Analytic Scaling Formulas for Crossed Laser Acceleration in Vacuum

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Abstract. Using an analytic result of Sprangle et al for the longitudinal accelerating field experienced by a relativistic electron from two crossed Gaussian lasers in vacuum, we derive a new approximation for the accelerating potential valid for any energy provided the electron is traveling near the speed of light. The expression is applicable when the laser crossing angle is small and the electron’s distance from the focal point is small compared to the Rayleigh length of the focused lasers. Both these conditions are the usual ones for high energy particle acceleration in a bounded accelerating structure near a laser focus. We use our expression for the accelerating potential to obtain formulas for the maximum energy gain per slip length, the optimal crossing angle, and the corresponding optimal slip distance. In the high energy limit, the optimal crossing angle is $\sqrt{2}$ times the diffraction angle and the gradient is 1.65 times the product of the individual laser field strength and the laser diffraction angle. Near the optimal crossing angle, the gradient increases approximately linearly with the ratio of crossing angle to diffraction angle.

1. Introduction

The purpose of this note is to discuss the optimization of laser acceleration of electrons in vacuum with respect to key parameters determined in experiments such as LEAP at Stanford University and E-163 at SLAC [1]. In particular, we present new analytic scaling formulas for crossed laser acceleration valid for any energy provided the electron travels at nearly the speed of light. The prototypical vacuum laser acceleration configuration we consider was analyzed by Sprangle et al [2,3] and is shown in Figure 1. In this scheme a pair of linearly polarized laser beams with Gaussian transverse profiles, having the same frequency and equal strength $E_0$, are focused and intersected in vacuum. The first laser propagates along the $z_1$ axis and the second laser propagates along the $z_2$ axis, where the $z_1$ axis and the $z_2$ axis are rotated by the crossing angles of $\theta$ and $-\theta$, respectively, with respect to the $z$ axis. The phases of the lasers are such that the transverse electric fields cancel along the axis while the axial fields add. Properly phased electrons injected along the $z$ axis can be accelerated by the net axial component of the laser field.

![Figure 1. Coordinate system and electric fields for two intersecting laser beams.](image)

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ARDB Note
The analysis by Sprangle et al was done assuming that the focused Gaussian laser propagates as a nearly unidirectional wave of finite cross section described by a field \( E_t = \mathcal{E}(x,y,z) \exp(ikz-i\omega t) \), and \( E_t \gg E_z \). Under this assumption the usual vector wave equation can be replaced by the scalar Helmholtz equation for the transverse field \( \mathcal{E}(x,y,z) \), and the longitudinal field is determined later by demanding \( \text{div} \ E = 0 \). The Helmholtz equation is solved in the so-called “paraxial approximation” in which the field \( \mathcal{E}(x,y,z) \) is assumed to vary slowly along the \( z \) direction on the scale of a wavelength. The paraxial approximation is valid when the laser spot size (waist) \( w_0 \) is much greater than the wavelength \( \lambda \). For small crossing angles \( \theta \ll 1 \), the axial accelerating field, as seen by an electron traveling with velocity \( v_z \approx c \) along the \( z \) axis, was calculated to be

\[
E_z = -\frac{2E_0\theta}{1+\hat{z}^2} \exp\left(-\frac{\hat{z}^2(\theta/\theta_d)^2}{1+\hat{z}^2}\right)\cos\psi_t, \quad (1a)
\]

where \( \hat{z} = z/z_R \), \( z_R = \pi w_0^2 / \lambda \) is the Rayleigh length, \( \theta_d = w_0 / z_R \) is the laser diffraction angle,

\[
\psi_t = -\frac{\hat{z}^2}{(\gamma_v \theta_d)^2}\left(\frac{\theta}{\theta_d}\right)^2 \hat{z}^2 - 2 \tan^{-1}\hat{z} + \phi_0, \quad (1b)
\]

\( t = z/v_z \) has been assumed, \( v_z \approx 1 - 1/(2\gamma_z) \), \( \gamma_z = (1-v_z^2/c^2)^{-1/2} \), and \( \phi_0 \) is the arrival phase of the electron relative to the laser maximum field value (\( \phi_0 = 0 \)). In Fig. 2 the normalized electric field \( E_z/E_0 \) as a function of the position \( z/z_R \) is plotted for the parameters of the E-163 experiment [1] using nominal 60 MeV kinetic energy electrons and for the same parameters in the limit of high energy electrons (\( \gamma \to \infty \)).

