THEORY AND MEASUREMENTS OF EMITTANCE
PRESERVATION IN PLASMA WAKEFIELD ACCELERATION

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Abstract

In this dissertation, we examine the preservation and measurement of emittance in the plasma wakefield acceleration blowout regime. Plasma wakefield acceleration (PWFA) is a revolutionary approach to accelerating charged particles that has been demonstrated to have the potential for gradients orders of magnitude greater than traditional approaches. The application of PWFA to the design of a linear collider will make new high energy physics research possible, but the design parameters must first be shown to be competitive with traditional methods. Emittance preservation is necessary in the design of a linear collider in order to maximize luminosity. We examine the conditions necessary for circular symmetry in the PWFA blowout regime, and demonstrate that current proposals meet these bounds. We also present an application of beam filamentation which describes the process of beam parameter and emittance matching. We show that the emittance growth saturates as a consequence of energy spread in the beam. The initial beam parameters determine the amount of emittance growth, while the contribution of energy spread is negligible. We also present a model for ion motion in the presence of a beam that is much more dense than the plasma. By combining the model of ion motion and emittance growth, we find the emittance growth due to ion motion is minimal in the case of marginal ion motion. In addition, we present a simulation that validates the ion motion model, which is under further development to examine emittance growth of both marginal and pronounced ion motion. Finally, we present a proof-of-concept of an emittance measurement which may enable the analysis of emittance preservation in future PWFA experiments.

Preface

This dissertation is based in part on the experimental apparatus and data of experiments at FACET, and the work of collaborators on various projects at SLAC National Accelerator Laboratory.

I was responsible, in collaboration with Prof. Tor Raubenheimer, for developing the theories of radially-symmetric bubble formation, applying beam theory to emittance growth due to energy spread, and developing ion motion models.

I was also responsible for the development of SLACtrac and PlasmaTrace software packages, simulations regarding ion motion, and emittance measurement algorithms.

Data was collected by FACET collaborators, with assistance including beamline commissioning and monitoring by SLAC National Accelerator Laboratory researchers and employees.

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Chapter 1

Introduction

1.1 Particle Accelerators

Particle accelerators have grown in scale and scope since their inception to become crucial tools in science and technology. They touch areas of research such as protein structure research, which may give mankind insight into his own machinery. They are used to access higher energy scales in the search of more fundamental physics. They provide targeted cancer treatment that saves lives.

In its simplest form, a particle accelerator is a machine that increases a particle's speed and thus its energy. Charged particles have historically been the least complicated particles to accelerate due to their interaction with electromagnetic fields. Electrons in particular are useful, as they are relatively simple to generate and their low mass makes manipulation easier. A simple accelerator would consist of a particle source, accelerating components, and an interaction stage; accelerators have ranged in scale from several millimeters in the case of dielectric laser accelerators [1] to the 27 km Large Hadron Collider (LHC) at CERN [2].

1.1.1 Ring and Linear Accelerators

Particle accelerators today tend to fit into two categories: ring and linear accelerators. Ring accelerators are formed of a closed loop particles can be passed through

repeatedly. There are several major advantages to ring accelerators: they can be used as light sources for a large number of users simultaneously; particles can be stored in relative stability; and beam energy can be scaled up as particles are accelerated, and relatively higher energies can be reached. However, synchrotron radiation unavoidably caps the maximum energy possible for a given design, making this scheme difficult in particular for low-mass lepton acceleration. Rings have several processes, such as filamentation [3], that result in a beam with phase space matched to the its design; as a result, making phase space improvements can be difficult to achieve.

Linear accelerators (linacs) address some of these issues, at the expense of many ring advantages. Linacs' main advantage over rings is that they do not produce synchrotron radiation. Charged particles traveling in the curved trajectories of rings radiate an average power of:

$$\langle P \rangle \propto \frac{\beta^3 E_0^4}{R \rho m^3} \quad (1.1)$$

where $\beta = v/c$, E_0 is the beam energy, R is the average radius of curvature of the ring, and ρ is the local radius of curvature of the component of the ring, and m is the particle mass [4]. Synchrotron radiation implies that power requirements increase exponentially with the desired energy. As synchrotron radiation is inversely proportional to mass and radius, heavier particles and larger rings can mitigate this power requirement. This trade-off is inherent to rings; linacs on the other hand do not have curved trajectories. In linacs without significant synchrotron radiation, energy losses can be minimal for lepton acceleration. On the other hand, without curved trajectories bunches cannot be recycled and accelerated through a linac. Since particles make a single pass through a linac, their maximum final energy is a property of the linac design, and they must be generated and accelerated continuously. Linacs also provide one beam to experimenters at a time, and if it is to be shared it must be sent to different locations. Each bunch sent through a linac has unique conditions, and in practice variations in the linac can cause unacceptable variations in the bunch properties. However, a linac beam's phase space can be conserved, so phase space improvements can be made by modifying the beam source.

1.1.2 RF Acceleration

Acceleration in both ring and linear accelerators is typically achieved through radio-frequency (RF) technology ranging in frequency from 10 MHz to 100 GHz. RF power is introduced to metal cavities which have longitudinal accelerating resonant modes. Charged particles are timed so that these accelerating modes increase the particles' energy as they pass. The necessary technologies have been vastly improved upon since their introduction, including the introduction of superconducting RF (SCRF) technology which attempts to improve efficiency and gradient through cavity walls which have lower losses than traditional RF cavities.

Both normal conducting RF and SCRF have advantages and disadvantages. Both are limited by events that disrupt their stored fields, of which there are a plethora of phenomena such as field emission, breakdown, quenching, multipacting, etc. [5–8]. At best, current RF technology can support fields up to 1 GV/m, as no gradients have been reported above this level. Novel approaches to generating field gradients larger than 1 GV/m should be considered as an alternative to stringing an increasing number of 100 MV/m structures together in a linear collider. It may be possible to abandon ring accelerators entirely if sufficiently high gradients can be realized, as that would remove the need for many passes through lower-gradient accelerating structures.

1.1.3 Current Major Facilities

Two of the highest-profile accelerator projects are the Large Hadron Collider (LHC) in operation at CERN [9] and the International Linear Collider (ILC) which has been proposed and is under consideration [10]. These projects represent the highest energies and largest accelerator facilities in the world. Each has different strengths, and depend on technologies that are currently realizable. The LHC is a 27 km ring, which cost approximately \$6.5 B in hardware and research and development, with a maximum 13 TeV collision energy (center-of-mass system) [11]. The ILC has a base design of a 31 km superconducting RF linear accelerator with a center-of-mass energy of 500 GeV. Its value, including hardware and labor, was estimated in 2007 to be \$6.65 B [12]. Baer *et al.* [13] cite several advantages of the ILC over the LHC. The ILC

Table 1.1: *Large Hadron Collider design.* The Large Hadron Collider major design parameters give insight into the performance, cost, and complexity of running the facility [11].

Energy (protons)	13 TeV
Circumference	26.7 km
Cost	\$6.5 B
Number of Magnets	9593
Number of Dipoles	1232
Number of Quadrupoles	392
Number of Bunches (protons)	2808
Number of Protons per Bunch	1.2×10^{11}

has the capability of colliding electrons and positrons, while the LHC is restricted to colliding either protons or ions. Electrons and positrons have well-defined energies and are fundamental particles (leptons), which allow for more precision when measuring mass or energy. Events do not access QCD physics that would require filtering and could obscure physics.

Traditional accelerating technology appears to be reaching limits of the gradients that can be realized. As a result, large facilities must be constructed to reach energies of interest in high-energy physics (HEP). A revolutionary approach to creating gradients orders of magnitude larger than RF or SCRF technology could simultaneously allow HEP research to reach new energies, and make it less costly and difficult to engineer HEP facilities.

1.1.4 Pushing Frontiers

Particle accelerators have advanced in the energy, the number of usable particles of beams, and the brightness of the beams, which has led to new areas of research. Higher energies allow for the creation of more exotic and heavier particles and the study of the physics of their interactions. For instance, high-mass particles such as the Higgs boson at 125.09 GeV [2] could not be studied without beams with enough energy to create them, and enough interactions to be statistically certain they had been observed. The Linac Coherent Light Source (LCLS) operates with 15 GeV electrons

and a low-emittance beam (much less than 1 mm-mrad) to generate coherent X-ray light [14].

1.2 Plasma Wakefield Acceleration

This thesis explores several aspects of the relatively new phenomenon of plasma wakefield acceleration (PWFA) as described in this section. PWFA has the potential to provide gradients orders of magnitude larger than traditional technologies, and is being actively developed into a mature acceleration method at several institutions, including the Facility for Advanced Accelerator Experimental Tests (FACET) at the SLAC National Accelerator Laboratory. As described previously, high energy physics research continues to look for advances in accelerating gradients that can be used to design experiments with higher energies in order to probe for new physics. Higher accelerating gradients may translate to smaller facilities and lower costs than designs normal conducting RF or SCRF used by the ILC.

1.2.1 Concept Overview¹

In general, acceleration by wakefields in a plasma is the acceleration of charged particles by interaction with a “wake” of plasma electrons. Plasma wakes are characterized by a transverse oscillation of plasma electrons around a central axis. Ideally, wakes appear to move at the same velocity $v \approx c$ as the accelerated charged particles. There are two main methods for driving plasma electron motion to create a wake. Laser wakefield acceleration (LWFA) uses a short, high-power laser pulse to directly drive plasma electron motion [15, 16]. PWFA uses the fields of a driving bunch of charged particles to drive plasma electron motion [17] (see fig. 1.1a).

LWFA and PWFA differ only in the method of driving the wake, while the physics of the bunch accelerated by the wake (called the “witness” bunch) remains the same. That difference has the practical effect of timing. LWFA requires transverse alignment and timing of a laser pulse with an electron beam. Typical wakefield lengths require

¹Plasma wakefield acceleration encompasses many subregimes and physics. This section describes many of the features of PWFA which are relevant to this thesis.

Figure 1.1: *Diagrams of PWFA physics.* (a) PWFA scheme for driving a wake with an electron bunch, which accelerates a witness bunch, represented in the co-moving frame of the electrons traveling at $v \approx c$. Focusing/defocusing and accelerating/decelerating regions are labeled. Ions are represented by red + signs, electrons by blue – signs. Electron trajectories are indicated by blue arrows. On-axis total charge densities are represented by a pink and blue gradient [18]. (b) A simulation of PWFA shows plasma electron densities in blue, bunch electron densities in red, and is overlaid with accelerating electric field in red and bunch current in dashed blue [19].

10 μm precision in bunch-to-wake alignment, which imply timing on the 1 fs time scale; this is difficult to achieve with high-power lasers. PWFA timing of two electron bunches is fundamentally different. If two bunches can be shaped from one bunch, the need for timing two separately-generated bunches may not be necessary. However, the required precision in bunch shaping may not be realizable with current technology either. Additionally, the group and phase velocities of a laser pulse in plasma are $v_p = v_g = c\sqrt{1 - \omega_p^2/\omega^2}$, so it is possible for the timing between the laser and the witness bunch to change over long distances as they propagate, which is known as phase slippage [16]. PWFA may not have this issue as long as the drive and witness bunches have approximately the same energy.

Typically, the particles accelerated in PWFA are electrons, although some experiments have observed PWFA-like acceleration with a positron driver [20, 21]. The plasma can be generated any multitude of ways, although two methods currently predominate: field ionization, where plasma is created as the strong fields present in the drive bunch ionize a gas (and immediately after, excite a wake); or “pre-ionization,” where a laser pulse ionizes a gas without creating a wake [22]. An ultrarelativistic bunch of electrons has electric fields predominately in the radial direction. These fields drive free plasma electrons outward, while more massive ions remain stationary (see the blue arrows in fig. 1.1a). Once the drive bunch has passed, the electrons experience a predominantly restoring force from the ion background. They collapse, creating a region with a high electron density. The space charge and electrodynamics of the electrons and ions create longitudinal fields that can accelerate bunches. In fig. 1.1a, the accelerated witness bunch is placed at a location that experiences accelerating electric fields.

1.2.2 Linear vs. Blowout Regimes

The wake response can either result in linear or nonlinear plasma electron density perturbations. The “blowout” regime is characterized by nonlinear wakes. These wakes have a “bubble” defined as the volume of ions with no plasma electrons, within an “electron sheath” composed of a narrow, high-density layer of plasma electrons

that originated from within the bubble region. In blowout, a radially-symmetric ion bubble results in linear transverse focusing forces. The accelerating fields are constant in transverse locations in the blowout regime [23, 24]. The continuous focusing in the blowout regime can focus and guide bunches within the electron sheath. This keeps bunch density high, which makes driving wakes for long distances possible.

Linear PWFA wakes result are characterized by a gradient in the plasma electron density. In the linear PWFA regime, there is no ion bubble, and as a result transverse focusing is nonlinear. Nonlinear transverse focusing makes modeling the transverse focusing and beam guiding much more difficult. The magnitude of the accelerating fields also depends on the transverse location in the plasma wake. These effects make the linear PWFA regime more challenging than the blowout PWFA regime; we will only consider the blowout regime.

1.2.3 Efficiency and Energy Conservation

In PWFA, it is important to recognize that energy is conserved, and so this scheme is essentially a transformer that operates with some level of efficiency. Energy is transferred from the drive bunch to the witness bunch via the plasma. There are schemes that may improve the efficiency of this process through bunch shaping [25, 26].

1.2.4 Breakdown

PWFA has the advantage that it depends on a media that is already broken down; gradients have been observed in excess of 50 GV/m [27], with the potential for gradients several orders of magnitude larger than the current observations of < 1 GV/m RF gradients. While currently not a limiting factor in PWFA, further breakdown limits are in the form of second ionization of the plasma ions. In the case of hydrogen, where second ionization is impossible, other physics such as ion motion interfere with PWFA. As larger gradients are realized, higher energies can be achieved in shorter distances. This has the potential to reduce the apparatus, space, and ultimately the cost required to achieve ever-higher energies.

1.2.5 Ion Motion

Typically, the plasma ions are assumed to be stationary because their mass is much larger than the plasma electrons. However, there are scenarios where plasma ions could move significantly as an electron bunch passes by [28]. Very dense drive and witness beams can have fields strong enough to cause ion motion before they pass. If the ions are focused to high densities, the assumption of linear focusing forces becomes incorrect and these high density regions can further focus beams. While this may not be an issue for drive beams, this may create difficulties in collecting and focusing witness beams with properties desired for colliders and other applications.

1.2.6 Hollow-Channel PWFA

While not covered in this dissertation, the hollow-channel PWFA configuration should be mentioned, as it has the potential to address positron acceleration [4], ion motion, and certain types of head erosion [27]. The key concept of a hollow channel is a plasma distribution that looks like a pipe; ideally, the plasma and any neutral gas is not present within a certain radius about a longitudinal axis, which is then a region that is simply a vacuum. Assuming that the bubble radius is larger than the vacuum, blowout PWFA can continue mostly as described before.

There are several attractions to this scheme. As the beam propagates in a vacuum, the gradient will result purely from the high electron density region. There will be no focusing or defocusing, as there will be no ions within the vacuum region. This is an advantage when accelerating positrons, as an ion background would defocus a positron beam.

Creating a hollow channel is technically difficult. Plasma approaches including ring profile laser pulses or angled gas jets could create a hollow channel, but they remain in development and conceptual stages. Without any inherent focusing, drive and witness beams will expand as they propagate. External focusing may complicate the plasma dynamics and has not been studied. Forgoing external focusing could result in significantly shorter PWFA regions as drive beams expand and cease to drive PWFA, or witness beams expand outside of the PWFA bubble.

1.2.7 Gas source

In order for plasma acceleration to occur, a plasma must first exist, and for a plasma to exist, there must be a gas present. In simpler experiments, a gas is pumped into the beam chamber. The density can be determined from the ideal gas law, $PV = Nk_B T$ which becomes $n_p = N/V = P/k_B T$. A thermocouple and a pressure sensor measure the pressure and the temperature of the gas. Assuming 100% ionization, as is expected in beam-driven ionization and laser pre-ionization, this determines the plasma density n_p .

Alternatively, a metal gas can be created in a heat pipe oven [29]. A heat pipe oven consists of a vacuum pipe lined with a wire mesh, surrounded by heating elements. At either end of the oven, water cooling jackets are installed, which lower the temperature of the pipe significantly. A metal is contained inside the oven, which should be a gas at the high temperatures inside the vacuum pipe and a solid at the temperatures near the cooling jackets. The metal is also typically chosen for low first ionization and high second ionization to ensure that the plasma created is singly-ionized.

In operation, the heating elements are turned on and the volume is filled with an inert buffer gas. As the oven heats up, the metal melts and vaporizes. At higher temperatures, the apparatus enters oven mode as convection begins. The cooling water jackets keep the inert buffer gas below the metal's boiling point. As the metal gas encounters the buffer gas, it condenses and forms a liquid long the pipe walls. It flows along the wire mesh as a liquid. Toward the center of the oven, it is vaporized by addition of heat from the heating elements. The metal gas then is drawn by convection towards the cooling jackets, repeating the process.

The heat pipe oven has been shown to create an even gas density in the interior of the oven. At the cooling jackets, there is a ramp down as the gas cools and condenses. An example of measurements particular to FACET are shown in fig. 1.2.

1.2.8 Plasma Ionization

The plasma density is typically assumed to be equivalent to the gas density by assuming 100 % ionization. Different ionization processes such as field ionization or

Figure 1.2: *Heat pipe oven density profile for lithium².*

laser ionization may result in different plasma profiles. PWFA is usually assumed to be agnostic to the atomic element used in the plasma, as ions are assumed to only single-ionize and be stationary. However, if intense fields are created, possibly by the beam or by a laser, further ionization of the gas can occur. This depends on which elements are present in the gas, as different elements have different ionization energies for multiple-ionization. Also, ion mass can play a role in the degree of any ion motion present.

Beam-driven ionization can occur either through field ionization or collisional ionization. In tunnel ionization, the drive beam's relativistic fields allow electrons to escape atoms, similar to laser ionization. This type of ionization may be unpredictable due to subtle differences in shot noise that lead to different beam fields on a shot-by-shot basis. In collisional ionization, electrons in the drive beam undergo scattering with electrons; in all physics considered here, scattering is negligible.

Rather than rely on beam-driven ionization, another option is to ionize the gas using a high-energy laser. One way of achieving ionization is with a cone-shaped optic

²Private communication from N. Vafaei-Najafabadi and K. Marsh.

called an axicon is used to create a line focus from a flat-top profile. At higher intensities, the laser's electric field ionizes the various gases present [22]. Laser ionization has the advantage of greater stability. Recombination of plasma electrons and ions happens on a much longer time scale than the PWFA interaction, so the laser pulse is timed to come ~ 100 ps before the beam.

Research is ongoing to determine the shape of the plasma's up- and down-ramps, which are most likely different from the ramp shown in fig. 1.2. Whether using laser ionization or beam-driven ionization, there is some ambiguity in the length scales where ionization goes from a region of no ionization to full ionization. As the focusing force of the ions is dependent on the plasma density, this can significantly affect the match into the plasma and the transverse dynamics of the beam. Some laser ionization schemes appear to circumvent this problem by using a ramp length scale much smaller than any possible beam evolution.

1.3 PWFA Linear Collider Applications

One of the potential applications for PWFA is a linear collider made up of PWFA stages, typically referred to as PWFA-LC (see fig. 1.3 for the diagram of a proposal [30]). Proposals call for center-of-mass energies that are capable of producing Higgs bosons, up to several TeV [30–33]. The Compact Linear Collider (CLIC) [34] is the most comparable project to the PWFA-LC in terms of ambitions. In order to be competitive as a collider, a PWFA linac must be high energy, at a minimum 0.5 TeV, preferably on the order of 1 TeV to 10 TeV. It should accelerate positrons as well as electrons, for high-energy electron-positron physics experiments.

The rate of collisions being studied is essential as well. This is the product of the luminosity $\mathcal{L}$ and cross section σ . While cross section is dependent on the physics, the luminosity is generally dependent on the linac design and beam properties [35]:

$$\mathcal{L} = f \frac{n_1 n_2}{4\pi \sqrt{\epsilon_x \beta_x \epsilon_y \beta_y}} \quad (1.2)$$

where the equation represents collisions of gaussian bunches, f is the frequency of

Figure 1.3: Diagram of staged PWFA-LC. Figure from Delahaye *et al.* [30].

bunch collisions, n_1 and n_2 are the number particles in each respective bunch, ϵ_x and ϵ_y are the transverse emittances, and β_x and β_y are the transverse beta functions. In particular, for electron-positron colliders with “flat beams” where one dimension is much smaller than the other, luminosity can be recast as:

$$\mathcal{L} \approx \frac{137}{8\pi r_e} \frac{P_{wall}}{E_{cm}} \frac{\eta}{\sqrt{\epsilon_y \beta_y}} N_\gamma H_D \quad (1.3)$$

where P_{wall} is the wall plug power, η is the efficiency of transferring power from the wall plug to the beam, E_{cm} is the center of mass energy, $N_\gamma \approx 2\alpha r_e n / \sigma_x \sim 1$ is the number of beamstrahlung photons, and H_D is due to the bunches pinching as they cross. Mitigating the number of beamstrahlung photons while maximizing luminosity is what leads to the use of flat beams [35]. However, this must be balanced by AC power consumption. Overall efficiency in converting wall plug power into beam energy must be at minimum 5% or greater in order to avoid excessive power requirements and operational expenses.

As in any accelerator, the repetition rate may be increased to increase luminosity.

It is important to consider that higher repetition rates may increase plasma temperatures and decrease the plasma density. PWFA linear collider designs will need to plan for the logistics of cooling due to plasma heating while maximizing the luminosity.

1.3.1 Staged PWFA-LC

In order to reach the desired energies, multiple PWFA stages may be used. Multiple drive beams can be used to accelerate a single witness bunch in plasma stages chained together. Additionally, the energy spread must be relatively small if the beam is to be captured and focused to small transverse sizes at an interaction point.

Several proposals for a staged PWFA collider have been put forward [30–33]. A comprehensive set of parameters for a staged PWFA-LC was proposed by Seryi *et al.* [32] (see table 1.2a). An updated version of this type of collider was proposed by Adli *et al.* [33] and Delahaye *et al.* [30] (see table 1.2b).

1.3.2 ILC “Afterburner”

It may also be possible to apply plasma wakefield acceleration to the ILC [30, 36]. The proposals call for splitting each ILC bunch into a drive and witness bunch pair. Raubenheimer [36] proposes a drive-to-witness charge ratio of 3 : 1, while an updated proposal by Delahaye *et al.* [30] calls for a ratio of 2 : 1. These parameters can be found in table 1.3.

1.4 PWFA-LC Challenges

In order to evolve PWFA-LC from a concept into a well-engineered collider proposal, many finer details must be addressed. Some are more mundane, such as timing, alignment, and stability; some are conceptual.

Emittance and emittance dilution are a serious concern, as this directly affects the ability to focus beams, luminosity and the ability to use PWFA-accelerated beams in

Table 1.2: *Selected parameters for a staged PWFA linear collider.*(a) PWFA-LC parameters, driven by a CLIC-type beam from Seryi *et al.* [32].

Energy (CMS)	1 TeV
N_e , witness bunch	1×10^{10}
Normalized (?) horizontal emittance [mm-mrad]	2
Normalized (?) vertical emittance [mm-mrad]	0.05
Horizontal beam size (1σ) at IP [m]	1.4×10^{-7}
Vertical beam size (1σ) at IP [m]	3.2×10^{-9}
Bunch length (1σ) at IP [μ m]	10
Plasma density [$1/\text{cm}^3$]	1×10^{17}

(b) Updated PWFA-LC parameters from Delahaye *et al.* [30], Adli *et al.* [33].

Energy (CMS)	250 GeV	500 GeV	1 TeV	3 TeV	10 TeV
N_e , experimental bunch	1.00×10^{10}	1.00×10^{10}	1.00×10^{10}	1.00×10^{10}	1.00×10^{10}
Normalized horizontal emittance [mrad]	1.00×10^{-5}	1.00×10^{-5}	1.00×10^{-5}	1.00×10^{-5}	1.00×10^{-5}
Normalized vertical emittance [mrad]	3.50×10^{-8}	3.50×10^{-8}	3.50×10^{-8}	3.50×10^{-8}	3.50×10^{-8}
Horizontal beam size (1σ) at IP [m]	6.71×10^{-7}	4.74×10^{-7}	3.35×10^{-7}	1.94×10^{-7}	1.06×10^{-7}
Vertical beam size (1σ) at IP [m]	3.78×10^{-9}	2.67×10^{-9}	1.89×10^{-9}	1.09×10^{-9}	0.60×10^{-9}
Bunch length (1σ) at IP [m]	20.0×10^{-6}	20.0×10^{-6}	20.0×10^{-6}	20.0×10^{-6}	20.0×10^{-6}
Plasma density [$1/\text{cm}^3$]	2×10^{16}	2×10^{16}	2×10^{16}	2×10^{16}	2×10^{16}

Table 1.3: *Selected parameters for a ILC “afterburner”.*

Proposal	Early [36]		Updated [30]
Design Type	US Warm	US Cold	ILC
Drive/Witness Charge Ratio	3 : 1	3 : 1	2 : 1
N_e , drive bunch	1.1×10^{10}	1.5×10^{10}	1.33×10^{10}
N_e , experimental bunch	0.4×10^{10}	0.5×10^{10}	0.66×10^{10}
Normalized horizontal emittance [m-rad]	360×10^{-8}	960×10^{-8}	960×10^{-8}
Normalized vertical emittance [m-rad]	4×10^{-8}	4×10^{-8}	4×10^{-8}
Horizontal beam size (1σ) at IP [nm]	37	60	286
Vertical beam size (1σ) at IP [nm]	3.9	3.9	2.7
Bunch length (1σ) at IP [μm]	33	50	20
Plasma density [$1/\text{cm}^3$]	2×10^{16}	2×10^{16}	?

many scenarios. Expressions for luminosity for generic colliders, as well as electron-positron linear colliders lead to [35]:

$$\mathcal{L} \propto \frac{1}{\sqrt{\epsilon}} \quad (1.4)$$

Positron PWFA, while observed experimentally [20, 21], remains virtually undeveloped as a theory. It is desirable to use PWFA gradients in a collider, but without a fully-developed positron approach, it will not be possible to design a system with equal positron and electron energies. This would make it more difficult to efficiently achieve a high center-of-mass energy that is desirable for high energy physics, and so positron PWFA is an extremely important phenomena to develop in addition to electron PWFA.

Hollow-channel PWFA may be a step towards positron PWFA, and may avoid matching concerns in PWFA, but also faces large hurdles. Recent research has shown energy loss and a degree of channel-to-beam alignment [37]. However, blowout and acceleration have yet to be conclusively observed.

PWFA is a promising contribution to accelerator science, and none of these challenges appear insurmountable. Indeed, advances in all the various frontiers of PWFA

are encouraging, as an increasing number of potential issues are addressed.

1.5 Addressing PWFA Challenges

This dissertation focuses on the preservation and measurement of small transverse beam emittances in PWFA, in order to achieve high luminosity eq. (1.4). Previously, the blowout region of PWFA has been assumed to be symmetric [23, 24]; we introduce theory that places bounds on beam parameters in order to ensure that the bubble is circularly symmetric in chapter 2. By guaranteeing circular symmetry, we can assume linear uncoupled focusing fields and in turn the possibility of emittance preservation as a consequence of Liouville's Theorem. It has been unclear how beam emittance is affected once the beam is in the plasma, although some work has been done regarding plasma ramps with specific forms [38]; we apply beam filamentation theory to PWFA and show how the beam and emittance will become matched and the length over which that process occurs chapter 3. Ion motion has long been assumed to be a problem, but models have been limited to showing the distance over which ion motion becomes an issue [28]; we present a model for ion motion in the presence of dense beams and calculate the change in plasma density on axis up to the ion collapse region. We then apply the emittance match theory we developed and find the emittance growth of a beam with moderate ion motion. We are also able to simulate the plasma response to a beam and reproduce the features of the ion motion theory. Emittance measurements of PWFA have been nonexistent until experiments at FACET³; we present a proof-of-concept for emittance measurement which presents the possibility of robust error analysis, a feature not present in other PWFA emittance measurements (e.g. Vafaei-Najafabadi *et al.* [39]). These improvements on the state-of-the-art theory and practice of PWFA are the main impact of this dissertation.

³Emittance measurements of the scheme of ionization injection PWFA were made independently and in parallel using a similar method to the one presented here, albeit without error analysis, by Vafaei-Najafabadi *et al.* [39].

Chapter 2

Plasma and Flat Beam Dynamics

Examining the luminosity conditions of a linear collider for electrons and positrons leads to the conclusion that flat beams can be used to prevent beamstrahlung while optimizing luminosity (see section 1.3 and Particle Data Group, 29.6.1, 2014 [35]). Generally, flat beams are not needed to drive PWFA. However, it may be tempting to use the same accelerator to produce the drive and witness bunches. In particular, energy upgrade “afterburner” schemes typically propose splitting an existing linear collider bunch into two bunches, as Delahaye *et al.* [30] suggest for the ILC.

Lu *et al.* [24] have developed a theory for the nonlinear regime of PWFA, but it assumes a circular symmetry that is not present in flat beams. It is important to understand what conditions are necessary for flat beams to drive PWFA with circular symmetry because asymmetric wakes have a variety of issues. Asymmetric wakes, for instance, will have asymmetric focusing forces. Their focusing forces are nonlinear as well, and exhibit coupling that is not desirable in PWFA.

All of these conditions are undesirable from beam optics, phase space preservation, and acceleration standpoints. Each of these issues presents difficult conditions to understand and perform PWFA, and it is simpler to determine what is necessary to prevent the generation of asymmetric bubbles. As long as the majority of plasma electrons form a symmetric bubble, any asymmetry will be a perturbation on circular symmetry. Determining a bound on the fraction of plasma electrons with an asymmetric response to a flat beam provides a pragmatic condition for driving PWFA with

a flat beam.

In order to examine the source of any potential asymmetric plasma electron motion, the fields of the driving bunch should be examined. The driving bunch can be assumed to be relativistic, as it is in Lu *et al.* [24], which ensures that their electric fields are approximately transverse. A comparison of circularly-symmetric and elliptical drive beams illustrates their differences. Qualitatively, it is clear that at a significant distance from an elliptical beam, its fields are circularly symmetric. Quantifying that distance provides a bound on the relative size of the drive beam to the bubble radius where an asymmetric beam will drive a symmetric bubble.

2.1 Relativistic Beam Fields

In order to describe the plasma response to the beam, it is first useful to examine the fields of a relativistic bunch. An ultra-relativistic drive bunch has an electric field in the lab frame that is nearly entirely radial. This can be seen by transforming the drive bunch fields from the center-of-mass (COM) frame to the lab frame. A Lorentz transform of the COM fields $\vec{E}$ and $\vec{B}$ to the lab frame fields $\vec{E}'$ and $\vec{B}'$ [40], assuming that $\vec{E}_x \sim \vec{E}_y \sim \vec{E}_z$, with the beam velocity $\vec{\beta} = (v/c)\hat{z} \approx \hat{z}$ and $\gamma \gg 1$:

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \quad (2.1)$$

$$\vec{B} = 0 \quad (2.2)$$

$$\vec{E}' = \gamma E_x\hat{x} + \gamma E_y\hat{y} + E_z\hat{z} \approx \gamma E_x\hat{x} + \gamma E_y\hat{y} \quad (2.3)$$

$$\vec{B}' = -\gamma\frac{\beta}{c}E_y\hat{x} + \gamma\frac{\beta}{c}E_x\hat{y} \quad (2.4)$$

Then, beam fields are nearly entirely transverse, and the longitudinal coordinate can be ignored. Assuming the plasma electrons are initially at rest, plasma electron motion is entirely transverse. The magnetic fields $\vec{B}'$ do not contribute to the dynamics as long as plasma electrons are not significantly relativistic, as the Lorentz force from the magnetic field is proportional to their velocity.

