Linearized FEL Equations & Gain

To characterize amplification in the FEL, we linearize the FEL equations in perturbations due to the electromagnetic signal. Recall, the FEL equations are the equations of particle motion,

\[ \frac{d\theta_i}{dz} = k_w - \delta \frac{\omega}{c} \frac{1}{2\gamma_i^2} \left\{ 1 + \frac{a_w^2}{2} - a_w [JJ] \text{Re}[a \exp(i\theta_i)] \right\}, \]

\[ \equiv k_w + k_z - \frac{\omega}{v_z} \]

\[ \frac{d\gamma_i}{dz} = -\frac{1}{2} \left( \frac{\omega}{c} \right) a_w \gamma_i \beta_i [JJ] \text{Im}[a \exp(i\theta_i)], \]

\[ \equiv \gamma_{\parallel} \cdot \vec{E}_{\parallel} \]

\[ \beta_i = 1 - \frac{1}{2\gamma_i^2} \left( 1 + \frac{a_w^2}{2} \right), \]

where \( i \) is the particle index, and the eikonal equation,

\[ \left( \frac{d}{dz} + i \frac{2\pi}{k_z \Sigma I_0} \frac{1}{\gamma_{\parallel}} \right) a = i \frac{2\pi}{k_z \Sigma} a_w [JJ] \left( \frac{I}{I_0} \right) \frac{\exp(-i\theta)}{\gamma_{\parallel}}. \]

We will describe the particle motion by small perturbations to the “ballistic” or \( a=0 \) motion,

\[ \theta_i = \theta_{i0} + \theta_{i1}, \]

\[ \gamma_i = \gamma_{i0} + \gamma_{i1}, \]

where the unperturbed motion is just
\( \theta_{i0}(z) = \theta_i(0) + \Delta k_i z, \)
\( \gamma_{i0}(z) = \gamma_{i0}(0). \)

The initial detuning is

\[ \Delta k_i = k_i - \delta k - \frac{\omega}{c} \frac{1}{2 \gamma_i} \left\{ 1 + \frac{a_w^2}{2} \right\}. \]

The equations for the perturbations take the form

\[
\frac{d\theta_{i1}}{dz} = \Delta k' \gamma_{i1} + q_i \beta_{i0} \text{Re} \left[ a \exp(i \theta_{i0}) \right] \\
= \Delta k' \gamma_{i1} + q_i \beta_{i0} \text{Im} \left[ a \exp(i \theta_{i0}) \right],
\]
\[
\frac{d\gamma_{i1}}{dz} = -q_i \text{Im} \left[ a \exp(i \theta_{i0}) \right]
\]

where we abbreviate

\[ q_i = \frac{1}{2} \frac{\omega}{c} \frac{a_w}{\gamma_{i0} \beta_{i0}} \left[J J \right], \]
\[ \Delta k' = \frac{\partial \Delta k_i}{\partial \gamma_i} = \frac{\omega}{c} \frac{1}{\gamma_i} \left\{ 1 + \frac{a_w^2}{2} \right\}. \]

The eikonal equation takes the form
\[ \left( \frac{d}{dz} + i\nu \right) a = iQ \left\{ \exp(-i\theta) \right\} \]

\[ \approx iQ \left\{ -i\theta_{i0} \exp(-i\theta_{i0}) - \exp(-i\theta_{i0}) \frac{\partial(\gamma_{i0} \beta_{i0})}{\partial \gamma_{i0}} \gamma_{i1} \right\} \]

\[ \approx iQ \left\{ -i\theta_{i0} \exp(-i\theta_{i0}) - \frac{\exp(-i\theta_{i0})}{\gamma_{i0}^2 \beta_{i0}^3} \gamma_{i1} \right\} \]

where we abbreviate

\[ \nu = \frac{2\pi}{k_z \Sigma I_0} \left\{ \frac{1}{\gamma \beta} \right\}, \]

\[ Q = \frac{2\pi}{k_z \Sigma} \left\{ \frac{I}{I_0} \right\} a_n [JJ], \]

and have used

\[ \frac{\partial(\gamma_{i0} \beta_{i0})}{\partial \gamma_{i0}} = \frac{\partial}{\partial \gamma_{i0}} \gamma_{i0} \left\{ 1 - \frac{1}{2\gamma_{i0}^2} \left( 1 + \frac{a_w^2}{2} \right) \right\} \]

\[ = \frac{1}{\beta_{i0} \gamma_{i0}^2} \left( 1 + \frac{a_w^2}{2} \right) \]

\[ \approx \frac{1}{\beta_{i0}} \]

in the last line.

