Energy Efficiency in a Laser Driven Linear Accelerator
Part 1 – Single Bunch Beam

Beam power is the basic requirement of any high-energy linear collider application. Earlier calculations\(^1,2\) have shown that the charge that can be accelerated in a single bunch is small, and to reach interesting beam powers the beam pulse rate has to be high. This can be accomplished by recycling the laser energy, accelerating multiple beam bunches per laser pulse and operating the laser at a high repetition rate.

This note develops some of these ideas by starting with considerations of the efficiency for a beam single passage and then calculating the efficiency when the laser (and wakefield) energy at the end of the accelerating structure are recycled. Sections of this paper that will be written later will cover multiple bunches, with and without energy recycling, and other aspects of efficiency and beam power.

The first stages of this work benefited from a collaboration with Levi Schächter.

1. Single Bunch Efficiency

   \[ Z_{\text{int}} = \frac{G_0^2 \lambda^2}{P} \]  

   Note that \( G_0 \) and \( P \) are defined at any point along the length of the structure. The charge will radiate energy into this mode with a loss factor\(^3\)

   \[ \kappa = \frac{1}{4} \frac{c \beta_g}{1 - \beta_g} \frac{Z_{\text{int}}}{\lambda^2} \]  

   producing a decelerating gradient

   \[ G_F = \kappa q = \frac{q}{4} \frac{c \beta_g}{1 - \beta_g} \frac{Z_{\text{int}}}{\lambda^2} \]  

   \( \kappa \) is the loss factor per unit length of the structure. It is the beam-induced field in the accelerating mode, and, if the accelerating mode is narrowband, it is narrowband in frequency.

   In addition, there will be radiation into a wide frequency band from processes such as Cherenkov radiation. As an example, an effective impedance for this radiation has been found

---


by calculating the retarding field due to Cherenkov radiation in an infinite dielectric rod with a hole of radius $R$ in the center.\(^4\) The effective retarding gradient is

$$G_H = \frac{qcZ_H}{\lambda^2} \tag{1.4}$$

where

$$Z_H = Z_0 \frac{1}{2\pi (R/\lambda)^2} \tag{1.5}$$

$Z_H$ is defined in this way to show the dependence on the dimensionless parameter $R/\lambda$. This result holds for other geometries with $R$ being a characteristic separation from the particle to the structure boundary and a factor of order unity multiplying the expression for $Z_H$.\(^5\)

The beam-loaded gradient is

$$G = G_0 - G_F - G_H = \sqrt{\frac{PZ_{\text{int}}}{\lambda}} - \frac{qc}{\lambda^2} \left( Z_{H} + \frac{1}{4} \frac{\beta_g}{1-\beta_g} Z_{\text{int}} \right) \tag{1.6}$$

The increase in kinetic energy of the beam per unit length is

$$\frac{dU_{\text{kin}}}{dL} = qG \tag{1.7}$$

and the acceleration efficiency is

$$\eta = \frac{1}{P} \frac{\beta_g c}{1-\beta_g} \frac{dU_{\text{kin}}}{dL} \tag{1.8}$$

where $P/\beta_g c$ is the electromagnetic energy per unit length and the leading factor accounts for the fact that the electromagnetic wave is moving at $\beta_g$ while the particle is moving at $\beta=1$. The efficiency is an optimum when

$$q_{\text{opt}} = \frac{\lambda \sqrt{PZ_{\text{int}}}}{2c \left( Z_H + \frac{1}{4} \frac{\beta_g}{1-\beta_g} Z_{\text{int}} \right)} = \frac{G_0}{2 \left( cZ_H/\lambda^2 + \kappa \right)} \tag{1.9}$$

and for that charge the efficiency is

$$\eta_{\text{max}} = \frac{Z_{\text{int}} \beta_g}{4Z_H (1-\beta_g) + Z_{\text{int}} \beta_g} \tag{1.10}$$

