Crossing Angles In The Beam-Beam Interaction

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ABSTRACT
A Hamiltonian perturbation analysis of the beam-beam interaction with a horizontal crossing angle is performed. The beam-beam tune shifts and resonances that result from a crossing angle are determined.

I. INTRODUCTION

Many storage ring colliders are being designed to reach high luminosity through the use of a large number of closely spaced bunches. This introduces a potential problem of parasitic collisions near the interaction point, but these parasitic collisions can be avoided by having the beams cross at an angle rather than head-on (Figure 1). The contributions to the tune shifts and the beam-beam resonances introduced by a crossing angle are analyzed in this paper.

The beam-beam interaction with a crossing angle has been studied by a number of authors, [1] - [4]. This paper is closest to that of Sagan et al [2] which obtained some of the results presented here. The notation and method are discussed extensively in references [5] and [6].

II. PERTURBATION FORMALISM

The Hamiltonian of a particle in beam 1 is

$$H = H_0 - rac{N_r c}{\gamma} \tilde{V}_{BB}$$

where $H_0$ is the Hamiltonian of the transverse motion in the absence of the beam-beam interaction, $\tilde{V}_{BB}$ is the beam-beam potential, $N$ is the number of particles in beam 2, $r_c$ is the classical particle radius, and $\gamma$ is the energy in units of rest energy. The betatron motions in the absence of the beam-beam interaction can be written in terms of the action-angle variables $\{I_x, \psi_x\}$ and $\{I_y, \psi_y\}$ of the unperturbed Hamiltonian, $H_0$,

$$x_B = \sqrt{2I_x \beta_x} \cos \psi_x; \quad y_B = \sqrt{2I_y \beta_y} \cos \psi_y.$$  (1)

These expressions are used in a perturbation analysis of $\tilde{V}_{BB}$.

The beam-beam potential is

$$\tilde{V}_{BB} = \frac{2}{\pi \sigma_L^2} \sum_{n=-\infty}^{\infty} V_F(x,y,s) e^{-2(s-(nC+ct))^2/\sigma_L^2}.$$  (1)

The sum is over all turns and the variables in this equation are: $\sigma_L$ = the RMS bunch length of beam 2; $s$ = coordinate along the reference orbit; $C$ = the collider circumference; $c$ = speed of light; and $\tau$ = displacement of the collision point given in terms of the synchrotron oscillation amplitude, $\frac{x}{L}$, and tune, $Q_s$, by

$$\tau = \frac{x}{2} \cos(2\pi n Q_s).$$

The potential $V_F$ depends at the displacements of the particle from the center of beam 2

$$V_F = \int_0^{\infty} dq \frac{1}{\sqrt{(2\sigma_x^2 + q)(2\sigma_y^2 + q)}} \exp \left\{ -\frac{x^2}{2\sigma_x^2 + q} - \frac{y^2}{2\sigma_y^2 + q} \right\},$$

where $\sigma_x$ and $\sigma_y$ are the RMS transverse sizes of beam 2.

Using the expressions in eq. (1) as approximations for the betatron motions and assuming that the crossing is in the $x$-dimension, the potential can be rewritten

$$V_F = \frac{\sqrt{\pi}}{2\pi \sin 2\theta} \int_0^{\infty} dq \frac{1}{\sqrt{(2\sigma_y^2 + q)}} \exp \left\{ -\frac{2\beta_y I_y \cos^2 \theta_y}{2\sigma_y^2 + q} \right\} \times \left\{ \frac{(s \sin 2\varphi + \sqrt{2\beta_x I_x} \cos \theta_x)^2}{2\sigma_x^2 + q} \right\}.$$  (1)

Fourier transforming the $x$ expression with respect to $s$ gives

$$V_F = \frac{\sqrt{\pi}}{2\pi} \frac{1}{\sin 2\theta} \int_0^{\infty} dq \frac{1}{\sqrt{(2\sigma_y^2 + q)}} \exp \left\{ -\frac{2\beta_y I_y \cos^2 \theta_y}{2\sigma_y^2 + q} \right\} \times \left( \int_0^{\infty} e^{i\omega s} ds \exp \left\{ -\frac{\omega^2 (2\sigma_x^2 + q)}{4\sin^2 2\varphi} + \frac{i\omega \sqrt{2\beta_x I_x} \cos \theta_x}{\sin 2\varphi} \right\} \right).$$

Following the usual procedure of Fourier transforming $\tilde{V}_{BB}$ with respect to $\psi_x, \psi_y$ and $s$ gives an expression for $\tilde{V}_{BB}$ in terms of Fourier coefficients each of which is related to the resonance

$$pQ_x + rQ_y + mQ_s = n.$$
where \( p, r, m, \) and \( n \) are integers. Making a change of variables \( \zeta = \omega / \sin 2\phi \) this expression is

\[
\tilde{V}_{BB} = \frac{1}{C} \sum_{m,n,p,r} \int d\zeta \frac{U_{pr}(I_x, I_y, \zeta)}{p} \exp\left(-\frac{k_{prm} + \zeta \sin 2\phi)^2 \sigma_L^2}{8}\right) \\
\times \exp\left(i m J_m\left(k_{prm} / 2 \right)\right) \exp\left(i(p \psi_x + r \psi_y - 2\pi(n - mQ_x)s / C) \right)
\]

where

\[
U_{pr} = \frac{\sqrt{\pi}}{(2\pi)\sqrt{3}} \int_0^{\infty} \int_0^{\infty} dq \exp\left(-\frac{2\gamma \beta_x \cos^2 \theta_x}{2\sigma_y + q}\right) \\
\times \exp\left(-\frac{\gamma(2\sigma_y + q)}{4}\right) + i\gamma \sqrt{2\beta_x \cos \theta_x}
\]

and

\[
k_{prm} = 2\pi(n - mQ_x) / C + p(1/\beta_x - 2\pi Q_x / C) + r(1/\beta_y - 2\pi Q_y / C).
\]

The quantities \( \beta_x^* \) and \( \beta_y^* \) are the \( \beta \)-functions at the collision point, and \( Q_x, Q_y \) are the tunes in the absence of the beam-beam interaction.