To solve this system we first represent the eikonal as a sum over exponentials

\[ a(z) = \sum_m a_m \exp(\Gamma_m z). \]

Note the conditions on the coefficients,
\[ a(0) = \sum_m a_m, \]

and, assuming an initially unbunched beam,

\[
\left( \frac{da}{dz} + iva \right)_{z=0} = \sum_m (\Gamma_m + iv)a_m = 0.
\]

It is convenient to express the perturbed particle variables as

\[
\gamma_{i_1} = \frac{\tilde{\gamma}_{i_1} - \tilde{\gamma}_{i_1}}{2i},
\]

\[
\theta_{i_1} = \frac{\tilde{\theta}_{i_1} - \tilde{\theta}_{i_1}}{2i},
\]

where

\[
\frac{d\tilde{\gamma}_{i_1}}{dz} = -q_i a \exp(i\theta_{i_0}),
\]

\[
\frac{d\tilde{\theta}_{i_1}}{dz} = \Delta k \tilde{\gamma}_{i_1} + q_i \frac{\beta_{i_0}}{\gamma_{i_0}} ia \exp(i\theta_{i_0})
\]

Integrating the first of these we obtain,
\[ \tilde{\gamma}_{i_1} = \int_0^{dz} \left[ -q a \exp(i\theta_1) \right] \]

\[ = -q \sum_m a_m \left[ \int_0^{dz} \exp\{i\theta_i(0) + (\Gamma_m + i\Delta k)z \} \right] \]

\[ = -q \sum_m a_m \frac{\exp\{i\theta_i(0)\}}{\Gamma_m + i\Delta k_m} \left[ \exp\{(\Gamma_m + i\Delta k_m)z\} - 1 \right] \]

A similar calculation shows that

\[ \tilde{\theta}_{i_1} = -q \Delta k_m \sum_m \frac{a_m \exp\{i\theta_i(0)\}}{(\Gamma_m + i\Delta k_m)^2} \left[ \exp\{(\Gamma_m + i\Delta k_m)z\} - 1 - (\Gamma_m + i\Delta k_m)z \right] \]

\[ + q \frac{\beta_{10}}{\gamma_{10}} \sum_m \frac{a_m \exp\{i\theta_i(0)\}}{(\Gamma_m + i\Delta k_m)} \left[ \exp\{(\Gamma_m + i\Delta k_m)z\} - 1 \right] \]

Next, we substitute these results in the eikonal equation,

\[ \left( \frac{d}{dz} + i\nu \right) a = \sum_m (\Gamma_m + i\nu) \frac{a_m \exp(\Gamma_m z)}{a_m \exp(-i\theta_i)} \]

\[ = iQ \left( \frac{-i\theta_i \exp(-i\theta_1)}{\gamma_{10} \beta_{10}} - \frac{\exp(-i\theta_1)}{\gamma_{10} \beta_{10}^3} \gamma_{10} \right) \]

We will find that this amounts to an equation between two sums over exponentials and equating coefficients will then permit us to solve for the complex wavenumbers \( \Gamma_m \). In the course of this, we will require that non-exponential terms vanish, and this imposes the conditions

\[ \sum_m \frac{a_m}{(\Gamma_m + i\Delta k_m)} = 0, \]

\[ \sum_m \frac{a_m}{(\Gamma_m + i\Delta k_m)^2} = 0. \]
(This last constraint will turn out to be redundant.) So in computing,

$$\left\langle \frac{\exp(-i\theta_0)}{\gamma_0 \beta_0} (-i\theta) \right\rangle = \left\langle \frac{\exp(-i\theta_0)}{\gamma_0 \beta_0} \left( -\frac{j \theta_j - \theta_j^*}{2i} \right) \right\rangle$$

$$= \left\langle \frac{\exp(-i\theta_0)}{\gamma_0 \beta_0} \left( -\frac{1}{2} \theta_0 \right) \right\rangle$$

$$= \left\langle \exp(-i\theta_0) \left( \frac{1}{2} q \Delta k' \right) \sum_m a_m \exp[i\theta(0)] \left[ \exp \left\{ \left( \Gamma_m + i\Delta k \right) z \right\} - 1 - \left( \Gamma_m + i\Delta k \right) z \right] \right\rangle$$

$$+ \left\langle \exp(-i\theta_0) \left( \frac{i}{2} q \beta_0 \right) \sum_m a_m \exp[i\theta(0)] \left[ \exp \left\{ \left( \Gamma_m + i\Delta k \right) z \right\} - 1 \right] \right\rangle$$

$$= \frac{q \Delta k'}{2 \gamma_0 \beta_0} \sum_m a_m \exp \left( \Gamma_m^z \right) \left[ 1 - (\Gamma_m + i\Delta k) \exp \left( \Gamma_m + i\Delta k \right) \right]$$

$$+ \frac{-iq}{2 \gamma_0^2} \sum_m a_m \exp \left( \Gamma_m^z \right) \left[ 1 + (\Gamma_m + i\Delta k) \exp \left( \Gamma_m + i\Delta k \right) \right]$$

we zero the non-exponential terms in the last line, likewise, we find

$$\left\langle \frac{\exp(-i\theta_0)}{\gamma_0^3 \beta_0^3} \gamma_{11} \right\rangle = \left\langle \frac{\exp(-i\theta_0) \gamma_{11}}{\gamma_0^3 \beta_0^3} \right\rangle$$

$$= \left\langle \frac{iq}{2 \gamma_0^2 \beta_0^3} \sum_m a_m \exp \left( \Gamma_m^z \right) \right\rangle$$

