In this note we review the scalings for single-hole directional couplers, noting dimensions and tolerances appropriate for operation of WR10 waveguide at 91.392 GHz. A 40 dB single-hole coupler at W-Band entails hole diameters in the 15-20 mil range, depending on the coupler type. For coupling tolerances of ±5 dB, tolerances on diameter are a loose ±3 mil. Hole placement tolerances are tight, being just shy of 1 mil for directivity of 25 dB.

The theory of holes and obstacles in introduced in Jackson’s text,\(^1\) and specific formulae are given in Collin’s text.\(^2\) We briefly review coupler terminology, and then consider two kinds of couplers.

### Coupler Terminology

Generically a directional coupler may be depicted as in Fig. 1. It is a four-port device. An ideal directional coupler is symmetric upon reflection in the horizontal or vertical, is lossless and is perfectly matched. These conditions restrict the S-matrix to the form

\[
S = \begin{pmatrix}
0 & \alpha & j\sqrt{1 - \alpha^2} & 0 \\
\alpha & 0 & 0 & j\sqrt{1 - \alpha^2} \\
j\sqrt{1 - \alpha^2} & 0 & 0 & \alpha \\
0 & j\sqrt{1 - \alpha^2} & \alpha & 0
\end{pmatrix},
\]

for some parameter \(\alpha \leq 1\), and choice of reference planes.

![FIGURE 1. Illustration of a directional coupler.](image)

A real directional coupler is described by coupling \(C\) and directivity \(D\), defined with respect to the quantities indicated in the sketch: power incident on port 1, \(P_i\), transmitted power to port 2, \(P_t\), forward power on port 3, \(P_f\), and reverse power, \(P_r\), on port 4. The coupling and directivity are

\[
C = 10\log_{10} \left( \frac{P_t}{P_f} \right), \quad \text{(coupling)} \quad D = 10\log_{10} \left( \frac{P_f}{P_r} \right), \quad \text{(directivity)}
\]

The isolation is defined according to


\[ I = 10 \log_{10} \left( \frac{P_i}{P_r} \right). \] (isolation)

Notice that \( C + D = I \). Let us next review these quantities for two particular coupler geometries.

**Bethe Hole Coupler**

This coupler geometry is illustrated in Fig. 2 consisting of two parallel rectangular waveguides, operated in the TE\textsubscript{10} mode, and sharing a common broad wall. The waveguides are coupled by means of a hole of diameter \( 2r_0 \) centered a distance \( d \) from the edge.

![Fig. 2](image)

**FIGURE 2.** Bethe Hole coupler consisting of two rectangular waveguides, parallel and sharing a common broadwall.

With respect to a reference plane at the hole center, and for a small hole, and an infinitely thin guide, Collin derives results for the voltage components due to the hole (coefficients of the TE\textsubscript{10} propagating mode, in the waveguide mode expansion) as follows,

\[
A_j = A_2 = - \frac{j \omega \epsilon_0}{ab Y_w} \frac{2}{3} r_0^3 \sin^2 \left( \frac{\pi d}{a} \right),
\]

\[
A_3 = \frac{j \omega \mu_0 Y_w}{ab} \frac{4}{3} r_0^3 \left\{ \sin^2 \left( \frac{\pi d}{a} \right) + \frac{\pi^2}{\beta^2 a^2} \cos^2 \left( \frac{\pi d}{a} \right) \right\},
\]

\[
A_4 = \frac{j \omega \mu_0 Y_w}{ab} \frac{4}{3} r_0^3 \left\{ -\sin^2 \left( \frac{\pi d}{a} \right) + \frac{\pi^2}{\beta^2 a^2} \cos^2 \left( \frac{\pi d}{a} \right) \right\}.
\]

The subscripts correspond to the notation of Fig. 1, \( i.e., A_n \) is the increment due to the port to the voltage coefficient of the wave propagating out of port \( n \), for unit incident voltage in port 1. The characteristic admittance of the TE\textsubscript{10} mode is

\[
Y_w = \frac{1}{Z_w} = \frac{\beta}{\omega l c} \frac{1}{Z_0},
\]
with $\omega$ the angular frequency of the rf signal, the wavenumber

$$\beta = \sqrt{\frac{\omega^2 - \pi^2}{c^2 - a^2}},$$

and the impedance of free-space,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}},$$

and the speed of light,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

The coupling, directivity and isolation are

$$C = -20 \log_{10} |A_3|, \quad D = 20 \log_{10} \left| \frac{A_3}{A_4} \right|, \quad I = -20 \log_{10} |A_4|.$$