For a circularly-symmetric driver, plasma electrons are expelled radially away from the drive bunch as the drive bunch passes. Depending on the field strength, some or

all of the plasma electrons are ejected from a region around the drive bunch, leaving only a central region or column of positively-charged ions [24].

2.2 Summary of PWFA Model¹

The most complete PWFA blowout model to date is that of Lu *et al.* [24]. The 2D model makes several assumptions that simplify PWFA physics to produce tractable solutions to quantities such as the bubble radius, and conditions for the blowout regime.

2.2.1 Quasistatic Approximation

The first assumption is the PWFA quasistatic approximation [41], which is the assumption that the transverse motion of both drive and witness electrons evolves on a much longer time scale than the physics of the plasma. In this case, the beam distribution can be assumed to be fixed on the timescale of the plasma response. This can be written in terms of a coordinate transform from $(x, y, z, t) \rightarrow (x, y, \xi, s)$ and assumptions on the partial derivatives:

$$\xi \equiv z - ct \tag{2.5}$$

$$s \equiv ct \tag{2.6}$$

$$\partial_s \ll \partial_\xi \tag{2.7}$$

$$\partial_s^2 \ll \partial_\xi^2 \tag{2.8}$$

It can be shown that this approximation leads to a modified Lorentz Force for the plasma [42]:

$$c \frac{d\vec{P}_\perp}{d\xi} = -\frac{e}{1 - v_z} \left[\vec{E}_\perp + \left(\vec{v} \times \vec{B} \right)_\perp \right] \tag{2.9}$$

¹This section summarizes the theory and results of Lu *et al.* [24], which will be referenced and modified for flat-beam approximations in other sections. *This section does not include original work.* The equations are only modified to use MKS units instead of the normalized units used by the authors, for the sake of clarity.

2.2.2 Ring Model

The ring model developed by Lu *et al.* [24] assumes that particles do not cross each other, and that there is circular symmetry. Then, all of the relevant physics can be derived from Gauss' Law by determining the amount of enclosed charge for a given particle. In this model, the enclosed charge includes the ions, the plasma electrons, and the drive beam electrons of radius a . The electric field can then be found at a radius of $r_0 > a$ and used to find the equations of motion. For a non-relativistic plasma electron, with a negligible longitudinal velocity v_z , this is a differential equation:

$$\frac{d^2 r}{d\xi^2} = \frac{\omega_p^2}{c^2} \left(-\frac{r}{2} + \frac{r_0^2}{2r} + \frac{n_b a^2}{2n_p r} \right) \quad (2.10)$$

where the plasma frequency is $\omega_p^2 = n_p e^2 / \varepsilon_0 m$ [24]. The particle's maximum radius r_m is calculated by finding the potential for this force. Lu *et al.* [24] find a local minimum of the maximum radius $r_m(r_0)$ by differentiating $\frac{dr_m}{dr_0}$, but only when:

$$\frac{n_b}{n_p} \geq 1.792 \quad (2.11)$$

Finding a local minimum contradicts the assumption that particles never cross each other, and so this represents a condition for blowout. The blowout radius, at a given plasma density that meets this condition, is found by Lu *et al.* [24]²:

$$r_{\max} = 2.58 \frac{c}{\omega_p} \sqrt{\Lambda} \quad (2.12)$$

$$\Lambda \equiv \frac{\omega_p^2}{c^2} \int_0^\infty r \frac{n_b}{n_p} dr \quad (2.13)$$

In particular, Lu *et al.* [24] solve $r_{\max}$ for a uniform circular cylinder beam of radius a and density n_b , and beam with a Gaussian transverse distribution with transverse

²The equations are presented with the normalization used by Lu *et al.* [24] reversed.

width σ_r and peak density $n_{b,0}$:

$$r_{\max} \Big|_{(\text{uniform})} = 1.831 \frac{c}{\omega_p} \sqrt{\frac{n_b}{n_p}} a \quad (2.14)$$

$$r_{\max} \Big|_{(\text{Gaussian})} = 2.589 \frac{c}{\omega_p} \sqrt{\frac{n_{b,0}}{n_p}} \sigma_r \quad (2.15)$$

However, “simulations and trajectory calculations” for a Gaussian beam show a blowout radius of [24]:

$$r_{\max} \Big|_{(\text{Gauss, exp})} \approx 2 \frac{c}{\omega_p} \sqrt{\frac{n_{b,0}}{n_p}} \sigma_r \quad (2.16)$$

Lu *et al.* [24] note that this is most likely a consequence of the Gaussian longitudinal density variation; the ring model assumes a flat longitudinal density profile with the peak density of the Gaussian longitudinal profile.

2.3 Fields of Asymmetric Distributions

A simple model for a drive beam distribution is one that is Gaussian in x , y , and ξ . Most generally, the beam may be an ellipsoid with a charge distribution of:

$$\rho(x, y, \xi) = \frac{Q_{total}}{(2\pi)^{3/2} \sigma_x \sigma_y \sigma_\xi} e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{\xi^2}{2\sigma_\xi^2}} \quad (2.17)$$

where $s = ct$ and $\xi = z - ct$

where x , y , z , and t are lab frame coordinates. Correlations in x and y are not considered, because a simple rotation of the coordinate system uncouples the two planes. Correlations with ξ are also not considered, as they lead to much more complex sheath dynamics such as varying bubble shapes; at any rate, ξ correlations can be controlled via lattice adjustments.

A circular drive beam has circular symmetry, thus reducing the dimensionality of

the problem:

$$\rho(r, \xi) = \frac{Q_{tot}}{(2\pi)^{3/2} \sigma_r^2 \sigma_\xi} e^{-\frac{r^2}{2\sigma_r^2} - \frac{\xi^2}{2\sigma_\xi^2}} \quad (2.18)$$

$$= Q_{tot} f(\xi) \frac{1}{2\pi \sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}} \quad (2.19)$$

where $f(\xi) = \frac{1}{\sqrt{2\pi} \sigma_\xi} e^{-\frac{\xi^2}{2\sigma_\xi^2}}$

The relativistic assumption means that the longitudinal field components can be ignored. This distribution results in an entirely radial electric field that can be calculated by exploiting Gauss' Law:

$$\vec{E} = E_r \hat{r} \quad \int_{SA} \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \quad (2.20)$$

$$E_r(r, \xi) = \frac{Q_{total}}{(2\pi)^{3/2} \epsilon_0 \sigma_\xi r} e^{-\frac{\xi^2}{2\sigma_\xi^2}} \left(1 - e^{-\frac{r^2}{2\sigma_r^2}} \right) \quad (2.21)$$

However, if the beam is not circularly symmetric but elliptically symmetric, the assumptions leading to analytic equations for the sheath radius are not met [24]. The fields for a Gaussian ellipsoid of charge (eq. (2.17)) can be written [43]:

$$\begin{aligned} \tilde{E} = & \frac{Q_{total}}{4\pi\epsilon_0\sigma_{xi}\sqrt{(\sigma_x^2 - \sigma_y^2)}} \\ & \times \left[w \left(\frac{x + iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) - e^{\left[-\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} \right]} w \left(\frac{x \frac{\sigma_y}{\sigma_x} + iy \frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) \right] \end{aligned} \quad (2.22)$$

As can be seen in fig. 2.1, fields close to the bunch can be asymmetrical for elliptical distributions, while fields far from asymmetrical and round bunches converge.

Figure 2.1: *Gaussian beam distributions and fields.* Red ellipses represent the RMS area of beam distributions, which have the same linear charge density but different aspect ratios. Arrows indicate direction and magnitude of the electric field. The magnitude of the electric field is shown in color, and contours represent surfaces of constant electric field magnitude. The contour levels shown in both (a) and (b) are identical and can be directly compared to each other.

2.4 Consequences of Ellipsoidal Blowout

While it is possible to drive non-circular blowout, and analyze and simulate this regime, this is beyond the scope of this dissertation. However, it is important to recognize some of the key difficulties of non-circular blowout to underscore why it is not desirable and how it may be avoided.

One of the key assumptions in PWFA without ion motion is that ions have large mass and don't move on the timescale of the drive beam passing by. This means that, in a circularly symmetric PWFA bubble, there is a constant ion background. Due to the symmetry, Gauss' Law can be applied to find the electric fields within the bubble, in a polar coordinate system:

$$E_r 2\pi r l = \frac{n_p \pi r^2 l}{\epsilon_0} \quad (2.23)$$

$$E_r = \frac{n_p r}{2\epsilon_0} \quad (2.24)$$

Figure 2.2: *Electric potential of a uniform ellipse.* Uniform ellipses have asymmetric potentials. Figure by Siemann [44].

This shows that, in a circularly symmetric PWFA bubble, the focusing forces are linear, and independent of ξ .

An ellipsoidal drive beam must create bubbles that lie between the airplane-wing geometry of a planar beam and the circularly geometry of a radial beam. Any geometry other than circular creates focusing forces that are nonlinear and potentially coupled, as the symmetry argument of Gauss' Law does not apply. This can be seen by taking an elliptic bubble as an example, as developed by Siemann [44]. An ellipsoidal bubble has an asymmetric potential as shown in fig. 2.2. This implies that focusing forces will be different in the x and y directions. The final result shows [44]:

$$E_x(x, y) = 4xQ\sqrt{\frac{1}{ab^3}} \int_0^\infty \frac{dp}{p^{3/2}} e^{-a/bp} \operatorname{erf}\left(\sqrt{b/ap}\right) + \mathcal{O}(x^3, y^3, x^2y, \dots) \quad (2.25)$$

Fields are linear and uncoupled through second order, but become nonlinear at higher orders (see fig. 2.3). The linear focusing property of these fields satisfies Liouville's Theorem; if nonlinearities are small, phase space is preserved.

Figure 2.3: *Electric field of a uniform ellipse.* The electric field in x is shown at varying offsets in y . Fields are uncoupled in x and y to second order, although they are coupled at higher orders. The Taylor series plotted comes from eq. (2.25). Figure by Siemann [44].

However, asymmetric fields suggest that bubble lengths in x vs. y are different as the field strengths in x and y are different. The fields in an asymmetric bubble will be complex and depend in detail on the bubble amplitude. However, in most practical cases, we do not need to solve for an asymmetric bubble, as shown in the next section.

2.5 Ellipsoidal Drive Beam with Circular Blowout

In order to ensure that blowout results in a circularly-symmetric bubble, we calculate the largest asymmetry that will produce nearly circularly-symmetric fields throughout most of the bubble radius. This can be done by examining the ring model, its condition of eq. (2.11), and the resulting bubble radius of eq. (2.16). If these conditions are met, a drive beam will create a bubble with circular symmetry, and be consistent with the ring model of PWFA.

In order to find these conditions, the PWFA model assumptions as they relate to

asymmetric Gaussian beams must be understood. The blowout condition (eq. (2.11)) was derived from a model of a drive beam with a uniform cylinder of charge. This model, however, does not translate well to Gaussian beams, and less so to ellipsoidal Gaussian beams, as the Gaussian equations are arrived at primarily through dimensional analysis and not theory. For example, imagine a 1-dimensional blowout regime resulting from an infinitely-wide planar beam. In this limit, the blowout is no longer a nearly-spherical bubble. The ring model in this regime is not valid, and a 1D theory would need to be developed. Rather than proceeding with a 1D theory, conditions can be placed on the beam profile to ensure a nonlinear plasma response. Once this is met, further conditions can be placed on profile to ensure the majority of plasma electrons' trajectories are in regions with circularly-symmetric fields.

2.5.1 Blowout Condition

In order to be certain of blowout, Lu *et al.* [24] find that the average beam density n_b within a radius a must meet eq. (2.11), $n_b > 1.792n_p$. This condition can be applied to a Gaussian beam by considering a circularly symmetric region of radius a . The average beam density inside that region is:

$$\langle n_b \rangle \Big|_{(r < a)} = \frac{1}{\pi a^2} \int_0^a n_b(r) r dr d\theta \quad (2.26)$$

By inspection, if $n_b(r)$ is Gaussian with $n_b(0) = n_{b,0}$, then the maximum possible average density is $n_{b,0}$. The beam charge for $r > a$ will only serve to increase the blowout, so a sufficient condition (which overestimates the density necessary for blowout) is given by:

$$n_{b,0} > 1.792 n_p \quad (2.27)$$

2.5.2 Symmetric Bubble

In order to ensure the resulting bubble is symmetric, the fields must be radially symmetric for a large portion of the plasma electrons that originate from the bubble region. This can be accomplished in two steps. First, find the distance from the

center of an asymmetric distribution where fields are approximately symmetric. Then require this distance to be much smaller than the bubble radius.

Symmetric Fields

It is physically intuitive that the fields are symmetric for a radius ρ greater than some distance, as is seen in fig. 2.1. This can be seen by the distance ρ where the potential becomes independent of θ . The formula for the potential in a two-dimensional problem is given by Jackson [40]. We can set the scale of the problem by normalizing the radial coordinate to the Gaussian R.M.S. width σ_x , a la $\rho \rightarrow \rho/\sigma_x$. Then, using azimuthal symmetry, the integrand can be expanded [40]:

$$\Phi(\vec{\rho}) = \int_{SA} \frac{\rho(\vec{\rho}')}{4\pi\epsilon_0} \ln \left(\frac{1}{|\vec{\rho} - \vec{\rho}'|^2} \right) d\mathbf{a}' \quad (2.28)$$

$$\Phi(x, y, \xi) = \int \frac{\rho(\vec{\rho}')}{2\pi\epsilon_0} \left(\ln \left(\frac{1}{\rho} \right) + \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho'}{\rho} \right)^m \cos[m(\theta - \theta')] \right) d\mathbf{a}' \quad (2.29)$$

$$= \Phi_0 + \Phi_1 + \dots \quad (2.30)$$

$$\Phi_0 = \int \frac{\rho(\vec{\rho}')}{2\pi\epsilon_0} \ln \left(\frac{1}{\rho} \right) = \frac{\lambda}{2\pi\epsilon_0} \ln \left(\frac{1}{\rho} \right) \quad (2.31)$$

$$\text{where } \lambda = \int \rho(\rho, \theta, \xi) \rho d\rho d\theta = Q_{tot} f(\xi)$$

$$\Phi_m = \frac{1}{2\pi\epsilon_0} \int \rho(\vec{\rho}') \left(\frac{\rho'^{m+1}}{\rho^m} \right) \cos[m(\theta - \theta')] d\rho' d\theta' \quad (2.32)$$

The obvious substitution of $\cos[m(\theta - \theta')] = \cos(m\theta) \cos(m\theta') + \sin(m\theta) \sin(m\theta')$ over the integration interval $(0, 2\pi)$ shows that only the even m terms will remain, as the odd terms are odd functions over the interval, while the charge distribution is an even function. Showing the $m = 2$ term is small necessitates that subsequent terms are even smaller.

Evaluating the $m = 2$ term using the charge distribution in eq. (2.18) is possible if a polar coordinate system is used and the substitutions $u = \rho/\sigma_x$ and $w = \sigma_y/\sigma_x$ are made, keeping the assumption that $\sigma_x > \sigma_y$, i.e. $0 < w < 1$. Noting that

$$\cos^2 \theta'/2 + \sin^2 \theta'/2w^2 > 0:$$

$$\Phi_2 = \frac{1}{2\pi\epsilon_0\rho^2} \int \rho(\rho', \theta', \xi) \rho'^3 \cos[2(\theta - \theta')] d\rho' d\theta' \quad (2.33)$$

$$= \frac{1}{2\pi\epsilon_0\rho^2} f(\xi) \frac{Q_{tot}}{2\pi\sigma_x\sigma_y} \int e^{-\left[\frac{\rho'^2 \cos^2 \theta'}{2\sigma_x^2} + \frac{\rho'^2 \sin^2 \theta'}{2\sigma_y^2}\right]} \rho'^3 \cos[2(\theta - \theta')] d\theta' d\rho' \quad (2.34)$$

$$= Q_{tot} f(\xi) \frac{\sigma_x^2}{4\pi^2\epsilon_0\rho^2 w} \int e^{-\left[\frac{u^2 \cos^2 \theta'}{2} + \frac{u^2 \sin^2 \theta'}{2w^2}\right]} u^3 \cos[2(\theta - \theta')] d\theta' du \quad (2.35)$$

$$= Q_{tot} f(\xi) \frac{\sigma_x^2}{4\pi^2\epsilon_0\rho^2 w} \int e^{-\frac{u^2}{2w^2} [w^2 \cos^2 \theta' + \sin^2 \theta']} u^3 \cos[2(\theta - \theta')] d\theta' du \quad (2.36)$$

This separates the integral into u' and θ' , which makes it simple to evaluate the u' portion of the integral:

$$\Phi_2 = Q_{tot} f(\xi) \frac{\sigma_x^2}{4\pi^2\epsilon_0\rho^2 w} \frac{4w^4}{2} \int_0^{2\pi} \frac{\cos[2(\theta - \theta')]}{(w^2 \cos^2 \theta' + \sin^2 \theta')^2} d\theta' \quad (2.37)$$

$$= Q_{tot} f(\xi) \frac{\sigma_x^2 w^3}{2\pi^2\epsilon_0\rho^2} \int_0^{2\pi} \frac{\cos 2\theta \cos 2\theta' + \sin 2\theta \sin 2\theta'}{[(w^2 - 1) \cos^2 \theta' + 1]^2} d\theta' \quad (2.38)$$

$$= Q_{tot} f(\xi) \frac{\sigma_x^2 w^3}{2\pi^2\epsilon_0\rho^2} \times \left(\int_0^{2\pi} \frac{\cos 2\theta \cos 2\theta'}{\left[\frac{w^2-1}{2} \cos 2\theta' + \frac{w^2+1}{2}\right]^2} d\theta' + \int_0^{2\pi} \frac{\sin 2\theta \sin 2\theta'}{\left[\frac{w^2-1}{2} \cos 2\theta' + \frac{w^2+1}{2}\right]^2} d\theta' \right) \quad (2.39)$$

On inspection, the second integral is odd over the period $(0, \pi)$, so the integral is zero. The functions are trigonometric integrals of θ , even over the range $(0, 2\pi)$, so

the range can be changed from $(0, 4\pi)$ to $(0, \pi)$:

$$\Phi_2 = Q_{tot} f(\xi) \frac{2\sigma_x^2 w^3 \cos 2\theta}{\pi^2 \varepsilon_0 \rho^2} \frac{1}{2} \int_0^{4\pi} \frac{\cos \theta'}{\left[\frac{w^2-1}{2} \cos \theta' + \frac{w^2+1}{2} \right]^2} d\theta' \quad (2.40)$$

$$= Q_{tot} f(\xi) \frac{\sigma_x^2 w^3}{\pi^2 \varepsilon_0 \rho^2} \cos 2\theta \int_0^\pi \frac{\cos \theta'}{\left[\frac{w^2-1}{2} \cos \theta' + \frac{w^2+1}{2} \right]^2} d\theta' \quad (2.41)$$

$$= Q_{tot} f(\xi) \frac{\sigma_x^2 w^3}{\pi^2 \varepsilon_0 \rho^2} \frac{4}{(w^2 - 1)^2} \cos 2\theta \int_0^\pi \frac{\cos \theta'}{\left[\cos \theta' + \frac{w^2+1}{w^2-1} \right]^2} d\theta' \quad (2.42)$$

Substituting $x = \cos \theta'$ and inverting the integral interval yields:

$$\Phi_2 = Q_{tot} f(\xi) \frac{4\sigma_x^2}{\pi^2 \varepsilon_0 \rho^2} \frac{w^3}{(w^2 - 1)^2} \cos 2\theta \int_{-1}^1 \frac{x}{(x - c)^2 \sqrt{1 - x^2}} dx \quad (2.43)$$

$$\text{where } c = \frac{1 + w^2}{1 - w^2} \text{ and } c > 1$$

Symbolic evaluation and algebra reduce to:

$$\Phi_2 = \left(\frac{Q_{tot} f(\xi) \sigma_x^2 (1 - w^2)}{2\pi \varepsilon_0} \right) \frac{\cos 2\theta}{\rho^2} \quad (2.44)$$

It is worthwhile to note that the correction to Φ given by eq. (2.44) falls off as $1/\rho^2$, as compared with the $-\ln \rho$ dependence of Φ_0 . The $\cos 2\theta$ term shows a sinusoidal modification to the potential at radius ρ , which intuitively matches an ellipse-like perturbation.

Differentiating to obtain the field gives:

$$\vec{E} = -\nabla \Phi = -\nabla(\Phi_0 + \Phi_2 + \dots) = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \dots \quad (2.45)$$

Performing this differentiation leads to the first-order and third-order field components (the second-order field component is identically zero):

$$\vec{E}_1 = -\nabla\Phi_0 = \frac{Q_{tot}f(\xi)}{2\pi\epsilon_0\rho}\hat{\rho} \quad (2.46)$$

$$\vec{E}_3 = -\nabla\Phi_2 = \left(\frac{Q_{tot}f(\xi)\sigma_x^2(1-w^2)}{2\pi\epsilon_0}\right) \left[\frac{2\cos 2\theta}{\rho^3}\hat{\rho} + \frac{2\sin 2\theta}{\rho^3}\hat{\theta}\right] \quad (2.47)$$

$$\vec{E}_3 = \left(\frac{Q_{tot}f(\xi)\sigma_x^2(1-w^2)}{\pi\epsilon_0}\right) \left[\frac{\cos 2\theta}{\rho^3}\hat{\rho} + \frac{\sin 2\theta}{\rho^3}\hat{\theta}\right] \quad (2.48)$$

Requiring the magnitude of the $\vec{E}_1$ term to be much larger than the $\vec{E}_3$ term yield constraints on ρ :

$$\frac{Q_{tot}f(\xi)}{2\pi\epsilon_0\rho} \gg \left(\frac{Q_{tot}f(\xi)\sigma_x^2(1-w^2)}{\pi\epsilon_0}\right) \frac{1}{\rho^3} \quad (2.49)$$

$$\rho^2 \gg \left[2\sigma_x^2(1-w^2) = 2(\sigma_x^2 - \sigma_y^2)\right] \quad (2.50)$$

$$\rho \gg \left[\sigma_x\sqrt{2(1-w^2)} = \sqrt{2(\sigma_x^2 - \sigma_y^2)}\right] \quad (2.51)$$

It is of interest to note that $\rho \gg \sqrt{2}\sigma_x$ always satisfies eq. (2.51) where $w \ll 1$. An aspect ratio equivalent to a 10% relaxation in the condition on ρ , occurs for

$$\rho \gg (90\%)\sqrt{2}\sigma_x \quad (2.52)$$

$$0.9\sqrt{2}\sigma_x = \sigma_x\sqrt{2(1-w_{10\%}^2)} \quad (2.53)$$

$$w_{10\%} = \sqrt{1-0.9^2} = 0.436 \quad (2.54)$$

This indicates a regime where the beam is flat (aspect ratio of $0 < w < 0.436$, i.e. anything flatter than approximately a 1:2 ratio) that requires distances of $\rho \gg \sqrt{2}\sigma_x$ to find circularly-symmetric fields (fig. 2.4). Rounder beams have circularly-symmetric fields at significantly closer distances.

Imposing a 10% limit of the magnitude of the electric fields at a radius ρ allows us to solve for the smallest radius where fields are approximately radial. Using eqs. (2.49)

Figure 2.4: *Circular symmetry limit.* Plot of the condition on ρ , normalized to σ_x , as found in 2.51. The point at $w = 0.4396$ represents the separation of the flat-beam regime and the round-beam regime.

to (2.51), the radius ρ becomes

$$\rho_{min} = \sigma_x \sqrt{20(1 - w^2)} = 4.47 \sigma_x \sqrt{1 - w^2} \quad (2.55)$$

Comparison to Bubble Radius

Following the dimensional analysis of Lu *et al.* [24], the bubble radius $r_{max}(\Lambda)$ is shown in eq. (2.12), and Λ in eq. (2.13). Applying these equations to an asymmetric drive beam requires a 2-D integral:

$$\Lambda = \frac{e\omega_p^2}{2\pi c^2} \iint \frac{n_b(q_1, q_2)}{n_p} dA \quad (2.56)$$

where $n_b(q_i)$ is the beam distribution that is a function of some coordinates q_1 and q_2 that are integrated over. In the fashion of Lu *et al.* [24], an asymmetric distribution

can be used and integrated over to get Λ :

$$n_b(x, y) = n_{b,0} e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}} \quad (2.57)$$

$$\Lambda = \frac{e\omega_p^2}{c^2} n_{b,0} \sigma_x \sigma_y \quad (2.58)$$

From eq. (2.12), this gives a bubble radius of:

$$r_{\max} \Big|_{(\text{Asymm. Gauss})} = 2.58 \sqrt{\frac{n_{b,0}}{n_p} \sigma_x \sigma_y} = 2.58 \sigma_x \sqrt{\frac{n_{b,0}}{n_p} w} \quad (2.59)$$

We should keep in mind that, similar to the symmetric Gaussian distribution, this is not entirely accurate, as it assumes that the beam has a uniform longitudinal distribution, which is not true. In the symmetric Gaussian case, the bubble radius was shown to be 20% less than expected [24].

Any non-circular perturbations on the plasma electrons and thus the ion column can be considered small as long as the majority of plasma electrons that originate inside the bubble radius experience first-order fields. Using eqs. (2.16) and (2.55), the condition for a circular bubble in the presence of an elliptical drive beam is then:

$$r_{\max} \gg \rho_{\min} \quad (2.60)$$

$$r_{\max} \lambda > \sigma_x \sqrt{20(1 - w^2)} \quad (2.61)$$

The value for λ can be chosen to reflect a certain ratio of the volume of plasma electrons experience symmetric fields. For example, $\lambda = 22.4\%$ implies that 95.0% of the plasma electrons that are ejected from the bubble region experience symmetric fields. Substituting eq. (2.59) gives:

$$2.58 \lambda \sigma_x \sqrt{\frac{n_{b,0}}{n_p} w} > \sigma_x \sqrt{20(1 - w^2)} \quad (2.62)$$

$$\Rightarrow n_{b,0} > \frac{3.00}{\lambda^2} \left(\frac{1 - w^2}{w} \right) n_p \quad (2.63)$$

2.6 Implications for a Linear Collider

The two proposed linear collider schemes (PWFA collider and ILC [30]) can be examined in terms of their emittances. Referencing work in future chapters, and making several assumptions, we can analyze these schemes to see if they meet the conditions for circularly-symmetric blowout. It is important to note now that this analysis only applies to drive beams. It is possible that witness beams may be large enough to cause some asymmetries in the tail of the bubble region in witness beam loading schemes; however, this analysis is not within the scope of this thesis.

2.6.1 Comparison to Bubble Radius using Emittance

Assuming matched beta functions (as covered in chapter 3, eq. (3.17)), the beam aspect ratio can be calculated from the emittance as follows:

$$w \equiv \frac{\sigma_y}{\sigma_x} = \frac{\sqrt{\beta\epsilon_y}}{\sqrt{\beta\epsilon_x}} = \sqrt{\frac{\epsilon_y}{\epsilon_x}} \quad (2.64)$$

Given a Gaussian distribution, the peak beam density is:

$$n_{b,0} = \frac{N_e}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \quad (2.65)$$

The beam density as matched into the plasma can be determined from eq. (3.17):

$$\beta = \sqrt{\frac{2\gamma m_e c^2 \epsilon_0}{e^2 n_p}} \quad (2.66)$$

The spot size can be determined from eq. (3.3), given the definition of normalized emittance $\epsilon_n = \gamma\epsilon$:

$$\langle x^2 \rangle = \beta \frac{\epsilon_n}{\gamma} \quad (2.67)$$

$$\langle x^2 \rangle = \sqrt{\frac{2m_e c^2 \epsilon_0}{\gamma e^2 n_p}} \epsilon_n \quad (2.68)$$

Using these equations for σ_x and σ_y in eq. (2.65), as the distribution is Gaussian, we recover:

$$n_{b,0} = \frac{N_e}{(2\pi)^{3/2}} \sqrt{\frac{e^2\gamma}{2m_e c^2 \epsilon_0}} \frac{1}{\sqrt{\epsilon_{n,x}\epsilon_{n,y}\sigma_z}} \sqrt{n_p} \quad (2.69)$$

These two eqs. (2.64) and (2.69) can be used in eq. (2.63) to determine whether the blowout can be assumed to be circularly symmetric:

$$n_p < \left[\lambda^2 (2.24 \times 10^{-4}) \right] N_e^2 \gamma \left(\frac{e^2}{m_e c^2 \epsilon_0} \right) \left(\frac{1}{\sigma_z^2 \epsilon_{n,x}^2} \right) \left(\frac{w}{(1-w^2)^2} \right) \quad (2.70)$$

In most cases it is more intuitive to calculate $n_{b,0}$ from eq. (2.65) and compare with eq. (2.63).

2.6.2 PWFA Linear Collider

CLIC-Type Drive Beam

Earlier proposals for PWFA-LC consider driving a PWFA collider with beam parameters similar to the Compact Linear Collider (CLIC) [32]. The CLIC-style beam would be a round beam with transverse normalized emittances of $\sim 1 \times 10^{-4}$ m-rad. As this beam is circularly symmetric, the blowout must also be circularly symmetric.

Updated PWFA-LC

The proposal for a linear collider leaves the drive beam parameters as “free variables” which may be optimized to improve the witness energy spread, efficiency, and drive beam energy [30, 33]. While the witness beam is expected to be used in a collider, the drive beam is discarded. This means that while the witness beam should be flat to avoid as beamstrahlung at the interaction point, this is not an issue for the drive beam, so the drive beam may be symmetric. No further analysis is needed for drive beams, then, except to say that they should be checked for driving asymmetric wakes.

We may still address the temptation of using the same parameters from the witness in the drive beam. Using the parameters for a PWFA linear collider (see table 1.2), and assuming round blowouts, with matched beta functions $\beta = 2E_0\epsilon_0/e^2 n_p$ (as

covered in chapter 3, eq. (3.17)), the beam aspect ratio w can be calculated from the proposed emittance by eq. (2.64). The ratio of the peak beam density to the plasma density required to drive circularly-symmetric dynamics can be calculated from eq. (2.63). For all parameters, this is orders of magnitude above the blowout condition of eq. (2.27). All densities are above the minimum beam density threshold by at least an order of magnitude, which suggests asymmetry will not be a problem.

2.6.3 ILC “Afterburner”

The ILC afterburner concept is more pertinent as the proposal includes parameters for the drive bunch, which is not round (see table 1.3). It calls for splitting ILC bunches into drive and witness bunches, which are assumed to have the same emittance as the original ILC bunch. All of the ILC parameters also exhibit circular symmetry. The calculations from section 2.6.2 are applied and listed in table 2.1b for details.

