Modeling the Transverse Linear Optics of a Charged Particle Storage Ring

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Abstract

Charged particle storage rings play a key role in the frontiers of high-energy physics. For the most part, however, particle physics is largely inaccessible to undergraduate-level students, discouraging such students from taking an early interest in it. It is our premise, however, that the fundamental physics of storage rings can, in fact, be taught at this stage, fostering interest in accelerator physics. In this paper we present an introduction to the transverse linear optics of a charged particle storage ring suitable for second-year undergraduates. We begin by introducing transverse motion in the one-dimensional case to establish a familiarity with linear lattices. We then move on to our primary goal which is to analytically decompose a coupled two-dimensional system with a minimal amount of axiomatic treatment (the derivation of the decoupling matrix, for example). While this is by no means a comprehensive exposition on beam dynamics, it serves to provide meaningful insight into the fundamental physics involved in these machines for introductory students. It is our hope that this approach can be extended to other aspects of high-energy physics so that undergraduates may have a better glimpse at the frontiers of their field.

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1 Introduction

Charged particle storage rings such as PEP-II at the Stanford Linear Accelerator Center play key roles in particle physics and the fundamental understanding of matter. While particle physics is, to some degree, a relatively esoteric field, the prerequisite physics and mathematics to understand storage rings themselves are largely contained the undergraduate curricula of linear optics and linear algebra. Despite this, however, there are very few introductory resources available for undergraduates who wish to explore this aspect of high-energy physics.

In this paper, we present a pedagogical introduction to the transverse (primarily betatron) motion in particle storage rings suitable for a beginning undergraduate. We axiomatically make reference to Hamiltonian formalism for motivation when necessary, but otherwise remain grounded in the matrix representation of optics. Starting with basic definitions and concepts (storage rings, betatron motion, phase space), we explore the one-dimensional lattice, and then describe the analytical decomposition of a coupled two-dimensional lattice.

2 What is a Storage Ring?

Storage rings are circular paths where a beam of charged particles, such as electrons, are accelerated for radiation or collision. Particle beams are manipulated using magnets: dipole magnets are used to guide beams around curves, quadrupole magnets are used to focus the beam, and higher-order magnets are used to correct for nonlinear effects. Accelerating charges produce electromagnetic radiation and hence the centripetal acceleration around a storage ring is an effective source of radiation (typically in the x-ray range) for scientific research. Two storage rings can be combined to construct a recirculating particle collider. This was the case at SLAC's SPEAR storage ring—the precursor of its current PEP-II ring—where the collisions of positrons and electrons led to the discovery of the charm quark [1].

We assume that the charged particles in our rings have very small mass and hence can be accelerated to near-light speeds. In order to understand and control the particle beams, we must be able to characterize the motion of particles perpendicular to the direction of the general beam. The effects of slight differences in the longitudinal velocities of particles in a storage ring are not central to our understanding of transverse motion and are thus omitted.

3 Betatron Motion

In the first approximation of an “ideal” linear accelerator, the particle beam has a fixed cross section and travels with all of its momentum longitudinally along the beam line. In a more accurate representation, however, the particles have non-zero transverse momentum components and tend to diverge from the beam line, much like a flashlight beam spreads out into a cone. One's initial solution to this problem might be to place a sort of refractive optical element—effectively

a converging lens—along the beam path to focus the diverging “conical” beam into a “cylindrical” beam with only longitudinal momentum. This, however, is not possible, as each particle may have different transverse momenta and there would hence be no single optical element that could bring the diverging beam into focus. The solution to this problem is to use a series of converging and diverging optical elements. The beam’s transverse motion would then be a closed oscillation as the particles moved longitudinally down the beam line.

Accelerators use magnets to focus the beam’s charged constituents. Because the beam has two transverse degrees of freedom (x and y), these beam-focusing magnets are quadrupoles. The quadrupoles are focusing-defocusing, meaning they act as converging lenses along one axis and diverging lenses along the other. A simplified model of a beam line consists of an alternating series of x -focusing/ y -defocusing and x -defocusing/ y -focusing quadrupoles.

