Heat Diffusion - Continuation of ARDB-25

Equations (3) and (3') of ARDB-25 are

\[
T(x,t) = \frac{1}{\rho c \varepsilon \delta} \int_{-\infty}^{t} dt' e^{4\beta / \delta^2} \frac{dP(t')}{dA} \left\{ e^{-2x / \delta \varepsilon \text{erfc} \left( \frac{2 \sqrt{\beta}}{\delta} - \frac{x}{2 \sqrt{\beta}} \right)} + e^{2x / \delta \varepsilon \text{erfc} \left( \frac{2 \sqrt{\beta}}{\delta} + \frac{x}{2 \sqrt{\beta}} \right)} \right\}
\]

(3)

and

\[
T(x,t) = \frac{1}{\sqrt{\pi \rho c \varepsilon \delta}} \int_{-\infty}^{t} dt' e^{-x^2 / \delta \varepsilon} \frac{dP(t')}{dA}
\]

(3')

where

\[
\beta = \frac{k}{\rho c \varepsilon} (t - t') = \alpha(t - t')
\]

The power dissipation per unit area is related to the surface magnetic field, the surface resistance, and the impedance $Z_H$ defined by Perry Wilson\(^1\) by

\[
\frac{dP(t)}{dA} = \frac{R_s}{2} \hat{H}^2(t) = \frac{R_s}{2Z_H^2} G^2(t)
\]

where

\[
Z_H = \frac{G}{\hat{H}}
\]

The peak surface field would be used for calculating $Z_H$ of a complex structure.

Substituting this expression into eq. (3) gives

\[
T(x,t) = \frac{\pi Z_0 G_0^2}{2 \rho c \varepsilon \lambda Z_H^2} \int_{-\infty}^{t} dt' e^{4\beta / \delta^2} \left( \frac{G(t')}{G_0} \right)^2 \left\{ e^{-2x / \delta \varepsilon \text{erfc} \left( \frac{2 \sqrt{\beta}}{\delta} - \frac{x}{2 \sqrt{\beta}} \right)} + e^{2x / \delta \varepsilon \text{erfc} \left( \frac{2 \sqrt{\beta}}{\delta} + \frac{x}{2 \sqrt{\beta}} \right)} \right\}
\]

(4)

where $G_0$ is a normalizing gradient and the relation between surface resistance and skin depth

\[
R_s = \frac{\pi Z_0 \delta}{\lambda}
\]

has been used. Eq. (3') becomes

\(^1\) Perry Wilson, SLAC-PUB-7449, ARDB102
\[T(x,t) = \frac{R_s G_0^2}{2\sqrt{\pi \rho c \varepsilon Z_H^2}} \int_{-\infty}^{t} dt' e^{-x^2/4\beta} \left( \frac{G(t')}{G_0} \right)^2\]  

\[(4')\]

**Pulse Length Dependence**

For a square pulse of length \(T_p\) starting at \(t = 0\). The surface temperature at the end of the pulse is

\[T(x = 0, T_p) = \frac{\pi Z_0 G_0^2}{\rho c \varepsilon \lambda Z_H^2} \int_{0}^{T_p} dt' e^{4\alpha'/\delta^2} \text{erfc} \left( \frac{2\sqrt{\alpha t'}}{\delta} \right)\]

\[(5)\]

When skin depth \(\rightarrow 0\),

\[F(T_p) = \frac{\delta}{\sqrt{\pi \alpha}} \sqrt{\frac{T_p}{\rho c \varepsilon \lambda Z_H^2}} \]

\[(6)\]

and

\[T(x = 0, T_p) = \frac{\pi Z_0 \delta G_0^2}{\lambda} \sqrt{\frac{T_p}{\rho c \varepsilon \lambda Z_H^2}}\]

as expected

For copper \(k = 391\) W/m-K, \(\rho = 8.95 \times 10^3\) kg/m\(^3\), \(c_\varepsilon = 385\) J/kg-K and \(\alpha = 1.135 \times 10^{-4}\) m\(^2\)/s, and \(\delta = 0.22\) \(\mu\)m at W-band.

