1 Crystalline Beams

When completed, crystalline beam research should be much more extensive than what will be presented below. Basically we could be re-doing the entire solid state physics – melting point, phonons, specific heat, heat conduction, quantum effects, spin effects – just replacing molecular force by Coulomb force. It also has the added complication that charges can radiate.

For accelerators, we ask (a) how to cool the beam sufficiently for it to crystallize (i.e. how does the beam make the phase transition from a gaseous state to a solid state), and (b) what conditions must the accelerator fulfill so as not to destroy the crystal once it is formed. For possible applications, we ask (c) what if we collide two crystalline beams, or (d) how does a crystalline beam radiate in an undulator, in a solid crystal, or in another crystalline beam.

We will first find several types of beam crystals (crystal hunting) and examine some of their properties. The procedure is not systematic and does not provide an exhaustive hunt. (A systematic hunt would use group theory.) We will then make a preliminary examination of question (b) above. What we will talk about can at best be a very small fraction of the research this topic eventually can offer.

When sufficiently cooled, a crystalline beam should exhibit significant quantum mechanical effects. We will not discuss these effects here.

Crystalization has been observed in ion traps. There has been hint of observation at Novosibirsk for a real crystalline beam in an accelerator. Crystalline beams have been observed in computer simulations. There are current efforts – not an easy task – to design storage rings to observe crystalline beams. Hopefully, as beam cooling technology advances, an exciting research field of crystalline beams will open up in front of us.

1.1 1-D Infinite Crystal

The simplest crystal is a 1-D infinite line of equally-spaced charged particles. Assume there is an infinitely strong transverse focusing so that the crystal is 1-D in \( z \)-direction. Let the spacing between charges be \( a \).

This is a trivial lattice arrangement. But one can still ask what are the small-amplitude normal modes of this crystal. Let the longitudinal location of the \( n \)th particle be designated as \( z_n = na + \Delta_n \), where \( \Delta_n \) is a small displacement from its lattice site. Let each particle have mass \( M \) and charge \( Q \). The equation of motion of the \( n \)th particle is

\[
M \ddot{z}_n + \sum_{k=1}^{\infty} \frac{Q^2}{(z_n - z_{n+k})^2} - \sum_{k=1}^{\infty} \frac{Q^2}{(z_n - z_{n-k})^2} = 0, \quad \text{for all } n
\]

where the second and third terms on the LHS are Coulomb forces due to charges on each side of the \( n \)th particle. For small deviations from the lattice sites, we
linearize Eq.(1) to obtain
\[
M \ddot{\Delta}_n + \sum_{k=1}^{\infty} \frac{2Q^2}{k^3q^3} (2\Delta_n - \Delta_{n+k} - \Delta_{n-k}) = 0 \quad \text{for all } n \quad (2)
\]

The eigenmode solution to Eq.(2) is found to be
\[
\begin{align*}
\text{eigenvalue } & \lambda = \frac{M \omega^2 a^3}{2Q^2} = 4 \sum_{k=1}^{\infty} \frac{1}{k^3} \sin^2 \left( \frac{k\theta}{2} \right) \\
\text{eigenvector } & \Delta_n = \cos(n\theta + \phi) e^{-i\omega t} \quad (3)
\end{align*}
\]

where \( \omega \) is the eigenmode frequency. This eigenmode is characterized by the continuous variables \( \theta \) and \( \phi \). All modes with different \( \phi \) values are degenerate, i.e. they all have the same eigenvalue. The mode index \( \theta \) gives the snapshot phase in the mode pattern \( \Delta_n \) between adjacent lattice sites. The modes with \( \theta = 0 \) have all charges moving in unison and the mode frequency is \( \omega_0 = 0 \) as one would expect because there is no restoring force. The modes with \( \theta = \pi \) have charges moving with alternating displacements; they have the largest mode frequency given by\(^1\)
\[
\omega_{\pi} = \sqrt{7\zeta(3) \frac{Q^2}{Ma^3}} \quad (4)
\]

where \( \zeta(p) \) is the Riemann zeta-function (which occurs often in crystal beam calculations),
\[
\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}, \quad \zeta(3) \approx 1.20205 \quad (5)
\]

Figure 1.1 shows eigenvalue \( \lambda \) as a function of \( \theta \). When \( \theta = \pi \), we have \( \lambda = 7\zeta(3)/2 \approx 4.207 \).

1-D Finite Crystal

Now consider a beam with a finite number \( N \) of particles in a crystalline state. In the \( z \)-direction, there is linear focusing with spring constant \( k_z \). The first question we ask is what are the lattice locations \( \{z_n, n = 1, 2, \ldots, N\} \). This may be useful when considering cold ions in a trap. The total potential energy

\(^1\)We have used the fact that
\[
\sum_{k=1}^{\infty} \frac{1}{k^p} - \sum_{k=1}^{\infty} \frac{1}{k^{p} \cdot \text{even}} = \sum_{k=1}^{\infty} \frac{1}{k^p} - \sum_{k=1}^{\infty} \frac{1}{(2k)^p} = (1 - \frac{1}{2^p}) \sum_{k=1}^{\infty} \frac{1}{k^p} = (1 - \frac{1}{2^p}) \zeta(p)
\]

In particular, this implies \( \sum_{k=1, \text{odd}}^{\infty} \frac{1}{k^p} = \frac{3}{8} \zeta(3) \).
The lattice locations are determined by the condition that all particles in the beam are in equilibrium, or
\[
\frac{\partial V}{\partial z_n} = 0 \quad \text{for all } n \tag{7}
\]
Equivalently one can say that \( V \) reaches a local minimum as a function of all coordinates \( \{z_n, n = 1, 2, \cdots, N\} \).

Deviation from the equilibrium state is described by the equations of motion
\[
M\ddot{z}_n + \frac{\partial V}{\partial z_n} = 0 \quad \text{for all } n \tag{8}
\]
Eq.(8) can be used to find small amplitude normal mode frequencies of the crystalline beam.

\( N = 1 \): When there is only one ion in the beam, the lattice is trivial. We have \( V = k_z z_1^2 / 2 \) and Eq.(7) gives \( z_1 = 0 \). This charge is located at the origin. It oscillates in a potential with frequency \( \omega_z = \sqrt{k_z/M} \).

