On Polynomial Identities in Algebras Generated by Idempotents and Their \(*\)-Representations

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Among algebras generated by linearly dependent idempotents we look for algebras with polynomial identities. One of the way to find a polynomial identity in an algebra is to calculate a linear basis for this algebra, build a “big” family of representations and prove using the basis that it is a residual family. In the paper was found a linear basis for some algebra generated by idempotents.

1 Introduction

Algebras generated by linearly dependent idempotents are investigated in the paper. Among these algebras we look for algebras with polynomial identities, so called \(PI\)-algebras (see, for example, [1]). The theory of \(PI\)-algebras is well developed and gives additional information about algebras. So for applications it is important to determine that an algebra has a polynomial identity. One of the way to find a polynomial identity in an algebra is to calculate a linear basis for this algebra, build a “big” family of representations such that the supremum of dimensions of representations from the family is finite and prove that it is a residual family (see [2, Theorem 2] or [3]).

Algebras which we are interested in are

\[
Q_{n,\overline{\lambda}} = \mathbb{C}\langle q_1, \ldots, q_n \mid q_k^2 = q_k, \sum_{k=1}^{n} \lambda_k q_k = e \rangle,
\]

\(n \in \mathbb{N}, \overline{\lambda} \in \mathbb{C}^n\), and its factor-algebras \(Q_{n,\overline{\lambda}}/\{q_i q_j = q_j q_i = 0\}_{i=1}^{n}\) where \(i \neq j \in \{1, \ldots, n\}\).

In paper [4] a criterion was given when algebras \(Q_{n,\overline{\lambda}}\) are \(PI\)-algebras (note that the case \(\lambda_i = \frac{1}{\lambda} \) was studied in the paper [5]). We remind the respective results.

In the case \(n \leq 3\) all algebras \(Q_{n,\overline{\lambda}}\) are finite-dimensional and so they are \(PI\)-algebras. In the case \(n \geq 4\) all algebras \(Q_{n,\overline{\lambda}}\) are infinite-dimensional. Let \(\delta(\overline{\lambda}) = 1 - \frac{1}{2} \sum_{j=1}^{n} \lambda_j\). Then we have the following results.

**Theorem 1.** If \(\delta(\overline{\lambda}) \neq 0\) then the algebra \(Q_{4,\overline{\lambda}}\) is not a \(PI\)-algebra.

**Corollary 1.** When \(n \geq 5\) all algebras \(Q_{n,\overline{\lambda}}\) are not \(PI\)-algebras.

**Theorem 2.** If \(\delta(\overline{\lambda}) = 0\) then the algebra \(Q_{4,\overline{\lambda}}\) is an \(F_4\)-algebra, i.e.

\[
\sum_{\sigma \in S_4} (-1)^{p(\sigma)} v_{\sigma(1)} v_{\sigma(2)} v_{\sigma(3)} v_{\sigma(4)} = 0,
\]

for any elements \(v_1, v_2, v_3, v_4 \in Q_{n,\overline{\lambda}}\), where \(S_4\) is a symmetric group.
The Corollary 1 shows that when \( n \geq 5 \) algebras \( \mathbb{Q}_{n, \lambda} \) are not \( PI \)-algebras. But if we add some relations, for example, relations of commuting \( \langle q_i q_j = q_j q_i \rangle \) or orthogonality \( \langle q_i q_j = 0 \rangle \) for some generators, then some of obtained factor-algebras are \( PI \)-algebras and are infinite-dimensional.

In paper [6] for the algebra \( A = \mathbb{C} \langle a, b \mid a^3 + \alpha_1 a + \alpha_0 = 0, b^3 + \beta_1 a + \beta_0 = 0, (a + b)^3 + \gamma_1 (a + b) + \gamma_0 = 0 \rangle \), where \( \alpha_i, \beta_i, \gamma_i \in \mathbb{C} \). But first we give some argumentation to solve this problem. Consider algebras

\[
A' = \mathbb{C} \langle p_1, p_2, q_1, q_2, q_3 \mid p_k^2 = p_k, p_k q_k = 0, \sum_{k=1}^3 (p_k + 2q_k) = 3 \epsilon \rangle,
\]

where \( \mu_k, \nu_k \in \mathbb{C}, \mu_k \neq \nu_k, \sum_{k=1}^3 (\mu_k + \nu_k) = 3 \) (in the case of \( \mu_k = 1/3, \nu_k = 2/3 \) we have the algebra \( \mathcal{R} \)). To find an answer the question whether some family of representations is a residual family for corresponding algebras we need a linear basis of the algebra \( A \) or in some algebra which is isomorphic with this one.