The beam’s transverse motion in configuration space (x - and y -axes) can then be thought of as an ellipsoid that alternates between stretching and squeezing in the x - and y -directions. The centroid of this cross section can be measured using BPMs (Beam Position Monitors) along the beam line and follows the transverse oscillatory behavior dictated by the quadrupoles. This behavior is called the betatron oscillation of the beam.

4 Transverse Phase Space

In order to describe a particle beam’s motion effectively, one must be able to develop a mathematical abstraction for a storage ring. We can represent the optical elements of the beam by matrices that act upon a vector in 4-dimensional phase-space: the two-dimensional configuration space (x and y) combined with a two-dimensional momentum space (p_x and p_y). Note that here we are only considering a simplified linear model. We will not consider energy dispersion or effects of circular motion from dipole magnets. Higher-order magnets such as sextupoles and octupoles are used as “correction” magnets for non-linear effects, and will also not be discussed further in this model as such non-linearities are not considered. Finally, it is important to remember that all “positions” and “momenta” refer to the transverse phase space of a cross section of the particle as we approximate the longitudinal speed to uniformly be that of the speed of light.

5 The One-Dimensional Lattice

The basic elements of a one-dimensional linear lattice are listed below:

$$\vec{x} = \begin{bmatrix} x \\ p_x \end{bmatrix}, \quad F_x = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}, \quad F_y = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

Here $\vec{x}$ represents the phase space vector of the beam centroid (the “average particle”), F_x is a focusing element, F_y is a defocusing element, and D is a drift

element. Recall that a y -focusing quadrupole is equivalently x -defocusing, so while we are presently considering a one-dimensional case in the x -axis, we use a naming convention that will facilitate our generalization to the 2-dimensional case later. Note that our simplified lattice treats the focusing/defocusing as thin-lenses:

$$F_x \vec{x} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ p_x \end{bmatrix} = \begin{bmatrix} x \\ p_x - kx \end{bmatrix},$$

$$F_y \vec{x} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ p_x \end{bmatrix} = \begin{bmatrix} x \\ p_x + kx \end{bmatrix}$$

Here k is the normalized integrated strength of the magnet. In this model, quadrupoles act instantaneously on the beam (this is the optical thin-lens approximation), changing the charged particles' transverse momentum based on its off-axis position. Between quadrupoles, particles drift; their positions change based on their momenta, which remain constant:

$$D \vec{x} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ p_x \end{bmatrix} = \begin{bmatrix} x + Lp_x \\ p_x \end{bmatrix}$$

Here L is proportional to the longitudinal distance between quadrupoles.

By taking a composition of matrices, one can form a linear lattice that corresponds to a certain arrangement of optical elements. For example, the arrangement $M = DF_y DF_x$ corresponds to a one- [transverse] dimensional storage ring beamline (portion of a storage ring) composed of an x -focusing and x -defocusing quadrupole separated by some distance and with some distance of drift afterwards. If this beamline represents a complete turn around the ring, then the transfer matrix M is known as the ring's *one-turn map* at the given starting point. From this standpoint, one can map a particle's position in phase space at any number of rotations, t , by $\vec{x}_t = M^t \vec{x}_0$, where $\vec{x}_0$ is the particle's initial phase space position. This naturally leads to a discussion of orbit stability, which is discussed analytically below.

6 The Symplectic Condition

An important condition on such transfer matrices is that they are symplectic. This condition stems from the nature of such matrices as canonical transformations (nondissipative Hamiltonian systems) [2]. A proof can be found in reference [6]. The symplectic condition for a matrix B can be expressed as:

$$B^T S B = S, \quad \text{for} \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

In the 4×4 case, this can be expressed as:

$$B^T S_{4 \times 4} B = S_{4 \times 4}, \quad \text{for} \quad S_{4 \times 4} = \begin{bmatrix} S_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & S_{2 \times 2} \end{bmatrix} \quad (1)$$

An important property of symplectic matrices is that their determinant is necessarily equal to one. The proof for the 2×2 case is trivial; we omit the general case, which is discussed in reference [3].