**Figure 2.** Normalized electric field for the E-163 experiment using nominal 60 MeV electrons (upper curve; red) and in the high energy limit (lower curve; blue).
The phase velocity of the accelerating field is greater than the speed of light, and therefore the field slips ahead of the electron. An electron interacting with the laser fields over the entire region \( z = -\infty \) to \( +\infty \) will experience no net energy gain. But if the laser fields are terminated, say by boundaries, then net energy gain can occur. An electron injected \(-\pi/2\) in phase relative the field crest will gain energy until the phase slip changes by \( \pi \), at which point it begins to be decelerated. From Eq. (1b) near the focal point \((z<<z_R)\), the distance required for the electron to slip by \( \pi \) is

\[
z_s \approx \frac{\lambda \gamma_c^2}{(1 + \gamma_c^2/\gamma_z^2)}.
\]  

where \( \gamma_c = \left(\theta^2 + 2\theta_d^2\right)^{-1/2} \) defines a critical energy. When \( \gamma_z < \gamma_c \), slippage is dominated by the low velocity of the electron, and when \( \gamma_z > \gamma_c \), slippage is dominated by the phase shift due to laser diffraction.

Experimentally \( z_s \) determines the length of the accelerator cell, which is marked by boundaries to terminate the laser fields. In Fig. 2 these points correspond to the locations of the field zero crossings on either side of the origin.

The axial electric field can in principle be written as the gradient of an effective potential, \( E_z = -z_R^{-1} \partial U / \partial \zeta \), provided one can identify the potential from the integral \( U = -z_R \int_0^\zeta E_z d\zeta \). As far as the author knows, the general integral of Eq. (1) is not known in closed form. However in the special limit where \( \gamma_z \theta_d \to \infty \), one can use the trigonometric identity \( 2 \tan^{-1} \zeta = \tan^{-1} \left( \frac{2\zeta}{1-\zeta^2} \right) \) to rewrite \( \cos \psi \) as a product of sine and cosine functions and absorb the leading \( \left(1 + \zeta^2\right)^{-1} \) found in \( E_z \). This observation allowed Sprangle et al to identify the potential in the high energy limit as

\[
U(\zeta) = \frac{-4E_0}{k \theta} \left[ \exp \left( -\left( \frac{\theta}{\theta_d} \right)^2 \frac{\zeta}{1 + \zeta^2} \right) \sin \left[ \phi_0 - \left( \frac{\theta}{\theta_d} \right)^2 \frac{\zeta}{1 + \zeta^2} \right] \right] - \sin \phi_0
\]

where \( k = 2\pi/\lambda \), and the potential is defined to be zero at the origin. Note that high energy here actually means \( \gamma_z \theta_d >> 1 \). The normalized potential \( U/(E_0/k) \) is shown in Fig. 3 along with the numerical integral of the electric field (1) for the parameters of E-163.

![Figure 3](image-url)  

**Figure 3.** Normalized potential from the numerical integration of Eq. (1) for 60 MeV electrons (smaller curve; red) and the potential in the high energy limit from Eq. (3) (larger curve; blue).
2. Effective Accelerating Potential for Arbitrary Energy

Although useful as a limit, Eq. (3) does not allow one to understand the optimization of parameters for a real experiment at finite energy or to derive simple scaling results for arbitrary energy. The analytic integration of Eq. (1) to obtain the potential for arbitrary $\gamma_z$ is hampered by the fact that the phase $\psi_t$ is nonlinear in $\tilde{z}$. To make progress we note that one is only interested in $E_z$ and the potential $U$ over a slip length around the origin, i.e. in the range $[-z_s/2, +z_s/2]$. Dividing the slip length $z_s$ in Eq. (2) by $2z_R$ one finds

$$\frac{\tilde{z_s}}{2} = \frac{\pi / 2}{2 + \left(\frac{\theta}{\theta_d}\right)^2 + \frac{1}{\gamma_z^2 \theta_d^2}}$$

which is always less than 1. This suggests the following approximation for $E_z$ valid for a relativistic electron when $\tilde{z} << 1$, which is essentially all of the accelerating region,

$$E_z \approx -2E_0\theta \exp \left[-\left(\frac{\theta}{\theta_d}\right)^2 \tilde{z}^2\right] \cos \left[-\frac{\tilde{z}}{(\gamma_z \theta_d)^2} \left(\frac{\theta}{\theta_d}\right)^2 \tilde{z} - 2\tilde{z} + \phi_0\right].$$