Table 2.1: *Beam densities and symmetric blowout in a PWFA-LC and ILC “Afterburner”.*

(a) PWFA-LC beam densities and symmetric blowout from parameters proposed in Delahaye *et al.* [30], Seryi *et al.* [32], Adli *et al.* [33].

Specification	Early Proposal [32]			Updated Proposal [30, 33]		
Energy (CMS)	1 TeV	250 GeV	500 GeV	1 TeV	3 TeV	10 TeV
Plasma density [1/cm ³]	1 × 10 ¹⁷	2 × 10 ¹⁶	2 × 10 ¹⁶	2 × 10 ¹⁶	2 × 10 ¹⁶	2 × 10 ¹⁶
N _e , witness bunch	1 × 10 ¹⁰	1 × 10 ¹⁰	1 × 10 ¹⁰	1 × 10 ¹⁰	1 × 10 ¹⁰	1 × 10 ¹⁰
Aspect ratio <i>w</i>	0.158	0.0592	0.0592	0.0592	0.0592	0.0592
Peak witness beam density <i>n</i> _{<i>b</i>,0} [1/cm ³]	8.36 × 10 ²¹	4.99 × 10 ²⁰	7.06 × 10 ²⁰	9.99 × 10 ²⁰	1.73 × 10 ²¹	3.16 × 10 ²¹
Minimum witness density [1/cm ³]	8.26 × 10 ¹⁸	4.51 × 10 ¹⁸	4.51 × 10 ¹⁸	4.51 × 10 ¹⁸	4.51 × 10 ¹⁸	4.51 × 10 ¹⁸
Circular symmetry?	Yes	Yes	Yes	Yes	Yes	Yes

Chapter 3

Emittance and Beam Matching

Colliders are more productive if they have high luminosity, increasing the rate of events. One path to increasing luminosity is decreasing emittance [35] (see eq. (1.2)):

$$\mathcal{L} = f \frac{n_1 n_2}{4\pi \sqrt{\epsilon_x \beta_x \epsilon_y \beta_y}} \quad (3.1)$$

Advanced beam sources have been designed to generate beams with small emittances, although there may be potential for using PWFA regimes to produce low-emittance beams [39]. The focus of PWFA as an acceleration method, then, is to ensure that emittance growth is minimized. Transverse emittance is a measure of the phase space area of the distribution of beam particles. Under Liouville's Theorem, phase space area is preserved under linear uncoupled optics. While there are a variety of higher-order effects that can affect transverse phase space such as betatron radiation [3], it is important to first examine the conditions that preserve emittance, and significant coupling that may contribute to emittance when projected into the transverse phase space plane. If these couplings are not reversed, this projection effectively results in emittance growth.

Of particular interest in PWFA is the coupling caused by the transverse focusing of the plasma. The linear focusing present in the PWFA blowout bubble breaks the symmetry in the equations of motion and results in a preferred coordinate system. As a result, the transverse distribution can become coupled to the energy distribution

as the focusing is energy-dependent. Additionally, ion motion can cause longitudinal variations in ion density, and therefore focusing. As a result, the transverse coupling is also dependent on the longitudinal position within the bunch. This means, practically speaking, that the transverse emittance will be a function of the longitudinal coordinate under significant ion motion.

3.1 Beam Distribution Parameters and Equations of Motion Formalisms

Beam matching and possible emittance growth is most simply understood by making direct comparisons of the equations of motion of a linear focusing plasma with the action-angle formalism of beams. This development of emittance preservation closely follows the derivation of beam filamentation in a traditional accelerator [3] with minor corrections, and then extending the results to PWFA regimes of interest.

3.1.1 Courant-Snyder Parameters

The transverse behavior of a collection of particles is described in detail by how its transverse distribution is affected by the particles' equations of motion. The distribution, in turn, can be described more simply by a set of parameters known as the Courant-Snyder parameters. For a given distribution in the transverse plane x , the parameters α , β , γ , and ϵ are defined as:

$$\epsilon_x \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} \quad (3.2)$$

$$\beta_x \equiv \frac{\langle x^2 \rangle}{\epsilon} \quad (3.3)$$

$$\alpha_x \equiv -\frac{\langle xx' \rangle}{\epsilon} \quad (3.4)$$

$$\gamma_x \equiv \frac{\langle x'^2 \rangle}{\epsilon} = \frac{1 + \alpha^2}{\beta} \quad (3.5)$$

Figure 3.1: *Courant-Snyder parameters of a distribution.* Gaussian particle distribution with $\beta = 10$ mm, $\alpha = -1$, and geometric emittance of $\epsilon = 1.26$ mm-mrad. The blue ellipse is drawn on a contour of $\rho(x, x')$ and encompasses an area of $\pi\epsilon$.

where $x' \equiv \frac{dx}{ds} = v \frac{dx}{dt}$ and $v \approx c$ for relativistic particles. In particular, a Gaussian distribution $\rho(x, x')$ can then be written as:

$$\rho(x, x') = \frac{1}{2\pi\epsilon} e^{-\gamma x^2 + 2\alpha x x' + \beta x'^2 / 2\epsilon} \quad (3.6)$$

This distribution and the Courant-Snyder definitions will be assumed unless otherwise specified, and is shown in fig. 3.1.

3.1.2 Action-Angle Formalism

Beam particle equations of motion in the presence of a focusing force $K(s)$ can be described under the action-angle formalism [3, 45], which is a solution of the pseudo-Hamiltonian:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}K(s)x^2 \quad (3.7)$$

where $K(s)$ is the focusing force, and H and $K(s)$ are normalized to mc^2 . The beam electrons' individual equations of motion can be written in terms of their action J and angle ϕ coordinates [3, 45] (with a sign correction to Raubenheimer *et al.* [3]), assuming a Gaussian distribution (eq. (3.6)):

$$x = \sqrt{2J\beta} \cos(\psi(s) + \phi) \quad (3.8)$$

$$x' = -\sqrt{\frac{2J}{\beta}} \left(\sin(\psi(s) + \phi) + \alpha \cos(\psi(s) + \phi) \right) \quad (3.9)$$

where β and α are given by

$$\psi(s) \equiv \int_0^s \frac{ds'}{\beta(s')} \quad (3.10)$$

$$\alpha \equiv -\frac{1}{2} \frac{d\beta}{ds} \quad (3.11)$$

Assuming a uniform distribution in ϕ (which can, in particular, be shown for Gaussian beams which have a distribution $\rho(J, \phi) = e^{-J/\epsilon}/2\pi\epsilon$), the initial emittance reduces to $\epsilon = \langle J \rangle$.

The beam's initial distribution can be described by these equations of motion for some initial emittance ϵ_i , beta function β_i , and alpha function α_i . In setting $s = 0$, $\psi(s = 0) = 0$; the beam's particles have coordinates J_i and ϕ_i which completely determine their equations of motion and their initial moments:

$$x_i = \sqrt{2J_i\beta_i} \cos \phi_i \quad (3.12)$$

$$x'_i = -\sqrt{\frac{2J_i}{\beta_i}} (\sin \phi_i + \alpha_i \cos \phi_i) \quad (3.13)$$

3.2 Linear Focusing Equations of Motion

The blowout regime consists of a circular bubble region of positively-charged ions with no plasma electrons. It can be shown, via Gauss's Law, that this region has a linear focusing force centered about its axis, which result in uncoupled equations of

motion (shown, for simplicity, in the x plane):

$$r'' = -kr \text{ where } k = \frac{\omega_p^2}{2\gamma c^2} = \frac{e^2 n_p}{2E\varepsilon_0} = \frac{e^2 n_p}{2\varepsilon_0} \frac{1}{E_0(1+\delta)} \quad (3.14)$$

$$\begin{aligned} x &= x_0 \cos(\sqrt{k}s) + \frac{x'_0}{\sqrt{k}} \sin(\sqrt{k}s) \\ x' &= -\sqrt{k}x_0 \sin(\sqrt{k}s) + x'_0 \cos(\sqrt{k}s) \end{aligned} \quad (3.15)$$

where $\omega_p^2 = n_p e^2 / \varepsilon_0 m$, $E = \gamma mc^2$ is the particle energy and E_0 is the average particle energy, and using the coordinate transformation $\delta \equiv (E - E_0)/E_0$, which will be useful later on.

These equations of motion can be directly related to the action-angle formalism's equations of motion. By inspection of eqs. (3.9), (3.10) and (3.15), in this coordinate system with constant focusing, the beta function β is constant, and $\alpha = 0$:

$$\begin{aligned} \psi(s) &= \int_0^s \frac{ds'}{\beta(s')} = \sqrt{k}s \\ &= s \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0} \frac{1}{1+\delta}} \approx s \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0}} \left(1 - \frac{\delta}{2}\right) + \mathcal{O}(\delta^2) \end{aligned} \quad (3.16)$$

In this system with constant focusing, the beta function $\beta(s)$ can be found exactly:

$$\begin{aligned} \beta^2 &= \frac{1}{k} = \frac{2E_0 \varepsilon_0}{e^2 n_p} (1+\delta) = \beta_0^2 (1+\delta) \\ \Rightarrow \alpha &= 0 \end{aligned} \quad (3.17)$$

At the start of the plasma, $s = 0$, the equations of motion are equal to the original distribution, $x(s=0) = x_i$ and $x'(s=0) = x'_i$. Keeping in mind that $\alpha = 0$:

$$\sqrt{2J\beta} \cos \phi = \sqrt{2J_i \beta_i} \cos \phi_i \quad (3.18)$$

$$-\sqrt{\frac{2J}{\beta}} \sin \phi = -\sqrt{\frac{2J_i}{\beta_i}} (\sin \phi_i + \alpha_i \cos \phi_i) \quad (3.19)$$

Solving these equations for the action J and angle ϕ of the electrons in the coordinate

system of the plasma (with further sign corrections to Raubenheimer *et al.* [3]) gives:

$$J = J_i \left[\left(\frac{\beta_i}{\beta} + \left(\alpha \sqrt{\frac{\beta_i}{\beta}} - \alpha_i \sqrt{\frac{\beta}{\beta_i}} \right)^2 \right) \cos^2 \phi_i - 2 \left(\alpha - \alpha_i \frac{\beta}{\beta_i} \right) \cos \phi_i \sin \phi_i + \frac{\beta}{\beta_i} \sin^2 \phi_i \right]$$

$$\tan \phi = \frac{\beta}{\beta_i} \tan \phi_i - \left(\alpha - \alpha_i \frac{\beta_i}{\beta} \right)$$
(3.20)

$$\left. \begin{aligned} J &= J_i \left[\left(\frac{\beta_i}{\beta} + \alpha_i^2 \frac{\beta}{\beta_i} \right) \cos^2 \phi_i + 2\alpha_i \frac{\beta}{\beta_i} \cos \phi_i \sin \phi_i + \frac{\beta}{\beta_i} \sin^2 \phi_i \right] \\ \tan \phi &= \frac{\beta}{\beta_i} \tan \phi_i + \alpha_i \frac{\beta_i}{\beta} \end{aligned} \right\} \text{ for } \alpha = 0$$
(3.21)

3.3 Matching Driven by Energy Spread

Up until this point, there has been no motivation for evaluating emittance in the coordinate system of the plasma. The emittance in this coordinate system can be written in terms of the moments:

$$\langle x^2 \rangle = 2 \langle J \beta \cos^2 \varphi \rangle \quad (3.22)$$

$$\langle x'^2 \rangle = 2 \left\langle \frac{J}{\beta} \sin^2 \varphi \right\rangle \quad (3.23)$$

$$\text{where } \varphi = \psi(s) + \phi = s \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0} \frac{1}{1 + \delta}} + \phi \quad (3.24)$$

It can now be shown that the beam distribution $f_\varphi(\varphi)$, becomes uniform with increasing s . As the distribution f_ϕ is $\phi \rightarrow (\phi \bmod 2\pi)$, the constant term can be absorbed so that the distribution f_ϕ has a mean $\langle \phi \rangle = 0$. Arbitrary constant terms can be absorbed into the final f_φ distribution, provided it is shown to be uniform on its own $(0, 2\pi)$ interval. The resulting coordinate transform leads to a convolution of the distributions f_ϕ and f_δ . Noting that $\left| \frac{\partial \phi}{\partial \varphi} \right| = 1$, and performing a change of

variables $t = \frac{s\delta}{2} \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0}}$ results in:

$$\varphi \approx -\frac{s\delta}{2} \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0}} + \phi \quad (3.25)$$

$$\begin{aligned} f_\varphi(\varphi) &= \int_{-\infty}^{\infty} f_\delta(\delta) f_\phi \left(\varphi + \frac{s\delta}{2} \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0}} \right) \left| \frac{\partial \phi}{\partial \varphi} \right| d\phi \\ &= s \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0}} \int_{-\infty}^{\infty} f_\delta \left(\frac{t}{s \sqrt{\frac{e^2 n_p}{2E_0 \varepsilon_0}}} \right) f_\phi(\varphi + t) dt \end{aligned} \quad (3.26)$$

As f_δ is a Gaussian, the broader f_ϕ is, the more uniform the distribution f_φ will become.

The relevant physical limit, then, is when the f_ϕ distribution is the narrowest. The nature of the distribution f_ϕ is governed by its own transform from eq. (3.21):

$$\begin{aligned} f_\phi(\phi) &= f_{\phi_i}(\phi_i(\phi)) \left| \frac{d\phi_i}{d\phi} \right| \\ &= \frac{\cos^2(\arctan(\tan \phi / A) + B)}{2\pi A \cos^2 \phi} \quad \text{where } A = \frac{\beta}{\beta_i}, B = \frac{\alpha_i}{A} \end{aligned} \quad (3.27)$$

Qualitatively, B is just a phase shift of the top term with the bottom term. This means that a discontinuity will appear at $\varphi = \pm\pi/2$ when B doesn't result in the numerator being equal to zero. This discontinuity dominates most distributions. When $B = 0$, the discontinuity does not appear. In this case, a narrow distribution is obtained for $A \gg 1$ and $A \ll 1$. In all these cases, the narrowest distributions that can be obtained are two delta functions separated by $\Delta\varphi = \pi$. A case where $\alpha_i = 0$ is simulated in fig. 3.2.

Taking this limit of two delta functions separated by π gives the least-uniform f_φ .

Figure 3.2: *Simulations of match and emittance saturation.* In order to demonstrate the beam becoming matched, a mismatch of $\beta/\beta_i = 10$ and $\alpha_i = 0$ with an energy spread $\sigma_\delta = 0.05$ is simulated, which is expected to be saturated after $N_\beta = 9.40$ betatron periods (eq. (3.33)). (a) The phase distribution $f_\phi(\phi)$ in the plasma becomes uniform along the plasma. As the distribution becomes more uniform (b) the beta function and (c) the emittance saturate.

The distribution due to one delta function is:

$$\begin{aligned}
 f_\phi(\phi) &= \frac{1}{(s/2)\sqrt{e^2 n_p/2E_0\epsilon_0}} \frac{1}{\sqrt{2\pi}\sigma_\delta} e^{-\phi^2/2((s\sigma_\delta/2)\sqrt{e^2 n_p/2E_0\epsilon_0})^2} \\
 &= \frac{1}{\sqrt{2\pi}\sigma_\phi} e^{-\frac{\phi^2}{2\sigma_\phi^2}} \quad \text{where } \sigma_\phi = \frac{s\sigma_\delta}{2} \sqrt{\frac{e^2 n_p}{2E_0\epsilon_0}}
 \end{aligned} \tag{3.28}$$

This shows that the width σ_ϕ of the distribution f_ϕ grows linearly with s . However, the distribution also wraps on an interval with a width of 2π , so as σ_ϕ increases, the distribution becomes more uniform. This can be understood as an infinite series of identical Gaussian distributions equally spaced a distance of $\Delta\phi = \pi$ apart, appropriately normalized. The location of the delta functions becomes irrelevant as the distribution becomes smooth:

$$\left. \begin{aligned}
 f_\phi(\phi) &= \sum_{n=-\infty}^{\infty} \frac{1}{2\sqrt{2\pi}\sigma_\phi} e^{-\frac{(\phi+\pi n)^2}{2\sigma_\phi^2}} \\
 &= \frac{1}{2\pi} \vartheta_3\left(\phi, e^{-2\sigma_\phi^2}\right)
 \end{aligned} \right\} \text{ for } 0 \leq \phi < 2\pi \tag{3.29}$$

where $\vartheta_n(z, q)$ is the Jacobi theta function [46]. The function can be said to be

uniform when the relative difference $\Delta\varepsilon$ between the maximum (at $\varphi = 0$) and the minimum (at $\varphi = \pi/2$) is small:

$$\Delta\varepsilon = \frac{f_\varphi(0) - f_\varphi(\pi/2)}{f_\varphi(0)} \quad (3.30)$$

$$= 1 - \frac{\vartheta_3\left(\pi/2, e^{-2\sigma_\varphi^2}\right)}{\vartheta_3\left(0, e^{-2\sigma_\varphi^2}\right)} \quad (3.31)$$

For practical purposes, a limit $\Delta\varepsilon = 5\%$ can be set. Numerically solving eq. (3.31) gives $\sigma_\varphi \approx 1.48$. Using eq. (3.28), we can find the length where filamentation has saturated to a uniform distribution in φ as s_{sat} , and in the number of betatron oscillations in the plasma via eqs. (3.24) and (3.33):

$$s_{\text{sat}} = \frac{2.95}{\sigma_\delta} \sqrt{\frac{2E_0\epsilon_0}{e^2 n_p}} \quad (3.32)$$

$$s_{\text{sat}}[N_\beta] = \frac{0.470}{\sigma_\delta} \quad (3.33)$$

Once a uniform distribution in φ has been obtained, the projected emittance ε can be calculated from moments. Recalling that $\alpha = 0$ implies $\langle xx' \rangle = 0$, and expanding $\langle \beta(\delta) \rangle$ and $\langle 1/\beta(\delta) \rangle$ from eq. (3.17) about $\delta = 0$ gives:

$$\begin{aligned} \langle x^2 \rangle &= \langle 2J\beta \cos^2 \varphi \rangle = \langle J\beta \rangle \\ &= \left\langle \frac{J_i}{2} \left(\beta_i + \alpha_i^2 \frac{\beta^2}{\beta_i} + \frac{\beta^2}{\beta_i} \right) \right\rangle = \left\langle \frac{J_i}{2} \right\rangle (\beta_i + \langle \beta^2 \rangle \gamma_i) \\ &= \frac{\epsilon_i}{2} (\beta_i + \beta_0^2 \gamma_i) \end{aligned} \quad (3.34)$$

$$\begin{aligned} \langle x'^2 \rangle &= \left\langle \frac{2J}{\beta} \sin^2 \varphi \right\rangle = \left\langle \frac{J}{\beta} \right\rangle \\ &= \left\langle \frac{J_i}{2} \left(\frac{\beta_i}{\beta^2} + \frac{\alpha_i^2}{\beta_i} + \frac{1}{\beta_i} \right) \right\rangle = \left\langle \frac{J_i}{2} \right\rangle \left(\beta_i \left\langle \frac{1}{\beta^2} \right\rangle + \gamma_i \right) \\ &\approx \frac{\epsilon_i}{2} \left(\beta_i \frac{1 + \sigma_\delta^2}{\beta_0^2} + \gamma_i \right) \end{aligned} \quad (3.35)$$

Note that in the evaluation of $\langle 1/\beta^2 \rangle$, higher order moments have been ignored. The final emittance can be found by applying eq. (3.21). For $\sigma_\delta^2 \ll 1$, as is the case for typical accelerator beams, the energy spread does not contribute significantly to the emittance:

$$\epsilon = \sqrt{\langle J\beta \rangle \left\langle \frac{J}{\beta} \right\rangle} \approx \epsilon_i \frac{\beta_i}{2\beta_0} \left(1 + \frac{\beta_0^2 \gamma_i}{\beta_i} + \frac{\sigma_\delta^2}{2} \right) \left\{ \begin{array}{l} \text{where } \sigma_\delta^2 \ll 1 \\ \approx \frac{\epsilon_i}{2} \left(\frac{\beta_i}{\beta_0} + \frac{\beta_0(1 + \alpha_i^2)}{\beta_i} \right) \end{array} \right. \quad (3.36)$$

3.4 Implications of Emittance Saturation

This result shows that match and emittance saturation is a consequence of finite energy spread in PWFA. The saturation length is inversely proportional to the energy spread. This phenomenon must be considered when measuring the initial beam match, as the beam distribution changes quickly; any measurement that relies on the beam behavior in the plasma will quickly become washed out. Emittance saturation must also be considered when injecting a beam into a plasma, as any mismatch will result in emittance growth.

All of the parameters subscripted by i arise solely from the initial beam distribution. This distinction is important, as many of these parameters are conventionally associated with machine parameters, which may not match the beam distribution.

There are several key assumptions in the derivation of saturation. First, the initial beam has a distribution that can be described by a uniform distribution in its angle coordinate ϕ . This is true for Gaussian distributions, and other distributions typically of interest, and is fairly general. Second, the energy spread is assumed to be Gaussian and small. As most accelerator beams are under $\sigma_\delta = 10\%$, this is a reasonable assumption. Third, a complete, circularly-symmetric blowout PWFA regime is assumed to provide linear focusing. Finally, acceleration is not considered in this derivation.

Several practical considerations are apparent. The beam becomes matched to the same beta function value in both planes. If the initial beam beta parameters are

matched to the plasma, the emittance will be preserved. However, the emittances in orthogonal planes are not required to be identical. It is possible, therefore, to preserve the transverse emittance of a flat beam, provided it is focused so that the beta functions are matched. While it is clearly desirable to match beta functions, it is also desirable to match initially in with a waist, so that $\alpha_i = 0$ in order to prevent emittance growth. The emittance may saturate much more quickly, if the initial distribution is more smooth than the case of the two delta functions. The details of the saturation process are determined by the distribution in φ .

The saturation length from eq. (3.33) can be found for several schemes of interest. In recent results at FACET, the average beam energy is 22.0 GeV, with typical plasma densities of $n_p = 5.0 \times 10^{16}/\text{cm}^3$ and energy spreads of $\sigma_\delta = 2\%$ [19], which gives a saturation length of $s_{\text{sat}} = 1.03$ m. As a result, it is plausible for the beam to saturate by the end of the chamber in FACET, depending on the acceleration distance and initial match. A PWFA linear collider has been proposed with beam energies up to 500 GeV and plasma densities of $2.0 \times 10^{16}/\text{cm}^3$ [30, 33]. Assuming a similar energy spread, the saturation length is $s_{\text{sat}} = 7.75$ m. This implies saturation may occur over several stages.

These analytical calculations and simulations show that PWFA Courant-Snyder match and emittance saturation is similar to that in periodic accelerators, as PWFA has a natural beta function, and the final emittance is driven by filamentation. It is shown that energy spread, while driving the physics of match saturation, does not contribute directly to the emittance of a PWFA beam. Future improvements are possible by considering the effect of acceleration on emittance in the presence of a linear focusing force, in both adiabatic and non-adiabatic regimes.

Chapter 4

Ion Motion

In the model of PWFA developed by Lu *et al.* [24] (known as a “ring” model because the derivation depends on the total charge contained within a ring), ions are assumed to be too massive to move appreciably as the drive bunch passes by. However, very dense beams have the potential to cause ion motion as they pass by ions [28]. As the ions move, variations in the plasma density will lead to fields that vary along the bunch and may not be linearly focusing. Ion motion can be a significant issue in a high-energy linear collider, as emittance preservation and matching depends on the uniform, linear focusing fields from ions.

In the ring model of the blowout regime, the beam density is higher than the plasma density (Lu *et al.* [24], eq. (2.11)). When the beam density is much higher than the plasma density ($n_b \gg n_p$), we may assume that the ion equations of motion are driven entirely by the beam fields. Ions are assumed to be initially at rest, and it will be shown that this assumption only breaks at fusion-like temperatures. The portions of the beam in nonlinear focusing fields may undergo large emittance growth over the accelerating distances that are on the order of 10 cm to 1 m.

4.1 Introduction to Ion Motion Model

The motion of ions in the presence of a relativistic Gaussian drive beam can be described by the Lorentz Force. In order to simplify the model, it can be assumed

that the ion motion occurs within the blowout bubble, so that plasma electrons do not contribute to the dynamics. It can also be assumed that the drive bunch has almost entirely transverse fields (see section 2.1). With these assumptions, the Lorentz Force can be described by the fields of the drive beam and the ions. If the drive beam is much denser than the ions, the ion field contributions can be neglected:

$$\frac{d\vec{P}_\perp}{dt} = -e \left[\vec{E}_\perp + \left(\vec{v} \times \vec{B} \right)_\perp \right] \quad (4.1)$$

Additionally, with non-relativistic ion velocities, the Lorentz Force equation can be approximated as:

$$\frac{d\vec{P}_\perp}{dt} \approx -e\vec{E}_\perp \quad (4.2)$$

This equation can be represented by a potential and treated as a central force problem. The solutions for the ion motion can be recovered and describe some macroscopic properties of ion motion under these assumptions, and inform some possible implications of ion motion. The beam is assumed to be relativistic, so that the beam electrons do not move significantly as they pass the ions.

This model's assumptions are similar to the quasistatic approximation (section 2.2.1, Sprangle *et al.* [41]), which separate the beam electron equations of motion from the ion equations of motion, as the electrons evolve on a much longer timescale than the ions. In particular, the quasistatic approximation's coordinate system defined in eqs. (2.5) and (2.6) is useful:

$$\xi \equiv ct - z \quad (4.3)$$

$$s \equiv ct \quad (4.4)$$

In this model, which does not consider the evolution of the beam, $s = 0$.

4.2 2-Dimensional Ion Motion

The Lorentz Force approximation in eq. (4.2) depends on the electric field of the drive bunch. Two different regimes of ion motion can be explored- that of round beams and flat beams. Round beams will be explored first, and the general methods of the model are developed here.

4.2.1 Equations of Motion

The fields of a Gaussian bunch can be calculated using Gauss's Law. First, assume the electron bunch is a uniform cylinder with a radial charge distribution given by:

$$\rho(\vec{r}, \xi) = -en_{b,0}e^{-\frac{r^2}{2\sigma_r^2}} \text{ where } n_{b,0} = \frac{N_e}{2\pi L_\xi \sigma_r^2} \text{ and } 0 < \xi < L_\xi \quad (4.5)$$

where N_e is the total number of electrons in the bunch. Applying Gauss's Law gives:

$$E_r 2\pi r l = \frac{l}{\epsilon_0} \int_0^{2\pi} \int_0^r \rho(r') r' dr' d\theta \quad (4.6)$$

$$E_r = -\frac{en_{b,0}\sigma_r^2}{\epsilon_0} \left(\frac{1 - e^{-r^2/2\sigma_r^2}}{r} \right) \quad (4.7)$$

Via the Lorentz force of eq. (4.2), and changing variables via $\xi = ct$:

$$F_r = m_i c^2 \frac{d^2 r}{d\xi^2} = e E_r \quad (4.8)$$

$$r'' = -\kappa_r \left(\frac{1 - e^{-r^2/2\sigma_r^2}}{r} \right) \text{ where } \kappa_r = \frac{e^2 n_{b,0} \sigma_r^2}{\epsilon_0 m_i c^2} = \omega^2 \frac{\sigma_r^2}{c^2} \quad (4.9)$$

where $\omega = n_{b,0}e^2/\epsilon_0 m_i$ is analogous to the plasma frequency, m_i is the mass of an ion and $r'' = \frac{d^2 r}{d\xi^2}$. The change of coordinates $u = r/\sqrt{2}\sigma_r$ completely removes the σ_r dependence from the differential equation:

$$u'' = -\kappa_u \left(\frac{1 - e^{-u^2}}{u} \right) \text{ where } \kappa_u = \frac{\kappa_r}{2\sigma_r^2} = \frac{\omega^2}{2c^2} \quad (4.10)$$

4.2.2 Small Amplitudes and Ion Collapse

As suggested by Rosenzweig *et al.* [28], at small amplitudes ion trajectories show that ions become focused to a point in a phenomena coined as “ion collapse.” At low temperature, the ions can be assumed to be at rest, i.e. no initial velocity. In this case, $r(0) = r_0$ and $r'(0) = 0$. The resulting differential equation describes simple harmonic motion for the limit $r_0 \ll \sigma_r$ [28]:

$$r'' = -kr \text{ where } k = \frac{\kappa_r}{2\sigma_r^2} = \kappa_u = \frac{\omega^2}{2c^2} \quad (4.11)$$

$$r = r_0 \cos(\Delta\phi) \text{ where } \Delta\phi = \sqrt{k}\xi = \frac{\omega}{c\sqrt{2}}\xi \quad (4.12)$$

In this case, at $\xi = \pi/2\sqrt{k}$, the ions within a small radius of the bunch have all collapsed to $r = 0$, notably causing a change in focusing due to a large nonlinearity in density near $\xi = 0$.

4.2.3 General Potential

It is additionally possible to solve for trajectories at large radii where $r_0 \gg \sigma_r$. As this is a central force problem, it is useful to derive the potential for the equation of motion:

$$\vec{F} = m_i \frac{d^2\vec{r}}{dt^2} = m_i c^2 r'' = -\vec{\nabla}V(r) = -\left[\frac{\partial}{\partial r} V(r)\right] \hat{r} = e\vec{E} \quad (4.13)$$

$$V(r) = -\int e\vec{E} dr = m_i c^2 \kappa_r \int \frac{1 - e^{-r^2/2\sigma_r^2}}{r} dr \quad (4.14)$$

Letting the constant of integration be $C = -m_i c^2 \kappa_r \ln(\sqrt{2}\sigma_r)$, these equations simplify to:

$$r'' = -\frac{1}{m_i c^2} \frac{\partial V}{\partial r} \quad (4.15)$$

$$V(r) = m_i c^2 \kappa_r \left[\ln\left(\frac{r}{\sqrt{2}\sigma_r}\right) - \frac{1}{2} \text{Ei}\left(-\frac{r^2}{2\sigma_r^2}\right) \right] \quad (4.16)$$

Figure 4.1: *Analytic and approximate 2D ion potential.* The potential for $mc^2\kappa_u = 1$. The small- and large-amplitude limits are clearly discernible.

With a change of variables $u = r/\sqrt{2}\sigma_r$ with $dr = \sqrt{2}\sigma_r du$, this becomes:

$$u'' = -\frac{1}{m_i c^2 2\sigma_r^2} \frac{\partial V}{\partial u} \quad (4.17)$$

$$V(u) = m_i c^2 \kappa_r \left[\ln u - \frac{1}{2} \text{Ei}(-u^2) \right] \quad (4.18)$$

The potential $V(0)$ can be found, which is conveniently correct for both $u = 0$ and $r = 0$:

$$\begin{aligned} V(0) &= \lim_{u \rightarrow 0^+} V(u) \\ &= -m_i c^2 \kappa_r \frac{\gamma_{EM}}{2} \end{aligned} \quad (4.19)$$

where γ_{EM} is the Euler-Mascheroni constant¹.

¹Given definitions [47] of $\text{Ei}(x)$ and $\lim_{x \rightarrow 0^+} \frac{e^{2\text{Ei}(-x)}}{x^2} = e^{2\gamma_{EM}}$ limit for the potential can be derived.