The algebra \( A' \) and an algebra

\[
A'' = \mathbb{C} \langle x_1, x_2, x_3 \mid x_1 + x_2 + x_3 = e, x_k(x_k - \mu_k)(x_k - \nu_k) = 0 \rangle
\]

are isomorphic. A map \( x_k \mapsto \mu_k p_k + \nu_k q_k \) is a corresponding isomorphism. A map \( x_k \mapsto x_k' + (\mu_k + \nu_k)/3 \) gives isomorphism between the algebra \( A'' \) and an algebra

\[
A''' = \mathbb{C} \langle x_1', x_2', x_3' \mid x_1' + x_2' + x_3' = 0, f_k(x_k) = 0 \rangle,
\]

where \( f_k \) are polynomials such that \( \deg f_k = 3 \) and sum of roots is zero. So this algebra and the algebra \( A \) are isomorphic.

### 2 A linear basis in the algebra \( A \)

We introduce a homogeneous lexicographical order on words in alphabet \( \{a, b\}: a < b \). Using the Diamond Lemma (for definitions of the notions \emph{Gröbner basis, reductions, compositions, growth of an algebra} etc. see review [7] and references therein) we will prove the following theorem which describes a linear basis in the algebra \( A \). We will denote reductions by symbol \( \rightarrow \).

**Theorem 4.** The set of words

\[
\{a^{\sigma_1}(ba^2)^n(ba)^m b^{\sigma_2} \mid \sigma_1, \sigma_2 \in \{0, 1, 2\}, n, m \in \mathbb{N} \cup \{0\} \}
\]

is a linear basis of the algebra \( A \).
Proof. Let us introduce notations

\[ p = -\alpha_1 a - \alpha_0, \]
\[ q = -\beta_1 b - \beta_0, \]
\[ s = \gamma_1(a + b) + \gamma_0, \]
\[ r = s + p + q, \]
\[ \Lambda = bab + ba^2 + ab^2 + aba + a^2b. \]

Then \((a + b)^3 + \gamma_1(a + b) + \gamma_0 \rightarrow b^2a + \Lambda + r\). So we have an ideal \(I\) generated by elements \(a^3 - p, b^3 - q, b^2a + \Lambda + r\). Let \(G\) be the set of this elements. The main words of the set \(G\) are \(b^3, b^2a\) and \(a^3\). So we have 7 compositions:

\[ b \cdot b^2 \cdot b, b^2 \cdot b \cdot b^2, b \cdot b^2 \cdot a, b^2 \cdot b \cdot ba, a \cdot a^2 \cdot a, a^2 \cdot a \cdot a^2 \text{ and } b \cdot ba \cdot a^2. \]

The first composition is a sub-word of the second, the third is a sub-word of the fourth and the fifth is a sub-word of the sixth, so by the Triangle Lemma (see for example [7, sec. 2.10]) it is enough to calculate only the first, the third, the fifth and the seventh compositions. The first and the fifth compositions are reduced to 0 because \([b, q] = 0\) and \([a, p] = 0\).

By reductions of the third composition we get 0:

\[
\begin{align*}
\text{first composition:} & \quad \Lambda + br + qa^2 = b^2ab + b^2a^2 + bab + baba + ba^2b + br + qa \\
           & \rightarrow (b^2a + bab + ba^2)b + (b^2a + bab)a + br + qa \\
           & \rightarrow -ab^2 + a^2b + br) + (ba^2 + ab^2 + aba + a^2b + r)a + br + qa \\
           & \rightarrow -a(b^2a + bab + ba^2 + ab^2 + aba) - bp + [q, a] + [b, r] - ra \\
           & \rightarrow a(a^2b + r) - bp + [q, a] + [b, r] - ra \\
           & \rightarrow [b + a, r] - [a, q] - [b, p] = [b + a, p + q] - [a, q] - [b, p] = 0.
\end{align*}
\]

The seventh composition gives a new element:

\[
\begin{align*}
\text{seventh composition:} & \quad \Lambda a^2 + ra^2 + b^2p \rightarrow baba^2 + ab^2a^2 + a^2ba^2 + bap + abp + ra^2 + b^2p \\
           & \rightarrow baba^2 + a(b^2a + aba)a + \{bp, a\} + ra^2 + b^2p \\
           & \rightarrow baba^2 - ababa - abp - a^2b^2a - pba - ara + \{bp, a\} + ra^2 + b^2p \\
           & \rightarrow baba^2 + ababa + a^2(\Lambda + r) - ara + \{bp, a\} + ra^2 + b^2p \\
           & \rightarrow baba^2 - ababa + a^2bab + a^2ba^2 + \{bp, a\} + ra^2 + b^2p \\
           & \rightarrow baba^2 - ababa + a^2bab + a^2ba^2 + \{b^2, p\} + \{b, ap\} + \{a^2, r\} - ara.
\end{align*}
\]

We introduce notations \(\Sigma = \{a^2, r\} - ara\) and \(\Omega = \{b^2, p\} + \{b, ap\}\) and add the element

\[ baba^2 - ababa + a^2bab + a^2ba^2 + \Omega + \Sigma \]

to the set \(G\).

So we obtain new compositions: \(b \cdot ba \cdot ba^2, b^2 \cdot ba \cdot ba^2, bab \cdot a^2 \cdot a\) and \(bab \cdot a \cdot a^2\). Again the first composition is a sub-word of the second and the third composition is a sub-word of the forth. And we need only to calculate only first and third ones.