![Figure 1: F(T\(_p\)) for copper using eq. (5) (dashed) and the approximation in eq. (6) (solid). The difference at large values of T\(_p\) is ~ 9.4\times10^{-11}.](image-url)
Alternate Derivations of eq (4')

When the skin depth \( \delta \rightarrow 0 \),

\[
\frac{d^2 \mathcal{P}(t)}{d A dx} = R_s \hat{H}^2(t) \delta(x) = \frac{R_s G_0^2}{Z_H^2} \left( \frac{G(t)}{G_0} \right)^2 \delta(x)
\]

where there is a factor of two to account for the adiabatic boundary condition. Substituting into the expression for temperature

\[
T(x, t) = \frac{1}{\rho c_\varepsilon} \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx' \frac{d^2 \mathcal{P}(x', t')}{d A dx'} G(x - x', t - t')
\]

\[
= \frac{R_s G_0^2}{2\sqrt{\pi} Z_H^2 \rho c_\varepsilon} \int_{-\infty}^{t} dt' e^{-x^2/\beta} \left( \frac{G(t')}{G_0} \right)^2
\]

Switch to the procedure of Butkovskiy\(^2\). This marvelous book gives the standardizing functions and Green's functions for classes of differential equations with different boundary conditions. Then there is no need to include the factor of two in the power density equation because the adiabatic boundary condition is accounted for in the Green's function

\[
\frac{d^2 \mathcal{P}(t)}{d A dx} = \frac{R_s}{2} \hat{H}^2(t) \delta(x) = \frac{R_s G_0^2}{2Z_H^2} \left( \frac{G(t)}{G_0} \right)^2 \delta(x). \quad (A1)
\]

The standardizing and Green's functions with the adiabatic boundary condition are

\[
w(x, t) = \frac{1}{\rho c_\varepsilon} \frac{d^2 \mathcal{P}}{d A dx} \quad (A2)
\]

and

\[
G(x, x', t) = \frac{1}{\sqrt{4\pi \alpha t}} \left\{ \exp \left( -\frac{(x - x')^2}{4\alpha t} \right) + \exp \left( -\frac{(x + x')^2}{4\alpha t} \right) \right\} \quad (A3)
\]

These results are given on pages 55 and 56. Combining equations (A1) -(A3) gives the same results as above.

\(^2\) Anatoliy G. Butkovskiy, *Green's Functions and Transfer Functions Handbook*, Ellis Horwood Ltd.
**Displaced Source**

Assume a delta-function power source located at \( x = L \) with the adiabatic boundary condition at \( x = 0 \). The standardizing function and Green's function in eqs. (A2) and (A3) are unchanged while the power in eq. (A1) is changed only by \( \delta(x') \) being replaced by \( \delta(x' - L) \). For a square pulse of length \( T_p \) starting at \( t' = 0 \),

\[
T_2(x, T_p) = \frac{R_sG_0^2}{2Z_H\rho c_2} \frac{T_p}{\sqrt{\pi}} \int_0^{T_p} dt' \exp\left(-\frac{(x-L)^2}{4\alpha_2 t'}\right) + \exp\left(-\frac{(x+L)^2}{4\alpha_2 t'}\right)
\]

The first term corresponds to the source itself and the second term to the image source that has been shifted away from the face to \( x = -L \). The subscript "2" is part of the solution of the thin coating problem posed below. Performing the integral\(^3\) gives an expression for the temperature in terms of the incomplete gamma function

\[
T_2(x, T_p) = \frac{R_sG_0^2}{8Z_H\rho c_2\sqrt{\pi}} \left| x - L \right| \left[ \left( \frac{x - L}{2\alpha} \right)^2 \right] \left[ \left( \frac{x + L}{2\alpha} \right)^2 \right] + \left( \frac{x - L}{2\alpha} \right)^2 \left( \frac{x + L}{2\alpha} \right)^2
\]

The incomplete gamma functions can be evaluated as in the appendix to give

\[
T_2(x, T_p) = \frac{R_sG_0^2}{4\sqrt{\pi}Z_H^2k_2} \left( \frac{x - L}{\sqrt{4\alpha_2 T_p}} \right)^2 \left( \frac{x + L}{\sqrt{4\alpha_2 T_p}} \right)^2 - \sqrt{\pi} \left[ \left( \frac{x - L}{\sqrt{4\alpha_2 T_p}} \right) \text{erfc} \left( \frac{x - L}{\sqrt{4\alpha_2 T_p}} \right) + \left( \frac{x + L}{\sqrt{4\alpha_2 T_p}} \right) \text{erfc} \left( \frac{x + L}{\sqrt{4\alpha_2 T_p}} \right) \right]
\]

The temperature for different values of \( L \) are shown in Figure 2. The effect of displacing the source is to reduce the maximum temperature. The image source is moved from the surface; effectively the heat is spread widely by diffusion after it is reflected by the boundary.

