 20° - 30° in galactic longitude, were not predicted. We wish to develop a model for the gamma-ray emission of the Fermi Bubbles. To do so, we assume that second order Fermi acceleration is responsible for the high-energy emission of the bubbles. Second order Fermi acceleration requires charged particles and irregular magnetic fields—both of which are present in the disk of the Milky Way galaxy. By solving the steady-state case of the transport equation, I compute the proton spectrum due to second order Fermi acceleration. I compare the analytical solutions of the proton spectrum to a numerical solution. I find that the numerical solution to the transport equation converges to the analytical solution in all cases. The gamma-ray spectrum due to proton-proton interaction is compared to Fermi Bubble data (from Ackermann et al. 2014), and I find that second order Fermi acceleration is a good fit for the gamma-ray spectrum of the Fermi Bubbles at low energies with an injection source term of $S = 1.5 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$. I find that a non-steady-state solution to the gamma-ray spectrum with an injection source term of $S = 2 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$ matches the bubble data at high energies.

1. Introduction

The Fermi Large Area Telescope (LAT) began to observe the gamma-ray sky at energies over 30 MeV in 2008 (Atwood et al. 2009). Until 2010, the gamma-ray sky was known to contain point sources and diffuse emission. Point sources may be galactic (e.g., supernovae, pulsars, binary systems) or extragalactic (e.g., active galactic nuclei (AGN), gamma-ray bursts (GRBs), blazars). Diffuse emission is more extended than the angular resolution of the instrument; therefore, it is not considered a point source. The diffuse emission of the Milky Way galaxy is primarily produced by cosmic rays

(CRs) interacting with the interstellar radiation field (ISRF) or the interstellar medium (ISM) (Strong, Moskalenko, Ptuskin 2007). CRs are taken to be predominantly protons and helium.

In 2010, however, an addition to the diffusion emission was observed: an unexpected gamma-ray structure, termed the Fermi Bubbles (Su, Slatyer, Finkbeiner 2010). The structure extends 55° in galactic latitude (Ackermann et al. 2014) and 20° - 30° in galactic longitude. The bubbles are symmetric with respect to longitude and with respect to the galactic center (GC) of the Milky Way galaxy. The hard spectrum and high-energy gamma-rays of

*SULI summer intern at Kavli Institute for Particle Astrophysics and Cosmology
Home Institution: University of California, Santa Cruz

the bubbles are only visible at higher energies. Gamma-ray production in the galactic ridge is dominated by Inverse Compton (IC) scattering, bremsstrahlung emission, and neutral pion decay (Strong, Moskalenko, Ptuskin 2007); once the gamma-ray emission from the aforementioned processes is subtracted from the gamma-ray sky, the Fermi Bubbles are visible.

The ISRF allows for the presence of soft gamma-rays, which creates an environment for IC scattering. A gamma-ray is produced in IC scattering as a high-energy electron collides with a low-energy photon (γ_{soft}); in this process, the electron transfers some of its energy to the photon, and as a result, the photon gains energy. Below is the IC process:

$$e_{CR}^- + \gamma_{soft} \longrightarrow e_{CR}^- + \gamma. \quad (1)$$

The cosmic ray electron (e_{CR}^-) on the right-hand side (RHS) of Equation (1) has lost energy and the gamma-ray (γ) on the RHS has gained energy.

Bremsstrahlung emission is caused by the acceleration of an electron in the presence of a stationary nucleus (or ion) due to the nucleus's (or ion's) electric field:

$$e_{CR}^- + N_{gas} \longrightarrow e_{CR}^- + N_{gas} + \gamma. \quad (2)$$

The cosmic ray electron on the RHS of Equation (2) has lost energy. The gas nucleus (N_{gas}) on the RHS does not lose energy in the interaction because it is heavy.

Neutral pions decay into two gamma-rays:

$$\pi^0 \longrightarrow \gamma + \gamma.$$

Neutral pions are produced by the interaction of cosmic ray protons (p_{CR}) with a gas nucleus:

$$p_{CR} + N_{gas} \longrightarrow \pi^0 + X. \quad (3)$$

In Equation (3), X represents an arbitrary product of the reaction aside from the neutral pion. Examine a more specific case of (3), as shown below:

$$p_{CR} + N_{gas} \longrightarrow \pi^0 + p + p.$$

Take the gas nucleus to be a proton (p_{gas}).