\( N = 2 \): When there are two charges, Coulomb repulsion between them separate them apart, while the external focusing keeps them together to reach an
equilibrium lattice arrangement. In this case,

\[ V = \frac{k_z}{2} (z_1^2 + z_2^2) + \frac{Q^2}{|z_1 - z_2|} \]  \hspace{1cm} (9)

and Eq. (7) gives two equations whose solution is

\[ z_1 = -z_2 = \left( \frac{Q^2}{4k_z} \right)^{1/3} \]  \hspace{1cm} (10)

One notes that the lattice size is characterized by the quantity

\[ (Q^2/k_z)^{1/3} \]  \hspace{1cm} (11)

There are two normal modes, a + mode in which both particles oscillate in phase, and a − mode in which the two particles oscillate out of phase. The small amplitude mode frequencies are found to be

\[ \omega_{z+} = \sqrt{\frac{k_z}{M}}, \quad \omega_{z-} = \sqrt{\frac{3k_z}{M}} \]  \hspace{1cm} (12)

\( N = 3 \) The lattice locations are

\[ z_1 = -z_3 = \left( \frac{5Q^2}{4k_z} \right)^{1/3}, \quad z_2 = 0 \]  \hspace{1cm} (13)

The small amplitude normal mode frequencies are determined by

\[
\begin{vmatrix}
\frac{M\omega^2}{k_z} + \frac{14}{5} & -\frac{8}{5} & -\frac{1}{5} \\
-\frac{8}{5} & -\frac{M\omega^2}{k_z} + \frac{21}{5} & -\frac{8}{5} \\
-\frac{1}{5} & -\frac{8}{5} & -\frac{M\omega^2}{k_z} + \frac{14}{5}
\end{vmatrix} = 0 \]  \hspace{1cm} (14)

\[ \Rightarrow \omega_z = \sqrt{\frac{k_z}{M}} \sqrt{\frac{3k_z}{M}}, \quad \sqrt{\frac{29k_z}{5M}} \]  \hspace{1cm} (15)

The three normal modes respectively corresponds to (1) all three particles move in phase with the same amplitude, (2) particle 1 and 3 move out of phase, while particle 2 is stationary, and (3) particles 1 and 3 move in phase with same amplitude, while particle 2 moves opposite to them with twice the amplitude.

\( N = 4 \) We find the lattice sites numerically,

\[ z_1 = -z_4 \approx 1.4368 \left( \frac{Q^2}{k_z} \right)^{1/3} \]  \hspace{1cm} (16)
\( N = 5 \)

\[
\begin{align*}
    z_1 &= -z_5 \approx 1.7429 \left( \frac{Q^2}{k_z} \right)^{1/3} \\
    z_2 &= -z_4 \approx 0.8221 \left( \frac{Q^2}{k_z} \right)^{1/3} \\
    z_3 &= 0
\end{align*}
\]

These lattices are shown in Fig. 2 for \( N = 1 \) to 5.

**Exercise 1** What are the mode frequencies for the cases of \( N = 4 \) and \( N = 5 \) discussed above?

**Solution** The mode frequency \( \omega \) is determined by \( \det \left[ \left( -\frac{Mz^2}{k_z^2} - 1 \right) I + \frac{2Q^2}{k_z} A \right] = 0 \), where matrix \( A \) has the elements

\[
A_{nm} = \begin{cases} 
    \frac{1}{|z_j - z_n|^3} & \text{if } m = n \\
    -\frac{1}{|z_m - z_n|^3} & \text{if } m \neq n
\end{cases}
\]

\( (18) \)

The fact that \( A \) is symmetric assures real eigenvalues.

**Exercise 2** Show that the ratio of the two terms in the potential energy Eq. (6) is 1:2.
Solution

\[
\frac{1}{2} \sum_{n=1}^{N} k_z z_n^2 = \frac{1}{2} \sum_{n=1}^{N} z_n (k_z z_n) = \frac{1}{2} \sum_{n=1}^{N} z_n \left( \sum_{j \neq n} Q^2 (z_n - z_j) \right) = \frac{1}{2} \sum_{n \neq j} z_n Q^2 (z_n - z_j) \frac{1}{|z_n - z_j|^3}
\]

By switching \( j \leftrightarrow n \),

\[
\frac{1}{2} \sum_{n=1}^{N} k_z z_n^2 = -\frac{1}{2} \sum_{n \neq j} z_j Q^2 (z_n - z_j) \frac{1}{|z_n - z_j|^3}
\]

By adding the above and dividing by 2,

\[
\frac{1}{2} \sum_{n=1}^{N} k_z z_n^2 = \frac{1}{4} \sum_{n \neq j} \frac{Q^2 (z_n - z_j)^2}{|z_n - z_j|^3} = \frac{1}{4} \sum_{n \neq j} \frac{Q^2}{|z_n - z_j|} = \frac{1}{2} \sum_{n < j} \frac{Q^2}{|z_n - z_j|}
\]

Q.E.D. \(^2\)

Exercise 3 Show that as \( N \to \infty \), the 1-D finite lattice length \( \propto (N^2 Q^2/k_z)^{1/3} \). Fitting to the \( N = 5 \) case, we find that the full length of the lattice is

\[
L \approx 1.2 \left( \frac{N^2 Q^2}{k_z} \right)^{1/3}
\]

1.3 Planar Crystal

In this case, we consider an infinitely long crystal without \( z \)-focusing, but confined transversely by a focusing in the \( y \)-direction with spring constant \( k_y \). Focusing in the \( x \)-direction is assumed to be infinite; the crystal is in the \( y-z \) plane.

An important difference from the 1-D lattices is that now some lattice configurations can be unstable. When the linear charge density (characterized by \( a \), the \( z \)-spacing between adjacent charges) is low, the transverse focusing is strong enough to overcome the Coulomb repulsion and to confine the crystal to a 1-D configuration, just like the 1-D infinite crystal studied earlier. As \( a \) becomes shorter, the 1-D configuration can become unstable. To illustrate this, consider a crystal with lattice sites \( z_n = na \) but undergoing an oscillation with a pattern of a zig-zag in which all particles at sites with \( (n = \text{even}) \) move in one direction in \( y \), while all particles with \( (n = \text{odd}) \) move in the opposite direction. The unperturbed lattice is sketched in Fig.3(a).