For a different charge the efficiency and gradient are

$$\eta = \eta_{\text{max}} \frac{q}{q_{\text{opt}}} \left( 2 - \frac{q}{q_{\text{opt}}} \right) \tag{1.11}$$

$$G = \sqrt{\frac{PZ_{\text{int}}}{\lambda}} \left( 1 - \frac{q}{2q_{\text{opt}}} \right) \tag{1.12}$$

As an example, for the photonic band gap fiber accelerator of X. E. Lin \(^6\) \(Z_{\text{int}} = 19.5 \Omega\), \(\lambda = 1.05 \mu m\), \(\beta_g = 0.60\), and \(Z_H \sim 130 \Omega\) when the crystal is approximated as solid dielectric and variation of the dielectric constant with frequency is ignored. The damage threshold for a 10 cm long fiber is estimated to be \(P = 7.4 kW\), \(^6\) and the optimum charge and the efficiency at that power are \(q \sim 5 fC\), and \(\eta_{\text{max}} \sim 0.05\), respectively. This example shows that the maximum efficiency can be rather high, but the accelerated charge is low. This is a consequence of the proportionality \(q_{\text{opt}} \propto \lambda\) in eq. (1.9), and it will be true for any laser driven accelerator in general.

The charge at which the beam induced voltage in the fundamental is one-half of the unloaded gradient

\[
q_F = \frac{G_0}{2k}
\]  

(1.13)

is a quantity that will appear latter. It is also given by eq. (1.9) with \(Z_H = 0\), and using that expression one can show

\[
q_F = \frac{q_{\text{opt}}}{\eta_{\text{max}}}
\]  

(1.14)

1.2 Energy Conservation for a Single Pass

Consider a single passage through the fiber where the bunch enters at \(t = 0\). The laser pulse must extend over the time range

\[
T_0 = - \frac{L}{c} \left( \frac{1 - \beta_g}{\beta_g} \right) \leq t \leq 0
\]  

(1.15)

at the input to overlap the beam while it is in the fiber. The input power is

\[
P_{\text{in}}(t) = \frac{G_0^2 \lambda^2}{Z_C} s(t) = P_{\text{in}} S(t)
\]  

(1.16)

where \(G_0\) is the value of the gradient. The time dependence is given by

\[
S(t) = \Theta \left( t + \frac{L}{c} \left( \frac{1 - \beta_g}{\beta_g} \right) \right) - \Theta(t);
\]  

(1.17)

where \(\Theta\) is the step function.

Assuming that there are no losses in the fiber, the power at the exit of the fiber is

\[
P_{\text{out}}(t) = \left| \frac{G_0 - qE_W^F}{Z_{\text{int}}} \right|^2 \lambda^2 s(t - L/\beta_g c) = P_{\text{out}} S(t - L/\beta_g c)
\]  

(1.18)

where \(E_W^F\) is the beam induced field in the fundamental mode per unit charge. Energy conservation applied to this single passage is

\[
U_{\text{in}} + \int_{-\infty}^{\infty} P_{\text{in}}(t) dt = U_{\text{out}} + qG_H L + \int_{-\infty}^{\infty} P_{\text{out}}(t) dt
\]  

(1.19)

---

where \( U_{in} \) and \( U_{out} = U_{in} + \Delta U_{kin} \) are the initial and final kinetic energies, respectively, and the second term of the right-hand-side accounts for the radiation into higher modes. Using eq. (1.7) and performing the integrals gives

\[
qL(G_0 - G_F) = \left( P_{in} - P_{out} \right) \frac{L}{c} \frac{1 - \beta_g}{\beta_g} \tag{1.20}
\]

Substituting

\[
q(G_0 - G_F) = \frac{1}{c} \frac{1 - \beta_g}{\beta_g} \frac{\lambda^2}{Z_{int}} \left( 2qG_qE^F_L - \left( qE^F_L \right)^2 \right) \tag{1.21}
\]