To model the gamma-ray emission of the Fermi Bubbles, we assume that the emission is due to second order Fermi acceleration, an acceleration mechanism in which charged particles interact with moving irregular magnetic fields. In a head-on collision, the particle travels with a velocity directionally opposite to the velocity of the magnetic irregularity before collision; the charged particle gains energy after this interaction. Conversely, in a following collision, the particle and the magnetic irregularity travel in the same direction before collision and the charged particle loses energy (Fermi 1949). Within the magnetic irregularity, particles diffuse, or undergo a continuous random walk. On average, second order Fermi acceleration causes particles to gain energy because head-on collisions are favored. The energy gain of the particles (in our case, protons), is second order in $\frac{V}{c}$ (in other words, $\Delta E \propto [\frac{V}{c}]^2$), where V is the velocity of the magnetic field and c is the speed of the particle (Longair Vol. 2).

I begin with a solution to the heat equation in Section 2; the heat equation is a simplified version of the transport equation. Next, I solve the transport equation for second order Fermi acceleration in Section 3. The proton spectrum given by the analytical solution to the transport equation is compared to the numerical solution of the proton spectrum (provided by Dr. P. Mertsch) in Section 4. In Section 5, we compare the numerical solution of the gamma-ray spectrum of the Fermi Bubbles due to proton-proton ($p-p$) interaction to data (from Ackermann et al. 2014).

2. Analytical Solution to the Heat Equation

Examine the most general transport equa-

tion in momentum-space:

$$\begin{aligned} \frac{\partial f(\vec{x}, \vec{p}, \vec{t})}{\partial t} = & \vec{\nabla} D \vec{\nabla} f(\vec{x}, \vec{p}, \vec{t}) - \vec{\nabla} \cdot (\vec{v} f(\vec{x}, \vec{p}, \vec{t})) \\ & + \frac{1}{p^2} \frac{\partial}{\partial p} D_{pp}(p) \frac{\partial}{\partial p} f(\vec{x}, \vec{p}, \vec{t}) p^2 \\ & + \frac{1}{p^2} \frac{\partial}{\partial p} (\dot{p} p^2 f(\vec{x}, \vec{p}, \vec{t})) \\ & - \frac{p}{3} (\vec{\nabla} \cdot \vec{v}) - \frac{f(\vec{x}, \vec{p}, \vec{t})}{\tau}. \end{aligned} \quad (4)$$

The first term on the RHS of Equation (4), $\vec{\nabla} D \vec{\nabla} f(\vec{x}, \vec{p}, \vec{t})$, describes spatial diffusion, the second accounts for spatial advection, the third models momentum diffusion, the fourth describes energy losses, the fifth describes adiabatic losses and gains, and the last term models particle escape and other losses (e.g., decay).

Take a specific case of Equation (4)—the heat equation—given by:

$$\frac{\partial}{\partial t} f(x, t) = D \frac{\partial^2}{\partial x^2} f(x, t). \quad (5)$$

Note that Equation (5) models spatial diffusion. The diffusion coefficient D is independent of x and t . Assume the following initial condition: $f(x, 0) = f_0(x)$. To solve Equation (5) over an infinite interval, employ Fourier transforms. Denote the Fourier transform of $\tilde{f}(k, t)$ as $f(x, t)$, and define the inverse Fourier transform of $f(x, t)$ as $\tilde{f}(k, t)$, as shown below:

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \tilde{f}(k, t) dk \quad (6)$$

$$\tilde{f}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x, t) dx. \quad (7)$$

The Fourier transform of the heat equation (Equation (5)) is given by:

$$\frac{d}{dt} \tilde{f}(k, t) + D k^2 \tilde{f}(k, t) = 0. \quad (8)$$

Note that Equation (8) is a separable ordinary differential equation involving differentiation with respect to a single variable. Solving Equation (8) yields:

$$\tilde{f}(k, t) = \tilde{f}_0(k) e^{-D k^2 t}. \quad (9)$$

Substitute the inverse Fourier transform given in Equation (9), into Equation (6), which defines the Fourier transform. The solution to the resulting equation is given by:

$$f(x, t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f_0(x') \exp\left(\frac{-(x - x')^2}{4Dt}\right) dx' \quad (10)$$

The mathematical details leading up to Equation (10) can be found in Appendix A. We see that $f(x, t)$ is a function of its initial condition. If the initial condition, $f_0(x')$, was given by a Dirac delta function, the solution to $f(x, t)$ would be as follows:

$$f(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-(x - x')^2}{4Dt}\right) dx'$$