4.2.4 Small-Amplitude Limit

There are two limits to the potential (see fig. 4.1): $r < \sigma_r$ and $r > \sigma_r$, which correspond to $u < 2^{-1/2}$ and $u > 2^{-1/2}$. In the small- r limit, the potential can be expanded in a Taylor series:

$$V(u) \Big|_{u \ll \frac{1}{\sqrt{2}}} \approx V(0) + m_i c^2 \kappa_r \frac{u^2}{2} + \mathcal{O}(u^4) \quad (4.20)$$

Equivalent to the solutions cited earlier in eqs. (4.11) and (4.12), this immediately gives a harmonic oscillator solution:

$$u'' = -\kappa_u u \quad (4.21)$$

$$u(\xi) = u_0 \cos \left(\sqrt{\frac{e^2 n_{b,0}}{2\varepsilon_0 m_i c^2}} \xi \right) + u'_0 \sqrt{\frac{2\varepsilon_0 m_i c^2}{e^2 n_{b,0}}} \sin \left(\sqrt{\frac{e^2 n_{b,0}}{2\varepsilon_0 m_i c^2}} \xi \right) \quad (4.22)$$

For now, the cold-ion limit gives $u'_0 = 0$. The transition from ion motion to the thermal limit will be explored in a subsequent section.

4.2.5 Large-Amplitude Limit

The large- r limit requires more nuance. Motion is characterized by a consistent shape for all initial conditions where $r'(0) = 0$, and a length scale that goes as $\lambda \propto r_0 / \sqrt{\kappa_r}$. These two problems can be treated separately, but both depend on the assumption that at large r the potential can be approximated by:

$$V(u) \Big|_{u \gg \frac{1}{\sqrt{2}}} = m_i c^2 \kappa_r \ln u \quad (4.23)$$

Period of Motion

As this is a central force problem, the function is periodic in ξ . Assuming non-relativistic ions gives $\frac{1}{2} m_i v^2 = V(r_0) - V(r)$. The period can be calculated by integrating the velocity one cycle. Since there is symmetry in the problem, it is sufficient

to integrate over one fourth of the period:

$$\frac{T}{4} = \sqrt{\frac{m_i}{2}} \int_0^{r_0} \frac{1}{\sqrt{E(r_0) - V(r)}} dr \quad (4.24)$$

Changing variables to u gives:

$$\frac{T}{4} = \sigma_r \sqrt{m_i} \int_0^{u_0} \frac{1}{\sqrt{E(u_0) - V(u)}} du \quad (4.25)$$

If the initial radius r_0 is far enough away, the majority of the motion occurs in the regime where $V(u)$ can be approximated by $V(u) = m_i c^2 \kappa_r \ln u$. The quantity $E(u_0) - V(u) = V(r_0) - V(r)$. It is then possible to separate the integral into pieces, where u_s separates the large-amplitude regime from the rest of the potential:

$$T = 4\sigma_r \sqrt{m_i} \left[\int_0^{u_s} \frac{1}{\sqrt{E(u_0) - V(u)}} du + \int_{u_s}^{u_0} \frac{1}{\sqrt{m_i c^2 \kappa_r \ln \left(\frac{u_0}{u} \right)}} du \right] \quad (4.26)$$

Assuming the second integral is significantly larger than the first, this simplifies to:

$$T \approx \frac{4\sigma_r}{c\sqrt{\kappa_r}} \int_{u_s}^{u_0} \frac{1}{\sqrt{\ln \left(\frac{u_0}{u} \right)}} du \quad (4.27)$$

$$T \approx 4\sqrt{\pi} \frac{\sigma_r u_0}{c\sqrt{\kappa_r}} \operatorname{erf} \left[\sqrt{\ln \left(\frac{u_0}{u_s} \right)} \right] \quad (4.28)$$

$$\lambda \approx 4\sqrt{\pi} \frac{\sigma_r u_0}{\sqrt{\kappa_r}} \operatorname{erf} \left[\sqrt{\ln \left(\frac{u_0}{u_s} \right)} \right] \quad (4.29)$$

For initial conditions where $u_0 \gg u_s$, this simplifies to:

$$\begin{aligned} T &\approx \frac{\sqrt{2\pi}}{\omega} \left(\frac{r_0}{\sigma_r} \right) \\ &\approx \sqrt{2\pi} \sqrt{\frac{m_i \varepsilon_0}{e^2 n_{b,0}}} \left(\frac{r_0}{\sigma_r} \right) \end{aligned} \quad (4.30)$$

$$\begin{aligned} \lambda &\approx \frac{\sqrt{2\pi} c}{\omega} \left(\frac{r_0}{\sigma_r} \right) \\ &\approx \sqrt{2\pi} \sqrt{\frac{m_i \varepsilon_0}{e^2 n_{b,0}}} \left(\frac{r_0}{\sigma_r} \right) c \end{aligned} \quad (4.31)$$

Scaled Trajectory

Ions that start at large amplitudes have trajectories with similar shapes, although different scales. This becomes intuitively obvious when considering the potential at large amplitudes goes simply as $V(r) \propto \ln r$. Showing it more rigorously requires an assumption that the trajectory has the form:

$$r = r_0 f \left(\frac{\eta \xi}{r_0} \right) \quad (4.32)$$

assuming that $f(x)$ is a periodic function. The argument goes as $\propto 1/r_0$ from what was found regarding the period. It is assumed that $|f(0)| \leq 1$, so that the function $f(x)$ is bounded and can be scaled to the appropriate amplitude given the initial conditions. Also, due to the periodic nature of differential equation, it is sufficient to describe $f(x)$ on the interval $(0, \lambda/4)$. The shape of the motion in all other quadrants of the period must be the same.

It is now possible to insert $r(\xi)$ into the differential equation describing the equations of motion:

$$r'' = -\kappa_r \left(\frac{1 - e^{-r^2/2\sigma_r^2}}{r} \right) \quad (4.33)$$

$$\frac{1}{r_0} f''(\xi \eta / r_0) = -\kappa_r \left(\frac{1 - e^{-r_0^2 f^2(\xi \eta / r_0) / 2\sigma_r^2}}{r_0 f(\xi \eta / r_0)} \right) \quad (4.34)$$

We can now show that the shape of the trajectory is similar everywhere except near the origin. For locations where $r > \sigma_r$, then $r_0^2 f^2 > 2\sigma_r^2$, and the exponential term approaches zero. Simplifying:

$$\frac{\eta^2}{r_0} f''(\xi\eta/r_0) = -\kappa_r \left(\frac{1}{r_0 f(\xi\eta/r_0)} \right) \quad (4.35)$$

$$f'' = - \left(\frac{\kappa_r}{\eta^2} \right) \frac{1}{f} \quad (4.36)$$

So it is clear that f satisfies the differential equation for points on the trajectory where $r_0^2 f^2 = r^2 > 2\sigma_r^2$. Also, values of κ_r can be transformed by absorbing into η , reducing the problem that needs to be solved to one second-order differential equation:

$$f'' = -\frac{1}{f} \quad (4.37)$$

Note, of course, that the initial conditions are $f(0) = 1$ and $f'(0) = 0$.

As we have not solved this differential equation at this point, it is not yet clear that this will be a significant portion of the trajectory. However, it is only necessary to solve the equations of motion once, in the limit where $r \gg \sigma_r$, to gain further insight. This can be done numerically, to fully define the problem, in a unitless system which can be rescaled to any system desired.

Recognizing that the system is periodic, and that the period can be solved, it becomes clear that f can be expressed by a Fourier series with only non-zero cosine terms where n is odd:

$$g(r) = \sum_{n=1}^{\infty} a_n \cos \left(n \frac{\sqrt{2\pi}}{2} r \right) \quad (4.38)$$

$$\text{where } a_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\sqrt{2\pi}} f(r) \cos \left(n \frac{\sqrt{2\pi}}{2} r \right) dr$$

Numerical integration and Fourier series expansion confirms that only odd terms exist. Using the parameters $r_0 = 1$, $\kappa = 1$, and $\sigma_r = 0.01$, the Fourier series approximates the numerical integration to the level of $|f(r) - g(r)| \leq 0.109$ retaining up to the

Figure 4.2: *2D ion trajectory integration and Fourier expansion.* Trajectory shape (a) and expansions error (b) for large amplitudes, with parameters $r_0 = 1$, $\kappa = 1$, and $\sigma_r = 0.01$. Note that the first-order expansion is given by $a_1 = 1.07$; while not an expansion, a model more consistent with initial conditions would be $a_1 = 1$. The shaded region $(-3\sigma_r, 3\sigma_r)$ shows where the trajectory shape is not expected to be identical even in the large-amplitude limit.

$n = 1$ term. Retaining up to $n = 17$ yields $|f(r) - g(r)| \leq 0.0091$. (See fig. 4.2a.)

Figure 4.2 shows the shaded region $|r| < 3\sigma_r$ where the assumption that the differential function reduces to $f'' = -1/f$ breaks down. As this region is small, the large-amplitude limit is a small perturbation on the numerically-calculated trajectory shape. In general, this region is expected to have a relatively small effect.

At $r = 3\sigma_r$, the ion has kinetic energy nearly entirely due to the logarithmic portion of the potential. The kinetic energy is:

$$\text{KE}_{r=3\sigma_r} = V(r_0) - V(3\sigma_r) \quad (4.39)$$

$$\text{KE}_{r=3\sigma_r} = m_i c^2 \kappa_r \left[\ln \left(\frac{r_0}{3\sigma_r} \right) - \frac{1}{2} \text{Ei} \left(-\frac{r_0^2}{2\sigma_r^2} \right) + \frac{1}{2} \text{Ei} (-9/2) \right] \quad (4.40)$$

$$\text{KE}_{r=3\sigma_r} = m_i c^2 \kappa_r \left[\ln \left(\frac{r_0}{3\sigma_r} \right) - \frac{1}{2} \text{Ei} \left(-\frac{r_0^2}{2\sigma_r^2} \right) + 0.00104 \right] \quad (4.41)$$

For r in the interval $(3\sigma_r, r_0)$, the second and third terms add up to, at most, $0.11m_i c^2 \kappa_r$.

The potential energy can be calculated as well:

$$\text{PE}_{r=3\sigma_r} = V(3\sigma_r) - V(0) \quad (4.42)$$

$$\text{PE}_{r=3\sigma_r} = m_i c^2 \kappa_r \left[\ln \left(\frac{3}{\sqrt{2}} \right) + \frac{\gamma_{EM} - \text{Ei}(-\frac{9}{2})}{2} \right] \quad (4.43)$$

$$\text{PE}_{r=3\sigma_r} = 1.04 \times m_i c^2 \kappa_r \quad (4.44)$$

If the kinetic energy in this region is much larger than the potential energy, the potential energy will affect the velocity very little. The velocity will remain nearly unchanged as the ion crosses the center of the bunch, and the trajectory will remain nearly the same even as parameters vary. This is true provided that the first term in the kinetic energy is larger than 1:

$$\ln \left(\frac{r_0}{3\sigma_r} \right) \gg 1.04 \quad (4.45)$$

$$r_0 \gg 8.49\sigma_r \quad (4.46)$$

This is consistent with all other modeling and can be used as the defining limit for large-amplitude motion. A simple model (including only a_1 in the Fourier series) for large-amplitude motion can be given as:

$$\begin{aligned} r &= r_0 \cos \left(\frac{2\pi}{\lambda} \xi \right) = r_0 \cos \left[\sqrt{2\pi} \frac{\omega}{c} \left(\frac{\sigma_r}{r_0} \right) \xi \right] \\ &= \cos \left[\sqrt{2\pi} \sqrt{\frac{n_{b,0} e^2}{\varepsilon_0 m_i c^2}} \left(\frac{\sigma_r}{r_0} \right) \xi \right] \end{aligned} \quad (4.47)$$

4.2.6 Ion Motion Numerical Integration

While in many scenarios it is useful to have a mathematical model, it is simpler to numerically integrate the equations of motion directly for each particle for an electron

Figure 4.3: *General 2D integrated ion trajectories and density.* Numerical integration of the ion equations of motion show ion trajectories and their resulting densities. Note how larger-amplitude trajectories have longer periods, causing ions to form a high-density cone similar to a shock wave. (a) The shaded region shows where $-\sigma_r < r < \sigma_r$.

bunch that is a radially-symmetric cylinder with a Gaussian transverse profile. For a reasonable number of particles, an ion density profile rapidly emerges. In fig. 4.3, we can see the evolution of wavelength from small trajectories to large trajectories produces peak densities similar to a shock wave.

4.3 1-Dimensional Ion Motion

The approach can be used for flat beams by reducing the dimensions of the problem. The 1-D assumption is good for flat elliptical beams with $\sigma_y \ll \sigma_x$, for ions with $x \ll \sigma_x$. The Gaussian beam distribution in ξ can be approximated by $f(\xi) = (1/\sqrt{2\pi}\sigma_\xi)e^{-\xi^2/(2\sigma_\xi^2)}$ with a uniform distribution having the same charge and variance:

$$f(\xi) = \begin{cases} \frac{1}{2\sqrt{3}\sigma_\xi} & \text{for } -\sqrt{3}\sigma_\xi < \xi < \sqrt{3}\sigma_\xi \\ 0 & \text{otherwise} \end{cases} \quad (4.48)$$

Figure 4.4: An approximation of a Gaussian by a step function.

In this configuration, the beam distribution is given by:

$$\rho(x, y, \xi) = \begin{cases} -\frac{eN_e}{4\sqrt{3}\pi\sigma_x\sigma_y\sigma_\xi} e^{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}} & \text{for } -\sqrt{3}\sigma_\xi < \xi < \sqrt{3}\sigma_\xi \\ 0 & \text{otherwise} \end{cases} \quad (4.49)$$

where $x \ll \sigma_x$, the dependence on x is removed. Ion motion is only relevant where there are electrons, and so the ξ dependence is dropped. This gives the charge density distribution:

$$\rho(x, y, \xi) = -\frac{eN_e}{4\sqrt{3}\pi\sigma_x\sigma_y\sigma_\xi} e^{-\frac{y^2}{2\sigma_y^2}} \quad (4.50)$$

$$\rho(x, y, \xi) = -en_{b,0}e^{-\frac{y^2}{2\sigma_y^2}} \text{ where } n_{b,0} = \frac{N_e}{4\sqrt{3}\pi\sigma_x\sigma_y\sigma_\xi} \quad (4.51)$$

Applying Gauss's Law gives:

$$E_y 2lw = \frac{lw}{\varepsilon_0} \int_{-y}^y \rho(y') dy' \quad (4.52)$$

$$E_y = -\sqrt{\frac{\pi}{2}} \frac{en_{b,0}\sigma_y}{\varepsilon_0} \operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right) \quad (4.53)$$

As before, using the Lorentz force equation with $\xi = ct$ and substituting $\nu = y/\sqrt{2}\sigma_y$:

$$F_y = m_i c^2 \frac{d^2 y}{d\xi^2} = m_i c^2 y'' = e E_y = -\frac{\partial V}{\partial y} \quad (4.54)$$

$$y'' = -\frac{1}{m_i c^2} \frac{\partial V}{\partial y} \quad (4.55)$$

$$y'' = -\kappa_y \operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right) \quad \text{where } \kappa_y \equiv \sqrt{\frac{\pi}{2}} \frac{e^2 n_{b,0} \sigma_y}{\varepsilon_0 m_i c^2} = \sqrt{\frac{\pi}{2}} \frac{\omega^2 \sigma_y}{c^2} \quad (4.56)$$

$$\nu'' = -\frac{1}{2m_i c^2 \sigma_y^2} \frac{\partial V}{\partial \nu} \quad (4.57)$$

$$\nu'' = -\frac{\kappa_y}{\sqrt{2}\sigma_y} \operatorname{erf}(\nu) \quad (4.58)$$

The initial conditions are again $y(0) = y_0$ and $y'(0) = 0$ for a cold plasma.

4.3.1 General Potential and Limits

Again, it is useful to find a potential for the problem. Integrating the force gives a potential:

$$V(y) = m_i c^2 \kappa_y \int \operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right) dy \quad (4.59)$$

$$V(y) = m_i c^2 \kappa_y \left[y \operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right) + \sqrt{\frac{2}{\pi}} \sigma_y e^{-\frac{y^2}{2\sigma_y^2}} \right] \quad (4.60)$$

Figure 4.5: *Analytic and approximate 1D ion potential.* The potential for $mc^2\kappa = 1$ and $\sigma_r = 1$. The small- and large-amplitude limits are shown. The transition region is nearly linear, which implies that a full solution is possible.

A change of variables $\nu = y/\sqrt{2}\sigma_y$, similar to the 2-D case, scales the problem:

$$V(\nu) = m_i c^2 \kappa_y \left[\sqrt{2}\sigma_y \nu \operatorname{erf}(\nu) + \sqrt{\frac{2}{\pi}}\sigma_y e^{-\nu^2} \right] \quad (4.61)$$

$$V(\nu) = m_i c^2 \kappa_y \sigma_y \sqrt{\frac{2}{\pi}} \left[\sqrt{\pi} \nu \operatorname{erf}(\nu) + e^{-\nu^2} \right] \quad (4.62)$$

$$V(\nu) = m_i c^2 k \left[\sqrt{\pi} \nu \operatorname{erf}(\nu) + e^{-\nu^2} \right] \quad (4.63)$$

where $k = \kappa_y \sigma_y \sqrt{\frac{2}{\pi}} = \frac{\omega^2}{c^2} \sigma_y^2$

There is nothing to be gained by setting the constant of integration to anything other than $V(0) = 0$. Finding $V(0)$ is simple:

$$V(0) = m_i c^2 k \quad (4.64)$$

Small-Amplitude Limit

There are again two limits for this potential: $y < \sigma_y$ and $y > \sigma_y$, or $\nu < 1/\sqrt{2}$ and $\nu > 1/\sqrt{2}$. In the small-amplitude limit, the potential and differential equation can again be approximated by a Taylor series to find a term responsible for harmonic motion:

$$V(\nu) \approx V(0) + m_i c^2 k \nu^2 + \mathcal{O}(\nu^4) \quad (4.65)$$

$$V(y) \approx V(0) + m_i c^2 k \frac{y^2}{2\sigma_y^2} + \mathcal{O}(\nu^4) \quad (4.66)$$

$$\nu'' \approx -\frac{k}{\sigma_y^2} \nu \quad (4.67)$$

$$y'' \approx -\frac{k}{\sigma_y^2} y = -\frac{e^2 n_{b,0}}{\varepsilon_0 m_i c^2} y = -\frac{\omega^2}{c^2} y \quad (4.68)$$

Note this is different from the wavelength of the 2D small-amplitude equation (eq. (4.21)) by only a factor of $\sqrt{2}$. This results in the harmonic oscillator solution:

$$y = y_0 \cos \left(\sqrt{\frac{e^2 n_{b,0}}{\varepsilon_0 m_i c^2}} \xi \right) + y'_0 \sqrt{\frac{\varepsilon_0 m_i c^2}{e^2 n_{b,0}}} \sin \left(\sqrt{\frac{e^2 n_{b,0}}{\varepsilon_0 m_i c^2}} \xi \right) \quad (4.69)$$

Large-Amplitude Limit

In the large-amplitude limit, where $y \gg \sigma_y$, on inspection of eqs. (4.56) and (4.60) the potential and differential equations become:

$$V(y) \approx m_i c^2 \kappa_y y \quad (4.70)$$

$$y'' \approx -\kappa_y y \quad (4.71)$$

In this case, there is a closed-form solution, which makes everything simpler:

$$y = -\frac{1}{2} \kappa_y \xi^2 + y_0 \text{ where } 0 \leq \xi \leq \frac{\lambda}{2} \quad (4.72)$$

Assuming the period is mostly due to the large-amplitude motion as before, and solving eq. (4.72) for the wavelength gives:

$$y(\xi) \Big|_{\xi=\frac{\lambda}{4}} = 0 \quad (4.73)$$

$$-\frac{1}{2}\kappa_y \frac{\lambda^2}{16} + y_0 = 0$$

$$\lambda \approx 4\sqrt{2}\sqrt{\frac{y_0}{\kappa_y}} \quad (4.74)$$

$$\lambda \approx \frac{2^{11/4}}{\pi^{1/4}} \sqrt{\frac{\varepsilon_0 m_i}{e^2 n_{b,0}}} \sqrt{\frac{y_0}{\sigma_y}} c \quad (4.75)$$

Proving the shape is consistent follows the same method as used previously. The trajectory is assumed to have the form:

$$y = y_0 f\left(\frac{\eta\xi}{\sqrt{y_0}}\right) \quad (4.76)$$

Checking if this is a solution where $y_0 f \gg \sqrt{2}\sigma_y$ produces:

$$\eta^2 f'' = -\kappa \quad (4.77)$$

$$f'' = -\frac{\kappa}{\eta^2} \quad (4.78)$$

This is familiar, and allows the problem to be solved once (see eq. (4.72) for solution) and rescaled. As seen in fig. 4.6, the trajectory is much closer to a cosine model than the 2-D case shown in fig. 4.2. Expanding up to $n = 5$ gives similar error when compared with the $n = 17$ expansion in 2-D.

4.4 Approximate Model

The best approach to a full model is to recognize that most trajectories follow sinusoidal paths to a very good approximation. By modeling all trajectories as sinusoidal, the only physics remaining is to determine each individual wavelength. This can be

Figure 4.6: *Large amplitude 1D ion trajectory integration and Fourier expansion.* Trajectory shape (a) and expansions error (b) for large amplitudes, with parameters $r_0 = 1$, $\kappa = 1$, and $\sigma_r = 0.01$. The first-order expansion is given by $a_1 = 1.03$; while not an expansion, a model more consistent with initial conditions would be $a_1 = 1$. The shaded region $(-3\sigma_r, 3\sigma_r)$ shows where the trajectory shape is not expected to be identical even in the large-amplitude limit.

Figure 4.7: *General 1D integrated ion trajectories and density.* Numerical integration of the ion equations of motion show ion trajectories and their resulting densities. Comparing with fig. 4.3, the shape has changed subtly, but the main characteristics including a “shock-front” feature and increasing ion density remain. (a) The red shaded region shows where $-\sigma_r < r < \sigma_r$.

done by first approximating the potential and then integrating to find the period of the motion.

4.4.1 2-Dimensional Model

The 2-D model is characterized by a potential that is concave up (rising exponentially) for small amplitudes and concave down (rising logarithmically) for large amplitudes. There must be some inflection point between the two, and thus a region that can be approximated by a linear function. This suggests a method of approximation: if the small-amplitude function is always larger than the large-amplitude function, it is possible to find a line that lies tangent to both curves and connects the two. This line then serves as a patch function between the two limits:

$$\frac{V_p(r)}{m_i c^2 \kappa} = m_i r + b \text{ where } r_1 < r < r_2 \quad (4.79)$$

Equations (4.19), (4.20) and (4.23) and using a change of variables $u = r/\sqrt{2}\sigma_r$ then give rise to the inequality:

$$V(0) + m_i c^2 \kappa \frac{r^2}{4\sigma_r^2} > m_i c^2 \kappa \ln r \quad (4.80)$$

$$-\frac{\gamma_{EM}}{2} + \ln(\sqrt{2}\sigma_r) + \frac{r^2}{4\sigma_r^2} > \ln r \quad (4.81)$$

$$-\frac{\gamma_{EM}}{2} + \frac{r^2}{4\sigma_r^2} > \ln \left(\frac{r}{\sqrt{2}\sigma_r} \right) \quad (4.82)$$

$$-\frac{\gamma_{EM}}{2} + \frac{u^2}{2} > \ln u \quad (4.83)$$

Plotting this transcendental inequality shows it is true for all $r > 0$ (fig. 4.8), so this method of approximation is valid for all possible parameters of ion motion.

Continuing to use the useful substitution $u = r/\sqrt{2}\sigma_r$, and factoring out $m_i c^2 \kappa$, allows us to translate the potentials into simpler forms and solve. The small-amplitude

Figure 4.8: *General validity of patched model.* The transcendental inequality (eq. (4.83)) is true for all $u > 0$.

potential (eq. (4.20)) and its derivative become:

$$V_s(u) = -\frac{\gamma_{EM}}{2} + \ln(\sqrt{2}\sigma_r) + \frac{u^2}{2} \quad (4.84)$$

$$V'_s(u) = u \quad (4.85)$$

The large-amplitude potential (eq. (4.23)) and its derivative are:

$$V_l(u) = \ln(u\sqrt{2}\sigma_r) \quad (4.86)$$

$$V'_l(u) = \frac{1}{u} \quad (4.87)$$

The patch function becomes:

$$V_p(u) = m\sqrt{2}\sigma_ru + b \quad (4.88)$$

$$V'_p(u) = m\sqrt{2}\sigma_r \quad (4.89)$$

where m is the slope of the function and b is the offset and must be solved for. The derivative at both the left endpoint $V'_s(u_1)$ and right endpoint $V'_l(u_2)$ are set equal:

$$u_1 = \frac{1}{u_2} \quad (4.90)$$

Additionally, the patch derivative is given by $m\sqrt{2}\sigma_r = \Delta V/\Delta r$. Setting that equal to $V'_s(u_1)$ makes it possible to solve:

$$u_1 = \frac{\ln(u_2\sqrt{2}\sigma_r) - \left[-\frac{\gamma_{EM}}{2} + \ln(\sqrt{2}\sigma_r) + \frac{u_1^2}{2}\right]}{u_2 - u_1} \quad (4.91)$$

$$u_1 = \frac{\ln u_2 + \frac{\gamma_{EM}}{2} - \frac{u_1^2}{2}}{u_2 - u_1} \quad (4.92)$$

$$u_1 = \frac{\ln(1/u_1) + \frac{\gamma_{EM}}{2} - \frac{u_1^2}{2}}{\frac{1}{u_1} - u_1} \quad (4.93)$$

$$1 - u_1^2 = -\ln u_1 + \frac{\gamma_{EM}}{2} - \frac{u_1^2}{2} \quad (4.94)$$

$$\ln u_1 - \frac{u_1^2}{2} = \frac{\gamma_{EM}}{2} - 1 \quad (4.95)$$

$$\ln u_1^2 - u_1^2 = \gamma_{EM} - 2 \quad (4.96)$$

$$-u_1^2 e^{-u_1^2} = -e^{\gamma_{EM}-2} \quad (4.97)$$

$$u_1^2 = -W(-e^{\gamma_{EM}-2}) \quad (4.98)$$

There are two possible solutions using different branches of the Lambert W -function. Because $u_2 = 1/u_1$ and we require that $u_2 > u_1$, then $u_1 < 1$. This leaves one solution, $u_1 = \sqrt{-W(-e^{\gamma_{EM}-2})} = 0.581$. Then, converting back to r coordinates gives:

$$r_1 = u_1\sqrt{2}\sigma_r = 0.822\sigma_r \quad (4.99)$$

$$r_2 = u_2\sqrt{2}\sigma_r = \frac{\sqrt{2}\sigma_r}{u_1} = 2.43\sigma_r \quad (4.100)$$

Finally, solving for m is simple:

$$m = \frac{u_1}{\sqrt{2}\sigma_r} = \frac{0.411}{\sigma_r} \quad (4.101)$$

Solving for b takes a little algebra:

$$V_p(r_2) = V_l(r_2) \quad (4.102)$$

$$mr_2 + b = \ln r_2 \quad (4.103)$$

$$b = \ln r_2 - mr_2 = \ln \left(\frac{\sqrt{2}\sigma_r}{u_1} \right) - 1 \quad (4.104)$$

$$b = 0.889 \ln \sigma_r - 1 \quad (4.105)$$

This gives a final potential approximation of:

$$V(r) = mc^2\kappa \begin{cases} -\frac{\gamma_{EM}}{2} + \ln(\sqrt{2}\sigma_r) + \frac{r^2}{4\sigma_r^2} & r < 0.822\sigma_r \\ 0.411\frac{r}{\sigma_r} + 0.889 \ln \sigma_r - 1 & 0.822\sigma_r \leq r < 2.43\sigma_r \\ \ln r & r \geq 2.43\sigma_r \end{cases} \quad (4.106)$$

$$V(u) = mc^2\kappa \begin{cases} -\frac{\gamma_{EM}}{2} + \ln(\sqrt{2}\sigma_r) + \frac{u^2}{2} & u < u_1 \\ u_1 u + \ln \left(\frac{\sqrt{2}\sigma_r}{u_1} \right) - 1 & u_1 \leq u < \frac{1}{u_1} \\ \ln(u\sqrt{2}\sigma_r) & \frac{1}{u_1} \leq u \end{cases} \quad \text{where } \begin{matrix} u_1 = \sqrt{-W(-e^{\gamma_{EM}-2})} \\ = 0.581 \end{matrix} \quad (4.107)$$

The period of the motion can be found as was done previously, by integrating the velocity via eq. (4.25) and using the $u = r/\sqrt{2}\sigma_r$ change of variables, and factoring out $mc^2\kappa$:

$$\frac{T}{4} = \sqrt{m}\sigma_r \int_0^{u_0} \frac{1}{\sqrt{E(r_0) - V(u)}} du \quad (4.108)$$

$$T = \frac{4\sigma_r}{c\sqrt{\kappa}} \int_0^{u_0} \frac{1}{\sqrt{E_n(r_0) - V_n(u)}} du \quad (4.109)$$

The subscript for $E_n = E/mc^2\kappa$ and $V_n = V/mc^2\kappa$ indicates these quantities have been normalized. As the potential V is split into three parts, the integral must also be

Figure 4.9: *Analytic and patched potential.* The potential and its approximation with $mc^2\kappa = 1$ and $\sigma_r = 1$. All approximations will have this shape, as all cases can be reached by a change of scale.

split into three parts. Each integral can be solved in turn, substituting the constant $E(r_0)$ at the end. Where r_0 falls determines which integrals will be included, and how much. The first part is given by:

$$T_1 = \frac{4\sigma_r}{c\sqrt{\kappa}} \int_0^{u_0} \frac{1}{\sqrt{E_n + \frac{\gamma_{EM}}{2} - \ln(\sqrt{2}\sigma_r) - \frac{u^2}{2}}} du \quad (4.110)$$

$$T_1 = \frac{4\sqrt{2}\sigma_r}{c\sqrt{\kappa}} \int_0^{u_0} \frac{1}{\sqrt{2E_n + \gamma_{EM} - \ln(2\sigma_r^2) - u^2}} du \quad (4.111)$$

Making the substitution $A^2 = 2E_n + \gamma_{EM} - \ln(2\sigma_r^2) > 0$:

$$T_1 = \frac{4\sqrt{2}\sigma_r}{c\sqrt{\kappa}} \int_0^{u_0} \frac{1}{\sqrt{A^2 - u^2}} du \quad (4.112)$$

$$T_1 = \frac{4\sqrt{2}\sigma_r}{c\sqrt{\kappa}} \sin^{-1} \left(\frac{u_0}{A} \right) \quad (4.113)$$

$$T_1 = \frac{4\sqrt{2}\sigma_r}{c\sqrt{\kappa}} \sin^{-1} \left(\frac{r_0}{\sigma_r \sqrt{4E_n + 2\gamma_{EM} - 2\ln(2\sigma_r^2)}} \right) \quad (4.114)$$

The second part is given by:

$$T_2 = \frac{4\sigma_r}{c\sqrt{\kappa}} \int_{u_1}^{u_0} \frac{1}{\sqrt{E_n - \ln\left(\frac{\sqrt{2}\sigma_r}{u_1}\right) - 1 - u_1 u}} du \quad (4.115)$$

Substituting $A = E_n - \ln(\sqrt{2}\sigma_r/u_1) - 1$ gives:

$$T_2 = \frac{4\sigma_r}{c\sqrt{\kappa}} \int_{u_1}^{u_0} \frac{1}{\sqrt{A - u_1 u}} du \quad (4.116)$$

$$T_2 = \frac{4\sigma_r}{c\sqrt{\kappa}} \int_{A-u_1^2}^{A-u_0 u_1} x^{-1/2} dx \quad (4.117)$$

$$T_2 = \frac{4\sigma_r}{c\sqrt{\kappa}} \left(\frac{\sqrt{A - u_1^2} - \sqrt{A - u_0 u_1}}{u_1} \right) \quad (4.118)$$

The third part is simpler as $u_0 > 1/u_1$, so the energy must be of that case. This has already been solved in eqs. (4.26), (4.27) and (4.29):

$$T_3 = 2\sqrt{2\pi} \frac{r_0}{c\sqrt{k}} \times \operatorname{erf} \left[\sqrt{\ln\left(\frac{r_0}{r_2}\right)} \right] \quad (4.119)$$

These can be stitched together to form a continuous function for the period of motion.