Consider the \( y \)-motion of the charge located at \( n = 0 \). The equation of motion is

\[
M \ddot{y} + y \left[ k_y - 4Q^2 \sum_{k=1, \text{odd}}^{\infty} \frac{1}{(4y^2 + k^2 a^2)^{3/2}} \right] = 0
\]

\(^2\)Thank to Jeff Holmes for this nice proof.
For small $y$, this becomes

$$M \ddot{y} + y \left[k_y - \frac{7Q^2}{2a^3} \zeta(3)\right] = 0 \quad (21)$$

The $y$-motion of the crystal would be unstable if

$$k_y < \frac{7\zeta(3) Q^2}{2 a^3} \quad \text{or} \quad a < \left(\frac{7\zeta(3) Q^2}{2 k_y}\right)^{1/3} \quad (22)$$

i.e., when the focusing is overcome by the Coulomb repulsion.

One may ask if the crystal is necessarily stable when condition (22) is not satisfied. The answer is we have not proved it. What we have proved is that the crystal is stable against one type of perturbation – the type with a zig-zag pattern. For the crystal to be stable, it must be stable against all possible perturbations, and we have not done that. However, the zig-zag pattern (we...
might call this the transverse $\pi$-mode) turns out to be the least stable pattern, so Eq.(22) is the necessary and sufficient condition for the 1-D lattice to be stable.

What happens to the unstable crystal when Eq.(22) is satisfied? The answer is that it can not stay 1-D. The crystal has a zig-zag pattern. Designate this pattern by two layers of lattice sites

$$\begin{align*}
(y, z) = & \begin{cases} 
(b, 2na) & , \quad n = -\infty \text{ to } \infty \\
(-b, (2n + 1)a) & , \quad n = -\infty \text{ to } \infty 
\end{cases}
\end{align*}$$

(23)

where $2b$ is the $y$-spacing between the two sets of sites, and is yet to be determined. This zig-zag pattern is shown as Fig.3(b).

To determine $b$, we apply the condition that the lattice is in equilibrium. The forces have been computed in Eq.(20). The equilibrium condition is found to be

$$\frac{k_y a^3}{4Q^2} = \sum_{k=1, \text{odd}}^{\infty} \frac{1}{4(\frac{b}{a})^2 + k^2}$$

We need to solve it for $b/a$. There is a solution if and only if condition (22) is satisfied. The solid curve in Fig.1.3 shows $b/a$ as a function of $k_y a^3/Q^2$.

Having a solution for $b/a$, however, does not guarantee the stability of the zig-zag crystal. When $a$ decreases further, the zig-zag crystal gives way to a configuration with three lines of charges. Let the lattice sites be indexed as

$$\begin{align*}
(y, z) = & \begin{cases} 
(b, (4n + 1)a) & , \quad n = -\infty \text{ to } \infty \\
(0, 2na) & , \quad n = -\infty \text{ to } \infty \\
(-b, (4n - 1)a) & , \quad n = -\infty \text{ to } \infty 
\end{cases}
\end{align*}$$

(25)
This lattice is sketched in Fig. 3(c). The equilibrium condition is found to be

\[
\frac{4k_ya^3}{5Q^2} = \sum_{k=1, \text{odd}}^{\infty} \frac{1}{[4(b/a)^2 + k^2]^{3/2}}
\]  

(26)

The dashed curve in Fig. 1.3 shows the solution of \( b/a \) for the 3-layered planar crystal.

We conclude that when \( (k_ya^3/Q^2) > (7\zeta(3)/2) \approx 4.207 \), the planar crystal is 1-D. When \( (7\zeta(3)/2) > (k_ya^3/Q^2) > (35\zeta(3)/32) \), the crystal has the zig-zag configuration. When \( (35\zeta(3)/32) > (k_ya^3/Q^2) \), a 3-layered planar crystal takes over. As \( a \) decreases more, of course, the crystal becomes increasingly complex, etc.

**Exercise 4** (a) Can you think of possible 2-layered arrangements other than the zig-zag? For example, how about the arrangement

\[
(y, z) = \begin{cases} 
(b, na) & , \quad n = -\infty \text{ to } \infty \\
(-b, na) & , 
\end{cases}
\]

(27)
as shown in Fig. 3(d)? What is the value for \( b/a \)? Under what conditions is this crystal stable? (b) Can you think of possible 3-layered arrangements other than that considered in Fig. 3(c)?

**Solution** (a) Unstable against sliding the top row relative to the bottom row of particles.

### 1.4 3-D Infinite Cubic Crystal

Before we specialize to cubic crystal, let’s consider a 3-D crystal in general. Let the lattice site of the \( n \)th particle be \( \vec{\ell}_n \). Let the position of the \( n \)th particle be slightly perturbed from the lattice site with

\[
\vec{r}_n = \vec{\ell}_n + \vec{\Delta}_n
\]

(28)

The Coulomb force seen by the \( n \)th particle is

\[
\vec{F} = Q^2 \sum_{j \neq n} \frac{\vec{r}_n - \vec{r}_j}{|\vec{r}_n - \vec{r}_j|^3}
\]

(29)

The fact that \( \vec{\ell}_n \) are lattice sites requires that the Coulomb force in equilibrium must vanish, i.e.,

\[
\sum_{j \neq n} \frac{\vec{\ell}_n - \vec{\ell}_j}{|\vec{\ell}_n - \vec{\ell}_j|^3} = \vec{0} \quad \text{for all n}
\]

(30)