3. Analytical Steady-State Solution to the Transport Equation

Second order Fermi acceleration is a purely stochastic process; therefore, given the following transport equation in momentum-space (Becker, Le, & Dermer 2006^[3]):

$$\begin{aligned} \frac{\partial f(p, t)}{\partial t} = & \frac{1}{p^2} \frac{\partial}{\partial p} \left[p^2 \left(D_{pp}(p) \frac{\partial f(p, t)}{\partial p} \right. \right. \\ & \left. \left. - A(p) f(p, t) \right) \right] - \frac{f(p, t)}{\tau_{esc}(p)} + \frac{S(p, t)}{4\pi p^2}, \end{aligned} \quad (11)$$

$A(p)$, which represents systematic acceleration, is zero. Take $f(p, t)$ to be the phase space density, $D_{pp}(p)$ to be the momentum diffusion coefficient (this causes energy gains and losses), $t_{esc}(p)$ to be the mean particle escape time, and $S(p, t)$ to be a source term (Becker, Le, & Dermer 2006^[3]). If we take the source term to be a Dirac delta function, the transport equation (11) for second order Fermi acceleration becomes

$$\begin{aligned} \frac{\partial f(p)}{\partial t} = & \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_{pp}(p) \frac{\partial f(p)}{\partial p} \right) \\ & - \frac{f(p)}{\tau_{esc}(p)} + S \frac{\delta(p - p_0)}{4\pi p^2}. \end{aligned} \quad (12)$$

The first term on the right-hand side (RHS) represents momentum diffusion/transport. The coefficient of the source term, S , gives the number

of particles injected per GeV per cm^3 per year. Because it will be easier to work with the particle spectrum, note the following:

$$\begin{aligned}\Psi(p) &= 4\pi p^2 f(p) \\ \Rightarrow f(p) &= \frac{\Psi(p)}{4\pi p^2}.\end{aligned}\quad (13)$$

$\Psi(p)$ is the particle density and $f(p)$ is the phase space density. Solving for the steady-state solution, in which $\frac{\partial f(p)}{\partial t} = 0$, and using Equation (13), Equation (12) becomes:

$$\begin{aligned}0 &= \frac{\partial}{\partial p} \left(p^2 D_{pp}(p) \frac{\partial}{\partial p} \left(\frac{\Psi(p)}{p^2} \right) \right) - \frac{\Psi(p)}{\tau_{esc}(p)} \\ &\quad + S\delta(p - p_0).\end{aligned}\quad (14)$$

Let $\frac{\partial}{\partial p} \left(\frac{\Psi(p)}{p^2} \right) = \Psi'(p)$. Evaluating the partial derivatives in Equation (14) yields the following steady-state transport equation:

$$\begin{aligned}0 &= \left(D_{pp}(p) \right) \Psi''(p) + \left(D'_{pp}(p) - \frac{2D_{pp}(p)}{p} \right) \Psi'(p) \\ &\quad + \left(\frac{2D_{pp}(p)}{p^2} - \frac{2D'_{pp}(p)}{p} - \frac{1}{\tau_{esc}(p)} \right) \Psi(p) \\ &\quad + S\delta(p - p_0).\end{aligned}\quad (15)$$

Now solve for the $q = 2$ case, which gives the following assumptions:

$$\begin{aligned}D_{pp}(p) &= D_0 \left(\frac{p}{p_0} \right)^q = \frac{1}{\tau_{acc,0}} \left(\frac{p}{p_0} \right)^q \\ \xrightarrow{q=2} D_{pp}(p) &= \frac{1}{\tau_{acc,0}} \left(\frac{p}{p_0} \right)^2 \\ \tau_{esc}(p) &= \tau_{esc,0} \left(\frac{p}{p_0} \right)^{q-2}\end{aligned}\quad (16)$$

$$\xrightarrow{q=2} \tau_{esc}(p) = \tau_{esc,0} \quad (17)$$

$$\text{as } p \rightarrow \infty, \Psi(p) \text{ remains infinite.} \quad (18)$$

Equation (17) shows that the $q = 2$ case gives a momentum-independent escape time. Noting the momentum-independent escape time in the $q = 2$ case (Equation (17)) and the relationship

between the diffusion coefficient and momentum (Equation (16)), and $p_0 = 1$, Equation (15) becomes:

$$0 = \Psi''(p) - \frac{1}{p^2} \left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right) \Psi(p) + \frac{\delta(p - p_0) \tau_{acc,0}}{p^2}.\quad (19)$$

Examine the homogeneous case of Equation (19), which is linear in $\Psi(p)$:

$$\Psi''(p) - \frac{1}{p^2} \left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right) \Psi(p) = 0.\quad (20)$$

