Analytic and Simulation Studies of a W-Band Planar Ubitron

David H. Whittum

This is the sixth note in a series summarizing the theory for low-voltage ubitron design at W-Band. The interest in this note is to synthesize the considerations of the previous notes to arrive at a ubitron design parameter set taking into account all features previously discussed. In the course of this we compare the predictions of the analytic dispersion relation and reduced "1D" simulations.

(This note is written in cgs units.)

Introduction

In this note theoretical studies of a W-Band Planar Ubitron are described for the purpose of summarizing the considerations behind an optimized tube design. The performance goal for the first tube is 5 MW in a 1 μs pulse; here we arrive at an optimized parameter set corresponding to a 480 kV, 290 A beam and other parameters as listed in the conclusions.

1D Ubitron Model

We will refer to the models discussed in this note as "1D" models. However, it will quickly become clear that these models take into account all the important 3D aspects of the device as they pertain to gain, for the case of perfect current transmission. Such a model is only the beginning of the simulation work, particularly since current transmission is critical; a later note will discuss particle-in-cell simulations. To fashion the "1D" model we need merely synthesize the work of the previous notes. Our optimization will proceed over a rather large parameter space, and much of the discussion following consists of defining the various parameters, and relating them to gain.

Waveguide

The first dimensional choices we make are the interior dimensions of the rectangular waveguide. The width in $x$ is $a$ and the height in $y$ is $b$, with $a > b$. The width is larger in $x$ since this is the direction of wiggle motion, and a large dimension here favors good current transmission over a range of wiggler field settings. We operate in TE_{01} mode and we neglect waveprofile modification in our treatment here (but not the PIC simulation). Related waveguide quantities are

\[ \Sigma = \frac{1}{2} ab, \]  
\[ k_{\perp} = \frac{\pi}{b}, \]  
\[ k_z = \left( \frac{\omega}{c} \right)^2 - k_{\perp}^2 \right)^{1/2}. \]

Also related to this mode are the squared beam wavenumber,

\[ k_b^2 = \frac{4\pi}{\Sigma} \left( \frac{I}{I_0} \right) = \frac{8\pi}{ab} \left( \frac{I}{I_0} \right), \]

with $I$ the beam current, and $I_0 = mc^3/e = 17.0$ kA. The normalization for power flowing
in the mode is
\[ P_{rf} = P_0 \frac{ab}{16\pi c} \left( \omega \kappa_z \right) |\tilde{a}|^2, \]
in terms of the complex eikonal \( \tilde{a} \). The constant \( P_0 = m^2 c^5 / e^2 = 8.701 \text{ GW} \). The system of equations above has an integral, corresponding to conservation of energy, and this may be expressed as
\[ \Delta P_{tot} = \frac{mc^2}{e} I(\gamma(z) - \gamma(0)) + P_{rf}(z) - P_{rf}(0), \]
This quantity is a constant of the motion and equal to zero absent numerical error. An estimate of numerical error in the code is then \( \Delta P_{rf}/P_{rf} \). This error will be quite small, particularly compared to errors implicit in the model (e.g., relativistic approximation).

We will assume 1 kW input power for these runs; this would correspond, for example, to use of an EEV MG5335 magnetron (rated at 2.5 kW). We do this while hoping of course for the eventual development of a higher power input source. However, it appears more prudent to assess wiggler lengths and related quantities based on the worst case.

**Envelope**

The waveguide we have chosen represents an aperture through which we must pass the beam. We consider a uniform density elliptical beam with semi-axes \( X \) and \( Y \). We fix
\[ X = \frac{a}{4}, \quad Y = \frac{b}{4}, \]
as a conservative means of insuring good current transmission. This corresponds to the beam filling 1/4 of the waveguide aperture. We must supplement this caveat with a statement concerning emittance and focusing. In our studies here, we will be scanning wiggler field settings at fixed beam size. This amounts to treatment of emittance as a variable. Later, in the course of PIC simulations informed by these studies, we will fix emittances at reasonable values, and assess actual envelope and current behavior over a range of magnet settings.

Envelope behavior is investigated in detail in a previous technical note.\(^1\) The matched beam envelope corresponds to an equilibrium relating the injected edge emittances to the beam edge dimensions.
\[ \varepsilon_x = k_{\beta x} X^2, \quad \varepsilon_y = k_{\beta y} Y^2, \]
with effective squared betatron wavenumbers,
\[ k_{\beta x}^2 = k_x^2 - \frac{2K}{X(X + Y)}, \quad k_{\beta y}^2 = k_y^2 - \frac{2K}{Y(X + Y)}. \]
These space-charge reduced focusing terms are determined from the focusing due to the wiggler and the quadrupole,
\[ k_x^2 = \frac{1}{\gamma^2 \beta_z mc^2} \frac{\partial B_y}{\partial x}, \quad k_y^2 = \frac{1}{2} \left( \frac{eB_w}{\gamma^2 \beta_z mc^2} \right)^2, \]
and the normalized perveance
\[ 1 \quad D. H. Whittum, "Beam Envelope in a Planar Ubitron Amplifier", ARDB TN 112. \]