To calculate the first composition we use that

\[ \Omega = \{b^2, p\} + \{b, ap\} = \alpha_1(\{b^2, a\} + \{b, a^2\}) + 2\alpha_0(b^2 + ba + ab) \]
\[ \rightarrow -\alpha_1(bab + aba + r) + \alpha_0(2b^2 + ba + ab). \]

Thus we have

\[
\begin{align*}
\text{bab} \cdot a^2 \cdot a : & \quad -\underbrace{ababa^2 + a^2bab + a^2bp + \Omega a + \Sigma a + babp}_{\rightarrow -\alpha_1(bab + aba + r) + \alpha_0(2b^2 + ba + ab)} \\
& = -a(a^2bab + a^2ba^2 + \Omega + \Sigma) + a^2bp + \Omega a + \Sigma a + babp \\
& \rightarrow pba^2 + pbab + a^2bp + babp + \{\Omega, a\} + \{\Sigma, a\} \\
& = \alpha_1(abab + abab + a^2ba + bab - \{bab + aba + r, a\}) \\
& + \alpha_0(bab + 2bab + a^2b + \{2b^2 + ba + ab, a\}) + \{a^2r + ra^2 - ara, a\} \\
& \rightarrow -\alpha_1\{r, a\} - 2\alpha_0(b^2a + \Lambda) + ra^2 + rp - ara^2 + pr + ara^2 - a^2ra \\
& \rightarrow -\alpha_1\{r, a\} - 2\alpha_0r + \{r, p\} = 0.
\end{align*}
\]

The third composition also does not give a new element into the set \( G \):

\[
\begin{align*}
b \cdot ba \cdot ba^2 : & \quad (bab + ba^2 + ab^2 + aba + a^2b + r)ba^2 - b(-ababa + a^2bab + a^2ba^2 + \Omega + \Sigma) \\
& = ba(b^2a + bab + ab^2 + aba + a^2b)a + ababa^2 + a^2b^2a^2 \\
& - ba^2(b^2a + bab + ba^2 + aba) + (aqa^2 + rba^2 - b\Omega - b\Sigma) \\
& \rightarrow -ba(ba^2 + r)a + ababa^2 + a^2b^2a^2 + ba^2(ab^2 + a^2b + r) \\
& + (aqa^2 + rba^2 - b\Omega - b\Sigma) \\
& \rightarrow -(bab^2 - ababa)a + a^2b^2a^2 \\
& + (aqa^2 + rba^2 - b\Omega - b\Sigma - barab + bp^2 + bpa + ba^2r) \\
& \rightarrow (a^2bab + a^2ba^2 + \Omega + \Sigma)a + a^2b^2a^2 \\
& + (aqa^2 + rba^2 - b\Omega - b\Sigma - barab + bp^2 + bpa + ba^2r) \\
& = a^2(b^2a + bab + ba^2)a \\
& + (aqa^2 + rba^2 - b\Omega - b\Sigma - barab + bp^2 + bpa + ba^2r + \Omega a + \Sigma a) \\
& \rightarrow -a^2(ab^2 + aba + a^2b + r)a \\
& + (aqa^2 + rba^2 - b\Omega - b\Sigma - barab + bp^2 + bpa + ba^2r + \Omega a + \Sigma a) \\
& \rightarrow -p(b^2a + ba^2 + aba) \\
& + (aqa^2 - rba^2 - b\Omega - b\Sigma - barab + bp^2 + bpa + ba^2r + \Omega a + \Sigma a - a^2ra) \\
& \rightarrow aqa^2 - rba^2 - b\Omega - b\Sigma - barab + bp^2 + bpa + ba^2r + \Omega a + \Sigma a - a^2ra \\
& + pbab + pab^2 + pa^2b + pr \\
& = aqa^2 - rba^2 - barba + bp^2 + bpa + ba^2r - a^2ra + pbab + pab^2 + pa^2b + pr \\
& + (b^2pa + pb^2a + bapa + apba - qp - bp^2 + b^2ap - bapb) \\
& + (a^2ra + rp - ara^2 - ba^2r - b^2a^2 + baba) \\
& = \eta(b^2a + bab + ab^2 + aba + a^2b + r) \\
& + aqa^2 + rba^2 + bapa - qp + rp - ara^2 - b^2a^2 \\
& \rightarrow pba^2 + aqa^2 + rba^2 + bapa - qp + rp - ara^2 - bra^2 \\
& = \eta(b, p) + [a, q] + [r, b]a^2 + qa^3 - qp + rp - ara^2 \\
& \rightarrow \eta([s, b] + [s, a] - [q, a] - [s, a])a^2 - ara^2 + rp = -([r, a] + ar)a^2 + rp \rightarrow 0.
\end{align*}
\]

Then by the Diamond Lemma the main words of the Gröbner bases \( G \) are \( b^3, a^3, b^2a, bab^2 \), so they are disallowed and the theorem is proved. \( \square \)
Corollary 2. *Algebras* \(A\) *are infinite-dimensional and have the quadratic growth.*

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