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3Strictly speaking, a 3-D infinite crystal beam cannot exist. Coulomb force from all other charges makes lattice sites intrinsically unstable – the summation (29) diverges. But here we study it as a curiosity.
For small deviations $\vec{\Delta}_n$, the linearized equation of motion (assumed to be nonrelativistic) for the $n$th particle is

$$M\ddot{\vec{\Delta}}_n = Q^2 \sum_{j \neq n} \frac{\vec{\ell}_n + \vec{\Delta}_n - \vec{\ell}_j - \vec{\Delta}_j}{|\vec{\ell}_n + \vec{\Delta}_n - \vec{\ell}_j - \vec{\Delta}_j|^3}$$

$$\approx Q^2 \sum_{j \neq n} \frac{1}{|\vec{\ell}_n - \vec{\ell}_j|^3} \left[ \Delta_n - \Delta_j - 3(\vec{\ell}_n - \vec{\ell}_j) \cdot (\vec{\Delta}_n - \vec{\Delta}_j) \right]$$

(31)

In a cubic crystal with lattice dimension $a$, we have $\vec{\ell}_i = i\vec{a}$, and Eq.(31) becomes

$$M\ddot{\vec{\Delta}} = Q^2 a^3 \sum_{\vec{k} \neq \vec{0}} \frac{1}{|\vec{k}|^3} \left[ 2 \vec{\Delta}_n - \vec{\Delta}_{n-\vec{k}} - \vec{\Delta}_{n+\vec{k}} - 3\vec{k} \cdot (2\vec{\Delta}_n - \vec{\Delta}_{n-\vec{k}} - \vec{\Delta}_{n+\vec{k}}) \right]$$

(32)

An eigenmode has a plane wave pattern

$$\vec{\Delta}_n = \vec{\theta} \cos(\vec{n} \cdot \vec{\theta} + \phi)e^{-i\omega t}$$

(33)

where $\vec{\theta}$ and $\phi$ serve as mode indices. The mode frequency $\omega$ is determined by

$$\frac{M\omega^2 a^3}{Q^2} = \sum_{\vec{k} \neq \vec{0}} \frac{1}{|\vec{k}|^3} \left[ 1 - \cos(\vec{k} \cdot \vec{\theta}) \right] \left( 3 \left( \frac{\vec{k} \cdot \vec{\theta}}{|\vec{k}|^2 |\vec{\theta}|^2} \right)^2 - 1 \right)$$

(34)

For example, when $\vec{\theta} = \theta\hat{z}$, or $\theta\hat{y}$, or $\theta\hat{x}$, we have

$$\lambda(\theta) = \frac{M\omega^2 a^3}{2Q^2} = \sum_{k_z=1}^{\infty} (1 - \cos k_z \theta) F(k_z)$$

$$F(k_z) = \sum_{k_z=-\infty}^{\infty} \sum_{k_y=-\infty}^{\infty} g(k_x, k_y)$$

$$g(k_x, k_y) = \frac{2k_x^2 - k_y^2}{(k_x^2 + k_y^2 + k_z^2)^{5/2}}$$

(35)

The double summation in $F(k_z)$ is rather subtle because of the cancelation among terms, and must be handled carefully. Figure 1.4 shows $\lambda(\theta)$. The cubic lattice is stable (provided it can be formed in the first place – see the previous footnote).
Figure 5: Eigenvalue $\lambda$ versus eigenmode index $\vec{\theta}$ for a cubic crystal, when $\vec{\theta} = \theta \hat{x}$, or $\theta \hat{y}$, or $\theta \hat{z}$.

\[
= \sum_{k_x=0}^{\infty} \sum_{k_y=0}^{\infty} \left[ g(k_x, k_y) + g(k_x + 1, k_y) + g(k_x, k_y + 1) + g(k_x + 1, k_y + 1) \right] \\
-4 \int_{k_x}^{k_x+1} dx \int_{k_y}^{k_y+1} dy g(\alpha_x, \alpha_y)
\]

where in the second step we have added artificially a term which sums up to zero to help the convergence,

\[
= -4 \int_0^{\infty} dx \int_0^{\infty} dy \frac{2k_x^2 - \alpha_x^2 - \alpha_y^2}{(\alpha_x^2 + \alpha_y^2 + k_z^2)^{3/2}} = 0
\]

The integral in Eq.(36) can be performed and we have a summation which converges much faster,

\[
F(k_z) = \sum_{k_x=0}^{\infty} \sum_{k_y=0}^{\infty} \left[ g(k_x, k_y) + g(k_x + 1, k_y) + g(k_x, k_y + 1) + g(k_x + 1, k_y + 1) \right] \\
-4\left[ h(k_x, k_y) - h(k_x + 1, k_y) \right] - \left[ h(k_x, k_y + 1) - h(k_x + 1, k_y + 1) \right] \]

\[
h(k_x, k_y) = \frac{k_x k_y (k_x^2 + k_y^2 + 2k_z^2)}{(k_x^2 + k_y^2)(k_x^2 + k_z^2)(k_y^2 + k_z^2)(k_x^2 + k_y^2 + k_z^2)^{1/2}}
\]

It turns out that $F(k_z)$ decreases rapidly with increasing $k_z$: $F(1) = 0.3274645, F(2) = 5.5496 \times 10^{-4}, F(3) = 1.03 \times 10^{-6}$, and all higher values of $k_z$ practically vanish.
Exercise 5 Verify that Eqs.(33-34) describe eigenmodes for the equation of motion (32).

Exercise 6 Following similar approach of the text, find the eigenmode pattern and eigenfrequencies for the 2-D infinite square crystal with lattice sites \((x, y) = (n_x a, n_y a), n_{x,y} = \pm \) integers.

Solution

\[
\frac{M \omega^2 a^3}{2Q^2} = \sum_{k_z=1}^{\infty} (1 - \cos k_z \theta) \sum_{k_y=-\infty}^{\infty} \frac{2k_z^2 - k_y^2}{(k_y^2 + k_z^2)^{5/2}}
\]

Exercise 7 We concluded that the cubic lattice is stable after analyzing the case \(\vec{\theta} = \hat{z} \theta \). Does this conclusion hold when we consider an arbitrary mode index \(\vec{\theta} \)?

Exercise 8 Repeat the analysis for a rectangular crystal with \(\vec{l}_n = (n_x b, n_y b, n_z a)\), where \(a\) and \(b\) are the longitudinal and transverse lattice periods. Is this crystal stable?

1.5 Planar Crystal With \(z\) Focusing

We next add back \(z\)-focusing to the planar crystal discussed earlier. The beam now has a finite number of \(N\) particles. The crystal is still in the \(y-z\) plane. Let \(k_y\) and \(k_z\) be the spring constants.