4.4.2 1-Dimensional Model

The 1-D model cannot be approximated in the same manner, as the large-amplitude region is not concave but linear. In this case, it is possible to pick a right end-point at a value where the large-amplitude equations take over, and choose a left end-point that makes the line segment a tangent to the small-amplitude region.

4.5 Ion Collapse and Temperature

Ion collapse places a serious bound on the problem of PWFA: at longer length scales, in particular where the ion density spikes, the highly nonlinear fields will disrupt the beam (figs. 4.3 and 4.7). At first glance, a high temperature plasma looks like a solution to the problem of ion collapse, as at high temperatures the ion density spikes would be smoothed out. However, we will show that it takes temperatures on the order of those required for fusion reactions in order to achieve a reasonably-smooth plasma.

At shorter length scales, ion motion will result in a change in plasma density. Changes in plasma density correlate to changes in the matched beta function and possible emittance growth as was seen in chapter 3.

4.5.1 On-Axis Ion Density

The full details of the ion densities as a result of ion motion will determine the focusing forces. However, it is possible to calculate the increase in ion density from the small-amplitude motion for a region near the axis, before the first nonlinear density spike. The details between round and flat beams are subtly different, but the method is the same.

Round Symmetry

In this case, the longitudinal ion density arises by treating the small-amplitude equations of motion $r(\xi) \propto u(\xi)$ given in eqs. (4.12) and (4.22), with $\theta = \theta_0$, as a change

of variable on the on-axis density function $n(\xi, r, \theta) = n_{i,0}$. The number of particles in a given area initially ($\xi = 0$) is:

$$\begin{aligned} N_i \Big|_{\xi=0} &= n_i \Big|_{\xi=0} r_0 dr_0 d\theta_0 \\ &= f_{i,0} dr_0 d\theta_0 \text{ where } f_{i,0} = n_{i,0} r_0 \end{aligned} \quad (4.120)$$

After the transformation, at $\xi = \xi_f$, the number of particles in a given area corresponding to the coordinate transform is:

$$\begin{aligned} N_i \Big|_{\xi=\xi_f} &= n_i \Big|_{\xi=\xi_f} r_0 dr_0 d\theta_0 \\ &= f_{i,f} dr d\theta \text{ where } f_{i,f} = n_i(\xi) r \end{aligned} \quad (4.121)$$

where $n_{i,f}(r)$ is the final ion density. Performing a change of variables gives:

$$r(\xi) = r_0 \cos(k_{\xi,2D} \xi) \quad \text{where } k_{\xi,2D} = \sqrt{\frac{e^2 n_{b,0}}{2\varepsilon_0 m_i c^2}} \quad (4.122)$$

$$\begin{aligned} f_{i,f} &= n_{i,0} r_0 |J| = n_{i,0} r_0 \frac{\partial r_0}{\partial r} \\ &= \frac{n_{i,0} r}{\cos^2(k_{\xi,2D} \xi)} \end{aligned} \quad (4.123)$$

The ion density is:

$$n_i(\xi) = \frac{n_{i,0}}{\cos^2(k_{\xi,2D} \xi)} \quad (4.124)$$

Flat Symmetry

A similar derivation for flat symmetry can be performed as well. Interpreting the equations of motion from eq. (4.69) in coordinate transformation gives:

$$y(\xi) = y_0 \cos(k_{\xi,1D} \xi) \quad \text{where } k_{\xi,2D} = \sqrt{\frac{e^2 n_{b,0}}{\varepsilon_0 m_i c^2}} \quad (4.125)$$

$$\begin{aligned} n_i(\xi) &= n_{i,0} |J| = n_{i,0} \frac{\partial y_0}{\partial y} \\ &= \frac{n_{i,0}}{\cos(k_{\xi,1D} \xi)} \end{aligned} \quad (4.126)$$

4.5.2 Emittance Growth

It is possible to evaluate the emittance growth where there is marginal ion motion, so that the beam length L is not long enough to reach the point where the ion density diverges:

$$L < \left(\frac{\pi}{2}\right) k_{\xi} \quad (4.127)$$

These new densities will cause the beam to be mismatched, as the plasma density is no longer one value but varies continuously as a function of ξ . Different mismatches will result in emittance growth, as in chapter 3. Ideally, the injected beam would have a distribution with β correlated to ξ that would compensate for the beam density $n_i(\xi)$ and result in no initial mismatch for any position ξ :

$$\beta^*(\xi) = \sqrt{\frac{2E_0 \varepsilon_0}{n_i(\xi) e^2}} \quad (4.128)$$

$$\alpha^*(\xi) = 0 \quad (4.129)$$

In this case, emittance growth would be negligible, as a result of eq. (3.36).

General Theory

A more realistic scenario is to assume that the initial beam has no correlation in β^* with ξ , as this is a typical scenario in beam transport design. It is possible to

attempt to match the beam's β^* to the varying density produced by ion motion in order to minimize emittance growth. The emittance analysis shown in eqs. (3.34) to (3.36) can be adapted for varying plasma density. The moments can be calculated by integrating over ξ , assuming a uniform beam distribution in ξ , and exchanging the plasma density n_p for the ion density $n_i(\xi)$ in the beta function found in eq. (3.17), with an expansion about $\delta = 0$:

$$\begin{aligned} \left\langle \frac{1}{\beta^2} \right\rangle &= \frac{1}{L} \int_0^L \int_{-\infty}^{\infty} \left[\frac{1}{\beta(\delta; \xi)^2} \right] f_\delta(\delta) d\delta d\xi \\ &= \frac{1}{L} \int_0^L \int_{-\infty}^{\infty} \left[\frac{1}{\left(\frac{2E_0\epsilon_0}{e^2 n_i(\xi)} \right) (1 + \delta)} \right] f_\delta(\delta) d\delta d\xi \end{aligned} \quad (4.130)$$

$$\begin{aligned} &\approx \frac{1}{L} \int_0^L \int_{-\infty}^{\infty} \left[\frac{1 - \delta + \delta^2 + \delta^3 + \mathcal{O}(\delta^4)}{\left(\frac{2E_0\epsilon_0}{e^2 n_i(\xi)} \right)} \right] f_\delta(\delta) d\delta d\xi \\ &\approx \frac{1}{L} \int_0^L \left[\frac{1 + \sigma_\delta^2}{\left(\frac{2E_0\epsilon_0}{e^2 n_i(\xi)} \right)} \right] d\xi \approx \frac{1}{L} \int_0^L \frac{e^2 n_i(\xi)}{2E_0\epsilon_0} d\xi \\ \langle \beta^2 \rangle &= \frac{1}{L} \int_0^L \int_{-\infty}^{\infty} \beta(\delta; \xi)^2 f_\delta(\delta) d\delta d\xi \\ &= \frac{1}{L} \int_0^L \int_{-\infty}^{\infty} \left(\frac{2E_0\epsilon_0}{e^2 n_i(\xi)} \right) (1 + \delta) f_\delta(\delta) d\delta d\xi \quad (4.131) \\ &= \frac{1}{L} \int_0^L \left(\frac{2E_0\epsilon_0}{e^2 n_i(\xi)} \right) d\xi \end{aligned}$$

It is then possible to evaluate these integrals in the context of flat and round symmetries to find the emittance $\epsilon^2 = \langle J\beta \rangle \langle J/\beta \rangle$ in these scenarios. From eqs. (3.34)

to (3.36), the emittance is:

$$\begin{aligned}
\epsilon^2 &= \left[\left(\frac{\epsilon_i}{2} \right) \left(\beta_i + \langle \beta^2 \rangle \gamma_i \right) \right] \left[\left(\frac{\epsilon_i}{2} \right) \left(\beta_i \left\langle \frac{1}{\beta^2} \right\rangle + \gamma_i \right) \right] \\
&= \left(\frac{\epsilon_i^2}{4} \right) \left[\left(\frac{\beta_i}{\beta_0} \right)^2 + \frac{\langle \beta^2 \rangle}{\beta_0^2} (1 + \alpha_i^2) \right] \left[\beta_0^2 \left\langle \frac{1}{\beta^2} \right\rangle + \left(\frac{\beta_0}{\beta_i} \right)^2 (1 + \alpha_i^2) \right] \quad (4.132) \\
&= \left(\frac{\epsilon_i^2}{4} \right) \left[M^2 + \frac{\langle \beta^2 \rangle}{\beta_0^2} (1 + \alpha_i^2) \right] \left[\beta_0^2 \left\langle \frac{1}{\beta^2} \right\rangle + M^{-2} (1 + \alpha_i^2) \right]
\end{aligned}$$

where the beta match ratio is given by $M \equiv \beta_i/\beta_0$. While the final form appears strange for the moment, it will allow us to simplify the equation in the cases of round and flat beams. It is clear that any value for $\alpha_i \neq 0$ will result in emittance growth and is undesirable.

Circular Symmetry

Using the beam density from eq. (4.124), the averages $\langle 1/\beta^2 \rangle$ and $\langle \beta^2 \rangle$ can be calculated, where β_0 is the matched beta for the density $n_i(0)$:

$$\left\langle \frac{1}{\beta^2} \right\rangle \approx \frac{\tan(k_{\xi,2D} L)}{\beta_0^2 k_{\xi,2D} L} \quad (4.133)$$

$$\langle \beta^2 \rangle = \frac{\beta_0^2}{2} \left(1 + \frac{\sin(2k_{\xi,2D} L)}{2k_{\xi,2D} L} \right) \quad (4.134)$$

The emittance can then be found from eq. (4.132), replacing $\psi_{2D} \equiv k_{\xi,2D} L$, assuming that $\alpha_i = 0$ in the ideal case:

$$\epsilon^2 \approx \left(\frac{\epsilon_i^2}{4} \right) \left[M^2 + \left(\frac{1}{2} + \frac{\sin(2\psi_{2D})}{4\psi_{2D}} \right) \right] \left[\frac{\tan \psi_{2D}}{\psi_{2D}} + M^{-2} \right] \quad (4.135)$$

(a) Emittance growth factor due to circular symmetry

(b) Emittance growth factor due to flat symmetry

Figure 4.10: *Emittance growth due to marginal ion motion.* Ion motion is most sensitive to the beam parameter match ratio rather than ion motion in both circular and flat symmetries.

Flat Symmetry

Using the beam density from eq. (4.124), the averages $\langle 1/\beta^2 \rangle$ and $\langle \beta^2 \rangle$ can be calculated, where β_0 is the matched beta for the density $n_i(0)$, making the similar substitution $\psi_{1D} \equiv k_{\xi,1D} L$:

$$\left\langle \frac{1}{\beta^2} \right\rangle \approx \left(\frac{1}{\beta_0^2 \psi_{1D}} \right) \ln \left[\frac{\cos(\psi_{1D}/2) + \sin(\psi_{1D}/2)}{\cos(\psi_{1D}/2) - \sin(\psi_{1D}/2)} \right] \quad (4.136)$$

$$\langle \beta^2 \rangle = \beta_0^2 \left(\frac{\sin \psi_{1D}}{\psi_{1D}} \right) \quad (4.137)$$

The emittance can then be found from eq. (4.132), assuming that $\alpha_i = 0$ in the ideal case:

$$\epsilon^2 \approx \left(\frac{\epsilon_i^2}{4} \right) \left[M^2 + \left(\frac{\sin \psi_{1D}}{\psi_{1D}} \right) \right] \left[\left(\frac{1}{\psi_{1D}} \right) \ln \left[\frac{\cos(\psi_{1D}/2) + \sin(\psi_{1D}/2)}{\cos(\psi_{1D}/2) - \sin(\psi_{1D}/2)} \right] + M^{-2} \right] \quad (4.138)$$

Observations and Impact

Qualitatively, a few things can be noted. Ions that experience a different ion density may undergo emittance growth, but they will come to an equilibrium. Ion motion is not as significant an issue as match is, as seen in fig. 4.10. This is encouraging: as long as beams avoid regions of total ion collapse, it appears to be possible to avoid significant emittance growth. The beam can also be matched for all ξ and have no emittance growth if the spot size is correlated in ξ .

4.5.3 Temperature Effects

Up until this point, all models have assumed the ions are at zero temperature. Zero-temperature gas particles have no initial velocities. In both 1-D and 2-D models, ions undergoing small-amplitude motion have trajectories that all cross at a single point corresponding to a phase advance of $\pi/2$. In addition, models predict density discontinuities extending outwards in a shell. However, in a gas with nonzero

temperature, ions have random initial velocities. The stochastic nature of the ion distribution precludes complete ion collapse.

Small-amplitude ion motion can be modeled in order to predict the density profile including temperature effects. Assuming one of the basic tenets of classical thermodynamics that all possible states have equal probability in a system, the partition function can be used to determine probability distributions and macro variables. In particular, the average energy E_i of each degree of freedom i is a function of temperature:

$$E_i = \frac{1}{2}k_B T \quad (4.139)$$

While some degrees of freedom may be partially frozen out and thus not have a full $1/2k_B T$ of energy, ion motion is concerned with translational degrees of freedom. Given the system details of a large container when compared with an ion wavefunction, quantum effects are not important when dealing with translational motion. Translational motion due to temperature can be described by classical thermodynamics.

4.5.4 High-Temperature Limit

High temperatures are defined as the point at which the average ion velocity will be so high that any induced ion motion will be a negligible perturbation on ion densities. Fundamentally, this can be understood as the average thermal energy of a particle in the dimensions of motion of interest being much greater than the energy due to the potential from the equations of motion, for all ions of interest:

$$V(\vec{r}) - V(0) \ll n \frac{k_B T}{2} \text{ where } n \text{ is the number of dimensions} \quad (4.140)$$

While this describes analytically the ions where high temperatures make ion motion irrelevant, it is important to note that this potential is due to the electron beam. Ions at large radii can have arbitrarily large potentials, and thus have significant ion motion at arbitrarily-high temperatures.

However, the time for ions to arrive on axis increases with radius as well. It is

Dimensions	Regime	Formula
1-D	Exact	$T \gg \frac{e^2 n_{b,0} \sigma_y^2}{2\epsilon_0 k_B} \left[\sqrt{\pi} \nu \operatorname{erf}(\nu) + e^{-\nu^2} - 1 \right]$
1-D	Small-Amplitude	$T \gg \left[\frac{e^2 n_{b,0} \sigma_y^2}{2\epsilon_0 k_B} \nu^2 = \frac{e^2 n_{b,0}}{4\epsilon_0 k_B} y^2 \right]$
1-D	Large-Amplitude	$T \gg \sqrt{\frac{\pi}{8}} \frac{e^2 n_{b,0} \sigma_y}{\epsilon_0 k_B} y$
2-D	Exact	$T \gg \frac{e^2 n_{b,0} \sigma_r^2}{2\pi\epsilon_0 k_B} \left[\ln \left(\frac{r}{\sqrt{2}\sigma_r} \right) - \frac{1}{2} \operatorname{Ei} \left(-\frac{r^2}{2\sigma_r^2} \right) + \frac{\gamma_{EM}}{2} \right]$
2-D	Small-Amplitude	$T \gg \left[\frac{e^2 n_{b,0} \sigma_r^2}{2\pi\epsilon_0 k_B} \frac{u^2}{2} = \frac{e^2 n_{b,0}}{8\pi\epsilon_0 k_B} r^2 \right]$
2-D	Large-Amplitude	$T \gg \frac{e^2 n_{b,0} \sigma_r^2}{2\pi\epsilon_0 k_B} \left(\ln u + \frac{\gamma_{EM}}{2} \right)$

Table 4.1: *High-temperature limits in various ion motion models.* Various limits where ion motion is a perturbation on ion velocities due to high temperature. 1-D models: Exact potential, eq. (4.63). Small-amplitude potential, eq. (4.66). Large-amplitude potential, eq. (4.70). (Note: $\nu = y/\sqrt{2}\sigma_y$) 2-D Models: Exact potential, eqs. (4.16) and (4.19). Small-amplitude potential, eq. (4.20). Large-amplitude potential, eq. (4.23). (Note: $u = r/\sqrt{2}\sigma_r$.)

important to be clear on the effect being studied in order to determine whether ion motion will be a problem. For instance, in a single bunch of finite length, only some of the ions will arrive on-axis and affect the beam. Other ions will not arrive on-axis while the bunch is still present, and thus their motion is irrelevant for that specific process.

It is simple enough to derive the high-temperature limit in the small- and large-amplitude regimes using the small- and large-amplitude potentials as derived previously.

In the model, motion in orthogonal planes is decoupled. Motion can be described by ions with a distribution of initial speeds due to Maxwell-Boltzmann statistics:

$$P(p_y) \propto e^{-(p_y^2/2m_i)/k_B T} \quad (4.141)$$

$$P(v_y) = \sqrt{\frac{m_i}{2\pi k_B T}} e^{-\frac{m_i v_y^2}{2k_B T}} \quad (4.142)$$

$$P(y') = \sqrt{\frac{m_i}{2\pi k_B T}} e^{-\frac{m_i c^2 y'^2}{2k_B T}} \quad (4.143)$$

The speed distribution results in a phase space distribution of ions. Where before

all of the ions undergoing small-amplitude motion were expected to collapse to a single point, now the distribution in phase space must be accounted for. Small-amplitude motion is then a rotation in phase space under the equations of motion given by eq. (4.69):

$$y = y_0 \cos \left(\sqrt{\frac{e^2 n_{b,0}}{2\varepsilon_0 m_i c^2}} \xi \right) + y'_0 \sqrt{\frac{2\varepsilon_0 m_i c^2}{e^2 n_{b,0}}} \sin \left(\sqrt{\frac{e^2 n_{b,0}}{2\varepsilon_0 m_i c^2}} \xi \right) \quad (4.144)$$

With cold ions, $y'_0 = 0$, and ion collapse occurs for a phase advance of $\pi/2$, which occurs for:

$$\xi_{90^\circ} = \frac{\pi}{2} \sqrt{\frac{2\varepsilon_0 m_i c^2}{e^2 n_{b,0}}} \quad (4.145)$$

At this location, phase space is rotated 90° , which is equivalent to the coordinate transform:

$$y' = y \sqrt{\frac{e^2 n_{b,0}}{2\varepsilon_0 m_i c^2}} \quad (4.146)$$

This results in an ion distribution given by:

$$P(y) = \sqrt{\frac{m_i}{2\pi k_B T}} e^{-\frac{y^2 e^2 n_{b,0}}{4\varepsilon_0 k_B T}} \quad (4.147)$$

This gives the effective blurring of ion collapse for a given temperature, ion atomic species, and drive beam density. This can be used to model an upper limit on the density of ions at the ion collapse location.

4.6 Implications for a Linear Collider

During an initial PWFA acceleration section, if the beam is not already matched to the plasma, filamentation via energy spread causes the beam to become matched. This occurs both for uniform plasmas and plasmas with ion motion. If it is possible

to implement a broadband optics transfer lattice, with either an I or $-I$ transform for a broad range of energies, including the plasma ramps between stages, the beam will remain matched. This means that, theoretically, the beam's slice emittance will only grow in the first stage, as a function of ξ . As a result, the slice and projected emittances are expected to saturate through many stages of a PWFA linear collider. This scenario requires a model that includes beam electron motion, which is not modeled analytically in this section. Simulation is still the best approach to emittance saturation in long PWFA sections or multiple PWFA stages.

All parameter sets show that ion motion is potentially a problem under this model (see table 4.2). In each case, the beam density is 4 to 5 orders of magnitude larger than the plasma density, and the ion collapse wavelength is on the order of the bunch size, which means the first spike and nonlinearities will affect a significant portion of the bunch. In all cases considered here, the beam is flat, and so the 1D theory should be used. Hydrogen gas is also assumed, although heavier gasses will reduce ion motion and increase the ion motion wavelength, as $\lambda \propto \sqrt{m_i}$. A solution may involve using gas species with higher atomic mass, which in some cases may increase the wavelength to the extent that the ion density of eq. (4.126) is nearly uniform over the bunch length.

If it is possible to use heavier gas, we may enter the regime of marginal ion motion, where no part of the beam experiences ion collapse and the focusing forces are entirely due to ions that have undergone small-amplitude motion. If this is the case, the match into the plasma will be the most important factor in determining emittance growth, with ion motion being a relatively small effect.

4.6.1 PWFA Linear Collider

Parameters are explored for earlier proposals [32] and later proposals [30, 33] for a staged PWFA linear collider. Ion motion in the case of the drive beam is not particularly concerning. The bubble created is very large, and the beam would have to have emittance growth by orders of magnitude for the beam to escape the bubble, which is the only practical concern. Simulations should still be done to verify this is

not a problem. The drive beam is discarded, and so any effects on the drive beam are not particularly important. However, the witness beam will see the ion motion, and the issues of nonlinearities apply to it, and have been analyzed in table 4.2a.

4.7 ILC “Afterburner”

The ILC “afterburner” [30, 36] requires several assumptions. Assuming the drive and witness bunch are created from the same initial bunch may imply that their aspect ratios are similar, while their bunch length varies. This leads to the use of the flat beam theory as well. Previous configurations [36] and more recent updates [30] are shown in table 4.2b. In these cases, motion induced by the drive bunch may also be relevant due to the close spacing to the witness bunch. While not explored in this thesis, it should be noted that ion motion after the drive passes may interact with the witness bunch, and that should be simulated as well.

Table 4.2: *Beam densities and ion motion in a PWFA-LC and ILC “Afterburner”.* Calculations assume the plasma is formed from hydrogen gas.

Specification	Early Proposal [32]				Updated Proposal [30, 33]									
Energy	1 TeV				250 GeV		500 GeV		1 TeV		3 TeV		10 TeV	
Plasma density [1/cm ³]	1 × 10 ¹⁷				2 × 10 ¹⁶		2 × 10 ¹⁶		2 × 10 ¹⁶		2 × 10 ¹⁶		2 × 10 ¹⁶	
N _e , witness bunch	1 × 10 ¹⁰				1 × 10 ¹⁰		1 × 10 ¹⁰		1 × 10 ¹⁰		1 × 10 ¹⁰		1 × 10 ¹⁰	
RMS witness length [μm]	10				20		20		20		20		20	
Peak witness	8.36 × 10 ²¹				4.99 × 10 ²⁰		7.06 × 10 ²⁰		9.99 × 10 ²⁰		1.73 × 10 ²¹		3.16 × 10 ²¹	
Density n _{b,0} [1/cm ³]	15.7				64.0		53.9		45.3		34.4		25.5	
Ion wavelength [μm]	Yes				Yes		Yes		Yes		Yes		Yes	
Potential ion motion issue?	Yes				Yes		Yes		Yes		Yes		Yes	

Chapter 5

Simulations of Accelerators and Ion Motion

Some first-order simulations are useful for determining beam optics. These simulations can be used to determine how optics affect beam focusing, and are used in the linear least squares regression for single shot emittance measurements presented in chapter 8. For these measurements, dozens of transfer matrices need to be calculated per measurement. A simulation package called SLACtrac¹ was written in Python to allow easy access and manipulation of simulations and fast calculations.

The ion motion model explored in chapter 4 makes several approximations, including that the beam is always more dense than the plasma and can be modeled by a bunch with a uniform longitudinal distribution and a Gaussian transverse distribution. The most straightforward way to validate this model and test where its assumptions break down is by simulation. A new simulation package called Plasma-Trace¹ was created in order to test this model.

5.1 Simulation Approaches

There are simulation techniques used to explore particle acceleration and plasma wakefield acceleration. These techniques follow very different physics models, and so

¹See supplemental materials for source code.

there is no overlap between the two. Particle acceleration in either ring accelerators or linear accelerators follow a model where separate components can be simulated individually. These codes tend to follow a particle tracking approach, where individual particles' trajectories are simulated and recorded. Plasma wakefield acceleration is challenging enough to merit its own simulation efforts, which are independent of the rest of the accelerator components. PWFA simulations involve a large number of plasma particles interacting with beam particles, and follow approaches to reduce the number of operations. These simulations are in many cases follow a particle-in-cell (PIC) approach.

5.1.1 Particle Tracking Simulation

Particle accelerator simulation is generally done via particle tracking codes such as ELEGANT [48]. Particle trackers integrate particle motion in response to various external fields. An accelerator is broken into its components, and each component is simulated separately in the particle tracker. Particles within the beam are represented by “macro particles” where a single particle represents several particles. In this way, the total number of particles that must be simulated can be reduced. The particles each have 6 parameters: 3 transverse space coordinates, and 3 momentum coordinates. Each component of the simulation uses these particles' parameters to simulate the physics within each component and produces a final collection of macro particles.

5.1.2 Particle-in-Cell Simulation

Plasma wakefield acceleration is a much newer phenomenon, and has its own specialized approach. Generally, PWFA is examined through particle-in-cell (PIC) codes such as Osiris [49, 50] and QuickPIC [51, 52]. Beam particles may still be represented as macro particles that follow trajectories. However, in many cases, plasma particle physics is not easily represented by macro particles. In this case, the plasma volume is divided into 3D cubes, known as cells. In order to simulate plasma behavior, the amount of charge and flow of particles through the cells is calculated. The beam

particles' response to the plasma, and the plasma's response to the beam particles, is simulated by calculating forces at small time-steps.

5.2 SLACtrac

While ELEGANT is a well-tested accelerator simulation package [48], it does not integrate well with Python. ELEGANT simulations are configured through input files, and its transfer matrix behavior is undocumented, and its output files are difficult to read with Python. Emittance measurements require the calculation of beam transfer matrices and beta functions at varying energies and magnet settings. SLACtrac is designed to change the parameters of a linear accelerator model and quickly recompute the beam's transfer matrices and beta functions within Python. This makes it possible to tailor SLACtrac simulations within an analysis written in Python.

SLACtrac copies heavily from open-source code ELEGANT [48]. In particular the transfer matrix routines of individual elements such as dipoles, quadrupoles, and drifts were copied and adapted to Python. SLACtrac runs purely in Python, allowing the construction and computation of beamline parameters of a linac as necessary. At this time, SLACtrac does not perform particle tracking in addition to transfer matrix calculations, which precludes any simulation of beam effects such as coherent synchrotron radiation². SLACtrac also does not perform any repeating-structure or ring calculations: e.g. tunes, matched beta functions and emittance, etc. In this sense, SLACtrac is in many ways a limited port of ELEGANT to Python.

To make sure that SLACtrac did not have errors, it has been benchmarked against ELEGANT where possible. ELEGANT can produce beam transfer matrices and transferred Courant-Snyder beam parameters, and its calculations match SLACtrac's.

5.2.1 First-order Transfer Matrix Formalism

Many models of linear accelerator components have equations of motion which are linear in positions x , y , and z , angles $x' \equiv p_x/p_z$, $y' \equiv p_y/p_z$, and energy $\delta \equiv$

²It is possible for a user to create their own simulation sections in Python to address this issue.

$1 - (E/E_0)$, where p_i is the i -th momentum component, E is the particle energy, and E_0 is a reference energy for the model. Linear equations of motion relating initial coordinates $X_{i,0}$ to final coordinates X_i can be written explicitly as:

$$X_i = X_{j,0} R_{ij} \quad (5.1)$$

(Refer to table 5.1 for conventions.) Rather than use large equations or summation notation, these models can be converted to a matrix formalism of Brown [53], where the coordinates are represented by a vector $\mathbf{X}$ and equations of motion are represented by a matrix $\mathbf{R}(\dots)$:

$$\mathbf{X} = \mathbf{R}(\dots)\mathbf{X}_0 \quad (5.2)$$

The parameters for $\mathbf{R}(\dots)$ will depend on the parameters required by each component's model. The transverse coordinate system origin is defined by a reference trajectory for a particle with energy E_0 ($\delta = 0$). This particle's initial coordinates are defined as $\mathbf{X}_{i,0} = 0$. Models may expand about this energy, and their approximations should always be kept in mind.

Each accelerator component has a matrix, and they can be linked to transform coordinates from one location to another by chaining them together. For example, a beam passing through components in the order $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_n$, can be written:

$$\mathbf{X} = \mathbf{R}_n \cdots \mathbf{R}_2 \mathbf{R}_1 \mathbf{X}_0 \quad (5.3)$$

Brown [53] has developed $\mathbf{R}$ matrices for several types of accelerator components, which are sufficient for our purposes. The full form of these transfer matrices is not always necessary for emittance measurement, as some approximations are possible.

Drifts

A drift is a component with no external forces. The particles are assumed to experience no forces, including space-charge forces, which is a good approximation for

Index i	Coordinate $\mathbf{X}_i$	Definition	Units
1	x	Horizontal coordinate ($\perp$ to trajectory)	m
2	x'	Angle: p_x/p_z	rad
3	y	Vertical coordinate ($\perp$ to trajectory)	m
4	y'	Angle: p_y/p_z	rad
5	z	Longitudinal position	m
6	δ	Energy: $1 - (E/E_0)$	(Unitless)

Table 5.1: *Transfer matrix formalism conventions and definitions.* The subscripts for coordinates q_i and $\mathbf{R}_{ij}$ relate to the coordinate definitions shown. Variables p_i are the i -th momentum component, E is the particle energy, and E_0 is the reference energy.

relativistic beams. The resulting transfer matrix is given by [53]:

$$\mathbf{R}_{\text{drift}} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.4)$$

Dipoles

Brown [53, 54] has derived a general transfer matrix for a dipole with quadrupole and sextupole components. The general form is accessible in SLACtrac. However, is not required for most calculations. The FACET sextupole magnet can be approximated by a wedge magnet with no quadrupole or sextupole moments:

$$\mathbf{R}_{\text{wedge dipole}} = \begin{pmatrix} C & S/h & 0 & 0 & 0 & (1-C)/h \\ -hS & C & 0 & 0 & 0 & S \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ S & (1-C)/h & 0 & 0 & 1 & (\alpha-S)/h \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.5)$$

where $C = \cos \alpha$, $S = \sin \alpha$, $h = 1/\rho_0$, ρ_0 is the bending radius of the reference trajectory, and α is the bend angle of the reference trajectory [53, 54].

Quadrupoles

Quadrupoles can also be described by linear equations of motion³, and Brown [53, 54] has derived a transfer matrix, which is adapted here to the context of this dissertation:

$$\mathbf{R}_{\text{quadrupole}} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{\sin(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{k}L) & \frac{\sinh(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh(\sqrt{k}L) & \cosh(\sqrt{k}L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.6)$$

where k is the geometric focusing of the quadrupole, and L is the length of the quadrupole. The matrix for a quadrupole that is defocusing in x and focusing in y can be found by exchanging the blocks $\mathbf{R}_{1-2,1-2}$ with $\mathbf{R}_{3-4,3-4}$.