7 Normalized Phase Space

Any real matrix B with complex eigenvalues λ and $\bar{\lambda}$ is mathematically similar to a rotation matrix [4], that is:

$$R = A^{-1} M A \quad \text{or} \quad M = A R A^{-1}.$$

Where $A = [\text{Re}(\vec{v}) \quad \text{Im}(\vec{v})]$ for complex conjugate eigenvectors $\vec{v} = a \pm bi$; that is A is the matrix whose columns are the real and imaginary parts of the eigenvectors.

In the normalized phase space (the transformation to this space is given by the matrix A^{-1}) we can treat the betatron motion of the beam centroid as a simple rotation. For a general transfer matrix (not necessarily a one-turn map), we can express the centroid motion as:

$$M_{12} = A_2 R A_1^{-1} \quad (2)$$

This is true, however, only when the transfer matrix M has complex eigenvalues. For $M_{2 \times 2} = m_{ij}$ with eigenvalues λ , noting that $(m_{11}m_{22} - m_{12}m_{21}) = \det(M) = 1$, as M is symplectic, this means:

$$\begin{aligned} \lambda^2 - (m_{11} + m_{22}) + 1 &= 0 \\ \lambda &= \frac{1}{2} \left[(m_{11} + m_{22}) \pm \sqrt{(m_{11} + m_{22})^2 - 4} \right] \\ \lambda \in \mathbb{C} &\Rightarrow (m_{11} + m_{22})^2 < 4 \end{aligned}$$

What is the physical significance of the case when this does *not* hold true? If the matrix A is degenerate (noninvertible) then the rotation R cannot be constructed, and there can be no betatron oscillation. This means that the lattice is optically unstable.

When expressing the transfer matrix as $M = A R A^{-1}$, we have the freedom of choosing a gauge rotation $\tilde{R}$ such that $M = A \tilde{R} \tilde{R}^{-1} A^{-1}$. This is allowed since rotations commute, thus this is identically equivalent to 2. We can group this rotation with the A matrix so that: $M = (A \tilde{R}) R (\tilde{R}^{-1} A^{-1}) = \tilde{A} \tilde{R} \tilde{A}^{-1}$, where $\tilde{A}$ is also symplectic.

Before selecting our gauge $\tilde{R}$, note that as the beam centroid (representative of a “typical” particle) has a circular orbit in normalized phase space and thus its area is constant. This invariant is defined as $2\pi J$ by convention, thus the radius of the normalized phase space orbit is $r_{norm} = \sqrt{2J}$. A particle’s displacement in real space is proportional to both r_{norm} and β (where β is the amplitude of the particle’s betatron oscillation): $x_{amplitude} = \sqrt{2J\beta}$.

We choose a gauge such that the real space position is proportional to the phase space position and the point of maximal displacement in normalized space corresponds to the point of maximal displacement in real space. As $x_{norm} = r = \sqrt{2J}$, then $x_{real} = \sqrt{\beta} x_{norm}$. Thus we know $\tilde{A}$ has the form:

$$\tilde{A} = \begin{bmatrix} \sqrt{\beta} & 0 \\ a_{21} & a_{22} \end{bmatrix}$$

We can solve for a_{21} and a_{22} very easily, as when $x_{norm} = \sqrt{2j}$ and $p_{norm} = 0$, we know that $a_{21}x_{norm} = \frac{d}{dt}\sqrt{2j}\beta$. Further, $a_{22}\sqrt{\beta} = 1$ by the symplectic condition. Thus:

$$\tilde{A} = \begin{bmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{bmatrix} \quad (3)$$

Where $\alpha = -\frac{1}{2}\frac{d\beta}{dt}$. Further, it is noteworthy that we have here considered only the motion of a single particle (i.e. the beam centroid). In the actual beam, there is a gaussian distribution of particles with the mean particle having no betatron oscillation—having a normalized orbit corresponding to a singularity at zero. This is especially clear when one considers damping (from synchrotron radiation due to centripetal acceleration), causing betatron oscillations to be suppressed. We define the *emittance* of the beam as $\frac{1}{\pi}$ times the area of the normalized space circular orbit that represents a particle that is one standard deviation from the mean. The beam’s emittance is physically related to its real cross-sectional area.