This figure was calculated with MATLAB function "t2cu(x)" which evaluates eq. (B1) for copper except for the leading factor

\[
\frac{R_sG_0^2}{4\sqrt{\pi}Z_H^2k_2}
\]

If \( x \) and \( L \) are scaled to the diffusion distance, i.e.

\[
x_s = \frac{x}{\sqrt{4\alpha_2 T_p}}; L_s = \frac{L}{\sqrt{4\alpha_2 T_p}}
\]

then

\[
T_2(x_s, T_p) = \frac{R_sG_0^2}{4Z_H^2k_2} \left( \sqrt{4\alpha_2 T_p} \right)^2 \left[ \left( \frac{x_s - L_s}{\sqrt{4\alpha_2 T_p}} \right)^2 \right] \left[ \left( \frac{x_s + L_s}{\sqrt{4\alpha_2 T_p}} \right)^2 \right] - \sqrt{\pi} \left[ \left( \frac{x_s - L_s}{\sqrt{4\alpha_2 T_p}} \right) \text{erfc} \left( \frac{x_s - L_s}{\sqrt{4\alpha_2 T_p}} \right) + \left( \frac{x_s + L_s}{\sqrt{4\alpha_2 T_p}} \right) \text{erfc} \left( \frac{x_s + L_s}{\sqrt{4\alpha_2 T_p}} \right) \right]
\]

\(^3\) Gradshteyn & Ryzhik, 3.381.3
Figure 2: Temperature profiles for different values of L. Both x and L are normalized to the diffusion distance, $\sqrt{4\alpha T_p}$.

In the limit where $L \rightarrow 0$,

$$T_2(x, T_p) = \frac{R_s G_0^2}{2 Z_H^2 k_k} \sqrt{4\alpha_2 T_p} \sqrt{\pi} \left\{ \exp \left( -\frac{x^2}{4\alpha_2 T_p} \right) - \frac{x\sqrt{\pi}}{\sqrt[4]{4\alpha_2 T_p}} \text{erfc} \left( \frac{x}{\sqrt{4\alpha_2 T_p}} \right) \right\} \quad (B2)$$

At small $T_p$ and $x \rightarrow 0$ this reduces to

$$T_2(x = 0, T_p) = \frac{R_s G_0^2}{2 Z_H^2 k_k} \frac{\sqrt{4\alpha_2 T_p}}{\sqrt{\pi}}$$

as expected.

Note that this calculation can also be performed taking skin depth into account. The result is that the source and image move to $x = +L$ and $x = -L$, respectively, and the resultant equation equivalent to eq. (4) is

$$T(x, t) = \frac{\pi Z_0 G_0^2}{2 \rho c e \lambda Z_H^2} \int_{-\infty}^{t} t' e^{4\beta/\delta^2} \left( \frac{G(t')}{G_0} \right) ^2 e^{-2(x-L)/\delta} \text{erfc} \left( \frac{2\sqrt{\beta}}{\delta} - \frac{x-L}{2\sqrt{\beta}} \right)$$

$$+ e^{2(x+L)/\delta} \text{erfc} \left( \frac{2\sqrt{\beta}}{\delta} + \frac{x+L}{2\sqrt{\beta}} \right)$$
Consider two semi-infinite slabs of different materials joined together at the position where power is deposited; call this position $x = 0$. The power that is deposited will flow into the two materials keeping the temperature of the interface the same on both sides. A way to estimate this problem is to take an adiabatic boundary condition at $x = 0$ and treat the power as divided between the two materials such that the temperature condition is satisfied.