\(^1\) D. H. Whittum, "Beam Envelope in a Planar Ubitron Amplifier", ARDB TN 112.
\[ K = \frac{2(I_b / I_0)}{\beta_z^2 \gamma^2} , \]

with \( I_b = \pi a b e n_a c \beta_z \). The quantity,
\[
\frac{1}{\gamma_z^2} = 1 - \beta_z^2 ,
\]

and
\[
\beta_z = \left(1 - \frac{1 + a_w^2}{\gamma^2}\right)^{1/2}.
\]

The rms wiggler parameter is \( a_w = \hat{a}_w / 2^{1/2} \), where
\[
\hat{a}_w = \frac{eB_w}{m c k_w} ,
\]

\( m c^2 / e = 1.7 \text{ kG cm} \), and \( B_w \) is the peak on-axis wiggler magnetic field.

**Beam Quality**

Our interest in beam quality derives from the importance of the distribution in detuning from resonance. Recall that phase-slip or detuning for a particular electron is determined by the averaged axial drift velocity,
\[
\frac{d\theta}{dz} = k_z + k_w - \omega V_z ,
\]

and
\[
V_z \approx 1 - \frac{1}{2 \gamma^2} \left(1 + \gamma^2 \frac{\bar{V}_z^2}{c^2}\right) .
\]

Thus energy spread, as well as emittance contribute to the width of the distribution in detuning. True energy spread is governed by space-charge, and scales with the beam self-potential
\[
\frac{e \phi}{mc^2} = \frac{\pi}{2} \frac{I}{I_0} \left(\frac{Y}{X}\right) .
\]

This is of the order 0.2% for parameters we will consider, and is quite small. More significantly, emittance contributes an "effective energy spread" (by which is meant, a spread in detuning), and this may be determined from
\[
\delta \frac{V_z}{c} \approx \frac{1}{2 \gamma^2} \frac{\bar{p}_z^2}{m^2 c^2} = \frac{1}{2} \left( k_{\beta_x}^2 x^2 + k_{\beta_y}^2 y^2 \right) ,
\]

where the bars indicate an average over the betatron motion, and we are somewhat cavalier with non-relativistic corrections, as the effect we are considering is already a higher-order term in the motion. The average over the ensemble is
\[
\delta \left(\frac{V_z}{c}\right) \approx \frac{1}{4} \left( k_{\beta_x} \varepsilon_x + k_{\beta_y} \varepsilon_y \right) ,
\]

and this is roughly the half-width in the distribution. One could consider the distribution at
greater length, concern oneself with more elaborate models. However, one purpose for the PIC simulation is the rigorous accounting for such particulars. Our dispersion relation will assume a distribution in energy,

$$\gamma_c - \Delta \gamma < \gamma < \gamma_c + \Delta \gamma$$

or,

$$\frac{\delta V_c}{c} = \pm \frac{\Delta \gamma}{\gamma} = \pm \left\{ \frac{\delta \gamma}{\gamma} + \frac{1}{8} \left( k_{\parallel} e_{ex} + k_{\parallel} e_{ey} \right) \right\}$$

and evidently the combination of emittance and energy spread corresponds to an effective energy spread

$$\frac{\Delta \gamma}{\gamma} \approx \frac{\delta \gamma}{\gamma} + \frac{1}{8} \gamma^2 \left( k_{\parallel} e_{ex} + k_{\parallel} e_{ey} \right).$$

Typical numbers, as we will see, might correspond to $\gamma \approx 1.94$ (480 kV), $k_{\parallel} Y < k_{\parallel} X = 0.1$, and

$$\frac{\Delta \gamma}{\gamma} \approx 0.2\% + 0.9\%.$$ 

Thus we may expect fractional effective energy spread on the order of 1%. This is a noticeable correction, but does not significantly reduce peak power at saturation.