Here we have replaced $1 + \tilde{z}^2$ by 1 and linearized the arctangent function in Eq. (1) within this approximation. In Fig. 4 the normalized electric field (5) for the E-163 parameters is plotted along with the fields shown earlier in Fig. 2. The approximation is in excellent agreement with Eq. (1) in the central region of $\pi$ phase slip where the accelerator cell is located.

![Figure 4](image)

**Figure 4.** Normalized electric field from the approximation in Eq. (5) for the E-163 parameters (dashed; brown) compared with the exact field from Eq. (1) (middle curve; red) and high energy limit (lower; blue).

The integral of $E_z$ in Eq. (5) over the range $[0, \tilde{z}]$ can be expressed in terms of the real part of the error function $\text{erf}(w)$ with complex argument $w = u + iv$ [4]. The approximate effective potential is then
\[ U(\hat{z}) = \frac{2\sqrt{\pi} E_0}{k \theta_d} \exp(-v^2) \cdot \Re \left[ e^{i \phi} (\text{erf}(u + iv) - \text{erf}(iv)) \right], \]  

(6a)

where the quantities \( u \) and \( v \) are defined by

\[ u(\hat{z}) = \frac{\theta}{\theta_d} \hat{z} \quad \text{and} \quad v = \frac{\theta_d}{2\theta} \left[ 2 + \left( \frac{\theta}{\theta_d} \right)^2 + \frac{1}{\gamma^2 \theta_d^2} \right]. \]  

(6b)

Note that \( u(\hat{z}) \) is proportional to the longitudinal coordinate \( z \), and \( v \) is proportional to the phase shift per unit length in Eq. (5). From Eq. (4), we can write \( v = (\theta_d / 2\theta) \cdot \pi / \hat{z} \).

The error function is convenient to work with because so many approximations exist for it depending on the magnitudes of \( u \), \( v \), and \( w \). We anticipate that the optimal value of \( \theta \) for the maximum energy gain will be of order \( \theta_d \) because if the crossing angle is much greater than the diffraction angle the slip distance is too short for the electron to gain much energy, while if the crossing angle is too small the projected longitudinal electric field will not produce much energy gain. The analytic results in the next section will be consistent with this assumption. The error function in Eq. (6) can be greatly simplified under the conditions that \(|u| \ll 1\) and \(|w| \gg 1\). The first condition is always true within the slip region since \(|\hat{z}| \ll 1\) and \(\theta \) is of order \( \theta_d \), and the second condition is a statement that \( v \gg 1 \), which is valid in practice since \( v \geq 3/2 \) for all \( \theta \).

Performing an asymptotic approximation of the error function under these conditions yields

\[ U(\hat{z}) \approx -\frac{2E_0}{k \theta_d \nu} \left\{ \exp(-u^2) \cdot \left[ \left( 1 + \frac{1}{2v^2} \right) \sin(-2uv + \phi_0) + \frac{u}{v} \cos(-2uv + \phi_0) \right] - \left( 1 + \frac{1}{2v^2} \right) \sin \phi_0 \right\}, \]  

(7)

where terms up to order \( 1/v^3 \) and to first order in \( u/v \) have been retained. The \( 1/v^3 \) terms are essential for a good approximation. The error function potential (6) and the asymptotic approximation (7) are plotted in Fig. (5) along with the two potentials shown earlier in Fig. 3. These potentials are excellent approximations to the numerical integral in the central region of \( \pi \) phase slip where the accelerator cell is located.

\[ \text{Figure 5. Approximate potentials from Eqs. (6) (dashed; brown) and (7) (dash-dot; magenta) compared with the potentials plotted in Fig. 3.} \]
3. Analytic Scaling Formulas

Because there are only exponential and trigonometric functions in the potential of Eq. (7), it is very useful for deriving simple scaling formulas for crossed laser acceleration at arbitrary energy. The energy gain per unit charge is defined by

\[ W = -(U(z_{\text{final}}) - U(z_{\text{initial}})) \].