5.3 PlasmaTrace

OSIRIS and QuickPIC are particle-in-cell approaches to simulating plasma wakefield acceleration phenomena. However, they do not track plasma particles, which would make validating the ion motion model difficult. In some cases some PWFA physics can be ignored, such as the sheath physics, and simulate ion motion via particle tracking. PlasmaTrace was created for this purpose. By taking a particle tracking approach, several difficulties of PIC codes such as the redistribution of particles and the need of very large simulations to resolve small ion motion can be avoided.

³See section 7.1.2 for a derivation of the equations of motion in a quadrupole.

5.3.1 Version 1.0

PlasmaTrace is a particle tracking code that makes several assumptions about the physics inherent in simulating relativistic particle beams. This is designed to be directly relevant to ion motion in PWFA in the blowout regime. Much of these assumptions are similar to the quasistatic approximation (section 2.2.1), which is a useful frame of reference.

Assumptions

The physical assumptions allow for some simplifications in the simulation. We assume transverse electric fields, a beam that drives all motion, non-relativistic ions, and that ions can be simulated on a different timescale than the beam.

We assume that the relativistic beam particles have transverse fields that are much larger than their longitudinal fields, as was shown in section 2.1. As a result, the fields of beam particles are simply Coulomb fields in two dimensions. While this approximation is typically good, it can lead to numerical scattering. This can be prevented by representing macro particles by Gaussian charge with a width that creates a smooth beam charge distribution⁴. The solution for the electric fields of a Gaussian charge Q can be found by Gauss' Law:

$$\rho(r) = \frac{Q}{2\pi\sigma_r^2} e^{-\frac{r^2}{2\sigma_r^2}} \quad (5.7)$$

$$\oint_{SA} \vec{E} \cdot d\vec{A} = \frac{2\pi}{\epsilon_0} \int_0^r \rho(r') r' dr' \quad (5.8)$$

$$E_r(r) = \frac{Q}{\epsilon_0 r} \left(1 - e^{-\frac{r^2}{2\sigma_r^2}} \right) \quad (5.9)$$

This field is calculated for particles in the simulation.

The beam is assumed to be the source of all forces in the simulation. This is a good approximation provided the ions never reach densities higher than the bunch density; at these densities the ion contribution to the electric field will be greater than the beam. It also implies that the ion motion is initially entirely transverse, as

⁴See appendix B.

the only force is the electric field from the bunch, and it is radial.

The ions are assumed to be non-relativistic, which implies that the ion motion is approximately transverse as the beam's fields are approximately transverse. The fields are assumed to be of the form of eqs. (2.3) and (2.4):

$$\vec{E}' \approx \gamma E_x \hat{x} + \gamma E_y \hat{y} \quad (5.10)$$

$$\vec{B}' = \gamma \frac{\beta_{\text{beam}}}{c} (-E_y \hat{x} + E_x \hat{y}) \quad (5.11)$$

where primed fields are in the lab frame, and unprimed fields are in the rest frame of the beam. The unprimed fields are of similar magnitudes, so a highly relativistic beam has similar magnetic and electric field strengths. The Lorentz' force on the ions is:

$$\begin{aligned} \vec{F} &= e_{\text{ion}} \left(\vec{E}' + \vec{v}_{\text{ion}} \times \vec{B}' \right) \\ &\approx e_{\text{ion}} \gamma (E_x \hat{x} + E_y \hat{y}) \\ &\quad + e_{\text{ion}} \frac{\beta_{\text{ion}}}{c} \gamma \left[-v_z E_x \hat{x} - v_z E_y \hat{y} + (v_x E_x + v_y E_y) \hat{z} \right] \end{aligned} \quad (5.12)$$

If the ion velocities never approach c , the magnetic contribution drops and the force is approximately:

$$\vec{F} \approx e_{\text{ion}} \gamma (E_x \hat{x} + E_y \hat{y}) \quad (5.13)$$

With no force in the $\hat{z}$ direction, the ion motion is assumed to be entirely transverse.

The next assumption is that the beam evolves on a much longer timescale than the ions. If the ions focus the electrons much more weakly than the electrons focus the ions, the ion motion can be simulated separately from the electron motion. This effect can be understood in terms of the beam's rigidity. Electron beam momentum is given by $\gamma m_e c^2$, while non-relativistic ion motion is given by $1/2 m_i v^2$. Additionally, in the model given, electron beam density is much greater than ion density. This implies that the change of momentum of the electron beam is much less than the change of momentum of ions in a given time step.

Simulation Mechanics

The number of calculations for a simulation resolving all of the physics is expected to be high enough to warrant a high-performance computing solution. A parallel computing master-slave approach was taken, where the master process coordinates the simulation and slave processes perform the calculations the master process dictates. The slave processes can then hold a large number of particles and perform calculations on those particles simultaneously, thus reducing the wall-clock time to complete a simulation. By tracking individual beam particles and plasma ions, several difficulties that arise in PIC codes are trivially solved. In PIC codes, particles occasionally need to be redistributed between cores, for instance. With this method, redistribution is unnecessary. The simulation can retain the speed of PIC codes by communicating fields discretized on a grid. This keeps the number of calculations linearly proportional to the number of particles, which allows simulations to scale by simply adding more processes. By operating on discrete particles, it may be possible to develop a single instruction multiple data (SIMD) approach, which has the potential to speed up the algorithm by running on SIMD-capable hardware.

The parameters describing the simulation are passed to the slave processes, which generate a number of beam particles and ions with the distribution requested. The master process then directs the slaves to calculate the fields from the beam. The slaves generate the particles using the random number generation routines from the Gnu Scientific Library (GSL) [55].

As each slave only contains a subset of the particles in the beam, the fields need to be summed from all of the slaves. The master receives the fields, sums them, and then redistributes the field to all of the slaves. The fields are discretized transversely and longitudinally with N_x , N_y , and N_z points. In this way, the number of field evaluations $N_{\text{field eval}}$ is linear in the number of beam particles N_{beam} :

$$N_{\text{field eval}} = N_x N_y N_z N_{\text{beam}} \quad (5.14)$$

The slaves then push the ions using a first-order symplectic integration algorithm, using routines from the GSL [55] to interpolate the fields at the ions' locations. The

number of interpolations is also linear in the number of ions:

$$N_{\text{field interp}} = N_z N_{\text{ions}} \quad (5.15)$$

As the ions only move transversely, they are updated for each longitudinal step. The particles are then optionally written to disk.

Simulation Results

In order to check the theory presented in chapter 4, beams with a flat and a Gaussian longitudinal profile were simulated. The beams were round, and the initial distribution of ions generated was circular with a uniform distribution. The results have the length scale similar to that predicted by the theory and are shown in fig. 4.3.

The simulation window has intentionally been left wider than necessary to demonstrate some of the challenges of simulating ion motion with this approach. In order to see all of the densities at a given location, the ions that will reach that location must be initially generated. Much of the simulation area shown in fig. 5.1 has missing ions. White indicates bins with < 5 ions/bin. The area with valid densities is shown by the red rectangles in fig. 5.1. This simulation can be qualitatively compared with fig. 4.3. The model predicts the correct length scales and shape of the ion distribution.

5.3.2 Version 2.0

Rather than make approximations that are similar to the quasistatic approximation⁵, Version 2 seeks to replicate the QuickPIC quasistatic approach to fields and integration [51, 52]. Unlike QuickPIC, we find all current and charge densities on a grid by binning and averaging macroparticles. This approach lets us retain the physics of QuickPIC while simplifying pushing particles.

⁵See section 2.2.1 and Sprangle *et al.* [41], Huang *et al.* [51], An *et al.* [52].

(a) Flat longitudinal profile.

(b) Gaussian longitudinal profile

Figure 5.1: *Ion motion simulation without beam evolution.* The feedback of ion motion on the beam is not simulated. (a) Flat and (b) Gaussian longitudinal profiles are simulated, which have the same charge and standard deviation in ξ . Ion distributions are projected in y . White areas have less than 5 ions/bin, while the peak density is $\sim 4 \times 10^3$ ions/bin. Red rectangles are unaffected by missing ions, for comparison with theory.

Integrating Beam Particle Motion

An *et al.* [52] apply the quasistatic approximation ($\partial_s \ll \partial_\xi$, $\partial_s^2 \ll \partial_\xi^2$, $s \equiv ct$, $\xi \equiv ct - z$)⁶ to Maxwell's equations and show that a pseudo potential ψ can be found:

$$\nabla_\perp^2 \psi = -\frac{\rho}{\varepsilon_0} + c\mu_0 J_z \quad (5.16)$$

$$\vec{E}_\perp + c\hat{z} \times \vec{B}_\perp = -\vec{\nabla}_\perp \psi \quad (5.17)$$

eq. (5.17) is approximately proportional to the force on a particle with $\vec{v} \approx c\hat{z}$:

$$\begin{aligned} \vec{F} &\approx e \left(\vec{E}_\perp + c\hat{z} \times \vec{B}_\perp \right) \\ &\approx -e\vec{\nabla}_\perp \psi \end{aligned} \quad (5.18)$$

In this case, the ion particles are nearly stationary, while beam particles have a velocity $\vec{v} \approx c\hat{z}$. This force can be applied to the beam electrons by integrating in s . This pseudo potential can be numerically calculated by binning the ion and beam charge and current densities on a grid and then solving in Fourier space:

$$\psi = \mathcal{F}^{-1} \left[\frac{1}{k^2} \mathcal{F} \left(-\frac{\rho}{\varepsilon_0} + c\mu_0 J_z \right) \right] \quad (5.19)$$

The force can be calculated by the numerical gradient of ψ as in eq. (5.17).

Integrating Ion Motion

In order to calculate this gradient at every longitudinal position ξ in the beam, the ion motion must be integrated in ξ first. In order to preserve phase space, first-order symplectic integration is used. This approach means that momentum and spatial coordinates are updated one after the other, instead of simultaneously. In this section, momentum coordinates are known at whole-integer steps in $m + i$, and spatial coordinates are known at half-integer steps in $m + i/2$ [52].

⁶Note the convention actually used by An *et al.* [52] in the definition of ξ has a sign change from the definition $\xi \equiv z - ct$ that is claimed to be used.

An *et al.* [52] proceed to find equations for the transverse fields $\vec{E}_\perp$ and $\vec{B}_\perp$, using normalized units:

$$\vec{\nabla}_\perp^2 \vec{B}_\perp = \hat{z} \times \left(\frac{\partial}{\partial \xi} \vec{J}_\perp + \vec{\nabla}_\perp J_z \right) \quad (5.20)$$

$$\vec{E}_\perp = -\nabla_\perp \psi - \hat{z} \times \vec{B}_\perp \quad (5.21)$$

At this point, we are solving for $\vec{B}_\perp$ and $\vec{J}$ simultaneously. Once $\vec{B}_\perp$ is found, the transverse electric field $\vec{E}_\perp$ can be found from eq. (5.21). The longitudinal fields E_z and B_z are [52]:

$$\nabla_\perp^2 B_z = -\nabla_\perp \times \vec{J}_\perp \quad (5.22)$$

$$\nabla_\perp^2 E_z = -\nabla_\perp \cdot \vec{J}_\perp \quad (5.23)$$

In order to solve for $\vec{B}_\perp$ and $\vec{J}$ simultaneously, a numerical solution to a diffusion equation is used. The diffusion equation to apply, with iteration index l , is [52]:

$$\nabla_\perp^2 \vec{B}_\perp^{new} - \vec{B}_\perp^{new} = \hat{z} \times \left(\frac{\partial}{\partial \xi} \vec{J}_\perp + \nabla_\perp \cdot \vec{J} \right) - \vec{B}_\perp^{old} \quad (5.24)$$

where $\vec{B}_\perp^{new}$ is the $l + 1$ magnetic field, $\vec{B}_\perp^{old}$ is the l magnetic field, and $\vec{J}$ is the l particle currents. The initial derivatives of the current at an integration step are assumed to be the same as the last integration step [51, 52], which can be found by numerically differentiating the current⁷. The current at step $m + 1/2$ is then [51]:

$$\vec{J}_{\perp, l=0}^{m+1/2} = \vec{J}_\perp^m + \frac{\Delta \xi}{2} \left(\frac{\partial}{\partial \xi} \vec{J}_\perp^{m-1/2} \right) \quad (5.25)$$

where l is the iteration index, m is the ξ integration index. Once the magnetic field has been found, the electric field is found from eq. (5.21). The transverse electric and magnetic fields are used to integrate the particle initial velocities. This gives an updated $\vec{J}_{\perp, l}^{m+1/2}$. The iteration loop can be repeated as desired until $\vec{B}_\perp$ and $\vec{J}_\perp^{m+1/2}$ has converged.

⁷At the first time step, no current derivatives are known, so the derivative is assumed to be zero.

5.4 Future Work

SLACtrac is designed to be flexible and extensible. Many different directions are possible, depending on what is needed most. One of the most compelling applications of SLACtrac is simulating the difficulty of emittance measurements given a beamline design. If multiple scattering is implemented in SLACtrac, it can show the theoretical best emittance measurements possible given scattering locations. It is also possible to study the effects of various beam distributions on the measurement. This includes correlations of emittance- ξ , emittance- E , and other distribution correlations. It can be used to study noise and error sources in measuring emittance in order to develop a robust measurement.

PlasmaTrace is undergoing revision to implement the Version 2 design. While Version 1 showed that the theory of ion motion developed in chapter 4 is valid under its assumptions, the impact of ion motion on the beam has not been simulated. It is expected that the emittance and beam matching theory developed in chapter 3 will describe how the beam becomes matched to the ion densities predicted by Version 1.

Additionally, a fully self-consistent simulation that relies only on the quasistatic approximation has the potential to reveal how ion motion will affect the PWFA-LC and ILC proposals explored in table 4.2. This may give insight into the emittance growth that may occur in these proposals in particular. Simulation will reveal whether ion motion will create densities above the beam densities, violating the assumptions of the ion motion model. These issues are crucial in developing the PWFA-LC and ILC proposals.

Chapter 6

Beam Profile Measurements

The next few chapters, including this chapter and chapters 7 and 8, lay out a proof of concept for an experimental method for measuring the emittance of a single beam pulse or “shot” at FACET at the SLAC National Accelerator Laboratory. An ideal PWFA experiment should be repeatable in all aspects, in particular with reliable focusing, acceleration, emittance, and beam charge. Experience at FACET has proven otherwise, as various drifts in the accelerator system components and noise sources such as jitter in the laser path have significantly reduced the possibility of collecting stable data (see fig. 6.1). Identifying and addressing these issues remains a significant concern at FACET and potentially FACET II.

In the meantime, a single-shot emittance measurement has the potential to provide insight into PWFA physics in spite of these unstable conditions. The measurement developed in this thesis depends on measuring the beam distribution. This chapter is devoted to describing the physics of the measurement system.

All experimental observations of the beam in one way or another rely on imaging photons through a system of cameras. The beam passes through different media at various locations, and a physical process, such as scintillation, optical transition radiation, or Cherenkov radiation, produces light that is used to image the beam’s transverse distribution or “profile.” Beam emittance measurements don’t depend on the amount of charge, but on the beam moments (eq. (3.2)), so it is only important to recover a normalized charge distribution. It is necessary to understand whether

Figure 6.1: *Electron beam spectra.* Images of the beam have been integrated to show charge density as a function of energy, sorted by efficiency. While demonstrating the potential for high efficiency, the beam spectrum is far from stable and shows large variations in decelerated and accelerated charge. (Figure from Litos *et al.* [19].)

the imaged profile is proportional to the beam distribution, and the geometry of the system so that beam moments are measured properly.

Camera systems are designed to image this surface as an object plane via a lens onto a sensor. The lens is responsible for all optical effects such as light collection, and in some cases may introduce diffraction and bokeh depending on the lens's aperture, focus, and distance from the media. The sensor is responsible for recording the image plane. It is composed of a grid of pixels that bin the photons incident on them, counts them with some efficiency, and returns a number proportional to the number of photons. The sensor can impose some spatial and bin count resolution on the resulting image; the pixels have some physical size, and the photons are converted to a count number that has a finite step size. The sensor also contributes to the error of the physical process through physics such as the quantum nature of its binning and

the readout electronics.

6.1 Light Emission

As electrons pass through media, they may give off light through a variety of processes. Cherenkov radiation, optical transition radiation, and scintillation are used at FACET to observe where electrons pass through air gaps, thin metal foils and scintillating material. Scintillators used at FACET are up to 30 cm wide and have high resolution (≈ 12.5 lp/mm), a combination not possible with Cherenkov or optical transition radiation systems. As a result, only scintillators, specifically Kodak Lanex Fine screens which are 0.6 mm thick, are practical for the spectrometers used to measure single-shot emittance at FACET.

Scintillation is the process of converting energy from charged particles incident on a material into light. Kodak Lanex Fine is composed of phosphors of $\text{Gd}_2\text{O}_2\text{S}$ within a urethane binder [56]. GEANT4 simulations show that the energy deposited per electron is constant for electrons with energies from 5 MeV to 1 GeV [56, 57]. Lanex Fine has been shown to have a linear response from $\approx 7.5 \times 10^{-1}$ pC/mm² to > 100 pC/mm² [58]. For the energies of interest at FACET, electrons yield photons with an efficiency of $\eta_0 = (1.75 \pm 0.15) \times 10^9$ photons/sr/pC [58]. The total number of photons emitted $dN_{\gamma, \text{total}}$ for a differential area dA_0 , for $\theta \approx 0$, into the solid angle $d\Omega$ is:

$$N_{\gamma,0} = \eta_0 \sigma_{\text{beam}} dA_0 d\Omega \quad (6.1)$$

where σ_{beam} is the projected charge density (with units of C/m²) of the beam on the area dA_0 . The angle of incidence of the beam has no noticeable effect on the distribution of emitted photons [59].

The photons generated may not have an exactly Lambertian distribution. Greater surface densities and thinner screens deviate more significantly from Lambertian distributions, but these effects are negligible for angles near the perpendicular plane [59]. In all cases, cameras are aligned perpendicular to the screens they are imaging primarily so that the entire screen will be in focus. This alignment also ensures the photon

distribution will be approximately Lambertian for these small angles [56, 58, 59]:

$$dN_\gamma(\theta) = \cos \theta dN_{\gamma,0} = \eta_0 \sigma_{\text{beam}} \cos \theta dA_0 d\Omega \quad (6.2)$$

Generally, camera triggers are set to capture all of the scintillation light emitted from the Lanex. Noise sources inherent in the scintillation process come from the shot noise of the electron bunch and the conversion efficiency of the scintillator.

6.2 Lens

A lens refocuses light incident on it through refraction. Light is bent and typically imaged to a plane. As refraction is typically dependent on wavelength, chromatic aberration can occur. Fortunately, the emission spectrum of Lanex phosphors has a narrow bandwidth of (546 ± 20) nm [58], so chromatic aberration is not expected to be an issue.

Lenses have finite approximately-circular apertures, which may cause diffraction and limit the amount of light collected. Lenses at FACET have apertures that are not closed down to the point that resolution becomes diffraction-limited. The most significant effect of the aperture is light collection efficiency. Lens aperture is usually given by its f -stop numbers. A lens's f -stop N can be related to its effective aperture radius r and focal length f :

$$r = \frac{f}{2N} \quad (6.3)$$

A lens will also have some magnification, which can be found from the lens maker's formulas:

$$\frac{1}{f} = \frac{1}{o} + \frac{1}{i} \quad (6.4)$$

$$m = -\frac{i}{o} \quad (6.5)$$

$$\Rightarrow f = \frac{mo}{m-1} \quad (6.6)$$

The approximately circular aperture will collect light from an object a distance of o away with a solid angle of:

$$\Omega_{\text{ap}} \approx 4\pi \left(\frac{\pi r^2}{4\pi o^2} \right) = \frac{\pi m^2}{4(m-1)^2 N^2} \quad (6.7)$$

6.3 Sensor

A typical camera system has a grid of pixels that are square. Photons incident on each pixel creates electrons that are trapped in a potential well, with some quantum efficiency QE . The sensor electronics in the system then reads out the number of electrons for each pixel and reports a number representing how full the potential well is. This array of numbers representing the physical pixels makes up an image.

6.3.1 Imaging

Imaging a Lambertian light source with a lens results in an intensity that is independent of the viewing angle θ . The lens images a constant solid angle Ω into a given pixel. As the lens changes angle, the surface area corresponding to the solid angle changes. A square pixel of area l^2 with a given magnification m corresponds to an area A_{px} on the Lanex sheet:

$$A_{\text{px}} = \frac{l^2/m^2}{\cos \theta} \quad (6.8)$$

Assuming that the pixel area is small so that $A_{\text{px}} = dA_0$, the number of photons imaged by the lens onto a pixel can be found by integrating the photon distribution over the aperture:

$$N_{\gamma, \text{px}}(\theta) = \int_{\Omega_{\text{ap}}} \sigma_{\text{beam}} \eta_0 (\cos \theta) \left(\frac{l^2/m^2}{\cos \theta} \right) d\Omega \quad (6.9)$$

As the integral does not depend on the angle, the number of photons does not depend on the angle, and the integral becomes:

$$N_{\gamma,\text{px}} = \sigma_{\text{beam}} \eta_0 \left(\frac{l^2}{m^2} \right) \left(\frac{\pi m^2}{4(m-1)^2 N^2} \right) \quad (6.10)$$

$$\Rightarrow N_{\gamma,\text{px}} = \sigma_{\text{beam}} \eta_0 \left(\frac{\pi l^2}{4(m-1)^2 N^2} \right) \quad (6.11)$$

6.3.2 Converting Photons to Counts

Pixels in a sensor convert photons incident on them into electrons with some quantum efficiency of the number of electrons trapped $N_{e,\text{trap}}$ per incident photons $N_{\gamma,\text{incident}}$:

$$QE \equiv \frac{N_{e,\text{trap}}}{N_{\gamma,\text{incident}}} \quad (6.12)$$

The efficiency is dependent on the wavelength, but for Lanex the emission bandwidth is narrow so the efficiency can be treated as a single value. The electrons are counted up to some well depth. The well depth is determined by the device, and any electrons counted over this number result in a full well number — at this point the sensor is saturated. The number of counts N_{counts} is given by:

$$N_{\text{counts}} = N_{\gamma,\text{px}} QE \left(\frac{2^{N_{\text{bits}}}}{N_{e,\text{well}}} \right) \quad (6.13)$$

where N_{bits} is the bit depth the camera uses, and $N_{e,\text{well}}$ is the number of electrons in a full potential well in a pixel.

6.4 Error and Linearity

Several error sources exist. The beam itself has shot noise, which depends on the beam source and the various processes it goes through as it travels through the linac. The scintillator conversion efficiency represents a stochastic process and has some error, as does the quantum efficiency of the pixels. The room temperature can also

Figure 6.2: *Theoretical camera response for a new scintillator screen.* The theoretical response of a Hamamatsu camera (green) with a 100mm $f/2.8$ lens with $m = -0.043$ magnification. The camera has an unexplained noise floor much higher than the expected 0.13 counts. The theoretical response curve of another Lanex Fine scintillator screen is in blue, and a data point of 5×10^{-4} pC/mm² with 1.81 counts was compared to an independent calibration by S. Corde of 1 count, which agrees fairly well.

affect the scintillator conversion efficiency [58]. The number of electrons trapped in the well can be affected by thermal noise. The sensor electronics also may make errors when measuring the number of trapped electrons. A more subtle issue is that the linearity of Lanex Fine is untested above 100 pC/mm^2 [58], and Lanex saturation will not be as easy to detect as sensor saturation. Care should be taken when evaluating bunches with high charge densities. Screen damage has also been observed, where the Lanex response is reduced in “burned” regions where high charge densities have been incident on the screen for many shots.

Practically speaking, all of these errors have been dwarfed by unexplained camera behavior. Cameras are affected by various types of radiation generated in the linac, including bremsstrahlung when the beam passes through any material, X-rays generated in magnetic fields, etc. Camera images must be carefully checked for unexplained noise and signals. In some cases, such as X-ray noise, filtering or compensation for noise is well-motivated and should be performed if the images are to be used.

6.5 Implications for Emittance Measurements

Measurements for emittance rely on beam distribution shape, and notably exclude the distribution amplitude. The distribution shape can be directly measured if the imaging system response is linear. All processes leading to the final count on the image appear to be linear, with the exception of saturation and Lanex damage.

Camera saturation can be checked by noting when the camera signal recorded is at the upper limit of its bit depth. Fortunately, this is typically easy to check, as saturation leads to islands in the image where pixels have exactly the same value. Lanex saturation and damage must be checked visually. Lanex saturation can be observed by finding unexpectedly-uniform regions in images. Witness charge, however, is significantly below Lanex saturation, so this is not an issue.

Observing Lanex damage is more subtle. The location and shape of damaged regions may vary depending on where the beam was steered and how long the beam was left in a particular configuration. The boundaries of damaged regions are not necessarily sharp either. As a consequence, damage must be observed by comparing

a selection of shots and observing if there are constant regions with depressed signal while the areas exposed by the beam vary. This is particularly an issue when measuring high-charge drive beams. Fortunately, damage has not been an issue when measuring witness beams, as witness beams are imaged with separate regions of Lanex from the drive beams, and their charge density tends to be low.

Chapter 7

Spectrometer Measurements

The energy spectrometer at FACET is a key component of the single shot emittance measurements performed in this thesis. The equations of motion through the spectrometer will be used in the emittance calculations. The spectrometer consists of a pair of quadrupoles which focus the beam, a dipole which disperses the beam by its energy content, and a beam profile measurement screen (see fig. 7.1). The equations of motion can be described in terms of the quadrupole and dipole field strengths. The spectrometer can also be calibrated experimentally as a check on the theory and to determine the most accurate description of the measurement of the beam spectrum.

7.1 Beam Trajectories in Components

In order to convert the profile measurement axis y into an energy axis E , we must find $E(y)$. The equations of motion must be solved in each of the spectrometer components. The thin-lens approximations for dipoles and quadrupoles are valid for the FACET beamline, and can be used to simplify equations.

7.1.1 Dipole Trajectories

A strong dipole can be approximated by a uniform magnetic field with $\vec{B} = B_0\hat{x}$. The bend angle r and angle θ shown in fig. 7.2 can be calculated from the Lorentz force

Figure 7.1: *Spectrometer diagram.* The PWFA experimental chamber, focusing quadrupoles, spectrometer magnet, and profile measurement scintillator, with distances between elements labeled. An ideal profile measurement with quadrupoles imaging the accelerated charge energy is shown. The beryllium foil and aluminum window are shown in grey. (Not to scale.)

Figure 7.2: *Charged particle trajectory in a rectangular dipole magnet.* A magnet with length L and a field into the page within the gray area exerts a Lorentz force on an electron. In the limit of high energies, particles travel straight ahead, while low energy particles travel in an arc of radius r and angle θ . The scintillator screen location is shown as a dashed line. (Not to scale.)

for a particle traveling approximately perpendicular to the entrance of the magnet:

$$\frac{d\vec{P}}{dt} = e \left(\vec{v} \times \vec{B} \right) \quad (7.1)$$

The force is toward the center of a circle with radius r and shows radial acceleration. This particular solution is called synchrotron motion. If synchrotron radiation is disregarded by assuming that the bend angle is small, the equation of motion can be written:

$$\gamma m \frac{dv}{dt} = evB \quad (7.2)$$

The solution is circular motion with $\frac{dv}{dt} = v^2/r$, and r can be solved for $v \approx c$ for relativistic particles:

$$r = \frac{\gamma m v^2}{evB} \quad (7.3)$$

$$r \approx \frac{E}{ecB} \quad (7.4)$$

where $E = \gamma m_e c^2$ is the energy of the particle. A rectangular magnet of length L_m bends a particle through an angle θ :

$$\sin \theta \approx \frac{L_m}{r} \quad (7.5)$$

At the exit of the magnet the particle has an angle θ with respect to its original trajectory and a y location given by:

$$y_m = r - r \cos \theta \quad (7.6)$$

Once the particle has exited the magnetic field, it travels a drift distance L_d to the scintillator screen. The location on the scintillator screen $y_s = L_d \tan \theta + y_m$ is:

$$y_s = L_d \tan \theta + (r - r \cos \theta) \quad (7.7)$$

Figure 7.3: *Expected energy/distance calibration curve.*

Solving for $y_s(r, L_m, L_d)$:

$$y_s = L_d \frac{L_m}{\sqrt{r^2 - L_m^2}} + r - \sqrt{r^2 - L_m^2} \quad (7.8)$$

The resulting cubic equation can be solved analytically or numerically (see fig. 7.3, [60]). This is neither necessary nor ideal for most experimental measurements. The thin-lens approximation is valid, particularly for high energies. This simplifies the equations and is sufficient for broad spectrum measurements, e.g. Litos *et al.* [19]. For emittance measurements, the spectrum analyzed is fairly narrow and a local approximation is sufficient. However, eq. (7.7) is useful for showing the thin-lens approximation is valid for the FACET spectrometer.

Thin-Lens Approximation

The thin-lens approximation can simplify the bend equations of motion. The bend angle is on the order of 1 mrad to 10 mrad for a ~ 20 GeV beam, so expanding eq. (7.7)

gives:

$$y_s = L_d \theta + \mathcal{O}(\theta^2) \quad (7.9)$$

The thin-lens approximation for a dipole is valid for all energies of interest at FACET. If there is a small initial angle entering the magnet, the angles add. For small angles to first order, the length of the trajectory within the dipole does not vary. Small offsets will then add to the offset produced by the angles. If fringe fields are ignored as well, and eq. (7.5) is approximated as $\theta = L_m/r$, the result is:

$$y_s \approx y_0 + y'_0 L_m + L_d (y'_0 + \theta) \quad (7.10)$$

$$y_s \approx y_0 + y'_0 L_m + L_d \left(y'_0 + \frac{L_m}{r} \right) \quad (7.11)$$

7.1.2 Quadrupole Trajectories

Quadrupoles have fields that are focusing in one plane and defocusing in the other, as they must satisfy Maxwell's equations. Assuming $\vec{v} \approx c\hat{z}$, the field of a quadrupole that is focusing¹ in x and the Lorentz force (eq. (7.1)) are:

$$\vec{B} = B_0 (-y\hat{x} + x\hat{y}) \quad (7.12)$$

$$\begin{aligned} \gamma m \frac{d^2 x}{dt^2} &\approx -ecB_0 x \\ \gamma m \frac{d^2 y}{dt^2} &\approx ecB_0 y \end{aligned} \quad (7.13)$$

By examination, $\vec{B}$ satisfies Maxwell's equations $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. This can be simplified by introducing primed derivatives $q' = \frac{dq}{ds}$ and $q'' = \frac{d^2 q}{ds^2}$ and using the change of coordinates $s = ct$ so $ds = c dt$:

$$\left. \begin{aligned} x'' &\approx -kx \\ y'' &\approx ky \end{aligned} \right\} \quad \text{where } k \equiv \frac{ecB_0}{\gamma mc^2} = \frac{ecB_0}{E} \quad (7.14)$$

¹A quadrupole that is focusing in x is defocusing in y for $B_0 > 0$, and vice versa for $B_0 < 0$

The solutions can be written:

$$x(s) = x_0 \cos(\sqrt{k}s) + \frac{x'_0}{\sqrt{k}} \sin(\sqrt{k}s) \quad (7.15)$$

$$y(s) = y_0 \cosh(\sqrt{k}s) + \frac{y'_0}{\sqrt{k}} \sinh(\sqrt{k}s) \quad (7.16)$$

The thin-lens approximation assumes that $\cos(\sqrt{k}s) \approx 1$ and $\sin(\sqrt{k}s) \approx \sqrt{k}s$. The equations immediately after the quadrupole of length L can be written:

$$x(L) = x_0 + x'_0 L \quad (7.17)$$

$$x'(L) = -x_0 k L + x'_0 \quad (7.18)$$

$$y(L) = y_0 + y'_0 L \quad (7.19)$$

$$y'(L) = y_0 k L + y'_0 \quad (7.20)$$

The focal length of the quadrupole is:

$$f = \frac{1}{kL} = \frac{E}{ecB_0 L} \quad (7.21)$$

7.2 Broadband Energy Calibration²

In a private communication, S. Corde and E. Adli develop a calibration method for the energy axis for spectrometer profile measurements at FACET. The ideal beamline is shown in fig. 7.1. In practice, small angles and offsets may affect the final position $y(E)$ and must be accounted for. This is done by first setting a coordinate system where the trajectory of a particle coming out of the plasma is the x -axis. Then the trajectory of a particle with arbitrary energy E can be calculated.