The resonance \( pQ_x + rQ_y + mQ_s = n \) occurs for values of the tune where the phase is stationary, i.e.

\[
\frac{d}{ds}(p \psi_x + r \psi_y - 2\pi(n - mQ_s)s / C) = 0,
\]

and the average value of the beam-beam potential is given by the term in the series with \( p = r = m = n = 0 \). Perform a Taylor expansion in powers of \( \sin 2\phi \)

\[
\tilde{V}_{BB} = \tilde{V}_{BB}^{\sin 2\phi = 0} + \sin 2\phi \frac{\partial \tilde{V}_{BB}}{\partial \sin 2\phi}^{\sin 2\phi = 0} + \ldots.
\]

The first term in the Taylor series is

\[
\tilde{V}_{BB}^{\sin 2\phi = 0} = \frac{1}{C} \sum_{m,n,p,r} \int d\zeta \frac{U_{pr}(I_x, I_y, \zeta)}{p} \exp\left(-\frac{k_{prm} \sigma_L^2}{8}\right) \\
\times \exp\left(i m J_m\left(k_{prm} / 2 \right)\right) \exp\left(i(p \psi_x + r \psi_y - 2\pi(n - mQ_x)s / C) \right).
\]

The integral is

\[
\int d\zeta U_{pr}(I_x, I_y, \zeta) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} dq e^{i p \psi_x + r \psi_y - 2\pi(n - mQ_x)s / C} \\
\times \exp\left(\frac{2I_x \beta_y \cos^2 \theta_y}{2\sigma_y + q} - \frac{2I_x \beta_x \cos^2 \theta_x}{2\sigma_y + q}\right).
\]

The second term in the Taylor series is

\[
\frac{\partial \tilde{V}_{BB}}{\partial \sin 2\phi}^{\sin 2\phi = 0} = \frac{1}{C} \sum_{m,n,p,r} \int \zeta d\zeta U_{pr}(I_x, I_y, \zeta) \exp\left(-\frac{k_{prm} \sigma_L^2}{8}\right) i m \\
\times \left[ \frac{\gamma}{2} J_m\left(m - 1\right) - \frac{m}{k_{prm}^2} J_m\left(m - \frac{k_{prm} \sigma_L}{2}\right) \right] \\
\times \exp\left(i(p \psi_x + r \psi_y - 2\pi(n - mQ_x)s / C) \right).
\]

Equations (2) - (5) contain the results. These frightening-looking expressions can be interpreted to give useful information about the beam-beam interaction.

### III. DISCUSSION

#### A. Tune Shifts

The tune shifts as a function of amplitude are

\[
\Delta Q_x = -\frac{\gamma^c}{2\pi^2} \frac{\partial \tilde{V}_{BB}}{\partial I_x}; \Delta Q_y = -\frac{\gamma^c}{2\pi^2} \frac{\partial \tilde{V}_{BB}}{\partial I_y}
\]

where \( \tilde{V}_{BB} \) is given by eqs. (2) - (5) evaluated with \( p = r = m = n = 0 \).

The tune shifts are the same as for head-on collisions because there are no contributions from the second term in the Taylor series, the term proportional to \( \sin 2\phi \). This follows from the parity of the \( \theta_x \) integrands. In the case of \( \Delta Q_y \) the integral is
The crossing angle has introduced new beam-beam resonances that have odd horizontal order and Fourier expansion coefficients proportional to $\sin 2\phi$. There are both betatron, $m = 0$, and synchrobetatron, $m \neq 0$, resonances. The appearance of odd order betatron resonances can be understood because there is a phase shift of $\pi$ across the interaction region. The synchrobetatron resonances arise from modulation introduced by the synchrotron oscillations. They depend on the synchrotron amplitude and have zero Fourier expansion coefficient when $\tilde{\tau} = 0$.

C. Remarks

The crossing angle has not changed the tune shifts, so the beam-beam footprint, the area of the tune plane occupied by the beam, is the same as for head-on collisions. The effect of the crossing angle has been to introduce odd horizontal order resonances. These additional resonances could lower the beam-beam limit.

TeV33 is considering using both horizontal and vertical crossing angles. The vertical crossing angle will introduce odd order vertical resonances as well, and there is a still larger probability of a reduced beam-beam limit.

IV. REFERENCES


B. Beam-Beam Resonances

Possible resonances can determined from the parity of the integrands in eqs. (3) and (5). The integrand of eq. (3) is an even function of $\theta_x$ and an even function of $\theta_y$. The only allowed resonances for head-on collisions must have both $p$ and $r$ equal to even integers. The integrand of eq. (5) is an odd function of $\theta_x$ and an even function of $\theta_y$. The allowed resonances must have $r$ equal to and even integer and $p$ equal to an odd integer.