In this problem you will analyze an FEL based on an ideal helical wiggler field,
\[ \vec{B} = -\frac{2B_w}{k_w} \vec{\nabla}\{I_1(k_w r) \sin(k_w z - \phi)\}, \]
where \( r \) is the radial coordinate, and
\[ \vec{v} = \hat{\xi} \frac{\partial}{\partial \xi} + \hat{\eta} \frac{\partial}{\partial \eta} + \hat{\phi} \frac{1}{r} \frac{\partial}{\partial \phi}, \]
with polar coordinates in the transverse plane, \( x = r \cos \phi \), \( y = r \sin \phi \), or
\[ \hat{x} = \cos \phi, \sin \phi, \hat{y} = (-\sin \phi, \cos \phi). \]

(0) Confirm that this magnetic field is consistent with Maxwell’s Equations.

(1) Evaluate the wiggler field on-axis. You will want to use \( I_1(\xi) \approx \xi/2 \).

(2) Considering an almost monochromatic electromagnetic signal,
\[ \vec{A} = \frac{mc^2}{e} \text{Im}\{\tilde{e} a \exp(ik z - i\omega t)\}, \]
determine the polarization \( \tilde{e} \) (a complex vector in the \( x-y \) plane) that couples to wiggler induced beam motions. Determine the resonance condition in terms of \( a_w = eB_w / mc^2 k_w \).

(3) In the limit of a small-emittance beam derive equations governing the longitudinal motion of electrons in \( (\gamma, \theta) \) - define \( \theta \) - and the eikonal equation for \( a \). Make any approximations you please, but do note them. By comparison with the planar wiggler equations, determine the gain parameter \( \rho \).

(4) Determine the natural focusing in this wiggler (“\( k_n \)”), by considering a small perturbation to the zeroth-order motion you determined in the course of problem 3.

(5) Make a sketch of the beam motion in the \( x-y \) plane, and a sketch of the \( E \)-field vector. Then sketch the arrangement of magnets required to produce this field. Don’t spend more than 5 minutes on this part of the midterm.
Physics of Free Electron Lasers - Midterm - Helical Wiggler FEL - Solutions

In this problem we analyze an FEL based on an ideal helical wiggler field,

$$\vec{B} = -\frac{2B_w}{k_w} \hat{v} \{ I_i(k_w r) \sin(k_w z - \phi) \},$$

where $r$ is the radial coordinate, and

$$\hat{v} = \frac{1}{r} \left( \hat{z} \frac{\partial}{\partial z} + \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial}{\partial \phi} \right),$$

with polar coordinates in the transverse plane, $x=rcos\phi, y=rsin\phi$, or $\hat{r}=(\cos\phi, \sin\phi), \hat{\phi}=(-\sin\phi, \cos\phi)$.

Let’s check that this magnet field is consistent with Maxwell’s Equations. Clearly $\nabla \times \vec{B} = 0$ since curl of a gradient vanishes, to check that $\nabla \cdot \vec{B} = 0$, we need only compute

$$\nabla^2 \{ I_i(k_w r) \exp(i(k_w z - \phi)) \} = \left( \frac{\partial^2}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \{ I_i(k_w r) \exp(i(k_w z - \phi)) \}

= \exp(i(k_w z - \phi) \left( -k_w^2 + \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) I_i(k_w r)$$

But $I_i$ satisfies (see the Bessel Function handout),

$$\left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} - k_w^2 \right) I_i(k_w r) = 0.$$ 

Next, let’s evaluate the wiggler field on-axis.

$$B_z = -2B_w I_i(k_w r) \cos(k_w z - \phi)

= -B_w k_w r \cos(k_w z - \phi)$$

$$B_r = -2B_w I'_i(k_w r) \sin(k_w z - \phi)

= -B_w \sin(k_w z - \phi)$$

$$B_\phi = 2B_w \frac{I_i(k_w r)}{k_w r} \cos(k_w z - \phi)