Corde and Adli make several approximations and assumptions. They use the thin-lens approximation for quadrupoles and dipoles. If the beam is offset in a quadrupole, a small deflection can result. They also assume the quadrupoles and dipole are only

²This section follows the method outlined in a private communication with S. Corde and E. Adli, with minor adjustments noted.

offset from the beam trajectory and not tilted, which can be shown to be a good approximation.

The first quadrupole in the spectrometer is a focusing quadrupole in x , with length L_1 and magnetic field $B_0 = B_1 > 0$. Corde and Adli note that an offset Δ_1 in the vertical position of the first quadrupole produces a deflection. This can be understood by seeing that $y_0 = \Delta_1$ and $y'_0 = 0$ in eqs. (7.19) and (7.20):

$$y(L_1) = 0 \quad (7.22)$$

$$y'(L_1) = \Delta_1 k_1 L_1 \quad (7.23)$$

This shows the beam is deflected with an angle of Δ_1/f_1 , which leads to an offset and angle entering the second quadrupole:

$$y_{QS2} = \frac{\Delta_1}{f_1} D_2 \quad (7.24)$$

$$y'_{QS2} = \frac{\Delta_1}{f_1} \quad (7.25)$$

The second quadrupole is defocusing in x , with length L_2 and magnetic field $B_0 = B_2 < 0$. Corde and Adli also assume an offset Δ_2 in the second quadrupole, so that the total offset is $(\Delta_2 - y_{QS2})$. The coordinate must also be transformed back to the reference trajectory after finding the angle:

$$y(L_2) = y_{QS2} + y'_{QS2} L_2 \quad (7.26)$$

$$y'(L_2) = y'_{QS2} - k_1 (\Delta_2 - y_{QS2}) L_2 \quad (7.27)$$

$$y'(L_2) = \frac{\Delta_1}{f_1} + \frac{(\Delta_2 - y_{QS2})}{f_2} \quad (7.28)$$

The $y'_{QS2} L_2$ term is a correction to Corde and Adli. There is an angle y'_B and offset

y_B at the entrance of the bend of:

$$y_B = y_{QS2} + y'_{QS2}L_2 + \left[y'_{QS2} + \frac{(\Delta_2 - y_{QS2})}{f_2} \right] D_3 \quad (7.29)$$

$$y_B = \frac{\Delta_1}{f_1} (D_2 + L_2 + D_3) + \frac{(\Delta_2 - y_{QS2})}{f_2} D_3 \quad (7.30)$$

$$y'_B = \frac{\Delta_1}{f_1} + \frac{(\Delta_2 - y_{QS2})}{f_2} \quad (7.31)$$

The thin-lens approximations of eqs. (7.10) and (7.11) can be used to calculate the offset at the screen after passing through the dipole magnet with length L_m :

$$y_{\text{detector}} = y_B + y'_B (L_m + D_4) + \frac{L_m D_4}{r} \quad (7.32)$$

Technically, ignoring the lengths of the quadrupoles and the dipole as Corde and Adli do is incorrect. However, at this point, we can recognize that we can absorb L_2 into D_3 and L_m into D_4 . In order to keep a reference to the absorbed lengths, let $L_2 + D_3 = \bar{D}_3$ and $L_m + D_4 = \bar{D}_4$. The equation for y_{detector} becomes:

$$\begin{aligned} y_{\text{detector}} &= \frac{\Delta_1 (D_2 + \bar{D}_3)}{f_1} + \frac{\Delta_2 D_3}{f_2} - \frac{\Delta_1 D_2 D_3}{f_1 f_2} \\ &\quad + \frac{\Delta_1 \bar{D}_4}{f_1} + \frac{\Delta_2 \bar{D}_4}{f_2} - \frac{\Delta_1 D_2 \bar{D}_4}{f_1 f_2} + \frac{L_m D_4}{r} \\ &= \frac{\Delta_1 (D_2 + \bar{D}_3 + \bar{D}_4)}{f_1} + \frac{\Delta_2 (D_3 + \bar{D}_4)}{f_2} - \frac{\Delta_1 D_2 (D_3 + \bar{D}_4)}{f_1 f_2} + \frac{L_m D_4}{r} \end{aligned} \quad (7.33)$$

It is unclear whether or not this is what was intended by Corde and Adli. However, it reduces to the same final form that they used to find an energy axis calibration.

Corde and Adli point out that $1/f$ and $1/r$ are proportional to B/E :

$$\frac{1}{f} = kL = \frac{ecB_0 L}{E} \quad (7.34)$$

$$\frac{1}{r} = \frac{ecB}{E} \quad (7.35)$$

This is desirable from an experimental standpoint, as the fields of the quadrupoles $B_{i,0}$ are chosen to image a particular energy E_0 at the profile measurement screen. Selecting a new energy by scaling the magnetic fields with $x = E_{QS}/E_0$ ensures that particles with $E = E_{QS}$ have quadrupole focal lengths that are the same as the focal lengths tuned for $E = E_0$:

$$\left. \frac{1}{f} \right|_{E_{QS}} = \frac{ecB_{\text{new}}}{E_{QS}} = \frac{ecB_0 (E_{QS}/E_0)}{E_{QS}} = \frac{ecB_0}{E_0} = \left. \frac{1}{f} \right|_{E_0} \quad (7.36)$$

The quadrupole field strengths B_1 and B_2 are:

$$\begin{aligned} B_i(E_{QS}) &= B_{i,0} \frac{E_{QS}}{E_0} \quad \text{where } B_{i,0} \equiv B_i(E_0) \\ &= B_{i,0}x \quad \text{where } x \equiv \frac{E_{QS}}{E_0} \end{aligned} \quad (7.37)$$

Scaling the fields in this manner converts eq. (7.33) into:

$$y_{\text{detector}} = \theta D_4 + ax^2 + bx + c \quad (7.38)$$

where a and b are unknown because Δ_1 and Δ_2 are unknown, and c is an arbitrary offset of the detector coordinate system.

This can be leveraged to calibrate the spectrometer in several ways. Corde and Adli chose to turn the dipole off while calibrating. They then kept the FACET beam with a constant central energy of $E_0 = 20.35 \text{ GeV}$. By scaling the fields by x , they were able to observe the location y_{detector} of the beam via profile measurement. They were then able to modify eq. (7.38), as $\theta = 0$ when the dipole is off:

$$y_{\text{detector}} = ax^2 + bx + c \quad (7.39)$$

By fitting to the data, they recovered values for a , b , and c for quadrupoles.

The angle θ can be calibrated by turning off the quadrupoles and observing the amount Δy that the dipole magnet deflects a beam of energy $E_0 = 20.35 \text{ GeV}$ when

it is turned on:

$$\theta_0 = \frac{\Delta y}{D_4} \quad (7.40)$$

The value for D_4 is well-known from precise measurements performed by the SLAC metrology team.

When performing experiments, the dipole magnetic field is typically not changed. The bend angle for a particle with energy E is:

$$\theta(E) = \theta_0 \left(\frac{E_0}{E} \right) \quad (7.41)$$

The quadrupoles may be scaled to the energy E_{QS} , and the focal lengths for particles with energy E are:

$$\frac{1}{f_i} = \frac{ec (B_{i,0} E_{QS} / E_0) L_i}{E} = \frac{1}{f_{i,0}} \left(\frac{E_{QS}}{E} \right) \quad (7.42)$$

Equation (7.38) then becomes:

$$y_{\text{detector}}(E) = \theta_0 D_4 \left(\frac{E_0}{E} \right) + a \left(\frac{E_{QS}}{E} \right)^2 + b \left(\frac{E_{QS}}{E} \right) + c \quad (7.43)$$

We can invert to give $E(y_{\text{detector}})$:

$$E = -\frac{2\bar{a}}{\bar{b} \pm \sqrt{\bar{b}^2 - 4\bar{a}(c - y_{\text{detector}})}} \quad (7.44)$$

where

$$\bar{a} = a (E_{QS})^2$$

$$\bar{b} = b E_{QS} + \theta_0 D_4 E_0$$

7.3 Narrowband Energy Calibration

While this calibration is valid for a wide energy range under the thin-lens approximation, emittance measurement is typically concerned with a narrow energy/ y range. The calibration performed by Corde and Adli shows that the a and b terms are small

compared with the $\theta_0 D_4 (E_0/E)$ term. Additionally, this calibration must be performed frequently, as different linac tunings may result in different exit trajectories from the PWFA chamber in fig. 7.1. This trajectory determines the offsets Δ_1 and Δ_2 .

The a and b terms can be removed by making a few approximations and recalibrating c . Equation (7.43) can be rearranged:

$$y_{\text{detector}}(E) = (\theta_0 D_4 E_0 + b E_{QS}) \left(\frac{1}{E} \right) + a \left(\frac{E_{QS}}{E} \right)^2 + c \quad (7.45)$$

The simplest approximation is to note that Corde and Adli showed that $b \ll \theta_0 D_4$, and $E_0 = 20.35$ the same order of magnitude as E_{QS} , as most emittance bunches measured have been accelerated by 1 GeV to 10 GeV, so the b term can be dropped. The a term can be dropped with the same argument. The new equation is:

$$y_{\text{detector}}(E) = \frac{\theta_0 D_4 E_0}{E} + c \quad (7.46)$$

Rather than use the offset c from previous calibrations, a separate calibration can be obtained for individual shots. The quadrupoles are scaled to image the witness bunch energy at the profile monitor. The witness bunch is at a waist at the object plane of the spectrometer quadrupoles. It is shown in appendix C that the energy imaged to the profile monitor will have approximately the narrowest spot size. Finding the minimum width of the witness bunch on the spectrometer is the location c of the energy the quadrupoles are scaled for. This calibration method has the advantage of being much faster, simpler, and is shot-by-shot rather than statistical. If the beam is sufficiently stable, the calibration for c can be averaged over many shots in order to improve the calibration.

7.4 Resolution

The spectrometer resolution places lower limits on the emittance measurements. If the spectrometer causes beam emittance growth, it may not be possible to reconstruct the

initial emittance of the beam entering the spectrometer. Emittance measurement is performed by taking bins or slices in the energy/ y axis of the spectrometer profile and measuring profile widths of a beam. The resolution in energy/ y and x will determine the smallest possible energy slice in energy and profile width that can be measured.

There are three factors that can affect and limit resolution: effects on the beam, for instance when the beam passes through a foil and is scattered; optics effects such as the spot size achievable at the screen; and detector effects such as the screen's response function and camera resolution. In order to understand the final measurement, each factor's nature and magnitude are determined. The spectrometer resolution is given as the FWHM of the distribution of a single-energy beam.

7.4.1 Beam Effects: Multiple Scattering

If the beam is changed itself, independently of the optics functions while traveling through the linac, the emittance measured can change. Most mechanisms introduce random angle changes which affect beam divergence. Of particular interest at FACET is multiple scattering in foils, which is a significant source of beam emittance growth. Other phenomena that could contribute to emittance growth include coherent and incoherent synchrotron radiation, but they have not been experimentally observed at FACET and are not covered in this thesis.

The main source of multiple scattering occurs as the beam travels through flat pieces of metal, called “foils” or “windows,” that separate sections of the beamline with different pressures or may be used for profile measurement. Charged particles Coulomb scatter off the metal atoms' nuclei as they pass through a foil; multiple scattering occurs when particles are likely to scatter several times within the foil. The degree to which beam scattering affects emittance depends on the beam distribution at a foil, the atomic composition of the foil, and the foil thickness. Scattering adds random angles to the beam electrons, spoiling and thus increasing the emittance. The R.M.S. scattering angle is given as [35, 61–63]:

$$\theta_0 = \frac{13.6 \text{ MeV}}{\beta c p} z \sqrt{x/X_0} [1 + 0.038 \ln(x/X_0)] \quad (7.47)$$

where p is the momentum, βc is the velocity, z is the charge, and x/X_0 is the material thickness in radiation lengths. This is accurate to within 11% for the R.M.S. of Gaussian fits to the central 98% of the scattering distribution, provided the number of radiation lengths X/X_0 is between 10^{-3} and 100 [63].

7.4.2 Emittance and Courant-Snyder Effects

Reid [64] derives expressions for emittance growth and changes in β and α of a beam that undergoes multiple scattering upon passing through a foil. The effects on a waist downstream of the foil are considered. While the principles apply, at FACET the waist is on the upstream side of the foil instead of downstream. The emittance measurement recovers the apparent emittance ϵ and Courant-Snyder parameters β and α assuming there is no foil. It is of interest to show how these apparent values relate to the initial values when a foil is present.

Effect on Emittance

Since the scattering angle of individual collisions is stochastic and can be considered to be random, the R.M.S. scattering angle adds in quadrature with the existing beam divergence $\langle x' \rangle^2$. The beam divergence can be written:

$$x'_{\text{exit}} = x'_0 + x'_{\text{scatter}} \quad (7.48)$$

$$\begin{aligned} \langle x'^2 \rangle &= \langle x'^2_0 + 2x'_0 x'_{\text{scatter}} + x'^2_{\text{scatter}} \rangle \\ &= \langle x'^2_0 \rangle + \langle x'^2_{\text{scatter}} \rangle \\ &= \langle x'^2_0 \rangle + \theta_0^2 \end{aligned} \quad (7.49)$$

This agrees with Reid [64]. The initial beam divergence is given by $\langle x'^2_0 \rangle = \epsilon\gamma = \epsilon(1 + \alpha^2)/\beta$ (eq. (3.5)). Then:

$$\langle x'^2 \rangle = \epsilon_0 \frac{1 + \alpha^2}{\beta} + \theta_0^2 \quad (7.50)$$

The RMS spot size does not change significantly after passing through the foil due to small scattering angles and the relativistically short distance the electrons travel in the foil. The correlation between location and angle, $\langle xx' \rangle$, does not change:

$$\langle xx' \rangle = \langle x (x'_0 + x'_{\text{scatter}}) \rangle = \langle xx'_0 \rangle + \langle xx'_{\text{scatter}} \rangle = \langle xx'_0 \rangle \quad (7.51)$$

With eqs. (3.2) to (3.4) and (7.50) increased emittance then becomes:

$$\begin{aligned} \epsilon^2 &= (\epsilon_0 \beta_0) \left(\epsilon_0 \frac{1 + \alpha_0^2}{\beta_0} + \theta_0^2 \right) - \epsilon_0^2 \alpha_0^2 \\ &= \epsilon_0^2 + \epsilon_0 \beta_0 \theta_0^2 \\ \epsilon &= \epsilon_0 \sqrt{1 + \frac{\beta_0}{\epsilon_0} \theta_0^2} \end{aligned} \quad (7.52)$$

This new value for emittance also agrees with Reid [64].

Emittance growth due to multiple scattering is difficult to adjust for when measuring final emittance. Inverting Eq. 7.52, if the scattering contribution θ_0^2 and β_0 are known, becomes:

$$\epsilon_0 = \sqrt{\epsilon^2 + \frac{\beta_0^2 \theta_0^4}{4}} - \frac{\beta_0 \theta_0^2}{2} \quad (7.53)$$

It will be numerically difficult to evaluate for initial emittance ϵ_0 when it is on the order of scattering contributions or smaller, and any deviation from the theoretical multiple scattering distribution can introduce errors as well. Eq. 7.53 is only useful when:

$$\epsilon_0 \gg \frac{\beta_0 \theta_0^2}{2} \quad (7.54)$$

Effect on Courant-Snyder Parameters

Reid [64] shows that, as the beam moment $\langle x^2 \rangle$ is unchanged at the foil, eq. (3.3)

can be used to determine the new beta function:

$$\beta = \frac{\epsilon_0 \beta_0}{\epsilon} \quad (7.55)$$

$$\beta^2 = \frac{\beta_0^2}{1 + \frac{\beta_0}{\epsilon_0} \theta_0^2} \quad (7.56)$$

$$\beta = \frac{\beta_0}{\sqrt{1 + \frac{\beta_0}{\epsilon_0} \theta_0^2}} \quad (7.57)$$

The alpha function (eq. (3.4)) does not change either, since $\langle xx' \rangle$ remains unchanged [64]:

$$\alpha = \frac{\epsilon_0 \alpha_0}{\epsilon} = \frac{\alpha_0}{\sqrt{1 + \frac{\beta_0}{\epsilon_0} \theta_0^2}} \quad (7.58)$$

The waist location and minimum can be found by using the equations for how the Courant-Snyder parameters β , α , and γ behave in a drift [45]:

$$\beta(s) = \beta^* + \frac{(s - s^*)^2}{\beta^*} \quad (7.59)$$

$$\alpha(s) = -\gamma^* (s - s^*) \quad (7.60)$$

$$(7.61)$$

$$\gamma(s) = \gamma^* \quad (7.62)$$

where the starred variables are the values at $s = s^*$, which is called the waist location as this is the location of the smallest $\beta = \beta^*$ value. These equations can be solved for distance $\Delta s_{\text{no foil}}$ the waist is from the foil location and the minimum beta function β^* :

$$\Delta s_{\text{no foil}} = \frac{\beta_0 \alpha_0}{1 + \alpha_0^2} \quad (7.63)$$

$$\beta_{\text{no foil}}^* = \frac{\beta_0}{1 + \alpha_0^2} \quad (7.64)$$

The new waist (virtual if before the foil, real if after) is shifted closer to the foil:

$$\Delta s_{\text{foil}} = \frac{\beta_0 \alpha_0}{1 + \alpha_0^2 + \frac{\beta_0}{\epsilon_0} \theta_0^2} \quad (7.65)$$

The new minimum beta function, for small scattering, is:

$$\beta_{\text{foil}} = \frac{\beta_0}{1 + \alpha^2} - \frac{\beta_0}{2} \frac{1 - \alpha^2}{(1 + \alpha^2)^2} \left(\frac{\beta_0}{\epsilon_0} \theta_0^2 \right) + \mathcal{O} \left(\left[\frac{\beta_0}{\epsilon_0} \theta_0^2 \right]^2 \right) \quad (7.66)$$

The minimum spot size (virtual if the waist is before the foil, real if after) is larger, given by:

$$\langle x^2 \rangle_{\text{foil}} = \frac{\epsilon \beta}{1 + \alpha^2} = \frac{\epsilon_0 \beta_0}{1 + \alpha_0^2 \left/ \left(1 + \frac{\beta_0}{\epsilon_0} \theta_0^2 \right) \right.} \quad (7.67)$$

At this point, the scattering limit of eq. (7.54) can be included. If this condition is met, the waist shift becomes small, as does the minimum spot size change. Equation (7.54) is then a necessary and sufficient condition to ensure that scattering does not limit resolution or significantly change the apparent Courant-Snyder parameters.

7.4.3 Optics Effects

The beam spot size at the detector is one of the main limits of the spectrometer's ability to resolve different beam energies. The spectrometer's quadrupoles focus the beam to different sizes, depending on their settings and the beam's emittance and parameters. Since the quadrupoles have varying focusing strength with different beam energies, the resolution will depend on the optics functions at different energies.

Configuring the Optics

Optimizing the optical transfer system from the object plane to the appropriate measurement screen is necessary to be sure charge distribution and energy correlate as closely as possible. The betatron matrix transfer equations describe the beam optics

of the system at a given energy (assuming no cross terms between x and y planes):

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 \\ 0 & 0 & R_{33} & R_{34} \\ 0 & 0 & R_{43} & R_{44} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix} \quad (7.68)$$

The final spot size at the detector system, close to the design energy so that higher-order terms are negligible, is given by:

$$\langle x^2 \rangle = R_{11}^2 \langle x_0^2 \rangle + 2R_{11}R_{12} \langle x_0 x'_0 \rangle + R_{12}^2 \langle x_0'^2 \rangle \quad (7.69)$$

$$\langle y^2 \rangle = R_{33}^2 \langle y_0^2 \rangle + 2R_{33}R_{34} \langle y_0 y'_0 \rangle + R_{34}^2 \langle y_0'^2 \rangle \quad (7.70)$$

The y transfer equations should be optimized so that the betatron motion competes as little as possible with the dispersive motion due to the beam's energy:

$$y = R_{33}y_0 + R_{34}y'_0 + y(E) \quad (7.71)$$

The exact optimization chosen will depend on beam conditions at the object plane, per Eq. 7.70 and Eq. 7.71. In particular, PWFA experiments may have bunches that experience plasma gradients, which may lead to large steering forces. These forces can introduce large angles $\langle y'_0 \rangle$. In this case, setting $R_{34} = 0$ is necessary so that betatron motion does not dominate the beam position in y . In this scenario, Eq. 7.70 is not considered while optimizing, but can be used to determine the beam optics effect on spectrometer resolution, given an understanding of the beam's spot size $\langle y_0^2 \rangle$, divergence $\langle y_0'^2 \rangle$, and correlation $\langle y_0 y'_0 \rangle$ at the imaging plane.

The horizontal spot size $\langle x^2 \rangle$ should be minimized in order to increase the electron density at the screen. Higher electron densities mean larger and more precise measurements by detector systems. A typical PWFA assumption is that the beam exiting the plasma has a waist close to the plasma exit. Minimizing the spot size, then, means the correlation term $\langle x_0 x'_0 \rangle = 0$ and that the divergence $\langle x_0'^2 \rangle$ is greater than the spot size $\langle x_0^2 \rangle$. A good choice to minimizing the spot size is to set $R_{12} = 0$

provided there is no understanding of the behavior of R_{11} and R_{12} given the constraint of $R_{34} = 0$, and the beam divergence and spot size.

Resolution Limits

The conventional definition for electron-beam spectrometer resolution is the FWHM of the spot size created by a beam with no energy spread. For a Gaussian profile, the FWHM is given by $2\sigma\sqrt{2\ln 2}$:

$$Ae^{-\frac{x^2}{2\sigma^2}} = \frac{A}{2} \quad (7.72)$$

$$\Rightarrow x^2 = 2\sigma^2 \ln 2 \quad (7.73)$$

$$\Rightarrow 2x = 2\sigma\sqrt{2\ln 2} \quad (7.74)$$

The width of a bunch $\langle y^2 \rangle$ at the detector is easily understood by a combination of the emittance and beta function, or alternatively the transfer matrices and beam moments:

$$\langle y^2 \rangle = \sqrt{\epsilon_y \beta_y} \quad (7.75)$$

$$\langle y^2 \rangle = R_{33}^2 \langle y_0^2 \rangle + 2R_{33}R_{34} \langle y_0 y'_0 \rangle + R_{34}^2 \langle y'^2_0 \rangle \quad (7.76)$$

Assuming an initial Gaussian beam with known emittance and beta function, or alternatively beam moments $\langle x_0^2 \rangle$, $\langle x_0 x'_0 \rangle$, and $\langle x'^2_0 \rangle$, the final spot size can be determined. Once the spot size is determined, eq. (7.74) gives the y -resolution in terms of Δy . An offset Δy can be converted to ΔE by taking its partial derivative with respect to energy:

$$\Delta E = \left(\frac{\partial y}{\partial E} \right)^{-1} \Delta y \quad (7.77)$$

For the FACET beamline with nominal settings, the resolution will vary as a function of energy and emittance. This two-dimensional parameter space is represented in fig. 7.4a.

Figure 7.4: *Example of resolution at varying energy and emittance.*

7.5 Impact on Emittance Measurement

All of the beam parameters vary by experiment and frequently even from shot to shot. Anecdotal evidence suggests that experiments at FACET are very sensitive to the alignment of a pre-ionization laser and the beam distribution. As the beamline warms and cools over the day, components responsible for beam steering and shaping cause the beam angle and distribution to change. Similar thermal drifts occur in the laser equipment, as the laser needs to be realigned for power and pointing on the scale of an hour. A resonant vibration of approximately 0.1 Hz was also detected in the optics of the laser transport line, which resulted in alignment jitter of the laser in the plasma when running the experiment at unmatched frequencies³.

As a result, general conclusions about the emittance cannot be drawn, even about shots in the same dataset, for any of these effects. The beam β exiting the PWFA chamber is not well-known, and frequently varies from shot to shot depending on the length the PWFA bubble is sustained, which has been shown to be unstable. As a result of this instability, the distance from the waist to the foils varies by shot. The exit angle of the beam from the PWFA chamber is also not well-known, and

³Experiment frequencies varied at FACET. Due to the rate of drift in other components and the quantity of shots needed for studies, running the experiment at 0.1 Hz was not possible.

Window Material	Location	Thickness	Scattering (θ_0)
Beryllium	After IP	50 μm	5.28 μrad
Aluminum	Near Lanex	5 mm	115 (141) μrad

Table 7.1: *Foils with multiple scattering at FACET.* Description of windows in the FACET beamline that affect emittance measurement

so the offsets of the quadrupoles is not known. The emittance varies significantly from experiment to experiment, and shot to shot, so the effect of the foils must be reverse-engineered from the emittance measurement. As the PWFA experiments do not appear to be stable, it is not surprising that the emittance of the beam is not stable. Finally, the quadrupole focal lengths and the dipole strength are intentionally scanned in order to observe charge at different energies, so resolution and energy calibrations vary from dataset to dataset, and frequently within datasets.

These effects make single-shot calibrations necessary, which are obtained through the application of section 7.3 and eq. (7.46). These calibrations work fairly well, as will be seen in chapter 8. The energy resolution for a typical emittance measurement is also on the order of 100 MeV near the imaging energy, as seen from fig. 7.4a assuming normalized emittance on the order of $\epsilon_{n,x} \approx 150 \text{ mm-mrad}$ and imaging energies on the order of $E_0 \approx 22 \text{ GeV}$. This corresponds to $\Delta\delta \approx 4.5 \times 10^{-3}$, and would correspond to negligible errors in fig. 8.2. A similar analysis for the smaller emittances shown in fig. 8.7 shows minimal errors as well.

The R.M.S. scattering angle can be calculated for the beryllium foil and aluminum window shown in fig. 7.1. There are two foils downstream of the PWFA chamber, shown in fig. 7.1. Their parameters are shown in table 7.1. Generally, the aluminum window does not have a significant impact on measurements. The Lanex screens are placed as close to the aluminum window as possible in order to prevent any scattering from affecting the profile measurements.

Figure 7.5: Multiple Coulomb scattering for aluminum and beryllium.

Chapter 8

Emittance Measurement

As discussed at the beginning of chapter 3, high luminosity increases the rate of events in colliders, which ultimately makes data collection faster and the cost per event lower. High luminosity in a linear collider is inversely related to the beam emittance, which motivates accelerating low-emittance beams ([35], see eq. (1.2)):

$$\mathcal{L} = f \frac{n_1 n_2}{4\pi \sqrt{\epsilon_x \beta_x \epsilon_y \beta_y}} \quad (8.1)$$

In addition to deriving emittance growth phenomena and simulating it, experimental measurement is needed to demonstrate emittance preservation under plasma wakefield acceleration. As previously mentioned, PWFA at FACET is too unstable to perform a statistical measurement, and so a single-shot non-destructive emittance measurement is needed.

Measuring emittance in a single shot is only possible if all of the beam moments can be measured or fit for simultaneously (from eq. (3.2)):

$$\epsilon_x \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2} \quad (8.2)$$

It is straightforward to measure the beam profile $\langle x^2 \rangle$ by passing the beam through a scintillator and observing the profile (chapter 6). A fit or regression of the beam

widths as a function of energy can recover the emittance ϵ and Courant-Snyder parameters β and α .

Some techniques, such as the pepper pot method [65] and analysis of undulator radiation [65, 66], have been proposed to measure spot size $\langle x^2 \rangle$ and divergence $\langle x'^2 \rangle$ simultaneously, but they all are destructive, and in the case of undulator radiation measurement require a lot of space. None of these methods are suitable for PWFA experiments at FACET.

8.1 Three-Screen Measurement

The most attractive non-destructive methods involve passing the beam through scintillator screens which have negligible multiple scattering. One method of emittance measurement involves making at least three separate beam measurements at different locations. This would allow the user to trace the beam evolution as a function of the $s = ct$ coordinate. Placing screens at appropriate locations allows for the recovery of β^* , and by extension the emittance ϵ . The simplest example of this type of measurement would be making three width measurements in a length of the accelerator with no focusing components, known as a drift section, where the beam has been focused to a waist. The beta function $\beta(s)$ (eq. (3.3)) has a simple formula in a drift space (eq. (7.59)):

$$\beta(s) = \beta^* + \frac{(s - s^*)^2}{\beta^*} \quad (8.3)$$

where the waist is at $s = s^*$, and the value at the waist is β^* . Given the definition for β in eq. (3.3), the spot size is $\langle x^2 \rangle = \epsilon_x \beta_x(s)$. Three measurements is enough to determine the three unknowns β_x^* , s^* , and ϵ_x . However, if β^* is large and the distances are small, $\beta(s) \approx \beta^*$. In this case, the spot size becomes:

$$\langle x^2 \rangle \approx \epsilon_x \beta_x^* \quad (8.4)$$

Then it is only possible to determined the product $\epsilon_x \beta_x^*$.

Unfortunately, this is the case at FACET. There are no accessible locations far

enough apart available to make profile measurements to make an accurate emittance measurement. Upstream of the aluminum window, optical transition radiation (OTR) foils would need to be used, as Lanex would quickly become damaged by high charge densities due to small transverse beam sizes. These OTR foils would spoil emittance through multiple scattering. Lanex measurements significantly downstream of the aluminum window would be susceptible to the multiple scattering in the aluminum window. (See section 7.4.1).

8.2 Energy-Dependent Measurement

A single-shot measurement is performed by measuring many different beam energies simultaneously by using the FACET spectrometer system. The spectrometer quadrupole doublet focuses the beam via magnetic fields, and the Lorentz force of magnetic fields is proportional to the product $\vec{v} \times \vec{B}$. This means that quadrupole focusing is energy-dependent. If a beam has a broad enough spectrum, a range of focusing strengths is available in a single beam. By separating the beam in energy with the energy spectrometer, the beam profile width is measured as a function of energy. Assuming the emittance is not a function of energy, a linear least squares minimization can be used to extract ϵ , β , and α from the measurement of the widths $\sigma_x(E)$. A linear least squares minimization assumes a well-understood linear model, which exists in the transfer matrix formalism from section 5.2.1.