The solution of $\Psi''(p) - ap^{-2}\Psi(p) = 0$, where a is a constant, is given by:

$$\Psi(p) = A|p|^{k_1} + B|p|^{k_2},\quad (21)$$

where k_1 and k_2 are the roots of $k^2 - k - a = 0$. Solving for the roots:

$$\begin{aligned}0 &= k^2 - k - \left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right) \\ k &= \frac{1 \pm \sqrt{1 + 4 \left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right)}}{2} \\ k &= \frac{1}{2} \pm \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}.\end{aligned}\quad (22)$$

Note that the times $\tau_{acc,0}$ and $\tau_{esc,0}$ are always zero or positive. Therefore, $\sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}$ is always positive. The smallest value $\sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}$ can take on, in the case of $\frac{\tau_{acc,0}}{\tau_{esc,0}} = 0$, is $\frac{3}{2}$. Thus, by Equation (22), we have one positive root and one negative root. However, to satisfy Equation (18), Equation (21) must only contain the negative root. Dropping the magnitude of p in Equation (21) because the momentum is always positive, we have:

$$\Psi(p) = Ap^{\frac{1}{2} - \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}},\quad (23)$$

which gives the steady-state solution to the transport equation for $\Psi(p)$, the particle density. Solving the transport equation for $f(p)$ using Equation (13) gives:

$$f(p) \propto p^{-\frac{3}{2} - \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}}\quad (24)$$

A detailed account of this solution is given in Appendix B. We see that for the $q = 2$ case, the phase space density is only dependent on the ratio between the acceleration and escape times of the particle. Examine the case of $\frac{\tau_{acc,0}}{\tau_{esc,0}} = 4$:

$$f(p) \propto p^{-4}$$

Multiplying $f(p)$ by p^4 , as I have done in Figures (1a) and (1b), produces a flat spectrum.

4. Proton Spectra: Analytical and Numerical Solutions

The proton spectrum describes the phase space density $f(p)$ at a fixed spatial radius.

The numerical solution to the transport equation (Equation (12)), given by a code (by Dr. P. Mertsch) which employs a finite-difference approximation, can be tested using the analytical solution (Equation (24)). In Figure (1), the proton spectrum (Equation (24)) is plotted with the numerical solution. Note that the analytical solution (Equation (24)) is only valid for $p > p_0$. We see that the numerical solution converges to the analytical solution for $p > p_0$ in Figure (1b).

The analytical solutions to the transport equation given by Becker, Le, & Dermer^[3] are valid for all values of p . Note that the steady-state solutions to the transport equation given by $N_{ss}^G|_{q=2}$ in Figure (2) and $N_{ss}^G|_{q<2}$ in Figure (3) are Equations [58] and [56], respectively, in Becker, Le, & Dermer^[3]. The equations are reproduced below. Note that $\Psi(p)$ and $N(p)$ both represent particle density. Equation [58]:

$$N_{ss}^G(x)|_{q=2} = \frac{\dot{N}_0}{2x_0 D_* \sqrt{\lambda}} \left(\frac{x}{x_0} \right)^{(a+1)/2} \left(\frac{x_{max}}{x_{min}} \right)^{-\sqrt{\lambda}}. \quad (25)$$

Equation [56]:

$$N_{ss}^G(x) = \frac{\dot{N}_0}{(2-q)x_0 D_*} \left(\frac{x}{x_0} \right)^{(a+1)/2} (xx_0)^{(2-q)/2} \times I_{(\beta-1)/2} \left(\frac{\sqrt{\theta} x_{min}^{2-q}}{2-q} \right) K_{(\beta-1)/2} \left(\frac{\sqrt{\theta} x_{max}^{2-q}}{2-q} \right). \quad (26)$$

See Becker, Le, & Dermer^[3] for details.

In Figures (2b) and (3b), we see that the numerical solution converges to the steady-state analytical solution. The numerical solution for the $q < 2$ case converges to the analytical solution more quickly than the $q = 2$ case.

5. Gamma-Ray Spectra: Numerical Solution and Fermi Bubble Data

The code for the gamma-ray flux due to $p - p$ interaction may be found at (<https://github.com/niklask/cparamlib>). The gamma-ray spectrum is computed from the gamma-ray flux. The Fermi Bubble data is from (Ackermann et al. 2014).

We wish to find the source term, S , in Equation (12) such that the solution to the transport equation matches the Fermi Bubble data. The numerical solution matches Fermi Bubble data well for lower energies (Figure (4a)) with a source term equal to $1.5 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$. Figure (4b) is a good fit at high energies for the data in the non-steady-state case; here, we find a source term of $S = 2 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$.