Both sides of this equation have terms linear and quadratic in \( q \). Equating the linear terms gives

\[
E^F_L = \frac{1}{2} \frac{c\beta_g}{1 - \beta_g} \frac{Z_{int}}{\lambda^2} = 2\kappa \tag{1.22}
\]

where \( \kappa \) is the loss factor given in eq. (1.2). The factor of two that appears is the factor expected from the Fundamental Theorem of Beam Loading\(^7\) that says that the self-field is one-half of the beam induced field. Equating the terms that are quadratic in \( q \) gives

\[
G_F = \frac{1}{c} \frac{1 - \beta_g}{\beta_g} \frac{\lambda^2}{Z_{int}} qE^2_L \tag{1.23}
\]

This is an identity when the definition of \( \kappa \) from eq. (1.2) and the result in eq. (1.22) are substituted.

Definitions of powers in energy recycling. \( g \) is the power gain of the active medium, \( P_{act} = gP_{out} \). The summing junction is a reciprocal device with a reflection coefficient \( r \) and \( S \) matrix given in the text.

### 2. Energy Recycling for a Single Bunch

#### 2.1 No Resistive Losses:

Powers are defined in the figure. Eqs. (1.12) and (1.9) hold when there is energy recycling as long as \( P \) is the input power. I.e.

\[
G = \sqrt{\frac{P_{in}Z_{int}}{\lambda}} \left( 1 - \frac{q}{2q_{opt}} \right) \quad \text{and} \quad q_{opt} = \frac{\lambda \sqrt{P_{in}Z_{int}}}{2c} \frac{1}{Z_H + \frac{1}{4} \frac{\beta_g}{1 - \beta_g} Z_{int}} \tag{2.1}
\]

The efficiency is the kinetic energy gain of the beam divided by the energy added per cycle. There is energy added by the external laser and by the active medium in the feedback, and there is energy recovered from the output. The efficiency when \( g \geq 1 \) is

\[
\eta = \frac{qGL}{\left( \frac{L}{c} \left( 1 - \beta_g \right) \right) \left( P_L + P_{act} - P_{out} \right)}
\]

(2.2)

Definitions of incident (\( a \)) and reflected and transmitted (\( b \)) fields that are related by \( b = Sa \)

The beam/splitter combiner is a reciprocal device with an \( S \) matrix that describes it.\(^8\) \( S \) acts on fields that are defined in the figure. Assuming there is no back reflection and no transmission in unfavored directions, \( S \) is given by\(^9\)

\[
S = \begin{pmatrix} 0 & jr & \sqrt{1-r^2} & 0 \\ jr & 0 & 0 & \sqrt{1-r^2} \\ \sqrt{1-r^2} & 0 & 0 & jr \\ 0 & \sqrt{1-r^2} & jr & 0 \end{pmatrix}
\]

(2.3)

The quantity \( r \) is the magnitude of the reflection from arm 4 to arm 3 (and also from arm 1 to arm 2). The fields input on the splitter in arms 1 (the external laser) and 4 (the active medium) will interfere in arms 2 and 3. The fields in arms 2 and 3 depend on i) the amplitudes and relative phase of the fields in arms 1 and 4 and ii) the value of \( r \).

The input power to the accelerator \( P_{in} \) and the output power \( P_{out} \) are pulse functions in time, eqs. (1.16) and (1.18). The field input to the accelerator are

\[
b_3(t) = \sqrt{P_{in}}S(t)
\]

(2.4)

where the phase of the this field is real. The field out of the active medium is

\[
a_4(t) = \sqrt{gP_{out}}e^{i\phi}S(t - L/\beta_g c - T_D)
\]

(2.5)

where \( T_D \) is the delay in the active medium. This output power is the same on each pulse in the steady state solution, so

\[
a_4(t) = \sqrt{gP_{out}}e^{i\phi}S(t)
\]

(2.6)

---

\(^8\) Chapter 11 of A. E. Siegman, *Lasers* (University Science, Mill Valley, Ca, 1986)