**RF Space-Charge**

The formalism for rf space-charge has been described in a previous note.\(^2\) We set down in Appendix A, the particular space-harmonic fill factors for an elliptical beam of high aspect ratio.

$$\eta_n = 1 - e^{2|\kappa_n Y|} \left( \frac{e^{-|\kappa_n Y|} + e^{\kappa_n Y}}{1 + e^{2|\kappa_n Y|}} \sinh \left( \frac{\kappa_n Y}{|\kappa_n Y|} \right) \right).$$

For $Y = a/2$ this reduces to

$$\eta_n = 1 - \frac{\tanh \left( \frac{a}{2} \kappa_n a \right)}{\left( \frac{a}{2} \kappa_n a \right)}.$$ 

Typical numbers would be

$$\gamma = 1 + \frac{480kV}{511kV} = 1.94, \quad \beta_z \approx \sqrt{1 - \frac{1 + a^2}{\gamma^2}} \approx 0.83,$$

$$\kappa_1 = \frac{\omega}{g_\beta \gamma c} = \frac{2\pi}{0.33cm} \frac{1}{0.83 \times 1.94} \approx 11.8cm^{-1}.$$ 

For a beam extending over ±0.1 cm, and waveguide boundaries at ±0.2 cm, we have $\kappa_1 Y \approx 1.18, \ k_1 a \approx 2.37, \text{ and } \eta_1 = 0.58.$ The result would have been $\eta_1 \approx 0.62$ for waveguide boundaries at infinity. Thus the boundaries produce a small correction.

"1D" Model

With these quantities selected we may set down our "1D" model for longitudinal bunching:

\[
\frac{d\theta}{dz} = k_z + k_w - \frac{(\omega/c)}{\beta} \left\{ 1 - \frac{\tilde{a}_w f_2}{2(\gamma\beta)^2} \Re(\tilde{a}\exp(i\theta)) \right\},
\]

\[
\frac{d\gamma}{dz} = -\frac{(\omega/c)}{2\gamma\beta} f_1 \tilde{a}_w \Im(\tilde{a}\exp(i\theta)) - \hat{E}(\theta),
\]

\[
\frac{\partial \tilde{a}}{\partial z} = i\tilde{\nu}_{\tilde{a}} \left( \frac{1}{\gamma\beta} \right) + i\tilde{\nu}_{\tilde{w}} \left( \frac{f_1 \exp(-i\theta)}{\gamma\beta} \right),
\]

where \( \hat{E} = eE_z/mc^2 \) is

\[
\hat{E}(\theta) = -\frac{\nu_{sc}}{\langle \beta \rangle} \Im \left\{ \sum_{n=1}^{\infty} \frac{\eta_n}{n} \exp(in\theta) \exp(-in\theta) \right\},
\]

with

\[
\nu_{sc} = 8\pi \left( \frac{1/I_0}{k_w + k_z} \right) A_b,
\]

and \( A_b \) is the effective beam cross-section,

\[ A_b = \pi XY. \]

Other quantities are

\[
f_1 = J_0(\hat{\theta}) \left[ 1 + \frac{1}{4} \varepsilon \right] - J_1(\hat{\theta}), \quad f_2 = J_0(\hat{\theta}) \left[ 1 + \frac{3}{4} \varepsilon \right] - J_1(\hat{\theta}),
\]

\[ \hat{\beta} = \left\{ 1 - \frac{1 + a_w^2}{\gamma^2} \right\}^{1/2}, \quad \varepsilon = \left( \frac{a_w}{\gamma_0\tilde{\beta}_{0s}} \right)^2,
\]

\[ \hat{\theta} = \frac{(\omega/c)}{4k_w\tilde{B}_{0s}} \varepsilon = \frac{(\omega/c) a_w^2}{4k_w\gamma_0^2(\tilde{\beta}_{0s})}, \quad \hat{\nu} = 4\pi(1/I_0)/k_\nu a b.
\]

referred to elsewhere as \( \hat{\theta}_2 \).

This model is employed first by loading a selection of \( N \approx 128 \) particles with representative values of phase-space coordinates \( (\theta, \gamma) \). We will take a uniform distribution in angle, distributed over \([ -\pi, \pi ]\), and a uniform distribution \([ \gamma_\nu - \Delta \gamma, \gamma_\nu + \Delta \gamma ]\). This distribution is symmetrized for "quietness" as described in a previous technical note. The initial value for the eikonal amplitude is determined from the specified input power. The phase we take to be zero. Eventually studies of phase sensitivity will be of interest, and in this case, the phase of the amplified output eikonal contains the useful information, referred to the input phase. To run the simulation one wants to select values for the various parameters first, and for this we need to consult the dispersion relation. Indeed, we may just as well ask the same code to consult and solve the dispersion relation, plot the solution and perform a set of runs to compare numerical with analytic results. This code we will refer to as WUBII.
Dispersion Relation

We will employ what was previously referred to as the "Warm-Compton-With-Space-Charge" approximation to the full transcendental dispersion relation, reducing it to a cubic,

\[ \Gamma^3 + A_2 \Gamma^2 + A_1 \Gamma + A_0 = 0, \]