8 The Two-Dimensional Case: Coupling

The simplest extension of our lattice to two-dimension is the case where we treat the x and y spaces as invariant subspaces, i.e. the x position or momentum cannot influence or be influenced by the y position or momentum. We can express the four-dimensional lattice elements as:

$$\vec{x} = \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix}, F_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}, F_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There is, however, an extra complication: in general the x and y phase spaces are not invariant subspaces of the whole space and are said to be coupled. Coupling

occurs when the beam's eigenplanes (the x - and y -axes relative to the betatron oscillation) do not match the x - and y -axes of the machine. Qualitatively, the beam is tilted off-axis. Coupling may occur because of solenoidal elements in the lattice (such around the interaction point of PEP-II) or because of slight tilts in the magnets. It is a reasonable assumption that the eigenplanes are close enough to the machine axes that the x -axis is strongly correlated with eigenplane 1 and the y -axis is strongly correlated with eigenplane 2. We can introduce coupling into our lattice with the following skew matrix:

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -s & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & 0 & 1 \end{bmatrix}$$

Where s is proportional to the tilt of the eigenplanes (typically much smaller than the parameters for the other optical elements). This matrix changes the x momenta proportionally to the y position, and vice versa, effectively coupling the beam's eigenplanes to the machine's x - and y -axes.

Similar to the 2×2 case, we would like to decompose a 4×4 transfer matrix M in a way where the beam's transverse motion is a rotation in some normalized eigenplane (reference frame of the beam):

$$M = C \tilde{A} R \tilde{A}^{-1} C^{-1} \quad (4)$$

Where C now represents a “decoupling” matrix that transforms into the eigenspace of the beam. It has been shown by Teng and Edwards [7] that the form of C is:

$$C = \begin{bmatrix} I_{2 \times 2} \cos \phi & -S_{2 \times 2} W^T S_{2 \times 2} \sin \phi \\ -W \sin \phi & I_{2 \times 2} \cos \phi \end{bmatrix} \quad (5)$$

Where W is a yet-undetermined 2×2 matrix. For our purposes, this form can be taken axiomatically, though Conte's text [6] is suggested for further exploration.

Given a transfer map M a composition of lattice elements, we now analytically extract the lattice decomposition. As in the 2×2 case in equation (2), we begin by decomposing based on the real and imaginary parts of complex eigenvectors:

$$M = A R A^{-1}, \quad \text{where } A = [\text{Re}(\vec{v}_1) \quad \text{Im}(\vec{v}_1) \quad \text{Re}(\vec{v}_2) \quad \text{Im}(\vec{v}_2)]$$

Here $\vec{v}_1$ and $\vec{v}_2$ represent the two pairs of eigenvectors $v_1 = a_1 \pm b_1 i$ and $v_2 = a_2 \pm b_2 i$. Note that the matrix R is already invariant to the x and y subspaces:

$$R = \begin{bmatrix} R_x & 0 \\ 0 & R_y \end{bmatrix}, \quad \text{where } R_x \text{ and } R_y \text{ are standard } 2 \times 2 \text{ rotation matrices.}$$

Our next step is to convert A into a symplectic matrix $\hat{A}$, such that equation (1) holds. In order to do this, we use a gauge transformation T composed of a scalar gauge for each subspace:

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad (6)$$

Note that this truly is a gauge transformation:

$$\begin{aligned} M &= A^{-1}RA \\ &= (TA)^{-1}R(TA) \\ &= A^{-1}T^{-1}RTA \\ &= A^{-1}(T^{-1}RT)A \\ &= A^{-1}RA \end{aligned}$$

Because R is x - y invariant and T is a scalar gauge in each subspace.

$$\begin{aligned} \hat{A} &= TA \\ \hat{A}^T S \hat{A} &= S \\ (TA)^T S (TA) &= S \\ T^2 A^T S A &= S \\ A^T S A &= T^{-2} S \end{aligned} \quad (7)$$

Since $T^T = T$ and S is x - y invariant. We may further examine (8) by looking only at the x invariant subspace. The procedure for the y subspace is identical.

$$\begin{aligned} A_x^T S_{2 \times 2} A_x &= T_1^{-2} S_{2 \times 2} \\ A_x^T S_{2 \times 2} A_x &= \frac{1}{\lambda_1^2} S_{2 \times 2} \end{aligned} \quad (9)$$

$$\lambda_1^2 I = A_x^{-1} S_{2 \times 2}^{-1} (A_x^T)^{-1} S_{2 \times 2} \quad (10)$$