\(^9\) R. H. Siemann, ARDB 352, "Beam Splitter/Combiner", October 2003 that contains a calculation based on Siegman.
The external laser field is
\[ a_i(t) = \sqrt{P_L} e^{i\phi_i} S(t) \]  
(2.7)

The output fields are given by
\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 
\end{pmatrix} = S
\begin{pmatrix}
  \sqrt{P_L} e^{i\phi_i} S(t) \\
  0 \\
  0 \\
  \sqrt{gP_{out}} e^{i\phi_i} S(t) 
\end{pmatrix}
\]  
(2.8)

From this equation, the field in the load arm is
\[ b_2 = \left( j r \sqrt{P_L} + \sqrt{1-r^2} e^{i(\phi_i - \phi)} \sqrt{gP_{out}} \right) e^{i\phi_i} S(t) \]  
(2.9)

There will be no field in this arm if
\[ \phi_4 - \phi_1 = -\frac{\pi}{2} \]  
(2.10)

and the coating on the beam splitter is chosen such that
\[ \frac{r}{\sqrt{1-r^2}} = \frac{gP_{out}}{P_L} \]  
(2.11)

The field input to the accelerator structure is
\[ b_3(t) = \sqrt{P_{in}} S(t) = \left( \sqrt{1-r^2} \sqrt{P_L} + j r e^{i(\phi_i - \phi)} \sqrt{gP_{out}} \right) e^{i\phi_i} S(t) \]  
(2.12)

The phase relation given by eq. (2.10) for destructive interference in the load arm gives constructive interference at the accelerator input, and for that phase relation
\[ P_{in} = \left( \sqrt{1-r^2} \sqrt{P_L} + r \sqrt{gP_{out}} \right)^2 \]  
(2.13)

In addition, if \( r \) is given by eq. (2.11),
\[ P_{in} = P_L + gP_{out} \]  
(2.14)

I.e. all of the power flowing into the beam splitter/combiner flows into the accelerator as expected from energy conservation.

In terms of the single pass efficiency for the same \( P_{in} \), given by eq. (1.11), denoted here as \( \eta \),
\[ \eta = \frac{P_{in}}{P_L + (g-1)P_{out}} \eta_i \]  
(2.15)

The efficiency will be a maximum when \( r \) is given by eq. (2.11); then all of the power from the external laser and from the active medium is flowing into the accelerator. Taking this value for \( r \) and substituting for the powers and using eq. (1.11) gives
\[ \eta = \frac{P_{in}}{P_{in} - P_{out}} \eta_i = \frac{G_0^2}{4\kappa q(G_0 - \kappa q)} \eta_{max} \frac{q}{q_{opt}} \left( 2 - \frac{q}{q_{opt}} \right) \]  
(2.16)

Using the expression for \( q_F \), eq. (1.13), and the relationship in eq. (1.14) results in
\[ \eta = \frac{2-q/q_{opt}}{2-q/q_F} \]  
(2.17)
The efficiency is unity for zero charge, and it equals zero when \( q = 2q_{opt} \). When \( q = q_{opt} \), \( \eta \approx 0.5 \) as would be expected because as much energy is radiated as is gained by the beam.

### 2.2 Including Resistive Losses:

A straightforward way to include resistive losses is to assume that these losses are proportional to the input power

\[
P_{\text{loss}} = \Delta P_{in}
\]

Then eq. (2.2) becomes

\[
\eta = \frac{qG\beta_{g,c}}{(1 - \beta_g)(P_{in} - P_{out} + P_{\text{loss}})} = \frac{qG\beta_{g,c}}{(1 - \beta_g)(P_{in}(1 + \Delta) - P_{out})}
\]

The same calculation as that leading to eq. (2.17) can be performed. The result is

\[
\eta = \frac{q/q_F(2 - q/q_{opt})}{\Delta + q/q_F(2 - q/q_F)}
\]

The result is plotted below for different values of \( \Delta \).