For energy gain over a slip distance, we have \( \dot{z}_{\text{initial}} = -\dot{z}_s / 2 \), and \( \dot{z}_{\text{final}} = \dot{z}_s / 2 \). At these points, \( 2\mu v = \pm \pi / 2 \). Using Eq. (7), the energy gain per unit charge in a slip distance is

\[ W = -\frac{8E_0}{k\theta_d} \cos \phi_0 (r + 2r^3) \exp(-\pi^2 \gamma_z^2 / 4), \]

where

\[ r = \frac{\theta / \theta_d}{2 + \left( \frac{\theta}{\theta_d} \right)^2 + \frac{1}{\gamma_z^2 \theta_d^2}}. \]

The optimal crossing angle which maximizes the energy gain is determined by the condition \( \partial W / \partial (\theta / \theta_d) = 0 \), or using the chain rule \( (\partial W / \partial r) \cdot (\partial r / \partial (\theta / \theta_d)) = 0 \). Solving \( (\partial W / \partial r) = 0 \) yields one real root for \( r \), but when substituted into Eq. (9b), the corresponding solution for \( \theta / \theta_d \) is complex. The other partial derivative yields

\[ \frac{\partial r}{\partial (\theta / \theta_d)} = \frac{1}{2 + \left( \frac{\theta}{\theta_d} \right)^2 + \left( 1/\gamma_z^2 \theta_d^2 \right)^2} - \frac{2(\theta / \theta_d)^2}{\left( 2 + \left( \frac{\theta}{\theta_d} \right)^2 + \left( 1/\gamma_z^2 \theta_d^2 \right)^2 \right)^2} = 0, \]

which yields the optimal crossing angle to diffraction angle ratio

\[ \left( \frac{\theta}{\theta_d} \right)_{\text{opt}} = \sqrt{\frac{2 + \frac{1}{\gamma_z^2 \theta_d^2}}{\gamma_z^2 \theta_d^2}}. \]

The maximum energy gain per unit charge over a slip length is

\[ W_{\text{max}} = -\frac{4E_0}{k\theta_d} \cos \phi_0 \left[ \frac{1}{\left( \frac{\theta}{\theta_d} \right)_{\text{opt}}} + \frac{1}{2\left( \frac{\theta}{\theta_d} \right)_{\text{opt}}} \right] \exp \left[ -\frac{\pi^2 \gamma_z^2}{16\left( \frac{\theta}{\theta_d} \right)_{\text{opt}}^2} \right]. \]

In the limit \( \gamma_z \theta_d \rightarrow \infty \), the optimum crossing angle becomes \( \sqrt{2} \theta_d \), and the maximum energy gain is

\[ W_{\text{max}} (\gamma_z \theta_d \rightarrow \infty) = -\frac{5}{\sqrt{2}} \exp(-\pi^2 / 32) \frac{E_0 \cos \phi_0}{k\theta_d} \approx -2.6 \frac{E_0 \cos \phi_0}{k\theta_d}. \]

The optimal slip length (= accelerator cell length) corresponding to the optimal crossing angle is

\[ z_{s(\text{opt})} = \frac{\lambda}{4\theta_d^2} = \frac{\lambda}{2\theta_d^2} \left( \frac{\theta}{\theta_d} \right)_{\text{opt}}^2. \]

The optimal slip length divided by the Rayleigh length \( z_R = \pi^{-1} \theta_d^2 \lambda \) is

\[ z_{s(\text{opt})} = \frac{\pi / 4}{\left( 1 + \frac{1}{2\gamma_z^2 \theta_d^2} \right)^2}, \]

which is less than one for all \( \gamma_z \).
From Eqs. (12) and (14), one obtains the average gradient when the energy gain is maximized,

\[
G = \frac{W_{\text{max}}}{z_s(\text{opt})} = -\frac{4E_0\theta_d}{\pi}\cos\phi_0 \left[ (\theta / \theta_d)_{\text{opt}} + \frac{1}{2(\theta / \theta_d)_{\text{opt}}} \right] \exp \left[ -\frac{\pi^2}{16(\theta / \theta_d)_{\text{opt}}}^2 \right].
\]  