The quantity \( \Gamma \) is the exponential gain in the eikonal amplitude. The power gain \( G \) in decibels (dB) per unit length is given by \( G = 8.69 \Gamma \). The coefficients have been derived in a previous note.\(^3\)

\[
\begin{align*}
A_2 &= 2\kappa_c - \hat{\nu}, \\
A_1 &= \kappa_c^2 - 2\hat{\nu}\kappa_c - \frac{1}{4}\kappa^2 + N - \Gamma_{sc}^2, \\
A_0 &= M - \hat{\nu}(\kappa_c^2 - \Gamma_{sc}^2) + N\kappa_c + \frac{1}{4}\hat{\nu}\kappa^2.
\end{align*}
\]

Quantities are

\[
\begin{align*}
\kappa_c &= k_c + k_w - \frac{(\omega / c)}{\hat{\beta}}, \\
\kappa &= 2(\omega / c)\frac{\gamma^2}{(\gamma^2 - \gamma_{-}^2)^{1/2} \Delta \gamma}, \\
\hat{\beta} &= \left(1 - \frac{\gamma_{-}^2}{\gamma_0^2}\right)^{1/2}, \\
\gamma_0 &= (1 + a_w^2)^{1/2}, \\
M &= \frac{k_b^2}{4k_c} a_w^2 f_1^2 (\omega / c)^2 \frac{\gamma^2}{(\gamma^2 - \gamma_{-}^2)^{3/2} \Delta \gamma}, \\
N &= \frac{k_b^2}{2k_c} a_w^2 f_3^2 (\omega / c) \frac{\gamma}{(\gamma^2 - \gamma_{-}^2)^{3/2}}, \\
\hat{\Gamma}_{sc} &= \frac{v_{sc} \eta \bar{\eta} \tilde{e}}{2(\hat{\beta})}, \\
\Gamma_{sc}^2 &= \frac{(\omega / c) \gamma_{-}^2}{2(\hat{\beta}) (\eta \beta)} v_{sc} \eta \tilde{e}.
\end{align*}
\]

Other quantities are \( f_3^2 = f_1(f_1 + f_2)/2 \), \( f_4 = 0.5 f_1 / \tilde{f} \), \( f_5 = 0.5 f_2 / \tilde{f} \) with \( \tilde{f} = 0.5(f_1 + f_2) \).

Parameter Optimization via the Dispersion Relation

Employing the cubic dispersion relation for the exponential gain we may arrive at a parameter set for the ubitron. We keep in mind that output power at saturation for an untapered wiggler scales as

\[ P_{sat} = \rho P_{beam} \]

\(^3\)David H. Whittum, "Space-Charge Corrections in a Planar Ubitron Amplifier" ARDB Technical Note 111
where the beam power is

$$P_{beam} = \frac{mc^2}{e}(\gamma - 1)I,$$

and $\rho$ is the FEL Pierce parameter

$$\rho = \frac{1}{2k_w}M^{1/3} \approx \pi^{1/3} \left( \frac{I}{\gamma I_0} \right)^{1/3} \left( \frac{a_{rel}}{k_w^2 \Sigma} \right)^{2/3},$$

with $k_w = k_w + k_z - (\omega/c)$. Keeping in mind that this estimate applies to the case of no energy spread (or space-charge), we nevertheless employ it to gauge the power available at saturation. The more rigorously computed quantity, $\Gamma$, we will expect to agree closely with the results of simulation.

Keeping in mind the goal of employing permanent magnet based wiggler, we take note of the Halbach scaling (for NdFeB with Vanadium-Permendur) giving peak wiggler field in Tesla, in terms of the wiggler gap and period,

$$B_w = 3.44 \exp \left( -\frac{g}{\lambda_w} \left( 5.08 - 1.54 \frac{g}{\lambda_w} \right) \right).$$

Carr observes that the coefficient may be increased by a factor 1.13. This relation constrains our choice of waveguide dimension, assuming $g=b+2\Delta$, where $\Delta$ is the waveguide wall thickness.

To arrive at a working parameter set, we first optimized peak gain on a spreadsheet. This is somewhat faster than performing runs with a 1D code, and thus is helpful for putting the parameters in the right ballpark.

**1D Simulation: "WUBI1"**

In this way we arrive at an initial parameter set, that we may depict in the form of the input file for WUBI1, WUBI1.IN

```plaintext
$INPUTNML
BWCH=3., !plot P vs z at this Bw (kG), or close to
SCON=1., !=0 for spacecharge off, = 1 for on
EMITON=1., !=0 for emittance dgamma off, =1 for on
VOLTB=0.48, !beam voltage MV
DELP=1., !% spread in gamma +- 
XLW=1.16, !wiggler period in cm
AGUIDE=4.0, !guide width in cm
BGUIDE=0.4, !guide height in cm
FREQ=91.392e9, !frequency in Hz 
CURR=290., !beam current in A
ZMAX=100., !max length in cm
POWIN=1.e3, !input power W 
SEMINX=0.5, !0-1,ratio of x semi-axis to half guide width 
SEMIY=0.5, !0-1,ratio of x semi-axis to half guide height 
EMITNX=0.2, !cm-rad, will adjust focusing to obtain semix 
EMITNY=0.05, !cm-rad, will adjust focusing to obtain semiy 
SWITCH1=1., !=1, or 0 for debugging of dpsi/dz correction
$END
```
For this input file we have adopted an energy spread of 1%; this adds in quadrature with the 'effective energy spread" due to emittance.