Consider the case with \(N = 2\). At equilibrium, they are located at \((y, z) = (y_1, z_1)\) and \((-y_1, -z_1)\) respectively, where

\[
k_y y_1 = \frac{Q^2 y_1}{4(y_1^2 + z_1^2)^{3/2}}, \quad k_z z_1 = \frac{Q^2 z_1}{4(y_1^2 + z_1^2)^{3/2}}
\]

(40)

There are two possible solutions,

\[
(y_1, z_1) = \begin{cases} 
\left(\left(\frac{Q^2}{4k_y}\right)^{1/3}, 0\right) \\
\left(0, \left(\frac{Q^2}{4k_z}\right)^{1/3}\right)
\end{cases}
\]

(41)

We next analyze the stability around each of the possible solutions. Take the first case first. Let the two ions have displacements

\((y_1 + \Delta y_1, \Delta z_1)\) and \((-y_1 + \Delta y_2, \Delta z_2)\)

(42)

The \(y\)-motion of ion 1 is described by

\[
M \ddot{\Delta} y_1 + k_y (y_1 + \Delta y_1) - \frac{Q^2 (2y_1 + \Delta y_1 - \Delta y_2)}{\left[(2y_1 + \Delta y_1 - \Delta y_2)^2 + (\Delta z_1 - \Delta z_2)^2\right]^{3/2}} = 0
\]

(43)
By linearizing with respect to $\Delta$-quantities, and extend the analysis to the $y$- and $z$-motions of both ions, we find

\begin{align}
M\ddot{\Delta}y_1 + ky_1(2\Delta y_1 - \Delta y_2) &= 0 \\
M\ddot{\Delta}y_2 + ky_1(2\Delta y_2 - \Delta y_1) &= 0 \\
M\ddot{\Delta}z_1 + (k_z - \frac{k_y}{2})\Delta z_1 + \frac{k_y}{2}\Delta z_2 &= 0 \\
M\ddot{\Delta}z_2 + (k_z - \frac{k_y}{2})\Delta z_2 + \frac{k_y}{2}\Delta z_1 &= 0
\end{align}

(44)

Note that the $y$- and $z$-motions are decoupled from each other. The $y$-motion has two eigenmodes,

\[ \omega = \begin{cases} \sqrt{\frac{k_y}{M}}, & \text{0 mode} \\ \sqrt{\frac{3k_y}{M}}, & \text{\pi mode} \end{cases} \]

while for the $z$-motion,

\[ \omega = \begin{cases} \sqrt{\frac{k_z}{M}}, & \text{0 mode} \\ \sqrt{(k_z - k_y)/M}, & \text{\pi mode} \end{cases} \]

(45)

(46)

The $z$-motion is unstable if $k_z < k_y$. These four modes are shown in Fig.6.

Figure 6: There are 8 eigenmodes for a 2-particle planar crystal with $z$-focusing. Four of them with the two particles on the $y$-axis are shown here. (a) $y$-motion, 0 mode. (b) $y$-motion, $\pi$-mode. (c) $z$-motion, 0 mode. (d) $z$-motion, $\pi$-mode. The sheering mode (d) is unstable if $k_z < k_y$. The remaining four modes are with the two particles on the $z$-axis.

For the second set of equilibrium coordinates in (41), we find its $y$-motion is unstable if $k_y < k_z$. We thus conclude that when $k_y > k_z$, the crystal is oriented
along the $z$-axis. When $k_y > k_z$, the crystal is oriented along the $y$-axis. Ions tend to stay away from the strong focusing dimension in a crystalline beam.

There is also a degenerate case when $k_y = k_z$. The crystal can be oriented along an arbitrary direction in the $y$-$z$ plane, but it is barely stable. Any shearing motion would not be focussed.

### 1.6 Helical Crystal

Consider the case without longitudinal focusing and the crystal is again infinitely long in $z$-direction. Let there be transverse focusing with spring constants $k_x$ and $k_y$. We learned from Eq.(22) that if the longitudinal average particle spacing $a$ is sufficiently large, the crystal is 1-D. As the line density increases (i.e. as $a$ decreases), the 1-D crystal becomes unstable and more complicated crystal structures result.

Figure 1.6 shows one simulation result of possible crystal structures when $k_x = k_y = k$ and no longitudinal focusing [3]. The structure depends on the dimensionless linear particle density $\lambda = \frac{1}{a}(3Q^2/2k)^{1/3}$. As $\lambda$ increases, one sees 1-D, zig-zag, and helical, and multi-layered helical crystals, successively. If a longitudinal focusing is introduced, one may contemplate a buckey ball crystal, and crystals with multiple 3-D shells.

![Figure 7: Simulation results of lattice configurations as the linear particle density $\lambda$ is varied. For comparison, condition (22) for the instability of a 1-D crystal gives $\lambda > (3/7\zeta(3))^{1/3}$, which is consistent with the results here.](image)

### 1.7 Moving Crystal

One may pursue crystal hunting some more, perhaps using group theory to be systematic. But below we will study the topic of crystal beam motion in an
accelerator. Let us consider a crystal moving in the z-direction with speed \( \beta c \). To illustrate, we take a planar crystal which has a \( y \)-focusing spring constant (in the laboratory frame) \( k_y \), while \( k_x = \infty \) and \( k_z = 0 \).

First let us write down the electric field of a moving charge:

\[
\vec{E}(\vec{r}) = \frac{\gamma Q \vec{r}}{(x^2 + y^2 + \gamma^2 z^2)^{3/2}}
\]

which can also be written as

\[
E_x = -\frac{\partial \Phi}{\partial x}, \quad E_y = -\frac{\partial \Phi}{\partial y}, \quad E_z = -\frac{1}{\gamma^2} \frac{\partial \Phi}{\partial z}
\]

\[
\Phi = \frac{\gamma Q}{(x^2 + y^2 + \gamma^2 z^2)^{1/2}}
\]

Another charge moving with the same velocity along z-direction sees a Lorentz force

\[
\vec{F} = -\frac{Q}{\gamma^2} \nabla \Phi
\]

where we have used the fact that the magnetic and electric contributions cancel each other to introduce an extra factor of \( 1/\gamma^2 \) to the transverse Lorentz force.