= B_w \cos(k_w z - \phi).$$
and we make use of $I_1(\xi) = \xi/2$. It is also helpful to have expressions in Cartesian coordinates,

$$B_x = B_e \cos \phi - B_\phi \sin \phi$$

$$= -B_w \sin(k_wz),$$

$$B_y = B_e \sin \phi + B_\phi \cos \phi$$

$$= B_w \cos(k_wz).$$

\[ (2) \]

Considering an almost monochromatic electromagnetic signal,

$$A = \frac{mc^2}{e} \text{Im}\{\tilde{e}\alpha \exp(ik_z z - i\omega t)\},$$

we are asked to determine the polarization $\tilde{e}$ (a complex vector in the $x$-$y$ plane) that couples to wiggler induced beam motions. To do this, let's solve the equations of motion

$$\frac{dp_x}{dz} = -\frac{e}{cv_z} (v_y B_z - v_z B_y),$$

$$\frac{dp_y}{dz} = -\frac{e}{cv_z} (v_z B_y - v_y B_z),$$

$$\frac{dp_z}{dz} = -\frac{e}{cv_z} (v_x B_y - v_y B_x),$$

considering the zeroth order motion (“design orbit”).

$$\frac{dp_{x0}}{dz} = \frac{e}{c} B_y = \frac{eB_w}{c} \cos(k_wz),$$

$$\frac{dp_{y0}}{dz} = \frac{-e}{c} B_z = \frac{eB_w}{c} \sin(k_wz).$$

These can be integrated to give,

$$v_{x0} = \frac{p_{x0}}{m\gamma} = \frac{c}{\gamma} \frac{a_w}{\gamma} \sin(k_wz),$$

$$v_{y0} = \frac{p_{y0}}{m\gamma} = \frac{-c}{\gamma} \frac{a_w}{\gamma} \cos(k_wz).$$

Here

$$a_w = \frac{eB_w}{mc^2 k_w}.$$
\[
\frac{dp_z}{dz} = -\frac{e}{cv_z} \left( c \frac{a_w}{\gamma} \sin(k_wz)B_v \cos(k_wz) - c \frac{a_w}{\gamma} \cos(k_wz)B_v \sin(k_wz) \right) = 0.
\]

Thus \( p_z, \gamma \) and \( v_z \) are all constants of the zeroth-order motion. It will be helpful later to have the zeroth-order trajectory

\[
x_0 = -\frac{a_w}{\gamma \beta_k k_w} \cos(k_wz),
\]

\[
y_0 = -\frac{a_w}{\gamma \beta_k k_w} \sin(k_wz).
\]

Also useful to note that

\[
\frac{v_z}{c} = 1 - \frac{1}{2\gamma^2} - \frac{\beta_x^2}{2} - \frac{\beta_y^2}{2}
\]

\[= 1 - \frac{1}{2\gamma^2} (1 + a_w^2),\]

Let's now determine the condition for resonant transfer of energy to the wave. The electric field is just

\[
\vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} mc^2 \text{Im}\{\tilde{e}a \exp(ik_zz - i\omega t)\}
\]

\[= \frac{mc^2}{e} \text{Im}\left\{ i \frac{\omega}{c} \tilde{e} a \exp(ik_zz - i\omega t) \right\},\]

and the rate of change of energy is governed by

\[
\frac{d\gamma}{dz} = -\frac{e}{mc^2 v_z} \vec{E} \cdot \vec{v}
\]

\[= -\text{Im}\left\{ i \frac{\omega}{c} \frac{a}{v_z} \tilde{v} \exp(ik_zz - i\omega t) \right\}
\]

\[= -\text{Im}\left\{ i \frac{\omega}{c} \frac{a_w}{\gamma \beta_k k_w} \left[ \epsilon_+ \sin(k_wz) - \epsilon_+ \cos(k_wz) \right] \exp(ik_zz - i\omega t) \right\} \]
We would like to choose polarization so that the particles can do work on the fields and the choice \((\varepsilon_x, \varepsilon_y) = (1, i)\), gives

\[
\frac{d\gamma}{dz} = -\frac{\omega}{c} \frac{a_w}{\gamma \beta} \text{Im}(ae^{i\theta}),
\]

where \(\theta = (k_z + k_w)z - \omega \tau\), and resonance corresponds to, \(d\theta / dz = 0\), or

\[
v_z = \frac{\omega}{(k_z + k_w)}.
\]