The code reads \texttt{WUBI1.IN}, and employs the dispersion relation to scan wiggler field, locating the range of wiggler field over which gain occurs (or, up to about twice the resonant wiggler field, whichever is smaller). After this scan, it adopts a collection of 30 or so wiggler field settings, computes gain at each setting from the dispersion relation. Finally, at each wiggler setting, the code integrates the 1D model equations, using a 2nd-order Runge-Kutta integration, with 200 steps in $z$, and an ensemble of $N = 1152$ macroparticles. The code outputs the maximum power obtained at any $z$, for each wiggler setting. This is the saturation power for this wiggler setting, provided $Z_{\text{MAX}}$ has been chosen large enough. At the setting corresponding to maximum gain (as predicted from the dispersion relation) the code plots power versus $z$. At each wiggler setting, the code also tries to fit the power curve with an exponential, in a region corresponding to $100 \, P_{\text{in}} < P < 0.1 \, P_{\text{max}}$. If this inequality can't be satisfied, it tries to fit in the region $100 \, P_{\text{in}} < P < P_{\text{max}}$. (One could try to put a finer point on it.) For the fitting, we employ the routine, \texttt{fit}, from Numerical Recipes\footnote{W.H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, \textit{Numerical Recipes, The Art of Scientific Computing}, (Cambridge University Press, Cambridge, 1992) p. 659.} and perform a least-squares fit to $y = \ln(P)$ versus $z$. The quantity

$$\delta_{y} = \ln(P + \delta P) - \ln(P) = \frac{\delta P}{P} \ln(P),$$

provides a measure of the expected error in the numerical output, where $\delta P$ is the full error in the power integral. The routine fit returns the rms deviation in the fit, and this we will employ for the error bar in our gain plots. For power plots we employ $\delta P$ as the error bar (One could argue that $\delta P/2$ is a better choice; in fact, however, these quantities are for the most part checks of the coding.)

One item should be noted: the code treats the input beam size and emittance as fixed. Focusing is adjusted to accomplish this, even when the required focusing is unrealistic, although in this case, an error message is issued. This is for simplicity, and is suitable because we have checked out the parameters with the spreadsheet in advance.

As a check of the large quantity of algebra involved in deriving the cubic approximation to the dispersion relation, let us first consider an example where space-charge is reduced, corresponding to the "warm, Compton" approximation implicit in the cubic. We consider $\text{SEMIX}=\text{SEMIY}=1.0$, corresponding to an artificially large beam. As seen in Fig. 1, agreement is good. The deviations at high wiggler field can be made smaller by increasing the particle number.

Shown in Fig.2 is the result for gain versus wiggler field, for the parameters listed in the print-out of \texttt{WUBI1.IN} above. Fig. 3 depicts saturation power, and power error. Fig. 4 depicts the evolution of power in $z$ for several different wiggler field settings. Peak power is 5 MW, at 3.3 kG, with gain of 80 dB/m and saturation in 0.8 m for 100 W of input power.
**FIGURE 1.** Comparison of the analytic result of the “warm, Compton” cubic dispersion relation, with the simulation WUBI1. For this run the beam was artificially doubled in size, to reduce the effect of space-charge, for a straight comparison with the cubic approximation.

**FIGURE 2.** Comparison of the analytic result of the “warm, Compton” cubic dispersion relation, with the simulation WUBI1, for the working example parameters listed in the print-out of WUBI1.IN above.
FIGURE 3. Saturation power and power error from WUBI1, for the parameters listed in WUBI1.IN above.

FIGURE 4. Power evolution in z, from WUBI1, for the parameters listed in WUBI1.IN above, and several wiggler field settings. Peak gain corresponds to 3.3 kG.
Summary

This note has collected the results of the previous notes in this series, and summarized the design scalings for a low-voltage ubitron amplifier. The code WUBI1 will be useful in checking out the results of a more realistic PIC simulation, the subject of the next note.