6. Conclusion

I solve the transport equation for second order Fermi acceleration and find that the numerical solution converges to the analytical solution. Further, the numerical solution converges to analytical solutions to the transport equation provided by Becker, Le, & Dermer^[3]. The gamma-ray spectra produced by protons undergoing second order Fermi acceleration is a good fit for the Fermi Bubbles. The Fermi Bubble

(a) The numerical solution (green) after a simulated time of 0.3 years.

(b) The numerical solution after a simulated time of 15 years.

Figure 1: The proton spectrum given by Equation (24) (blue) plotted with the numerical solution (green) in momentum-space for the steady-state, $\frac{\tau_{acc,0}}{\tau_{esc,0}} = 4$ and $q = 2$ case.

(a) The numerical solution (green) after a simulated time of 0.6 years.

(b) The numerical solution (green) after a simulated time of 14.7 years.

Figure 2: The proton spectrum (light blue) given by Equation (25) plotted with the numerical solution (green) for steady-state, $\frac{\tau_{acc,0}}{\tau_{esc,0}} = 4$ and $q = 2$ case. Equation (25) is plotted in momentum-space using the same parameters and assumptions as the numerical solution.

data at high energies suggest that the bubbles may not be produced by a steady-state process. Uncertainties exist due to the unknown gas density of the Fermi Bubbles. We find a source term of $S = 1.5 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$ for lower energies of the Fermi Bubbles and a steady-state gamma-ray spectrum. We find a source term of

$S = 2 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$ for high energies of the bubbles and non-steady-state solution to the gamma-ray spectrum.

7. Acknowledgements

This work was supported in part by the U.S. Department of Energy, Office of Science, Office

(a) The numerical solution after a simulated time of 0.3 years.

(b) The numerical solution after a simulated time of 1.53 years.

Figure 3: The proton spectrum (dark red) plotted with the numerical solution (green) for $\frac{\tau_{acc,0}}{\tau_{esc,0}} = 4$ and $q \neq 2$. Equation (26) is plotted with the parameters as the numerical solution.

of Workforce Development for Teachers and Scientists (WDTS) under the Science Undergraduate Laboratory Internships Program (SULI) and was made possible thanks to the guidance of Dr. Philipp Mertsch. The author would also like to thank Enrique Cuellar for making the SULI interns feel welcome.

8. References

- [1] Ackermann et al. 2014. ApJ 793 64 *The Spectrum and Morphology of the Fermi Bubbles*.
- [2] Atwood, W. B. et al. 2009. ApJ 697, 1071
- [3] Becker, Peter A.; Le, Truong; Dermer, Charles D. 2006. ApJ 647, 539 *Time-Dependent Stochastic Particle Acceleration in Astrophysical Plasmas: Exact Solutions Including Momentum-Dependent Escape*.
- [4] Fermi, E. 1949, Phys. Rev., 75, 1169
- [5] Kamae, Tuneyoshi; Karlsson, Niklas; Mizuno, Tsunefumi; Abe, Toshinori; Koi, Tatsumi. 2006. ApJ 647, 692 *Parametrization of γ , $e^\pm$, and Neutrino Spectra Produced by $p-p$ Interaction in Astronomical Environments*.
- [6] Karlsson, Niklas. <https://github.com/niklask/cparamlib>.
- [7] Longair, Malcolm S. *High Energy Astrophysics 2nd Ed. Vol 1: Particles, photons, and their detection*. Cambridge University Press, New York, New York, 1992.
- [8] Longair, Malcolm S. *High Energy Astrophysics. 2nd Ed. Vol. 2: Stars, the Galaxy, and the interstellar medium*. Cambridge University Press, New York, New York, 1994.
- [9] Strong, Andrew W.; Moskalenko, Igor V.; Ptuskin, Vladimir S. Annual Review of Nuclear and Particle Science. 2007. Vol 57, 285
- [10] Su, Meng; Slatyer, Tracy R.; Finkbeiner, Douglas P. 2010. ApJ 724, 1044

Appendix

In this appendix I solve the heat equation (Section 2) and transport equation (Section 3) in detail.

(a) A source term of $S = 1.5 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$ after 14.4 years.

(b) A source term of $S = 2 \times 10^{-10} \text{ GeV}^{-1} \text{ cm}^{-3} \text{ yr}^{-1}$ after 6.3 years.

Figure 4: Fermi Bubble data (red) (Ackermann et al. 2014) plotted with numerical solution (blue) of the gamma-ray spectrum due to proton-proton interaction. The data matches the numerical solution well for lower energies (Figure (4a)). Figure (4b) is a good fit at high energies for the data in the non-steady-state case. Note that the errors bars show statistical errors.