With these results, the eikonal equation

$$\sum_m \left( \Gamma_m + iv \right) a_m \exp \left( \Gamma_m z \right) = iQ \left\langle \frac{q \Delta k'}{2 \gamma_0 \beta_0} \sum_m a_m \exp \left( \Gamma_m^z \right) \right\rangle + iQ \left\langle \frac{-iq}{2 \gamma_0^2} \sum_m a_m \exp \left( \Gamma_m^z \right) \right\rangle$$

$$- iQ \left\langle \frac{iq}{2 \gamma_0^2 \beta_0^3} \sum_m a_m \exp \left( \Gamma_m^z \right) \right\rangle$$
is reduced to an algebraic relation determined by equating coefficients,

\[
(\Gamma_m + i\nu) = iQ\left\langle \frac{q_i \Delta k}{2\gamma_0 \beta_{i0}} \left(\frac{1}{\Gamma_m + i\Delta k_i}\right) \right\rangle - iQ\left\langle \frac{\omega}{2\gamma_0^2} \left(1 + \frac{1}{\beta_{i0}^2}\right) \left(\frac{1}{\Gamma_m + i\Delta k_i}\right) \right\rangle.
\]

At this point, the brackets amount to an average over the initial energy distribution of the electrons. The primary dependence on energy is through the resonant denominators, thus one may to a good approximation simply evaluate the algebraic factors at the average \(\gamma\). This dispersion relation is most simply described in terms of dimensionless parameters, \(\rho, \hat{\rho}, \delta\), where

\[
(2\rho k_w)^3 = \frac{1}{2\gamma\beta} Qq\Delta k'
\]

or

\[
\rho^3 = \frac{\pi}{16} \frac{(\omega/c)^2}{k\Sigma k_w^3} \left(\frac{I}{I_0}\right) \left(\frac{1}{\gamma^2}\right) \left(\frac{\nu}{\beta^2}\right) \left(\frac{1 + a_w^2}{2}\right).
\]

and in the last line we have evaluated \(\rho\) on resonance. The second dimensionless parameter, \(\hat{\rho}\), arising from roughly equal contributions due to the relativistic mass effect, and the perturbation to the transverse motion due to signal-induced jitter, is
\[(2\dot{\rho}_w)^2 = \frac{Qq}{2\gamma^2} \left(1 + \frac{1}{\beta^2}\right)\]

\[= \frac{1}{2\gamma^2} \frac{2\pi}{k \Sigma} \left[ \frac{I}{I_0} \right] a_w [JJ] \frac{1}{2} \frac{\omega}{c} \frac{a_w [JJ]}{\gamma \beta} \left(1 + \frac{1}{\beta^3}\right)\]

\[= \frac{\pi (\omega / c)}{2} \left( \frac{I}{I_0} \right) \left( \frac{a_w [JJ]}{\gamma' \beta} \right)^2 \left(1 + \frac{1}{\beta^3}\right)\]

or

\[\dot{\rho}^2 = \frac{\pi (\omega / c)}{8} \frac{1}{k \Sigma k_w^2} \left( \frac{I}{I_0} \right) \left( \frac{a_w [JJ]}{\gamma' \beta} \right)^2 \left(1 + \frac{1}{\beta^3}\right)\]

\[= \frac{\pi}{4} \frac{1}{\Sigma k_w^2} \left( \frac{I}{\gamma' I_0} \right) \left( \frac{a_w [JJ]}{\gamma' \beta} \right)^2\]

\[= 2\rho^3\]

The dimensionless detuning is

\[\delta = \frac{\Delta k - \nu}{2k_w},\]

and this is in principle a particle-variable (a function of energy), but we will shortly specialize to the case of a mono-energetic (“cold”) beam. In terms of these variables, and the dimensionless complex wavenumber,

\[\zeta = \frac{\Gamma + i\nu}{2ik_w},\]

we arrive at the dispersion relation
\[ \zeta = -\rho^3 \left\{ \frac{1}{(\zeta + \delta)^3} \right\} + \hat{\rho}^3 \left\{ \frac{1}{(\zeta + \delta)} \right\}, \]

or, for a cold beam, the cubic,

\[ (\zeta + \delta)^2 \zeta - \hat{\rho}^2 (\zeta + \delta) = -\rho^3. \]

Making the approximation,

\[ \hat{\rho}^2 \approx 2\rho^3, \]

this can be reduced to a two-parameter cubic equation. The three roots of this equation determine the actual complex wavenumbers \( \Gamma_m \) and they determine the mode coefficients according to

\[ \sum_m a_m = a(0), \]
\[ \sum_m \zeta_m a_m = 0, \]
\[ \sum_m \frac{a_m}{\zeta_m} = 0. \]

Numerous useful results derive from analysis of this cubic, and we turn to this next.