Let us summarize our working design parameter set,

**Wiggler Parameters**

- Wiggler period $\lambda_w$ 1.16 cm
- Peak wiggler field $B_w$ 5.5 kG
- Minimum # of wiggler periods
- Conservative # of wiggler periods $N_w$
- Wiggler tolerances to be determined
- Desirable tuning range 3kG-7kG

**Beam Parameters**

- Voltage $V_b$ 480 kV
- Current $I_b$ 300 A
- Beam full width $2X$ 2 cm
- Beam full height $2Y$ 0.2 cm

**Waveguide & RF Parameters**

- Height, inner dimension $b$ 0.4 cm
- Width, inner dimension $a$ 4 cm
- Input power $P_{in}$ 1 kW
- Gain 40 dB
- Required match to output $VSWR < 1.002$

Taking into account the studies here and in the previous notes of this series, we...
may also summarize some of the planar ubitron design problems as follows

**Gun:** section of a cylindrical diode
480 kV, 300 A
beam is 2 cm x 0.2 cm at the anode
assume cathode loading of 100 A/cm²
convergence of x7.5 produces 750 A/cm² at anode
convergence of 8 corresponds to 40° half-angle
1.247 cm cathode radius,
0.154 cm anode radius
gun field too large, will arc -> 3D Gun Design

**Waveguide:** 4 cm x 0.4 cm
vertical clearance limited by eventual adoption of permanent magnets,
requires high laminarity in the vertical from the gun

**Wiggler:** pulsed "aircore", water-cooled
wiggler period of 1.16 cm
peak wiggler field on-axis: min 5.5 kG, 6-7 kG preferred
first two-half periods independently powered
last two-half periods independently powered

**Quadrupole:** "eight wire" (four-coil) dc electromagnet
water-cooled

**Input Signal:** require 2 kW source to keep gain under 40 dB,
prefer 50 kW source
launch mode via WR10 taper and corner reflector with beam entry slit cut off to RF,
multi-hole input coupler (Henke) should be studied

**Collector:** considering dispersing beam on side-wall
with the help of a short quad focusing in the vertical at 10 Hz, 150 J beam dispersed over 5 cm in z,
and 4cm in x corresponds to 40 W/cm²

**Output:** power in excess of 5 MW
for tube #1, horn to anechoic chamber
VSWR<1.002 to inhibit oscillation
phase and amplitude flatness remain to be studied
Problems for future work include, in addition to a write-up of PIC work thus far,

**Realistic Gun Design** - most likely a 3D gun design is required

**Code work:** higher mode diagnostics
additional analytic code checks
comparison of code with other experiments

**Code Studies:**
tolerance to fabrication errors
transmission tolerance to beam quality
phase error vs beam voltage error
launch of higher modes & gain
parasitic gain of higher modes
oscillation

**Magnet System:** ---layout, circuits, cooling---
number of wiggler segments
placement of cooling channels
electrical, mechanical design
including tolerances

**Collector Design**

**Other Items**
output coupler design
estimate spontaneous emission and noise figure
feedback circuits for phase-stabilization
consider other modulators
operation with modified 5045 gun, lower current
matching at input coupler
Appendix A - Space-Charge Fill Factors, Planar Beam

Much as in a klystron tube, a bunched beam in a ubitron is influenced by the repulsion of electrons, and this space-charge effect is also influenced by the waveguide geometry. For this work we consider the case of a sheet beam (width $X$ very large) and we calculate the axial electric field of a periodic bunch train, including waveguide corrections.

Maxwell's Equations may be reduced to a single wave equation for the axial electric field,

$$
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E_z = \frac{4\pi}{c} \left( \frac{\partial J_z}{\partial t} + c^2 \frac{\partial r}{\partial z} \right),
$$

driven by source terms including the axial current density $J_z$, and the charge density $r$. Since the beam is periodic in the phase $\theta$, we may expand the current density in a Fourier series,

$$
J_z(r_\perp, z, t) = \sum_{n=-\infty}^{\infty} J_n(r_\perp) e^{-j n \theta},
$$

and likewise for $r$ (related by continuity) and for the axial electric field. The $n$-th Fourier component of the field satisfies,

$$
\left( \nabla^2 - \kappa_n^2 \right) E_n(r_\perp) = -\frac{4\pi j \kappa_n}{\gamma \beta \kappa_n} J_n(r_\perp),
$$

$$
\kappa_n = \frac{n \omega}{\gamma \beta c}.
$$

The coordinate in the transverse plane of the waveguide is,

$$
r_\perp = (x, y, 0),
$$

and the Laplacian is

$$
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y^2}.
$$

Note that for a beam of infinite width infree-space, the result is

$$
E_n(r_\perp) \rightarrow e_n = \frac{4\pi}{\gamma \beta \kappa_n} \left[ \frac{I_n}{\Sigma_{sc}} \right],
$$

where $I_n$ is the $n$-th current harmonic,

$$
\frac{I_n}{I_b} = \left( e^{jn\gamma} \right),
$$

and $I_b$ the average beam current. For numerical purposes the term in brackets means simply,
\( \langle e^{in\phi} \rangle = \frac{1}{N} \sum_{m=1}^{N} e^{in\phi_m} \),

with \( N \) the number of particles in the simulation. The quantity \( \Sigma_{sc} \) is the beam cross-sectional area.