Note that this choice of polarization corresponds to

\[
A_x = \frac{mc^2}{e} \text{Im}\{a \exp(i\alpha)\}
\]

\[
= \frac{mc^2}{e} a_x \sin(\alpha + \varphi_x)
\]

\[
A_y = \frac{mc^2}{e} \text{Im}\{ia \exp(i\alpha)\}
\]

\[
= \frac{mc^2}{e} a_y \cos(\alpha + \varphi_y)
\]

and we abbreviate \(\alpha = k_z - \omega \tau\), \(a = a_x \exp(i\varphi_x)\).

(3) In the limit of a small-emittance beam the equations governing the longitudinal motion of electrons in \((\gamma, \theta)\) are

\[
\frac{d\theta}{dz} = (k_z + k_w) - \frac{\omega}{v_z}
\]

\[
= (k_z + k_w) - \frac{\omega}{c} \left(1 + \frac{1}{2\gamma^2}(1 + a_w^2)\right)
\]

\[
= k_w - \delta k - \frac{\omega / c}{2\gamma^2}(1 + a_w^2)
\]

and

\[
\frac{d\gamma}{dz} = -\frac{\omega}{c} \frac{a_w}{\gamma \beta} \text{Im}(ae^{i\theta}).
\]
The derivation of the eikonal equation for $a$ follows that in the review notes. We express the vector potential as

$$ \tilde{A} = \frac{mc^2}{2e} \left[ -ia \exp(i\alpha)\hat{x} + a \exp(i\alpha)\hat{y} + c.c. \right]. $$

Maxwell's equations in the eikonal approximation take the form

$$ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \tilde{A} = \frac{1}{2} e^{i\alpha} \left( \nabla^2 + \left( \frac{\omega}{c} \right)^2 k_z^2 + 2ik_z \left( \frac{\partial}{\partial \zeta} \right) \right) \left[ -ia \hat{x} + a \hat{y} \right] + c.c. $$

$$ = -\frac{4\pi}{c} \tilde{j} $$

The current density takes the form

$$ J_i = \sum_{\omega} -\frac{\varepsilon c}{\gamma} \delta(x-x_i)\delta(y-y_i)\delta(z-z_i) \left\{ a_{\omega} \sin(k_{\omega}z)\hat{x} - a_{\omega} \cos(k_{\omega}z)\hat{y} + a_{\omega} \sin(\alpha+\phi)\hat{x} + a_{\omega} \cos(\alpha+\phi)\hat{y} \right\} $$

performing averages over the signal wavelength and period, and making use of the "bucket average" (see notes),

$$ \left\langle \sum_{\omega \in e} \delta^3(\vec{r} - \vec{r}_i) f(\vec{r}, t) \right\rangle = \frac{1}{e c} \left( \beta^{-1} \delta^3(\vec{r}_i - \vec{r}_{i-1}) f(\vec{r}_i, t) \right) $$

we arrive at

$$ \frac{1}{2} \left( \nabla^2 + \left( \frac{\omega}{c} \right)^2 k_z^2 + 2ik_z \left( \frac{\partial}{\partial \zeta} \right) \right) \left[ -ia \hat{x} + a \hat{y} \right] = 4\pi \frac{1}{I_0} \left( \frac{\delta^3(\vec{r} - \vec{r}_{\omega})}{\gamma^2} \right) \left[ -\frac{a}{2i} \hat{x} - \frac{a}{2i} \hat{y} - \frac{a_{e^{-i\theta}}}{2i} \hat{x} - \frac{a_{e^{-i\theta}}}{2i} \hat{y} \right]. $$