A. Analytical Solution to the Heat Equation

Examine the heat equation in position-space, given by

$$\frac{\partial}{\partial t} f(x, t) = D \frac{\partial^2}{\partial x^2} f(x, t), \quad (27)$$

The coefficient D is independent of x and t . Assume the following initial condition: $f(x, 0) = f_0(x)$. To solve Equation (27) over an infinite interval, take the Fourier transform of the heat equation. Denote $f(x, t)$ as the Fourier transform of $\tilde{f}(k, t)$ and define $\tilde{f}(k, t)$ as the inverse Fourier transform of $f(x, t)$.

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \tilde{f}(k, t) dk \quad (28)$$

$$\tilde{f}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x, t) dx. \quad (29)$$

Substitute Equation (28) into the left-hand side (LHS) of Equation (27) using Equation (28). Note that e^{-ikx} is time-independent. Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} f(x, t) &= \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \tilde{f}(k, t) dk \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t} \tilde{f}(k, t) \right) e^{-ikx} dk \end{aligned} \quad (30)$$

Now substitute Equation (28) into the right-hand side (RHS) of Equation (27). Note that $\tilde{f}(k, t)$ is

position-independent. Thus,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial^2}{\partial x^2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \tilde{f}(k, t) dk \right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k, t) \left(\frac{\partial^2}{\partial x^2} e^{-ikx} \right) dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k, t) \left(-k^2 e^{-ikx} \right) dk
\end{aligned} \tag{31}$$

Therefore, using the partial derivatives given by Equation (30) and Equation (31), Equation (27) becomes:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} f(x, t) - D \frac{\partial^2}{\partial x^2} f(x, t) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t} \tilde{f}(k, t) \right) e^{-ikx} dk \\
&\quad + \frac{D}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(k^2 \tilde{f}(k, t) \right) e^{-ikx} dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\partial \tilde{f}(k, t)}{\partial t} + Dk^2 \tilde{f}(k, t) \right) e^{-ikx} dk \\
\Rightarrow 0 &= \frac{\partial}{\partial t} \tilde{f}(k, t) + Dk^2 \tilde{f}(k, t)
\end{aligned} \tag{32}$$

Equation (32) represents the Fourier transformed heat equation. Note that it is an ordinary differential equation and can be written as:

$$0 = \frac{d}{dt} \tilde{f}(k, t) + Dk^2 \tilde{f}(k, t) \tag{33}$$

Solving Equation (33) gives:

$$\begin{aligned}
\frac{d\tilde{f}(k, t)}{dt} &= -Dk^2 \tilde{f}(k, t) \\
\left(\frac{1}{\tilde{f}(k, t)} \right) d\tilde{f}(k, t) &= -Dk^2 dt \\
\int_{\tilde{f}_0(k)}^{\tilde{f}(k, t)} \frac{d\tilde{f}(k, t)}{\tilde{f}(k, t)} &= \int_0^t -Dk^2 dt \\
\ln \left(\frac{\tilde{f}(k, t)}{\tilde{f}_0(k)} \right) &= -Dk^2 t \\
\exp \left(\ln \left(\frac{\tilde{f}(k, t)}{\tilde{f}_0(k)} \right) \right) &= \exp \left(-Dk^2 t \right) \\
\tilde{f}(k, t) &= \tilde{f}_0(k) e^{-Dk^2 t}
\end{aligned} \tag{34}$$

Using the inverse Fourier transform given by Equation (34) in Equation (28) to solve for $f(x, t)$:

$$\begin{aligned} f(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left(\tilde{f}_0(k) e^{-Dk^2 t} \right) dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_0(k) e^{-Dk^2 t - ikx} dk. \end{aligned} \quad (35)$$

Using Equation (29) to define $f(\tilde{k}, 0)$,

$$\tilde{f}_0(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_0(x') e^{ikx'} dx' \quad (36)$$

Substituting Equation (36) in Equation (34), obtain:

$$\begin{aligned} f(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(x') \\ &\times \exp \left(-Dtk^2 + i(-x + x')k \right) dk dx'. \end{aligned} \quad (37)$$

Examine the following Gaussian integral,

$$\int_{-\infty}^{\infty} \exp \left(-ax^2 + bx \right) dx = \sqrt{\frac{\pi}{a}} \exp \left(\frac{b^2}{4a} \right) \quad (38)$$