For the case of the finite beam of height of uniform density, extending in \( y \) over \( \pm Y \), our problem takes the form,

\[
\left( \frac{\partial^2}{\partial y^2} - \kappa_n^2 \right) E_n(y) = -\kappa_n^2 e_n \begin{cases} 1; & |y| < Y \\ 0; & Y < |y| \end{cases}.
\]

The general solution has symmetry about \( y=0 \) and may be expressed as

\[
E_n(y) = e_n \begin{cases} 1 + A \cosh(|\kappa_n|y); & 0 < y < Y \\ Be^{k_r|y|} + Ce^{-|k_r|y}; & Y < y \end{cases},
\]

with coefficients \( A, B \) and \( C \) to be determined from the boundary and matching conditions. Continuity at the beam edge requires,

\[
1 + A \cosh(|\kappa_n|Y) = Be^{k_r|Y|} + Ce^{-|k_r|Y}, \quad A \sinh(|\kappa_n|Y) = Be^{k_r|Y|} - Ce^{-|k_r|Y},
\]

and the condition for a wall at \( y=G \), is that the axial field should vanish,

\[
0 = Be^{k_r|G|} + Ce^{-|k_r|G}.
\]

Solving these equations, we find,

\[
C = \frac{\sinh(|\kappa_n|Y)}{1 + \exp(-2|\kappa_n|G)}, \quad B = -\frac{\sinh(|\kappa_n|Y)}{\exp(2|\kappa_n|G) + 1},
\]

\[
A = -\exp(|\kappa_n|Y) \frac{\exp(-2|\kappa_n|Y) + \exp(-2|\kappa_n|G)}{1 + \exp(-2|\kappa_n|G)}.
\]

For 1D simulation purposes we will employ an average over the beam cross-section,

\[
\bar{E}_n = \frac{1}{2Y} \int_{-Y}^{Y} dy E_n(y) = e_n \left\{ 1 + A \frac{\sinh(|\kappa_n|Y)}{|\kappa_n|Y} \right\} = \eta_n e_n,
\]

and in the last line we define the "space-charge fill factor", a dimensionless quantity accounting for the reduction in cross-section averaged axial electric field from the infinite-beam result. The work of this Appendix amounts to a calculation of this term,
\[ \eta_n = 1 + A \frac{\sinh(\kappa_n|Y|)}{|\kappa_n|Y} = 1 - e^{\kappa_n|Y|} \left( \frac{e^{-2|\kappa_n|Y} + e^{-2|\kappa_n|G}}{1 + e^{-2|\kappa_n|G}} \right) \frac{\sinh(\kappa_n|Y|)}{|\kappa_n|Y}. \]

Our parameters correspond to
\[ \gamma = 1 + \frac{480kV}{511kV} = 1.94, \quad \beta_z \approx \sqrt{1 - \frac{1}{\gamma^2}} \approx 0.83, \]
\[ \kappa_1 = \frac{\omega}{\gamma^2} \approx \frac{2\pi}{0.33cm} \frac{1}{0.83 \times 1.94} \approx 11.8 \text{ cm}^{-1}. \]

For a beam extending over ±0.1 cm, and waveguide boundaries at ±0.2 cm, we have \( \kappa_1 Y \approx 1.18, \) \( \kappa_1 G \approx 2.37, \) and \( \eta \approx 0.58. \) The result would have been \( \eta \approx 0.62 \) for waveguide boundaries at infinity. Thus the boundaries produce a small correction.
Appendix B - 1D Ubitron Model, Relativistic Approximation

Deviations from the relativistic approximation (beam voltage much greater than 511kV) are significant for this 480kV beam. For this reason corrections due to the low voltage have been incorporated from the outset in our ubitron model. Nevertheless, for the sake of completeness, let us include in this Appendix the considerations corresponding to the relativistic limit.