The coupler is directional when $A_4$ is much smaller than $A_3$. Ideally, one would arrange that, at the operating frequency, $A_4 = 0$, and this requires particular placement for the hole,

$$\tan \left( \frac{\pi d}{a} \right) = \frac{\pi}{\beta a} = \left\{ \left( \frac{\omega a}{c \pi} \right)^2 - 1 \right\}^{-1/2}.$$

For example, for operation of WR10 at 91.392 GHz (0.3280 cm free-space wavelength), we have

$$a = \frac{10}{100} \text{ inch} = 0.254 \text{ cm}, \quad \beta = 14.626 \text{ cm}^{-1} = \frac{2 \pi}{0.4296 \text{ cm}},$$

so that

$$\tan \left( \frac{\pi d}{a} \right) \approx 0.8456, \quad d \approx \frac{0.70196}{\pi} a \approx 0.2234 a \approx 0.05675 \text{ cm}.$$

For other frequencies, the directivity may be expressed in terms of

$$\eta(\omega) = \frac{\beta a}{\pi} \tan \left( \frac{\pi d}{a} \right) = \left\{ \left( \frac{\omega a}{c \pi} \right)^2 - 1 \right\}^{1/2} \tan \left( \frac{\pi d}{a} \right),$$

as

$$D(\omega) = 20 \log_{10} \left| \frac{1 - \eta^2(\omega)}{1 + \eta^2(\omega)} \right|,$$

and thus the directivity is independent of the hole size. Meanwhile, the coupling may be expressed in terms of
\[ |A_3| = \frac{4}{3} \pi^2 \frac{r_0^3}{a^3 b \beta} \cos^2 \left( \frac{\pi d}{a} \right) \left[ 1 + \eta^2(\omega) \right], \]

as

\[
C(\omega) = -20 \log_{10} \left\{ \frac{4}{3} \pi^2 \frac{r_0^3}{a^3 b \beta(\omega)} \cos^2 \left( \frac{\pi d}{a} \right) \left[ 1 + \eta^2(\omega) \right] \right\}
\]

or

\[
C(\omega) = -20 \log_{10} \left\{ \frac{4\pi}{3} \frac{r_0^3}{a^2 b} \cos^2 \left( \frac{\pi d}{a} \right) \left\{ 1 + \eta^2(\omega) \right\} \left[ \frac{\omega^2 a^2}{c^2 \pi^2} - 1 \right]^{1/2} \right\},
\]

making the frequency dependence explicit. At the design frequency this is

\[
C = -20 \log_{10} \left\{ \frac{8}{3} \pi^2 \frac{r_0^3}{a^2 b} \left( 1 + \frac{\pi^2}{\beta^2 a^2} \right)^{-1} \right\}.
\]

For the case of interest \( b = a/2 \) and \( \beta a = 3.715 \), so that

\[
C = -60 \log_{10} \left\{ 2.022 \frac{r_0}{a} \right\},
\]

and the hole diameter may then be expressed as,

\[
2r_0 \approx 0.2513 \text{ cm} 10^{-C(\text{dB})/60}.
\]

Thus a 40 dB coupler requires a 0.054 cm (21 mil) diameter. For our applications we will be interested in (a) being able to test the coupler on the bench (b) employing the coupler to monitor initially 30 kW peak power, and (c) eventually monitoring 1 MW of power. To keep the diagnostic signal below the 1 kW level, we need a coupling lower than 30 dB, and so we select 40 dB as our provisional coupling. The range for the bench apparatus we require then to extend below 40 dB.

For work at the 30 kW level, we still have a very respectable 3 W signal.

For a tolerance of \( \pm 3 \text{ mil} \) on the diameter, the coupling ranges from 35 dB to 45 dB and this is perfectly adequate. The tolerance on hole center dimension \( d \) is not so loose. If we ask for directivity of 25 dB, we have a value for \( \eta \) of

\[
\eta \approx \frac{1 - 10^{-D_{(\text{dB})}/20}}{1 + 10^{-D_{(\text{dB})}/20}}^{1/2} \approx 0.945,
\]

and this corresponds to a hole-center position,

\[
d' = \frac{a}{\pi} \tan^{-1} \left( 0.945 \tan \left( \frac{\pi d}{a} \right) \right) = \frac{0.254 \text{ cm}}{\pi} \tan^{-1} (0.945 \times 0.8546) = 0.0549 \text{ cm},
\]

an error of 0.002 cm. This corresponds to a tight tolerance of \( \pm 0.7 \text{ mil} \) on the hole-center position.
Note that for a properly tuned resonant ring circuit, we would hope to find reverse power down by 20 dB from forward power. Thus a directivity of 25 dB corresponds to an error term in the diagnosed reverse power that is only 5 dB lower (\times 1/3 in power) than the actual reverse power, or an error voltage already 60% of the actual voltage. At this level reverse power could be monitored usefully to observe instances of breakdown, or other situations with mismatch, but that’s about all the reverse signal monitor would be good for. Plots of coupling and directivity versus frequency for a perfectly fabricated 40 dB coupler are given below.