Integrating Equation (37) with respect to k using Equation (38):

$$\begin{aligned} f(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_0(x') dx' \\ &\times \int_{-\infty}^{\infty} \exp \left(-Dtk^2 - i(x - x')k \right) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_0(x') \end{aligned} \quad (39)$$

$$\begin{aligned} &\times \left(\sqrt{\frac{\pi}{Dt}} \exp \left(\frac{-(x - x')^2}{4Dt} \right) \right) dx' \\ &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f_0(x') \exp \left(\frac{-(x - x')^2}{4Dt} \right) dx'. \end{aligned} \quad (40)$$

B. Analytical Solution to the Transport Equation

Second order Fermi acceleration is a purely stochastic process; therefore, given the following transport equation in momentum-space (Becker, Le, & Dermer 2006^[3]):

$$\frac{\partial f(p, t)}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left[p^2 \left(D_{pp}(p) \frac{\partial f(p, t)}{\partial p} - A(p) f(p, t) \right) \right] - \frac{f(p, t)}{\tau_{esc}(p)} + \frac{S(p, t)}{4\pi p^2}, \quad (41)$$

$A(p)$, which represents systematic acceleration, is zero. Take $f(p, t)$ to be the phase space density, $D_{pp}(p)$ to be the momentum diffusion coefficient (this causes energy gains and losses), $\tau_{esc}(p)$ to be the mean particle escape time, and $S(p, t)$ to be a source term (Becker, Le, & Dermer 2006^[3]). If we

take the source term to be a Dirac delta function, the transport equation (Equation (41)) for second order Fermi acceleration becomes

$$\frac{\partial f(p)}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_{pp}(p) \frac{\partial f(p)}{\partial p} \right) - \frac{f(p)}{\tau_{esc}(p)} + S \frac{\delta(p - p_0)}{4\pi p^2} \quad (42)$$

The first term on the right-hand side (RHS) represents momentum diffusion/transport. The coefficient of the source term, S , gives the number of particles injected per GeV per cm^3 per year. Because it will be easier to work with the particle spectrum, note the following:

$$\begin{aligned} \Psi(p) &= 4\pi p^2 f(p) \\ \Rightarrow f(p) &= \frac{\Psi(p)}{4\pi p^2} \end{aligned} \quad (43)$$

$\Psi(p)$ is the particle density and $f(p)$ is the phase space density. Solving for the steady-state solution, in which $\frac{\partial f(p)}{\partial t} = 0$, and using Equation (43), Equation (42) becomes:

$$\begin{aligned} 0 &= \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_{pp}(p) \frac{\partial}{\partial p} \left(\frac{\Psi(p)}{4\pi p^2} \right) \right) - \frac{\Psi(p)}{4\pi p^2 \tau_{esc}(p)} + S \frac{\delta(p - p_0)}{4\pi p^2} \\ &= \frac{1}{4\pi p^2} \left[\frac{\partial}{\partial p} \left(p^2 D_{pp}(p) \frac{\partial}{\partial p} \left(\frac{\Psi(p)}{p^2} \right) \right) - \frac{\Psi(p)}{\tau_{esc}(p)} + S \delta(p - p_0) \right] \\ &= \frac{\partial}{\partial p} \left(p^2 D_{pp}(p) \frac{\partial}{\partial p} \left(\frac{\Psi(p)}{p^2} \right) \right) - \frac{\Psi(p)}{\tau_{esc}(p)} + S \delta(p - p_0) \end{aligned} \quad (44)$$

Let $\frac{\partial}{\partial p} \left(\frac{\Psi(p)}{p^2} \right) = \Psi'(p)$. Evaluating the inner partial derivative of the first term in Equation (44) using the product rule gives:

$$\frac{\partial}{\partial p} \left(\frac{\Psi(p)}{p^2} \right) = \frac{\Psi'(p)}{p^2} - \frac{2\Psi(p)}{p^3} \quad (45)$$

Using the partial derivative defined in Equation (45), Equation (44) becomes:

$$0 = \frac{\partial}{\partial p} \left(D_{pp}(p) \Psi'(p) - \frac{2D_{pp}(p)\Psi(p)}{p} \right) - \frac{\Psi(p)}{\tau_{esc}(p)} + S \delta(p - p_0) \quad (46)$$

Let $\frac{\partial}{\partial p} \left(D_{pp}(p) \right) = D'_{pp}(p)$. Invoking the product rule on the first term in Equation (46) produces:

$$\frac{\partial}{\partial p} \left(D_{pp}(p) \Psi'(p) \right) = D_{pp}(p) \Psi''(p) + D'_{pp}(p) \Psi'(p) \quad (47)$$

and

$$\frac{\partial}{\partial p} \left(\frac{-2D_{pp}(p)\Psi(p)}{p} \right) = 2 \left(\frac{-D_{pp}(p)\Psi'(p)}{p} + \frac{D_{pp}(p)\Psi(p)}{p^2} - \frac{D'_{pp}(p)\Psi(p)}{p} \right) \quad (48)$$