From here it is trivial to extract T and $\hat{A}$ from equations (6) and (7). We want to decompose $\hat{A}$ into:

$$\hat{A} = C \tilde{A} \tilde{R} \quad (11)$$

Where C is the decoupling matrix 5, $\tilde{A}$ is a normalization matrix in “standard form” and $\tilde{R}$ is a gauge rotation.

$$\tilde{A} = \begin{bmatrix} A_x & 0 \\ 0 & A_y \end{bmatrix} \quad (12)$$

$$\tilde{R} = \begin{bmatrix} R_x & 0 \\ 0 & R_y \end{bmatrix} \quad (13)$$

Where A_x and A_y are 2×2 s of the form of equation (3) and R_x and R_y are plane rotations of angles θ_1 and θ_2 in the x - and y -eigenplanes, respectively. We now analytically determine the values of θ_1 and θ_2 , thereby determining the matrix $\tilde{R}$. Because of the nature of the invariant x and y subspaces, we can simplify our procedures with equation (11) by looking only at the multiplication of the upper-left and lower-right 2×2 sub-matrices of each component. Here we denote a sub-matrix using upper indices: M^{11} represents the upper-left sub-matrix of a matrix M , for example. We show the case for finding θ_1 ; θ_2 can be found similarly.

$$\begin{aligned} \hat{A}^{11} &= C^{11} \tilde{A}^{11} \tilde{R}_x \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \begin{bmatrix} \cos \phi & 0 \\ 0 & \cos \phi \end{bmatrix} \begin{bmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \begin{bmatrix} \sqrt{\beta} \cos \phi \cos \theta & \sqrt{\beta} \cos \phi \sin \theta \\ -\frac{\alpha}{\sqrt{\beta}} \cos \theta - \frac{1}{\sqrt{\beta}} \cos \phi \sin \theta & -\frac{\alpha}{\sqrt{\beta}} \sin \theta + \frac{1}{\sqrt{\beta}} \cos \phi \cos \theta \end{bmatrix} \\ \sqrt{\beta} \cos \phi \cos \theta &= a_{11} \quad (14) \end{aligned}$$

$$\sqrt{\beta} \cos \phi \sin \theta = a_{12} \quad (15)$$

Dividing equation (15) by equation (14):

$$\begin{aligned} \tan \theta &= \frac{a_{12}}{a_{11}} \\ \theta &= \arctan \left(\frac{a_{12}}{a_{11}} \right) \end{aligned}$$

Ultimately we get:

$$\tilde{R}_x = \begin{bmatrix} \cos \left(\arctan \left(\frac{a_{12}}{a_{11}} \right) \right) & \sin \left(\arctan \left(\frac{a_{12}}{a_{11}} \right) \right) \\ -\sin \left(\arctan \left(\frac{a_{12}}{a_{11}} \right) \right) & \cos \left(\arctan \left(\frac{a_{12}}{a_{11}} \right) \right) \end{bmatrix} \quad (16)$$

We now explicitly know $\tilde{R}$ from equations (13) and (16) as well as $\hat{A}$ from equation (7), from which we explicitly know the composition $(C\tilde{A})$. We now wish to decompose $(C\tilde{A})$ into explicit expressions for C and $\tilde{A}$. From equation (11) we get:

$$\begin{aligned}
\hat{A}\tilde{R}^{-1} &= C\tilde{A} \\
&= \begin{bmatrix} I_{2 \times 2} \cos \phi & -SW^T S \sin \phi \\ -W \sin \phi & I_{2 \times 2} \cos \phi \end{bmatrix} \begin{bmatrix} A_x & 0 \\ 0 & A_y \end{bmatrix} \\
&= \begin{bmatrix} A_x \cos \phi & -SW^T S A_y \sin \phi \\ -W A_x \sin \phi & A_y \cos \phi \end{bmatrix} \tag{17}
\end{aligned}$$