The simplest one-dimensional (“1D”) numerical model of the ubitron is derived after an average over the wiggle motion, and assuming a beam envelope uniform in $z$. The model variables consist of a complex phasor, $a$, describing the electromagnetic signal amplitude and phase, propagating with some $N$ particles distributed in phase $\theta = (k_w + k_y)z - \omega t$, and $\gamma = 1 + eV / mc^2$, with $eV$ the electron kinetic energy. Particles are initialized with phase angle distributed over $2\pi$, corresponding to an unbunched beam. Initial values of $\gamma$ are distributed over an interval $\pm \Delta \gamma$, and this interval includes the "effective energy spread" due to emittance (see below). The motion evolves according to,

$$
\frac{d\theta}{dz} = k_w - \delta k - \frac{\omega}{c} \frac{1}{2\gamma^2} \left\{ 1 + \frac{a_w^2}{2} - a_w [JJ] \text{Re} \left[ a \exp(i\theta) \right] \right\},
$$

$$
\frac{d\gamma}{dz} = -\frac{1}{2} \frac{\omega}{c} \frac{a_w}{\gamma \beta} [JJ] \text{Im} \left[ a \exp(i\theta) \right] - \frac{8\pi}{\Sigma_{nc}} \frac{c}{\omega} \left( \frac{1}{I_0} \right) \sum_{n=1}^{\infty} \text{Im} \left\{ \exp(in\theta)(\exp(-in\theta)) \right\}
$$

and

$$
\beta = 1 - \frac{1}{2\gamma^2} \left( 1 + \frac{a_w^2}{2} \right),
$$

is the average axial speed, up to a term of order $a$. The $\theta$-equation tells us how particles slip back and forth longitudinally, as they execute prescribed transverse motions. This longitudinal motion depends on the energy-variable, $\gamma$, and the $\gamma$-equation describes the work being done on the particle by the wave, and the sum representing the axial field due to space-charge. The beam cross-section $\Sigma_{nc}$, and the space-charge fill factors $\eta_n$ are discussed in Appendix A. The wiggler parameter $a_w$ is related to the vector potential according to

$$
A_w = \frac{mc^2}{e} a_w \cosh(k_w y) \sin(k_w z).
$$

and to the fields according to

$$
B_{wy} = \frac{mc^2}{e} k_w a_w \cosh(k_w y) \cos(k_w z) \quad B_{wz} = -\frac{mc^2}{e} k_w a_w \sinh(k_w y) \sin(k_w z).
$$

and the factor in front, $mc^2/e \approx 1.7G - cm$, insures that the wiggler parameter is dimensionless.

The "Bessel factor" is expressed in terms of

$$
\zeta = \frac{\omega}{ck_w} \frac{a_w^2}{8\gamma^2},
$$
and is given by,

\[ [JJ] = J_0(\xi) - J_1(\xi) \approx \left(1 - \frac{1}{4}\xi^2 + \frac{1}{64}\xi^4\right) - \left(\frac{1}{2}\xi - \frac{1}{16}\xi^3\right) \approx 1 - \frac{1}{2}\xi - \frac{1}{4}\xi^2. \]

The validity of the quadratic approximation can be checked by consulting Fig. A.1.

\[ \xi = \frac{\omega a^2}{c k_w 8 \gamma^2} \]

The Bessel Function Factor vs. Quadratic Fit

In the expressions above we abbreviate

\[ \delta k = \frac{\omega}{c} - k_c, \]

where,

\[ \frac{\omega}{c}^2 = k_\xi^2 + k_\xi^2, \]

and \( k_c \) is the cut-off wavenumber for the mode of interest. For the TE_{01} mode, \( k_c = \pi/b \), with the notation \( a>b \).

The eikonal \( a \) is initialized at \( z=0 \) and evolves according to

\[ \left(\frac{d}{dz} + \frac{2\pi i}{k_c \Sigma I_0} \left(\frac{1}{\gamma \beta}\right)\right) a = \frac{2\pi i}{k_c \Sigma} a_w [JJ] \left(\frac{I}{I_0}\right) \left(\frac{\exp(-i\theta)}{\gamma \beta}\right), \]

and the use of \( \beta \) in the denominator requires again the relativistic approximation (since there
is a term in $\beta$ of first order in $a$ that otherwise might not be negligible). In this result "slippage" is neglected (the partial derivative in $z$ is replaced with an ordinary derivative). The constant $I_0=mc^2/e^2=17.03$ kA. The quantity $\Sigma$ is the “mode-area”, $\Sigma=ab/2$ for the TE$_{01}$ mode of a waveguide of dimensions $a \times b$. (This model omits "wave-profile modification", otherwise known as "optical-guiding"). The rf power flowing through the waveguide at location $z$ is determined from,

$$P_{rf} = \frac{P_0}{8\pi} \Sigma \left( \frac{\omega}{c} k_z \right) |d|^2.$$

The constant $P_0=mc^3/e^2=8.701$ GW. The system of equations above has an integral, corresponding to conservation of energy, and this may be expressed as

$$\Delta P_{sol} = \frac{mc^2}{e} I(\gamma(z) - \gamma(0)) + P_{rf}(z) - P_{rf}(0),$$

This quantity is a constant of the motion and equal to zero absent numerical error. An estimate of numerical error in the code is then $\Delta P_{rf}/P_{rf}$. This error will be quite small, particularly compared to errors implicit in the model (e.g., relativistic approximation).