Using Equation (47) and Equation (48), Equation (46) becomes:

$$0 = \left(D_{pp}(p) \right) \Psi''(p) + \left(D'_{pp}(p) - \frac{2D_{pp}(p)}{p} \right) \Psi'(p) + \left(\frac{2D_{pp}(p)}{p^2} - \frac{2D'_{pp}(p)}{p} - \frac{1}{\tau_{esc}(p)} \right) \Psi(p) + S \delta(p - p_0) \quad (49)$$

Solve for the $q = 2$ case and assume the following:

$$D_{pp}(p) = D_0 \left(\frac{p}{p_0} \right)^q = \frac{1}{\tau_{acc,0}} \left(\frac{p}{p_0} \right)^q$$

$$\xrightarrow{q=2} D_{pp}(p) = \frac{1}{\tau_{acc,0}} \left(\frac{p}{p_0} \right)^2 \quad (50)$$

$$\tau_{esc}(p) = \tau_{esc,0} \left(\frac{p}{p_0} \right)^{q-2}$$

$$\xrightarrow{q=2} \tau_{esc}(p) = \tau_{esc,0} \quad (51)$$

$$\text{as } p \rightarrow \infty, \Psi(p) \text{ remains infinite} \quad (52)$$

Equation (51) shows that the $q = 2$ case gives a momentum-independent escape time. Noting Equation (50), Equation (49) becomes:

$$\begin{aligned} 0 &= \frac{p^2}{\tau_{acc,0}} \Psi''(p) + \left(\frac{2p}{\tau_{acc,0}} - \frac{2p}{\tau_{acc,0}} \right) \Psi'(p) + \left(\frac{2}{\tau_{acc,0}} - \frac{4}{\tau_{acc,0}} - \frac{1}{\tau_{esc,0}} \right) \Psi(p) + S\delta(p - p_0) \\ &= \frac{p^2}{\tau_{acc,0}} \Psi''(p) - \left(\frac{2}{\tau_{acc,0}} + \frac{1}{\tau_{esc,0}} \right) \Psi(p) + S\delta(p - p_0) \\ &= \Psi''(p) - \frac{1}{p^2} \left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right) \Psi(p) + S \frac{\delta(p - p_0) \tau_{acc,0}}{p^2} \end{aligned} \quad (53)$$

Examine the homogeneous case of Equation (53), which is linear in $\Psi(p)$:

$$\Psi''(p) - \frac{1}{p^2} \left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right) \Psi(p) = 0. \quad (54)$$

The solution of $\Psi''(p) - ap^{-2}\Psi(p) = 0$, where a is a constant, is given by:

$$\Psi(p) = A|p|^{k_1} + B|p|^{k_2} \quad (55)$$

where k_1 and k_2 are the roots of $k^2 - k - a = 0$.

Solving for the roots:

$$\begin{aligned} 0 &= k^2 - k - \left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right) \\ k &= \frac{1 \pm \sqrt{1 + 4\left(2 + \frac{\tau_{acc,0}}{\tau_{esc,0}} \right)}}{2} \\ k &= \frac{1}{2} \pm \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}} \end{aligned} \quad (56)$$

Note that the times $\tau_{acc,0}$ and $\tau_{esc,0}$ are always zero or positive. Therefore, $\sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}$ is always positive. The smallest value $\sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}$ can take on, in the case of $\frac{\tau_{acc,0}}{\tau_{esc,0}} = 0$, is $\frac{3}{2}$. Thus, by Equation (56), we have one positive root and one negative root. However, to satisfy Equation (52),

Equation (55) must only contain the negative root. Dropping the magnitude of p in Equation (55) because the momentum is always positive, we have:

$$\Psi(p) = Ap^{\frac{1}{2} - \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}}, \quad (57)$$

which gives the steady-state solution to the transport equation for $\Psi(p)$, the particle density. Solving for $f(p)$ using Equation (43):

$$\begin{aligned} f(p) &= \frac{Ap^{\frac{1}{2} - \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}}}{4\pi p^2} \\ f(p) &= \frac{A}{4\pi} p^{(\frac{1}{2} - 2) - \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}} \\ f(p) &\propto p^{-\frac{3}{2} - \sqrt{\frac{9}{4} + \frac{\tau_{acc,0}}{\tau_{esc,0}}}} \end{aligned} \quad (58)$$