8.2.1 Linear Least Squares Minimization Theory¹

In order to use the method of linear least squares minimization, its equations and their definitions must be presented and matched to beam measurements.

Linear least squares minimization [67] is possible for equations of the form:

$$y(x) = \sum_{k=0}^{M-1} a_k X_k(x) \quad (8.5)$$

¹The method of linear least squares by Press *et al.* [67] is summarized here, without modification.

In this equation, y_i are measurements with a corresponding standard deviation σ_i which may be repeated an arbitrary number of times N for a given input x_i . The parameters a_k are found by minimizing the χ^2 value [67]:

$$\chi^2 = \sum_{i=0}^{N-1} \left[\frac{y_i - \sum_{k=0}^{M-1} a_k X_k(x_i)}{\sigma_i} \right]^2 \quad (8.6)$$

Press *et al.* [67] define several intermediate matrices in the process of obtaining a solution. The $N \times M$ “design matrix” $\mathbf{A}$ is defined as:

$$A_{ij} = \frac{X_j(x_i)}{\sigma_i} \quad (8.7)$$

The N -length vector $\mathbf{b}$ is defined as:

$$b_i = \frac{y_i}{\sigma_i} \quad (8.8)$$

Finally, the solution vector $\mathbf{a}$ is defined as the parameters from eq. (8.5):

$$\mathbf{a} = \begin{pmatrix} a_0 \\ \vdots \\ a_{M-1} \end{pmatrix} \quad (8.9)$$

Differentiating χ^2 and finding the minimum leads to the solution for $\mathbf{a}$ [67]:

$$\left(\mathbf{A}^T \cdot \mathbf{A} \right) \cdot \mathbf{a} = \mathbf{A}^T \cdot \mathbf{b} \quad (8.10)$$

Solving for $\mathbf{a}$ gives:

$$\mathbf{a} = \left(\mathbf{A}^T \cdot \mathbf{A} \right)^{-1} \cdot \left(\mathbf{A}^T \cdot \mathbf{b} \right) \quad (8.11)$$

Finally, the covariance matrix $\mathbf{C}$ is given by [67]:

$$\begin{aligned} [\alpha] &\equiv \mathbf{A}^T \cdot \mathbf{A} \\ C_{jk} &\equiv 1 / [\alpha_{jk}] \end{aligned} \quad (8.12)$$

8.2.2 Matching Theory to Measurement

Equations of the form of eq. (8.5) can be derived when considering the spot size $\langle x^2 \rangle$ as a function of energy. For a given energy, the transfer matrix $\mathbf{R}(E)$ of the spectrometer can be calculated. In practice, SLACtrac is used to build this transfer matrix. For the FACET spectrometer, assuming there is minimal coupling to x from coordinates other than x and x' , and applying the definitions for β , α , and γ (eqs. (3.3) to (3.5)), the trajectory equations for x and $\langle x^2 \rangle$ can be written:

$$x = \mathbf{R}_{11}(E)x_0 + \mathbf{R}_{12}(E)x'_0 \quad (8.13)$$

$$\langle x^2 \rangle = (\mathbf{R}_{11}(E))^2 \langle x_0^2 \rangle + 2\mathbf{R}_{11}(E)\mathbf{R}_{12}(E)\langle x_0 x'_0 \rangle + (\mathbf{R}_{12}(E))^2 \langle x_0'^2 \rangle \quad (8.14)$$

$$\begin{aligned} \langle x^2 \rangle = & (\mathbf{R}_{11}(E))^2 (\epsilon_{y,0}\beta_{y,0}) + (2\mathbf{R}_{11}(E)\mathbf{R}_{12}(E)) (-\epsilon_{y,0}\alpha_{y,0}) \\ & + (\mathbf{R}_{12}(E))^2 \left(\epsilon_{y,0} \frac{1 + \alpha_{y,0}^2}{\beta_{y,0}} \right) \end{aligned} \quad (8.15)$$

In this case, for measurements $\langle x^2 \rangle_i$ with errors σ_i at energies E_i , the matrix $\mathbf{A}$ is:

$$\mathbf{A} = \begin{pmatrix} \frac{(\mathbf{R}_{11}(E_1))^2}{\sigma_1} & \frac{(2\mathbf{R}_{11}(E_1)\mathbf{R}_{12}(E_1))}{\sigma_1} & \frac{(\mathbf{R}_{12}(E_1))^2}{\sigma_1} \\ \vdots & \vdots & \vdots \\ \frac{(\mathbf{R}_{11}(E_N))^2}{\sigma_N} & \frac{(2\mathbf{R}_{11}(E_N)\mathbf{R}_{12}(E_N))}{\sigma_N} & \frac{(\mathbf{R}_{12}(E_N))^2}{\sigma_N} \end{pmatrix} \quad (8.16)$$

The vector $\mathbf{b}$ is given by:

$$b_i = \frac{\langle x^2 \rangle_i}{\sigma_i} \quad (8.17)$$

Then, the solution vector can be found via eq. (8.11). The solution vector yields:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \langle x_0^2 \rangle \\ \langle x_0 x'_0 \rangle \\ \langle x_0'^2 \rangle \end{pmatrix} \quad (8.18)$$

The 3×3 covariance matrix $\mathbf{C}$ can be found from eq. (8.12). The emittance can be found directly from its definition (eq. (3.2)):

$$\epsilon_x \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} \quad (8.19)$$

The variance of the emittance σ_ϵ^2 can be found more easily by examining the variance of the square of the emittance $\sigma_{\epsilon^2}^2$:

$$w = a_1 a_2 - a_3 \quad (8.20)$$

Following approximate error propagation, the variance of w , σ_w^2 , is given by:

$$\mathbf{J} = \begin{pmatrix} a_2 & a_1 & -1 \end{pmatrix} \quad (8.21)$$

$$\sigma_w^2 \approx \mathbf{J} \mathbf{C} \mathbf{J}^T \quad (8.22)$$

where the row vector $\mathbf{J}$ is the Jacobian. The standard deviation of emittance $\sigma_\epsilon = \sqrt{\sigma_w}$. This value can be calculated numerically for a given minimization.

8.3 Experimental Data

In order to demonstrate a single shot emittance measurement, a comparison of a quadrupole scan with the single shot method was performed on a single dataset. The comparison is a proof-of-concept for measuring emittance, and shows that a single-shot measurement returns similar values as a statistical measurement for a semi-stable experiment.

Analysis was also performed on a sister experiment to plasma wakefield acceleration called ionization injection PWFA. The accelerated charge is introduced to the plasma wake through another mechanism, typically field ionization from the drive beam. The beams have lower emittance and produce data that is cleaner than other experiments.

8.3.1 Benchmarking Single Shot Emittance Measurement

A comparison between a traditional quadrupole scan measurement and a single shot emittance measurement can be performed on a single dataset if the beam is stable. The analogous comparison should show similar results as a proof-of-concept of the single shot emittance measurement method.

Quadrupole Scan

A traditional quadrupole scan involves refocusing quadrupoles so the transfer matrix $\mathbf{R}$ downstream of a point changes in a predictable manner. As the transfer matrices change, the spot size changes, and can be measured:

$$\langle x^2 \rangle = R_{11}^2 \langle x_0^2 \rangle + 2R_{11}R_{12} \langle x_0 x'_0 \rangle + R_{12}^2 \langle x'^2_0 \rangle \quad (8.23)$$

The initial parameters $\langle x_0^2 \rangle$, $\langle x_0 x'_0 \rangle$, and $\langle x'^2_0 \rangle$ can be determined by the same linear least squares minimization as the single shot method (see section 8.2), and then used to find emittance ϵ_x .

An average image of the 49 shots with the quadrupoles focused at an energy $E = 20.35$ GeV was taken, and gives a minimum at pixel location of 452 pixels. The quadrupoles were varied from -2.9% to 2.0% of the central energy, with 49 shots per step. The 49 images were averaged together, and a strip 2 pixels wide was taken at the location of $E = 20.35$ GeV. The strips were fit with Gaussians to recover the variance $\langle x^2 \rangle$ of the beam. The initial beam parameters were then recovered per eq. (8.23) and a linear least squares fit. This gives a measurement of emittance at 153 mm-mrad, with Courant-Snyder parameters of $\beta = 0.136$ m and $\alpha = -0.194$.

Single Shot Emittance Measurement

Rather than scanning quadrupoles as described above, different energies are selected away from the in-focus energy at the pinch. These energies have their own transfer matrices $R_{ij}(E)$. This requires the ability to transform between energy and location on the Lanex screen.

Figure 8.1: *Gaussian fit to a slice of data.*

Figure 8.2: *Quadrupole data and its linear least squares minimization.*

Figure 8.3: *Single shot emittance measurements for benchmarking.* Measurements carried out for the main bunch, in order to benchmark the single shot emittance technique.

The dataset contains many shots at different quadrupole values. The energy axis was directly calibrated by observing the pinch location as a function of energy². Each shot was then analyzed separately for emittance and Courant-Snyder parameter fit. Each image was sliced into 11 strips, $\pm 0.8\%$ around the in-focus energy. These strips' average energies were calculated from the relative number of counts at different energies/pixel locations, as strips may be several pixels wide. The distribution of each slice was fit with Gaussians to determine their variance. Then, similar to a quad scan, the variance and energy are fit with emittance and Courant-Snyder parameters. The emittance results are compiled into a histogram shown in fig. 8.3. These measurements agree with quadrupole scan emittance measurements.

8.3.2 Accelerated Bunch Measurement

While the single-bunch and two-bunch PWFA experiments exhibited a high degree of instability, ionization injection experiments [39] showed a much more stable profile.

²See appendix C.

Emittance calculations from ionization data represents some of the lowest emittance shots observed at FACET and are useful for showing a proof of concept of this single-shot measurement.

Bunches had energies of approximately 22.8 GeV, which were imaged onto the scintillator screen (fig. 8.4). The images were then sliced vertically, as shown by the red box in fig. 8.4. The energy scale for this dataset was calibrated by finding the average pinch location of the shots. This was done by fitting Gaussians to the profile and finding the y location of the smallest width. This calibration was used for all of the shots in the dataset in this instance.

Then, each individual shot's slices were fit with Gaussians (fig. 8.5). The width of each slice is then a measurement as a function of its energy (fig. 8.6). This provides the data required for a linear least squares fit. The emittance is calculated for 7 shots in fig. 8.7. While the dataset has dozens of shots, only a small subset are suitable for emittance measurement. The small sample size makes analysis challenging at best.

8.4 Discussion of Impact, Technical Difficulties

Before any single shot emittance measurement can be relevant to emittance studies, the beam needs to be stable enough to obtain measurements on a significant fraction of shots in a dataset. This was, unfortunately, not the case at FACET. The minimal number of shots in these datasets is a result of the stability difficulties FACET faced. As mentioned in section 7.5, there are several phenomena that contribute to the difficulty of obtaining a stable measurement. Some experiments at FACET are designed to use a laser to pre-ionize the plasma [22], and this laser was found to have transverse jitter that was resonant with the beam orbit. The jitter amplitude was observed to vary from night to night. The beam transverse distribution was at times unstable and non-Gaussian. The fundamental stability at FACET will need to be improved before any measurement, single shot or otherwise, can be used. On the other hand, a single shot measurement will allow for measurements if stability can only be marginally improved.

Several factors play a large role in the limits of this measurement. In particular,

Figure 8.4: *Ionization injection bunch profile.* The bunch profile is shown, with contours to show its shape. The pinch energy is near 22.8 GeV, and the measurement range is approximately 0.8 GeV, which corresponds to a 3.5 % energy spread. The red box indicates an example energy slice.

Figure 8.5: *Example of a fit to a slice in energy.* A Gaussian is fit to the example slice shown in fig. 8.4. The fit includes a DC background parameter. While the fit recovers the mean μ_x , R.M.S. width σ_x , and background count level C_0 , the only parameter of interest is the width.

Figure 8.6: *Example linear least squares minimization for single-shot measurement.* The slice spot sizes are shown as a function of energy. The parameters recovered by the linear least squares minimization combined with the beamline model are used to reconstruct the expected spot sizes.

Figure 8.7: *Small sample of emittance measurements.* Shots range from approximately 12 mm-mrad to 20 mm-mrad normalized emittance. Note that this includes emittance contributions from the beryllium foil on the order of 10 mm-mrad.

the beryllium foil increases emittance at the level of 10 mm-mrad. This foil could not be removed from the FACET linac as it was necessary due to the configuration of vacuum pumping. This foil prevents the measurement of emittances moderately smaller than 10 mm-mrad. Future experiments such as FACET II will need to take the multiple scattering of foils into account if they are to succeed in measuring sub-mm-mrad emittances.

There are some significant advantages to the technique proposed here. This emittance measurement method has been used by others [39], but the approach of using curve fitting has made error analysis difficult. The proposed application of linear least squares allows for a rigorous error analysis of the measured parameters. This is highly desirable, in particular when measuring small emittances.

On the other hand, there are some potential difficulties. Any stray charge or unexpected focusing can disrupt the Gaussian fitting and lead to inaccurate results. Low emittances will be difficult to resolve and may lead to significant and high errors. Tails in Gaussian distributions are not fully understood, and may be related to emittance growth (see chapter 3 and Raubenheimer *et al.* [3]). These areas all present challenges for this emittance measurement technique that can be addressed as more data is collected.

There are areas of research that should be undertaken to improve and understand this method more fully. A basic assumption of the method is that the entire bunch has the same emittance for every energy bin, and neglects energy-emittance and energy-beta correlations. While the results of chapter 3 suggest that these correlations may be negligible, this has not been proven or demonstrated for PWFA.

This measurement technique remains a proof of concept. Once stability is addressed, many issues can be explored with this measurement. Emittance preservation has yet to be experimentally studied as a function of other PWFA parameters. While efforts to study emittance in the experiments of ionization injection and trojan horse injection have begun, these phenomena have yet to be closely observed. Emittance growth and matching presented in chapter 3 and PWFA models [24] have yet to be fully experimentally checked. The development of this technique presents an opportunity to advance the science of PWFA experiments significantly.

Chapter 9

Final Summary

Two new models for emittance growth and ion motion have been introduced, software has been developed to analyze emittance and ion motion, and a proof-of-concept emittance measurement has been demonstrated. This work is intended to support future analysis and development of PWFA experiments, upgrades to the ILC, and the design of a staged PWFA-LC.

Emittance preservation and beam matching will be important to all of these stages. In experiments at FACET, beam behavior has been consistent with matched beams, and surprisingly insensitive to variations in match conditions. The model put forth in chapter 3 offers an explanation for how this is likely to have happened. The saturation lengths of the ILC and the staged PWFA-LC should be considered in their designs.

Ion motion has the potential to impact the ILC and the PWFA-LC proposals. The model and theory developed in chapter 4 show that the match should be optimized in order to preserve emittance. Also, using a more massive gas species than hydrogen for the plasma source may make emittance growth due to ion motion negligible. It will be important to observe ion motion experimentally in order to measure its impact on acceleration. The formulas for the ion motion wavelength allow for the design of experiments that probe this regime.

Ion simulation software developed in chapter 5 validates the ion motion model in chapter 4. This software has the potential to assist in the proposals for the ILC and staged PWFA-LC by showing where ion motion may present a problem, and where

it will be insignificant.

The emittance measurement developed in chapters 6 to 8 presents a blueprint for an emittance measurement in future experiments. This technique may still be the best option for observing emittance growth at the FACET-II facility under development at SLAC, and be crucial in ensuring emittance preservation in experiments and future facilities.

Plasma wakefield acceleration is a promising technique for linear colliders and future experiments in understanding the fundamental fabric of our universe. The author hopes that the theories, software, and measurements described here may push the frontiers of science of accelerator physics forward and provide some insight where there was none before.

Appendix A

Beta Mismatch in a Plasma Channel

Mismatch in a plasma can be solved analytically. This is not immediately obvious, as the general form of the differential equation that governs beta evolution, where K is a function of any parameter (typically s) and $\beta' = \frac{d\beta}{ds}$ is given by:

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4 \quad (\text{A.1})$$

A.1 General Solution

In particular, a plasma channel can be assumed to have a constant K . Some numerical analysis suggests a solution of the form:

$$\beta(s) = A \cos(\kappa s + \phi) + D \quad (\text{A.2})$$

$$\beta'(s) = -A\kappa \sin(\kappa s + \phi) \quad (\text{A.3})$$

$$\beta''(s) = -A\kappa^2 \cos(\kappa s + \phi) \quad (\text{A.4})$$

Replacing the solution in the differential equation yields:

$$\begin{aligned} \cos^2(\kappa s + \phi) (-2A^2\kappa^2 + 4A^2K) + \sin^2(\kappa s + \phi) (-A^2\kappa^2) \\ + \cos(\kappa s + \phi) (-2A\kappa^2D + 8ADK) + 4D^2K = 4 \end{aligned} \quad (\text{A.5})$$

The only way to remove the dependence on s is to set the coefficients for $\cos^2$ and $\sin^2$ equal, and to set the coefficient for the $\cos$ term equal to zero. Upon some inspection, these conditions both yield the same equation:

$$-2A^2\kappa^2 + 4A^2K = -A^2\kappa \quad (\text{A.6})$$

This describes an oscillation in the value of β that is at twice the frequency of the particle oscillation about the focusing axis. In the matched case, $A = 0$. In the unmatched case where $A \neq 0$:

$$\kappa^2 = 4K \quad (\text{A.7})$$

Assuming some initial conditions $\beta(0) = \beta_0$ and $\beta'(0) = \beta'_0$, applying the conditions to simplify eq. (A.5), and applying $\kappa^2 = 4K$, we are left with three equations and three unknowns (A , ϕ , and D):

$$\beta_0 = A \cos \phi + D \quad (\text{A.8})$$

$$\beta'_0 = -A\kappa \sin \phi \quad (\text{A.9})$$

$$-4A^2K + 4D^2K = 4 \quad (\text{A.10})$$

Since $\beta > 0$, for eq. (A.8) to be greater than zero, $D > 0$. With $D > 0$, eq. (A.10) implies that $|A| < D$. Next, preparing eqs. (A.8) and (A.9) to allow a substitution $\sin^2 \phi + \cos^2 \phi = 1$ gives:

$$\beta'^2_0 + \beta_0 \times (4AK \cos \phi) = 4A^2K \sin^2 \phi + 4A^2K \cos^2 \phi + 4ADK \cos \phi \quad (\text{A.11})$$

This yields a solution for $\cos \phi$:

$$\cos \phi = \frac{\beta_0'^2 - 4A^2K}{4AK(D - \beta_0)} \quad (\text{A.12})$$

It is now possible to obtain $A(D)$ from eq. (A.8) via substitution:

$$\frac{\beta_0'^2 - 4A^2K}{4K(D - \beta_0)} + D = \beta_0 \quad (\text{A.13})$$

Solving yields:

$$A^2 = \frac{\beta_0'^2}{4K} + (D - \beta_0)^2 \quad (\text{A.14})$$

Substituting into eq. (A.10) gives:

$$-\frac{\beta_0'^2}{4} - K(D - \beta_0)^2 + D^2K = 1 \quad (\text{A.15})$$

Solving for D gives:

$$D = \frac{4 + 4K\beta_0'^2 + \beta_0'^2}{8K\beta_0} \quad (\text{A.16})$$

Solving eq. (A.10) for A and plugging in D gives:

$$A = \sqrt{D^2 - \frac{1}{K}} \quad (\text{A.17})$$

$$A = \sqrt{\frac{(4 + 4K\beta_0'^2 + \beta_0'^2)}{64K^2\beta_0^2} - \frac{1}{K}} \quad (\text{A.18})$$

Then, eqs. (A.2), (A.12), (A.16) and (A.18) represent a solution for $\beta(s)$. In all cases, the beta function evolves with a sinusoidal motion.

A.2 Specific Cases

Now that we know the general solution, we can examine several specific cases, including large and small amplitudes.

A.2.1 Large Amplitudes

The maximum amplitude occurs where $\beta'(s) = 0$. Allowing this to be an initial condition such that $\beta'_0 = 0$, and assuming that $\beta_0 \gg 1/\sqrt{k}$ allows us to approximate D (eq. (A.16)) and A (eq. (A.18)):

$$D = \frac{1 + K\beta_0^2}{2K\beta_0} \approx \frac{\beta_0}{2} \quad (\text{A.19})$$

$$A = \beta_0 - D \approx \frac{\beta_0}{2} \quad (\text{A.20})$$

This shows that large amplitude beta functions undergo a sinusoidal oscillation from approximately $\beta = 0$ to their maximum value.

A.2.2 Small Amplitudes

In the small-amplitude limit where $A \ll D$, from eq. (A.10):

$$\frac{1}{K} = D^2 - A^2 \approx D^2 \quad (\text{A.21})$$

As the matched beta is given by $\beta(s) = 1/\sqrt{K} = \beta_m$ (eq. (3.17)), then $D \approx \beta_m$. Therefore, the beta function undergoes a sinusoidal oscillation around β_m . The amplitude of the oscillation is found simply by finding $s_{\max}$ where $\beta'(s_{\max}) = 0$ and $\beta''(s_{\max}) < 0$. The amplitude is then $A = \beta(s_{\max}) - 1/\sqrt{k}$.

Appendix B

Simulation Smoothing

One of the considerations for simulating beam phenomena via macro-particles is that they carry a large amount of concentrated charge in one location. This gives rise to a discontinuities at a larger scale than is physical. In order to intuitively understand this, picture a volume of randomly-distributed electrons with a given density. At a small enough scale, individual electrons are resolved, while at a larger scale, the atoms are not resolved. However, if the electrons are replaced with macro particles giving the volume the same density, the macro particles will be resolved at a larger scale than the electrons.

This in particular becomes a problem when simulating physics without scattering. Once particles are resolved, individual scattering becomes possible. While scattering is a valid physical phenomena, the scattering cross section depends on the appropriate scale where individual electrons are scattered. This means that macro particle scattering simulated in this manner is both invalid and undesirable.

A solution can be found by recognizing that macro particles can be represented as individual Gaussian distributions of charge. This obviates the infinite fields found infinitesimally close to particles, which obviates scattering in general. As long as the width of each macro particle is large enough, the charge distribution represented by macro particles will be smooth. A discretized field can be constructed from the macro particles and must be smooth as well.

In order to implement this solution, an appropriate width must be determined.

Locally, the appropriate width to obtain a smooth distribution stems from the average distance to the nearest neighbor. In a similar manner as Chandrasekhar [68] but in two dimensions, the distance can be found. The probability of finding the nearest neighbor in two dimensions is given by:

$$w(r) = \left(1 - \int_0^r w(r') dr'\right) 2\pi r n_\sigma \quad (\text{B.1})$$

where n_σ is the average number of particles per unit surface area. From eq. (B.1) we derive:

$$\frac{d}{dr} \left[\frac{w(r)}{2\pi r n_\sigma} \right] = -w(r) \quad (\text{B.2})$$

$$\frac{w'(r)}{2\pi r n_\sigma} - \frac{w(r)}{2\pi n_\sigma r^2} = -w(r) \quad (\text{B.3})$$

$$w'(r) + w(r) \left[2\pi n_\sigma r - \frac{1}{r} \right] = 0 \quad (\text{B.4})$$

This first-order linear inhomogeneous differential equation can be solved via an integrating factor. The solution is given by:

$$w(r) = e^{-\pi n_\sigma r^2} r n_\sigma \quad (\text{B.5})$$

The average distance is then:

$$D = \int_0^\infty w(r) r dr \quad (\text{B.6})$$

$$D = \frac{1}{2\sqrt{n_\sigma}} \quad (\text{B.7})$$

The next issue is determining the width that will result in a smooth distribution.

The required width is found by applying eq. (3.33):

$$\sigma_r = 2.95 \times \left(\frac{D}{2\pi} \right) \quad (\text{B.8})$$

$$\sigma_r = \frac{2.95}{4\pi\sqrt{n_\sigma}} \approx \frac{0.23475}{\sqrt{n_\sigma}} \quad (\text{B.9})$$

B.1 Electrons

For electrons, the density is determined by the bunch density that must not contribute to scattering. Arbitrarily, it may be said that the bunch should appear to be smooth at a distance of 3σ from the bunch center. Quantitatively, given an elliptical distribution, this density is given as the surface at:

$$\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2} = \frac{(3\sigma)^2}{2\sigma^2} = \frac{9}{2} \quad (\text{B.10})$$

Given the parameters of number of macro particles $N_{\text{macro}} = Q/q$ with bunch charge Q and macro particle charge q , transverse RMS $\sigma_x = \sigma_y = \sigma_r$, and longitudinal RMS σ_ξ , the volumetric number density is given as:

$$n_{\text{vol}} = \frac{\rho}{q} = \frac{N_{\text{macro}}}{(2\pi)^{3/2}\sigma_r^2\sigma_\xi} e^{-9/2} \quad (\text{B.11})$$

The surface area density is then given by $n\Delta l$, where Δl is the slice width:

$$n_\sigma = \frac{N_{\text{macro}}}{(2\pi)^{3/2}\sigma_r^2\sigma_\xi} e^{-9/2} \Delta l \quad (\text{B.12})$$

The appropriate width to smear macro particles in order to ensure a smooth field is given by combining eqs. (B.9) and (B.12):

$$\sigma_{\text{macro}} = 0.9316 \frac{\sigma_r \sigma_\xi^{1/2} e^{9/4}}{\sqrt{N_{\text{macro}} \Delta l}} \quad (\text{B.13})$$

B.2 Ions

For ions, the density is determined by the lowest ion density that is desired to be smooth. In general, in the highly-localized ion distributions covered here, the ion density will never decrease below its initial ion density. Thus, the starting ion density can be used.

Appendix C

Spectrometer Imaging

For PWFA acceleration experiments, it is unknown whether some or many particles may exit the plasma with large angles. The quadrupoles are tuned so that any exit angles do not affect the distribution. In optics, this is called imaging, which implies that the light from the object plane is focused at the image plane where the measurement is performed.

Naively, we might expect the beam to have the smallest width when it is imaged, but this is not the case. As seen in chapter 8, the quadrupoles deliver a witness beam with an energy spread to a screen, and the distribution has a pinch. The naive assumption, then, is that the particles at the location of the pinch have the energy the quadrupoles are set to focus. While this is not true, the difference between the two is small.

Imaging is done by minimizing R_{12} and R_{34} from the transfer matrix. With two quadrupoles, these terms may be reduced to zero¹, resulting in a transfer matrix in the x -plane of

$$\mathbf{R}_{\text{image}} = \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix} \quad (\text{C.1})$$

Using this transfer matrix, the initial twiss coordinates can be propagated from

¹Private communication from E. Adli.

the plasma exit to the downstream lanex screen [53, 54]:

$$\Sigma_f = \mathbf{R}\Sigma_0\mathbf{R}^t \quad (\text{C.2})$$

$$\begin{pmatrix} \beta_f & -\alpha_f \\ -\alpha_f & \gamma_f \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & (-\alpha_0 = 0) \\ (-\alpha_0 = 0) & \gamma_0 \end{pmatrix} \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix} \quad (\text{C.3})$$

$$\Rightarrow \alpha_f = -R_{11}R_{21}\beta_0 \quad (\text{C.4})$$

This means that the focused energy does not form a waist at the lanex screen, which in turn means that the pinch in the profile is not the imaged energy.

To find which energy is in focus at the lanex screen, transfer matrices can be constructed for off-energy particles. If these transfer matrices result in an $\alpha_f = 0$ at the lanex screen, that energy is at a waist. Then, α_f^2 can be numerically minimized for a given input energy, thus solving for the energy that is in focus at the screen. If we assume that all energies have the same initial Twiss parameters ($\beta_0, \alpha_0 = 0$), whatever energy is in focus will have the smallest spot size at the screen. Visually, this is the pinch in the butterfly image on the lanex screen; physically, it is the energy with the smallest β value. The energy at the pinch is not the imaging energy of the quadrupoles.

The smallest β and α values can be found by numerically minimizing β^2 and α^2 respectively. Minimizing β^2 is the more useful, as this recovers the energy at which the spot size is the smallest.

In order to determine the energy of the minimum β , the imaging condition settings for the quadrupole magnets was found via simplex minimization of $R_{12}^2 + R_{34}^2$. The transfer matrices were then recalculated for different energies in this configuration. The transfer matrix is then used to calculate the resulting Twiss parameters given initial Twiss parameters of β_0 and $\alpha_0 = 0$. A simplex minimization routine was used to determine the energy at which the final value β^2 is minimized. This is the energy represented by the pinch on the screen, as per Fig. C.1. The pinch energy is located at $E_{\text{pinch}} = 1.0101E_{\text{img}}$ for all energies. Assuming the $\mathbf{R}$ matrices can be

Figure C.1: Averaged spot sizes with different energies in focus.

Taylor-expanded in energy, this makes sense, as shown below.

The result can be checked by numerically computing the derivative, which should be equal to zero, keeping in mind that $\alpha_0 = 0$.

$$\frac{d\beta}{dE} = \frac{d}{dE} \left[R_{11}^2 \beta_0 - 2R_{11}R_{12}(\alpha_0 = 0) + R_{12}^2 \frac{1 + \alpha_0^2}{\beta_0} \right] \quad (\text{C.5})$$

$$= 2R_{11}\beta_0 \frac{dR_{11}}{dE} + \frac{2R_{12}}{\beta_0} \frac{dR_{12}}{dE} \quad (\text{C.6})$$

Analytically, beta can be minimized by setting eq. (C.6) $\frac{d\beta}{dE} = 0$ and solving for energy, as the transfer matrix $\mathbf{R}$ is a function of the energy. It is difficult and not particularly enlightening to solve analytically, as the transfer matrix is a function of the entire spectrometer configuration. However, if the energy in focus is close to the

imaging energy, the $\mathbf{R}$ matrix elements can be expanded.

$$R_{ij} = R_{ij}(E = E_{\text{img}}) + \left(\frac{d}{dE} R_{ij}(E) \Big|_{E=E_{\text{img}}} \right) \Delta E \quad (\text{C.7})$$

Then, solving for “in-focus” energy in terms of the imaging $\mathbf{R}$ transfer matrices, per Eq. C.6, gives

$$(R_{11} + R'_{11}\Delta E) \beta_0^2 R'_{11} + (R_{12} + R'_{12}\Delta E) R'_{12} = 0 \quad (\text{C.8})$$

$$\Rightarrow \Delta E = -\frac{R_{11}R'_{11}\beta_0^2 + R_{12}R'_{12}}{R'^2_{11}\beta_0^2 + R'^2_{12}} \quad (\text{C.9})$$

Keeping in mind that $R_{12}(E = E_{\text{img}}) = 0$ by definition

$$\begin{aligned} \Delta E &= -\frac{R_{11}R'_{11}\beta_0^2}{R'^2_{11}\beta_0^2 + R'^2_{12}} \\ &= -\frac{R_{11}R'_{11}}{R'^2_{11} + R'^2_{12}/\beta_0^2} \end{aligned} \quad (\text{C.10})$$

This is in good agreement with Fig. C.1, disagreeing less than 2 MeV for values from approximately 16 GeV to 25 GeV. With an effective calibration value of 1.01% and beams at 20 GeV, it is expected that this effect is negligible for applications where resolutions < 200 MeV are used.

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