Using eq. (B2) at $x = 0$ gives for the two materials

$$
T_1(x = 0, T_p) = f_1 \frac{\sqrt{\alpha_1 \frac{R_s G_0^2}{\sqrt{T_p}}}}{k_1 \sqrt{\pi Z_{H}^2}} = T_2(x = 0, T_p) = f_2 \frac{\sqrt{\alpha_2 \frac{R_s G_0^2}{\sqrt{T_p}}}}{k_2 \sqrt{\pi Z_{H}^2}}
$$

where $f_1$ and $f_2$ are the fractions of energy deposited in each material. The ratio is

$$
\frac{f_1}{f_2} = \frac{k_1}{k_2} \sqrt{\frac{\alpha_2}{\alpha_1}} = \frac{k_1 \rho_1 c_1}{k_2 \rho_2 c_2}
$$

If the materials of interest are copper and diamond this ratio is close to one. The characteristic length is proportional to $\sqrt{\alpha} \sim 1.8$ for these two materials, so a diamond layer would have to be roughly two times as thick as a copper layer in the examples above.

<table>
<thead>
<tr>
<th>Material</th>
<th>Copper</th>
<th>Diamond</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$ (W/m-K)</td>
<td>391</td>
<td>660</td>
</tr>
<tr>
<td>$\rho$ (kg/m$^3$)</td>
<td>8.95x10$^3$</td>
<td>3.52x10$^3$</td>
</tr>
<tr>
<td>$c_e$ (J/kg-K)</td>
<td>385</td>
<td>509</td>
</tr>
<tr>
<td>$\alpha = k/\rho c_e$ (m$^2$/s)</td>
<td>1.135x10$^{-4}$</td>
<td>3.684x10$^{-4}$</td>
</tr>
<tr>
<td>$k\rho c_e$ (MKSA)</td>
<td>1.347x10$^9$</td>
<td>1.183x10$^9$</td>
</tr>
</tbody>
</table>
Appendix - Incomplete Gamma Functions

Write \( \Gamma(a,z) \) in terms of \( \gamma^* \) gives\(^4\)

\[
\Gamma(a,z) = \Gamma(a) - \gamma(a,z) = \Gamma(a) \left[ 1 - z^a \gamma^*(a,z) \right]
\]

\( \gamma^* \) is a single value analytic function of \( a \) and \( z \) that has the following recursion relation\(^5\)

\[
\gamma^*(a,z) = z \gamma^*(a+1,z) + \frac{e^{-z}}{\Gamma(a+1)}
\]

Substituting

\[
\Gamma(a,z) = \Gamma(a) \left[ 1 - z^{a+1} \gamma^*(a+1,z) - \frac{z^a e^{-z}}{\Gamma(a+1)} \right]
\]

Using the relation between \( \gamma^*(a,x) \) and \( P(a,x) \)\(^6\) gives

\[
\Gamma(a,z) = \Gamma(a) \left[ 1 - P(a+1,z) - \frac{z^a e^{-z}}{\Gamma(a+1)} \right]
\]

\[
= - \frac{\Gamma(a)}{\Gamma(a+1)} \left[ z^a e^{-z} - \Gamma(a+1)(1 - P(a+1,z)) \right]
\]

\[
= - \frac{1}{a} \left[ z^a e^{-z} - \Gamma(a+1)(1 - P(a+1,z)) \right]
\]

Look at the particular case where \( a = -1/2 \) and \( z = y \)\(^7\)

\[
\Gamma(-1/2,y^2) = 2 \left[ \frac{1}{\sqrt{y}} e^{-y^2} - \sqrt{\pi} (1 - P(1/2,y^2)) \right]
\]

\[
= 2 \left[ \frac{1}{\sqrt{y}} e^{-y^2} - \sqrt{\pi} (1 - \text{erf}(y)) \right]
\]

\[
= 2 \left[ \frac{1}{\sqrt{y}} e^{-y^2} - \sqrt{\pi} \text{erfc}(y) \right]
\]

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\(^4\) M. Abramowitz and L. Stegun, Handbook of Mathematical Function, eq. 6.5.4
\(^5\) M. Abramowitz and L. Stegun, Handbook of Mathematical Function, eq. 6.5.23
\(^6\) M. Abramowitz and L. Stegun, Handbook of Mathematical Function, eq. 6.5.4
\(^7\) M. Abramowitz and L. Stegun, Handbook of Mathematical Function, eq. 6.5.16