For sufficiently large \( a \) (\( a \) is in the laboratory frame), the equilibrium crystal is 1-D. To see how large \( a \) must be in order for this moving 1-D crystal to be stable, we follow what we did for the stationary planar crystal, and consider the perturbation of a zig-zag pattern. The \( y \)-motion obeys [Cf. Eq.(20)]

\[
M \gamma c^2 y'' + y \left[ k_y - \frac{4Q^2}{\gamma} \sum_{k=1, \text{odd}}^{\infty} \frac{1}{(4y^2 + \gamma^2 k^2 a^2)^{3/2}} \right] = 0
\]

It follows that a 1-D crystal is stable only if

\[
a > \left( \frac{7 \zeta(3)}{2} \frac{Q^2}{\gamma^4 k_y} \right)^{1/3}
\]

Similarly, we find the crystal has a zig-zag pattern if

\[
\left( \frac{7 \zeta(3)}{2} \frac{Q^2}{\gamma^4 k_y} \right)^{1/3} > a > \left( \frac{35 \zeta(3)}{32} \frac{Q^2}{\gamma^4 k_y} \right)^{1/3}
\]

**Exercise 9** Show Eqs.(51-52) by Lorentz transformation from the rest frame of the crystalline beam.

**Solution** In the beam rest frame, particle spacing is \( \gamma a \) and spring constant is \( \gamma k_y \).
Equations of Motion  

Equation (49) allows one to write down the equations of motion for one of the particles in a circular accelerator,

\[
x'' + K_x x + \frac{\partial V}{\partial x} = \frac{\delta}{\rho}
\]

\[
y'' + K_y y + \frac{\partial V}{\partial y} = 0
\]

\[
z' = -\frac{x}{\rho} + \frac{\delta}{\gamma^2}
\]

\[
\delta' = -\frac{\partial V}{\partial z}
\]

where

\[
V = \frac{Q^2}{Mc^2\gamma^2} \sum_k \frac{1}{[(x - x_k)^2 + (y - y_k)^2 + \gamma^2(z - z_k)^2]^{1/2}}
\]

with summation over all lattice sites \( k \) other than the site for the particle under consideration. Quantities \( K_x, K_y \) and \( \rho \) are periodic functions of \( s \) with period \( C = 2\pi R \). Quantities \( K_{x,y} \) are related to the spring constants \( k_{x,y} \) by \( K_{x,y} = k_{x,y}/M\gamma c^2 \). We have considered an unbunched beam in an accelerator without rf focusing. We have also ignored the curvature of the crystal beam conforming to the circular closed orbit of the accelerator.

The Hamiltonian for the above equations of motion is

\[
H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2\gamma^2} \delta^2 - \frac{x\delta}{\rho} + \frac{1}{2}(K_{x}\bar{x}^2 + K_{y}\bar{y}^2) + V
\]

In a weak focusing synchrotron, we have

\[
K_x = \frac{1 - n}{R^2}, \quad K_y = \frac{n}{R^2}, \quad \rho = R
\]

In a smooth approximation for an alternating gradient synchrotron, we may take

\[
K_x = \left(\frac{\nu_x}{R}\right)^2, \quad K_y = \left(\frac{\nu_y}{R}\right)^2, \quad \rho = R
\]

Lattice Sites  
Let the coordinates of the \( n \)th particle be designated as

\[
x_n = X_n + \alpha_n, \quad y_n = Y_n + \beta_n, \quad z_n = Z_n + \gamma_n, \quad \delta_n = \Delta_n + \sigma_n
\]

where \( X_n, Y_n, Z_n, \Delta_n \) are the lattice site coordinates, and \( \alpha_n, \beta_n, \gamma_n, \sigma_n \) are small deviations from the sites. Note that each lattice site is assigned not only three space coordinates \( X_n, Y_n, Z_n \), but also an energy deviation \( \Delta_n \). We expand \( V \) to quadratic terms in \( \alpha_n, \beta_n, \gamma_n \) to obtain

\[
V \approx \frac{Q^2}{Mc^2\gamma^2} \sum_{k \neq n} \frac{1}{R_{nk}} \times \]

16
\[
\left\{ \begin{array}{l}
1 - \frac{(\alpha_n - \alpha_k)(X_n - X_k) + (\beta_n - \beta_k)(Y_n - Y_k) + \gamma^2(\gamma_n - \gamma_k)(Z_n - Z_k)}{R_{nk}^2} \\
- \frac{(\alpha_n - \alpha_k)^2 + (\beta_n - \beta_k)^2 + \gamma^2(\gamma_n - \gamma_k)^2}{2R_{nk}^2} \\
+ \frac{3}{2} \left[ \frac{(\alpha_n - \alpha_k)(X_n - X_k) + (\beta_n - \beta_k)(Y_n - Y_k) + \gamma^2(\gamma_n - \gamma_k)(Z_n - Z_k)}{R_{nk}^2} \right]^2 \end{array} \right. 
\]

\[R_{nk} \equiv \left[ (X_n - X_k)^2 + (Y_n - Y_k)^2 + \gamma^2(Z_n - Z_k)^2 \right]^{1/2} \] (59)

Equation (53) then yields the equations for the lattice sites,

\[
X_n'' + K_x X_n - \frac{Q^2}{M c^2 \gamma^2} \sum_{k \neq n} \frac{X_n - X_k}{R_{nk}^3} = \Delta_n \rho \\
Y_n'' + K_y Y_n - \frac{Q^2}{M c^2 \gamma^2} \sum_{k \neq n} \frac{Y_n - Y_k}{R_{nk}^3} = 0 \\
Z_n' = - \frac{X_n}{\rho} + \Delta_n \gamma^2 \\
\Delta_n' = \frac{Q^2}{M c^2} \sum_{k \neq n} \frac{Z_n - Z_k}{R_{nk}^3} 
\] (60)

In order for the beam to crystalize, we must require the lattice site coordinates to be periodic with period \(C\).