(16)

In the limit \( r / \theta_d \to \infty \), this gradient becomes

\[
G_{\text{inf}} = -\frac{5\sqrt{2}}{\pi} \exp\left(-\frac{\pi^2}{32}\right) E_0 \theta_d \cos\phi_0 \approx -1.65 E_0 \theta_d \cos\phi_0.
\]

(17)

It should be stressed that \( G \) in Eq. (16) is not the maximum gradient that can be achieved in the accelerator cell, but rather is the particular value when the energy gain is maximized. In fact \( W \) tends to be nearly constant for \( \theta / \theta_d \) near the optimal value, while the slip length \( z_s \) tends to fall off rapidly with increasing \( \theta / \theta_d \). Hence one can increase the gradient by increasing \( \theta / \theta_d \) somewhat above the optimum and still achieve about the same energy gain in a shorter accelerator cell. For a given laser strength \( E_0 \), which may be limited by surface damage thresholds, the accelerating gradient can be increased in this way. One finds that the gradient increases approximately linearly with \( \theta / \theta_d \) within the approximations of our analysis. In Fig. 6 the normalized energy gain from Eq. (9), the slip length and gradient are plotted as a function of \( \theta / \theta_d \) to illustrate this relation for the E-163 parameters. We have not plotted the exact energy gain calculated from the numerical integration of Eq. (1) for comparison in Fig. 6 since the curve would lie directly over the approximation.

![Figure 6](image)

**Figure 6.** Normalized energy gain \( W/(-4E_0\cos\phi_0/k\theta_d) \), slip length \( z_s/z_R \), and normalized gradient (ratio of these quantities) as a function of the laser crossing angle ratio \( \theta / \theta_d \) for the E-163 parameters. The optimal value of \( \theta / \theta_d \) for maximal energy gain in a cell is 1.94, but larger gradients are possible by increasing the crossing angle with little reduction in the energy gain.

In Table 1 we summarize the E-163 experimental parameters from the proposal [1] and the optimal values for the key parameters calculated from our analysis above. It should be noted that the design crossing angle of 11.5 mrad originally chosen for E-163 was determined by using the Sprangle high energy limit for the potential (3) and the slip length (2) for a 60 MeV electron, and then varying the crossing angle to achieve a maximum energy gain. This was the only method available without an analytic formula valid for arbitrary energy. The predicted energy gain using the previous method overestimates the energy gain by 20%
According to our analysis, this can be seen graphically in Fig. 5 where the potential in the high energy limit is significantly larger than the finite energy potential near the turning points.

Table 1: Parameters for the E-163 laser acceleration experiment.

<table>
<thead>
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<th>Parameter</th>
<th>Value</th>
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<td>$w$</td>
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<td>$z_R$</td>
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<td>$\theta_d$</td>
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<tr>
<td>$\gamma_z$</td>
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<tr>
<td>$(\theta/\theta_d)_{opt}$</td>
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</tr>
<tr>
<td>$G_{inf}$</td>
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</table>

4. Conclusion

Using an analytic result of Sprangle et al for the longitudinal accelerating field experienced by a relativistic electron from two crossed Gaussian lasers in vacuum, we have derived a new approximation for the accelerating potential valid for any energy provided the electron is traveling near the speed of light. The optimal laser crossing angle for maximum energy gain in a slip length, and the corresponding slip length were determined. It was shown that in the high energy limit, the optimal crossing angle is $\sqrt{2}$ times the diffraction angle, and the gradient is approximately 1.65 times the product of the individual laser field strength and the laser diffraction angle. The energy gain function has a broad maximum around the optimal crossing angle, while the slip length falls off rapidly with angle. As a result in the neighborhood of the optimal value, the gradient increases almost linearly with crossing angle. The calculations described in this paper have been written into a Mathcad program, available from the author, which allows the user to vary the key parameters and plot the corresponding accelerating fields and potentials.

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REFERENCES