which reduces after the usual approximations to

$$ \left( \frac{d}{dz} + \frac{2\pi i}{k_z} \frac{1}{I_0} \left( \frac{1}{\gamma^2} \right) \right) a = \frac{2\pi i}{k_z} \sum I_0 \left( \frac{1}{\gamma^2} \right) \left( \frac{\exp(-i\theta)}{\gamma^2} \right) $$
We can determine the gain parameter $\rho$ by comparison with the planar wiggler FEL equations. The helical wiggler equations can be expressed as

\[
\frac{d\theta}{dz} = \Delta k_0 + 2(k_w - \delta k)\left(\frac{\gamma - \gamma_0}{\gamma_0}\right),
\]

\[
\frac{d\gamma}{dz} = -\frac{\omega}{c} \frac{\hat{a}_w}{2\beta} \text{Im}(\hat{a}e^{i\theta}),
\]

\[
\left(\frac{d}{dz} + i\nu\right)\hat{a} = \frac{2\pi i}{k_z \Sigma} \hat{a}_w \left(\frac{I}{I_0}\right) \left(\frac{\exp(-i\theta)}{\gamma\beta}\right),
\]

where $\hat{a}_w = a_w 2^{1/2}, \hat{a} = a 2^{1/2}$, and

\[
\Delta k_0 = k_w - \delta k = \frac{\omega I}{2\gamma_0^2} (1 + a_w^2).
\]

In this form the equations are identical to those for the planar wiggler, and thus the Pierce parameter must be

\[
\rho = \left(\frac{\pi}{8} \frac{1}{\Sigma k_w^2} \left(\frac{I}{\gamma^3 I_0}\right)\right)^{1/3} a_w^2.
\]

Note that for a given value of $a_w$, the gain parameter for a helical wiggler can be as much as 60% more than that for a planar wiggler.

(4) Next we determine the natural focusing in this wiggler ("$k_f$"), by considering a small perturbation to the zeroth-order motion we examined above.
\[
\frac{dp_{x1}}{dz} = -\frac{e}{cv_z}(v_{x0} + v_{x1})B_z,
\]
\[
\frac{dp_{z1}}{dz} = \frac{e}{cv_z}(v_{z0} + v_{x1})B_z
\]

We can neglect 2nd order terms in \(B_x, B_y\), as well as jitter in \(v_z\) in the limit \(k_p << k_w\).

In the same limit also, the \(v_{y1}, v_{x1}\) terms will average to zero over a wiggler period. In this limit, one is left with focusing due to the gradient in the solenoidal field,

\[
\frac{dp_{x1}}{dz} = -\frac{e}{cv_z}v_{y0}B_z
\]

\[
= -\frac{e}{cv_z} \left( -c \frac{a_w}{\gamma} \cos(k_w z) \right) \frac{\left( -B_w k_w \right) \left( x \cos(k_w z) + y \sin(k_w z) \right)}{k_w}
\]

In this expression, we substitute

\[
x = x_0 + x_1 = -\frac{a_w}{\gamma \beta k_w} \cos(k_w z) + x_1,
\]

\[
y = y_0 + y_1 = -\frac{a_w}{\gamma \beta k_w} \sin(k_w z) + y_1,
\]

and average over a wiggler period,

\[
\frac{d^2x_1}{dz^2} = \frac{1}{mc\gamma} \frac{dp_{x1}}{dz} = -\frac{a_w^2 k_w^2 \cos^2(k_w z)}{2\gamma^3 v_z} x_1 = -\frac{a_w^2 k_w^2}{2\gamma^2 \beta} x_1 = -k_p^2 x_1,
\]

with

\[
k_p = \frac{a_w k_w}{2^{1/2} \gamma \beta^{1/2}}.
\]

Symmetry suggests that focusing should be equal in both planes, and this can be checked by a similar calculation.

(5) To illustrate, we make a sketch of the beam motion in the x-y plane and the co-moving E-field vector,
The arrangement of magnets required to produce this field might look like that for a proposed UCSB FEL (http://sbfel3.ucsb.edu/2mv/undulator.html):
or they might look like the twistor magnet (thanks to Roger Carr!) below or the Mitsubishi wiggler depicted in the attached brochure.