In the smooth approximation, \(K_x, K_y, \rho\) as well as the lattice structure are independent of \(s\). The crystal rotates around the accelerator rigidly. All primed terms on the LHS of Eq. (60) vanish. One must have

\[
\Delta_n = \frac{\gamma^2}{\rho} X_n 
\] (61)

Equation (61) means that a particle with \(X_n \neq 0\) must be associated with a \(\Delta_n\) in such a way that the extra path length due to \(X_n\) is exactly compensated by the extra velocity of the particle due to \(\Delta_n\). Equation (60) then becomes

\[
(K_x - \frac{\gamma^2}{\rho^2})X_n - \frac{Q^2}{M c^2 \gamma^2} \sum_{k \neq n} \frac{X_n - X_k}{R_{nk}^3} = 0 \\
K_y Y_n - \frac{Q^2}{M c^2 \gamma^2} \sum_{k \neq n} \frac{Y_n - Y_k}{R_{nk}^3} = 0 \\
\sum_{k \neq n} \frac{Z_n - Z_k}{R_{nk}^3} = 0 
\] (62)

It is conceivable that a “liquid crystal” beam can be defined when a layer of lattice slips with respect to other layers by exactly an integer multiple of
the unit dimension of the lattice when the beam completes one revolution. The lattice is re-formed after each revolution. This possibility is not considered here.

1-D Lattice

A 1-D lattice trivially satisfies Eq.(62),

\[ X_n = Y_n = 0, \quad \Delta_n = 0, \quad Z_n = na \]  

(63)

Zig-zag Lattice in the \( y-z \) Plane

This lattice has \( X_n = 0 \) and \( \Delta_n = 0 \). Its \((Y,Z)\) site coordinates are given by Eq.(23), with [Cf. Eq.(24)]

\[
\frac{K_y Mc^2 \gamma^2 a^3}{4Q^2} = \sum_{k=1, \text{odd}}^\infty \frac{1}{4 \left( \frac{b}{a} \right)^2 + \gamma^2 k^2}^{3/2}  
\]  

(64)

Eq.(64) has solution only if [Cf. Eqs.(22) and (51)]

\[
a < \left( \frac{7}{2} \zeta(3) \frac{Q^2}{K_y M \gamma^5 c^2} \right)^{1/3}  
\]  

(65)

Zig-zag Lattice in the \( x-z \) Plane

This zig-zag crystal is more complicated than the one in the \( y-z \) plane because \( x \)-deviations necessarily involve energy deviations. Let the crystal sites have \( Y_n = 0 \) and

\[
(X_n, Z_n, \Delta_n) = \begin{cases} 
(b, 2na, \gamma^2 b/R) \\
(-b, (2n+1)a, -\gamma^2 b/R) 
\end{cases}  
\]  

(66)

Substituting into Eq.(62) yields the condition

\[
\frac{Mc^2 \gamma^2 a^3}{4Q^2} \left( K_x - \frac{\gamma^2}{\rho^2} \right) = \sum_{k=1, \text{odd}}^\infty \frac{1}{4 \left( \frac{b}{a} \right)^2 + \gamma^2 k^2}^{3/2}  
\]  

(67)

Eq.(67) has solution only if

\[
K_x > \frac{\gamma^2}{\rho^2}  
\]  

(68)

and

\[
a < \left( \frac{7}{2} \zeta(3) \frac{Q^2}{(K_x - \frac{\gamma^2}{\rho^2}) M \gamma^5 c^2} \right)^{1/3}, \quad \text{or} \quad k_x < \frac{\gamma^2}{\rho^2} + \frac{7}{2} \zeta(3) \frac{Q^2}{Mc^2 a^5 \gamma^5}  
\]  

(69)

In the smooth approximation (57), Eq.(68) means

\[ \nu_x > \gamma \]  

(70)
We thus conclude that the accelerator must be operated below transition in order for the crystalline beam to form. Physically this is so that the horizontal focusing force must be greater than the centrifugal force, i.e.

\[ K_x X > \frac{\Delta}{\rho} \quad (71) \]

When the particle line density is small enough that both Eqs.(65) and (69) are not satisfied, the crystal is 1-D. Take for example the case of a proton beam, and \( \rho = 10 \text{ m}, \gamma = 5, \nu_x = 10, K_x = \nu_x^2/\rho^2 = 1 \text{ m}^{-2} \), Eq.(69) says that the formation of 1-D crystal requires \( a > 1.4 \text{ \mu m} \), or the number of protons in the accelerator is less than \( 2\pi\rho/(1.4 \text{ \mu m}) = 4.5 \times 10^7 \). To store more protons in a 1-D crystal, one way is to increase \( \gamma \).

### 1.8 Perturbation from Lattice Sites

Assuming the lattice sites have been established, we next need to make sure the crystal is stable against small deviations like Eq.(58). Linearizing Eq.(53) around the lattice sites, the equations of motion become

\[
\begin{align*}
\alpha''_n + K_x \alpha_n - \frac{Q^2}{M c^2 \gamma^2} \sum_{k \neq n} \left\{ \frac{1}{R^3_{nk}} - \frac{3(X_n - X_k)^2}{R^6_{nk}} \right\} (\alpha_n - \alpha_k) \\
- \frac{3(X_n - X_k)(Y_n - Y_k)}{R^5_{nk}} (\beta_n - \beta_k) - \frac{3\gamma^2(X_n - X_k)(Z_n - Z_k)}{R^5_{nk}} (\gamma_n - \gamma_k) \right\} = \frac{\sigma_n}{\rho} \\
\beta''_n + K_y \beta_n - \frac{Q^2}{M c^2 \gamma^2} \sum_{k \neq n} \left\{ \frac{1}{R^3_{nk}} - \frac{3(Y_n - Y_k)^2}{R^6_{nk}} \right\} (\beta_n - \beta_k) \\
- \frac{3(X_n - X_k)(Y_n - Y_k)}{R^5_{nk}} (\alpha_n - \alpha_k) - \frac{3\gamma^2(Y_n - Y_k)(Z_n - Z_k)}{R^5_{nk}} (\gamma_n - \gamma_k) \right\} = 0 \\
\gamma'_n = -\frac{\alpha_n}{\rho} + \frac{\sigma_n}{\gamma^2} \\
\sigma'_n = \frac{Q^2}{M c^2} \sum_{k \neq n} \left\{ \frac{1}{R^3_{nk}} - \frac{3\gamma^2(Z_n - Z_k)^2}{R^6_{nk}} \right\} (\gamma_n - \gamma_k) \\
- \frac{3(X_n - X_k)(Z_n - Z_k)}{R^5_{nk}} (\alpha_n - \alpha_k) - \frac{3(Y_n - Y_k)(Z_n - Z_k)}{R^5_{nk}} (\beta_n - \beta_k) \right\} \quad (72)
\end{align*}
\]

Note that, in general, horizontal, vertical and longitudinal motions are coupled. An oscillation in one dimension excites oscillations in the other dimensions.