We can now extract the coupling angle ϕ by looking at the upper-left 2×2 sub-matrix of $(C\tilde{A})$, $A_x \cos \phi$. We first note that although we do not yet know the values of β_x and α_x in A_x , we do know that A_x takes the form of equation (3). Thus:

$$\begin{aligned}
\det(C\tilde{A})^{11} &= \left(\sqrt{\beta_x} \cos \phi\right) \left(\frac{1}{\sqrt{\beta_x}} \cos \phi\right) \\
\det(C\tilde{A})^{11} &= \cos^2 \phi
\end{aligned}$$

But as the top right element of $(C\tilde{A})^{11}$ is zero, the determinant is equivalent to the product of its two diagonal elements.

$$\phi = \arccos \sqrt{(C\tilde{A})_{11}^{11} (C\tilde{A})_{22}^{11}} \tag{18}$$

Using equations (3) and (18) in conjunction with equation (17) it is straightforward to calculate β_x and α_x .

$$\beta_x = \left(\frac{(C\tilde{A})_{11}^{11}}{\cos \phi} \right)^2 \tag{19}$$

$$\alpha_x = -\frac{\sqrt{\beta_x} (C\tilde{A})_{12}^{11}}{\cos \phi} \tag{20}$$

The case for β_y and α_y is identical. We now have completely described $\tilde{A}$ and the diagonal 2×2 sub-matrices of the right-hand side of equation (17). In order to fully decompose $(C\tilde{A})$, we must extract the elements of the 2×2 matrix W :

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

Given ϕ , β_x , and α_x from equations (18), (19), and (20) we can determine W analytically using only the lower-left 2×2 sub-matrix $(C\tilde{A})^{21}$. The system of equations using from the upper-right sub-matrix $(C\tilde{A})^{12}$ based on the y subspace is redundant. From equation (17):

$$\begin{aligned} (C\tilde{A})^{12} &= A_x (-W \sin \phi) \\ &= -\sin \phi \begin{bmatrix} w_{11}\sqrt{\beta_x} - w_{12}\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{w_{12}}{\sqrt{\beta_x}} \\ w_{21}\sqrt{\beta_x} - w_{22}\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{w_{22}}{\sqrt{\beta_x}} \end{bmatrix} \end{aligned}$$

The two rightmost elements can be easily solved for w_{12} and w_{22} :

$$\begin{aligned} w_{12} &= \frac{(C\tilde{A})_{12}^{21}\sqrt{\beta_x}}{\sin \phi} \\ w_{22} &= \frac{(C\tilde{A})_{22}^{21}\sqrt{\beta_x}}{\sin \phi} \end{aligned}$$

We can now solve for the other two elements in terms of w_{12} and w_{22} :

$$\begin{aligned} w_{11} &= \frac{1}{\sqrt{\beta_x} \sin \phi} \left[(C\tilde{A})_{11}^{12} - \frac{\alpha_x w_{12} \sin \phi}{\sqrt{\beta_x}} \right] \\ &= \frac{1}{\sqrt{\beta_x} \sin \phi} \left[(C\tilde{A})_{11}^{12} - (C\tilde{A})_{12}^{21} \alpha_x \right] \\ w_{21} &= \frac{1}{\sqrt{\beta_x} \sin \phi} \left[(C\tilde{A})_{21}^{12} - \frac{\alpha_x w_{22} \sin \phi}{\sqrt{\beta_x}} \right] \\ &= \frac{1}{\sqrt{\beta_x} \sin \phi} \left[(C\tilde{A})_{21}^{12} - (C\tilde{A})_{22}^{21} \alpha_x \right] \end{aligned}$$

With the W matrix specified, then C is specified by equation (5), and our decomposition is complete. We can now explicitly decompose the one-turn map M as prescribed in equation (4).

9 Conclusion

It is our wish that this paper will be able to provide an introduction to storage ring physics in a way that can inspire students to investigate accelerator physics and provide insight about the the fundamental way in which these machines operate. Our approach, of course, has been limited and leaves much to be explored, but we hope that it can be a launching point from which students may more easily transition towards more advanced textbooks. On a pedagogical level, we hope to have demonstrated that many of the topics in physics which are considered to be “too advanced” for undergraduates can at least be introduced at a fundamental level, lending a sense of insight about what is being done at the forefront of physical research.

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