**Modified Bethe Hole Coupler**

Next we consider a coupler formed by a centered hole in two crossed waveguides, as indicated in Fig. 3. Collin computes the coupling and directivity,

\[
\begin{align*}
C(\omega) &= -20 \log_{10} \left\{ \frac{4}{3} \frac{\beta r_0^2}{ab} \cos \theta \left[ 1 + \eta^2(\omega) \right] \right\}, \\
D(\omega) &= 20 \log_{10} \left| \frac{1 - \eta^2(\omega)}{1 + \eta^2(\omega)} \right|,
\end{align*}
\]

where

\[
\eta(\omega) = \frac{1}{2^{1/2}} \frac{\omega}{\cos \theta} c \beta.
\]

For our application, we require, at the design frequency,

\[
\cos \theta = \frac{1}{2^{1/2}} \frac{\omega}{c \beta} = \frac{1}{2^{1/2}} \frac{\lambda_s}{\lambda} \approx 0.4296 \text{ cm} \\
\approx 0.5280 \text{ cm} = 0.9261,
\]

or \( \theta = 22.17^\circ \). For a directivity of 25 dB, we have \( \eta = 0.945 \), corresponding to

\[
\cos \theta \approx \frac{0.9261}{0.945} \approx 0.98,
\]

or \( \theta = 11.48^\circ \). Thus the tolerance on the angle is quite loose. The coupling at the design frequency is

\[
C(\omega) = -20 \log_{10} \left\{ \frac{4}{3} \frac{\beta r_0^2}{ab} \cos \theta \left[ 1 + \eta^2(\omega) \right] \right\},
\]

\[
D(\omega) = 20 \log_{10} \left| \frac{1 - \eta^2(\omega)}{1 + \eta^2(\omega)} \right|,
\]
\[
C = -20 \log_{10} \left( \frac{8 \frac{r_0^3}{a^3} \beta a^2}{b \cos \theta} \right) = -60 \log_{10} \left( \frac{2.637 r_0}{a} \right)
\]
or
\[
2r_0 \approx 0.1927 \text{ cm} \times 10^{-C(\text{dB})/60}.
\]

Thus a 40 dB coupler requires a hole diameter of 0.0415 cm or 16 mils.

Dependence of coupling and directivity on frequency are depicted in Figs. 4 and 5, for both kinds of coupler. Results above 118 GHz (above cutoff for higher modes) are plotted mainly out of curiosity and should be discounted since the calculations assumed single-mode propagation.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Coupling versus frequency for 40 dB Bethe Hole couplers. Above a frequency of 118 GHz (free-space wavelength of 2.54 mm) the TE$_{20}$ and TE$_{01}$ modes propagate and these are not accounted for in the derivation of the coupling characteristics. Thus coupling depicted there should may be expected to deviate.}
\end{figure}

**Conclusions**

A 40 dB single-hole coupler at W-Band entails hole diameters in the 15-20 mil range, and tight hole placement tolerances just shy of 1 mil. It seems feasible to fabricate such a coupler, either by EDM (depth of penetration for the stub is an issue), or by batch preparation, depending on shop
time required and the eventual rejection rate. For purposes of layout of the beamline assembly for the NLCTA experiment, the Type II couplers entail a 22° angle.

For example, a Type I coupler might be assembled from a 2” WR10 straight, and a U E-Bend. Each piece would have a 21 mil diameter hole, centered 223.4 mils from the inner waveguide edge, in one broadwall. To assemble, one might use a stub insertable from the guide interior into the hole. The E-bend hole would then mate to this stub. Clamped, and with stub removed, the assembly might be amenable to diffusion bonding.

**FIGURE 5.** Directivity versus frequency for Bethe Hole couplers designed for operation of WR10 waveguide at 91.392 GHz. Above cutoff for higher modes (118 GHz) directivity may be expected to deviate from that depicted here.