#### 1-D Lattice

For the 1-D lattice (63), the \( y \)-motion is decoupled from the other two dimensions, with

\[
\beta''_n + K_y \beta_n - \frac{Q^2}{M c^2 \gamma^2} \sum_{k \neq n} \frac{1}{R^3_{nk}} (\beta_n - \beta_k) = 0 \quad (73)
\]
where $R_{nk} = a\gamma|n - k|$, while the $x$- and $z$-motions are coupled,
\[\alpha''_n + K_x\alpha_n - \frac{Q^2}{Mc^2\gamma^2} \sum_{k \neq n} \frac{1}{R^3_{nk}}(\alpha_n - \alpha_k) = \frac{\sigma_n}{\rho}\]
\[\gamma'_n = -\frac{\alpha_n}{\rho} + \frac{\sigma_n}{\gamma}\]
\[\sigma'_n = -\frac{2Q^2}{Mc^2} \sum_{k \neq n} \frac{1}{R^3_{nk}}(\gamma_n - \gamma_k)\]  \hspace{1cm} (74)

Eigenmodes of $y$-motion are, according to Eq.(73), described by
\[\beta_n = \cos(n\theta + \phi)e^{-i\omega s/c}, \quad \alpha_n = 0, \quad \gamma_n = 0, \quad \sigma_n = 0\]  \hspace{1cm} (75)
where $\theta, \phi$ are mode indices, and $\omega$ is the eigenmode frequency, [Cf. Eq.(3)]
\[(\frac{\omega}{c})^2 = K_y - \xi\]  \hspace{1cm} (76)

where
\[\xi = \frac{4Q^2}{Mc^2a^3\gamma^5} \sum_{k=1}^{\infty} \frac{1}{k^3} \sin^2\left(\frac{k\theta}{2}\right)\]  \hspace{1cm} (77)

The Coulomb force is defocusing for $y$-motion as seen by the fact that it shifts the mode frequencies down-ward as $\xi > 0$.

In order for the 1-D crystal to be stable against small $y$-perturbations, all modes in Eq.(76) must be stable. This means $K_y$ must be larger than $\xi$ for all possible values of $\theta$. This in turn requires exactly the opposite of condition (65). When (65) is satisfied, 1-D crystal is unstable, and the next stable crystal is of course the zig-zag crystal.

Eigenmodes in the $x$-$z$ motion are described by
\[\alpha_n = \cos(n\theta + \phi)e^{-i\omega s/c} \quad \gamma_n = iG\cos(n\theta + \phi)e^{-i\omega s/c} \quad \sigma_n = S\cos(n\theta + \phi)e^{-i\omega s/c}\]  \hspace{1cm} (78)

Substituting Eq.(78) into Eq.(74) gives
\[K_x - \left(\frac{\omega}{c}\right)^2 - \xi = \frac{S}{\rho}\]
\[G\frac{\omega}{c} = -\frac{1}{\rho} + \frac{S}{\gamma}\]
\[S\frac{\omega}{c} = 2G\gamma^2\xi\]  \hspace{1cm} (79)

We need to solve Eq.(79) for $S, G, \omega$. The solution is
\[S = \frac{2\gamma^2\xi\frac{1}{\rho}}{2\xi - \frac{\omega^2}{c}}\]
\[G = \frac{\omega}{2\xi - \frac{\omega^2}{c}}\]  \hspace{1cm} (80)
and the eigenfrequency $\omega$ satisfies

$$
\left( \frac{\omega^2}{c^2} - K_x + \xi \right) \left( \frac{\omega^2}{c^2} - 2\xi \right) = \frac{2\gamma^2 \xi}{\rho^2}
$$

$$
\Rightarrow \left( \frac{\omega}{c} \right)^2 = \frac{K_x + \xi}{2} \pm \frac{1}{2} \sqrt{\left( K_x - 3\xi \right)^2 + \frac{8\xi^2}{\rho^2} \gamma^2} \tag{81}
$$

In order for the $x$-$z$ motion to be stable, it is necessary that both solutions of $(\omega/c)^2$ in Eq.(81) are real and positive. This requires

$$
K_x > \frac{\gamma^2}{\rho^2} + \xi \tag{82}
$$

Since Eq.(82) must be satisfied for all $\theta, \phi$, the 1-D crystal is stable against $x$-$z$ motion only if

$$
K_x > \frac{\gamma^2}{\rho^2} + \frac{7}{2} \zeta(3) \frac{Q^2}{Mc^2a^3\gamma^2} \tag{83}
$$

which we recognize is just the opposite of Eq.(69), as it should.

Equation (80) indicates that the $x$-, $z$- and $\delta$-amplitudes have relative magnitudes of

$$
A_x : A_z : A_\delta = \rho(2\xi - \frac{\omega^2}{c^2}) : \frac{\omega}{c} : 2\gamma^2 \xi \tag{84}
$$

This relative amplitudes would be what one observes in an accelerator when an 1-D crystalline beam is executing an $x$-$z$ mode oscillation.

**Exercise 10** (a) Follow the text to derive the mode frequencies (76) for $y$-motion and (81) for $x$-$z$ motion. (b) Show Eq.(84). What happens to the oscillation amplitudes when $k_x$ is barely above the stability condition (83)? What happens when $k_x$ is much greater than the threshold value?

Zig-zag Lattice One may proceed to analyze the stability of the zig-zag lattices and the lattice in general in a similar fashion.

**References**