On Spectrum of Matrix-Valued Continuous Functions of a Family of Commuting Operators

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A spectrum of matrix-valued continuous functions of a family of self-adjoint commuting bounded operators on a Hilbert space is studied.

1 Introduction

Let \( \tilde{A} = \{A_i = A^*_i\}_{i=1}^m \subset L(H) \) be a family of self-adjoint commuting operators and \( \{E_i\}_{i=1}^m \) be a family of their spectral measures. A direct product of spectral measures is \( \tilde{E}(\alpha_1 \times \alpha_2 \times \ldots \times \alpha_m) = \prod_{i=1}^m E_i(\alpha_i) \), a measure on a measurable space. A support of decomposition of unit \( E \) is called \( \text{Supp} \ E = \bigcap \{\varphi \in \mathcal{F}: E(\varphi) = 1\} \), i.e. intersection of all closed sets of full measure. A common spectrum of a family of self-adjoint commuting bounded operators is called \( S(\tilde{A}) = S(\{A_i \mid i = 1, \ldots, m\}) := \text{Supp} \ E \), i.e. the support of product of spectral measures. By definition, the following conclusion may be drawn

\[
S(\tilde{A}) = \text{Supp} \ E \subseteq \prod_{i=1}^m \text{Supp} E_i = \prod_{i=1}^m \sigma(A_i),
\]

where \( \sigma(A_i) \) is a spectrum of operator \( A_i, i = 1, \ldots, m \). A continuous function of a family of self-adjoint operators \( \tilde{A} \) is

\[
f(\tilde{A}) = \int_{S(\tilde{A})} f(\lambda_1, \ldots, \lambda_m) d\tilde{E}(\Lambda),
\]

where \( \Lambda = (\lambda_1, \ldots, \lambda_m) \in S(\tilde{A}) \subset \mathbb{R}^m \) (see e.g. [1–4]).

Let us consider a matrix-valued continuous function \( \{f_{ij}(t_1, t_2, \ldots, t_m)\}_{i,j=1}^m \), where \( f_{ij}(t_1, t_2, \ldots, t_m) \in C(\mathbb{R}^m, \mathbb{C}) \). The result of this paper is the formula for spectrum of matrix-valued continuous functions of a family of self-adjoint commuting bounded operators.

2 On spectrum of matrix operators

Let \( G = \tilde{F}(\tilde{A}) = \{F_{ij}(\tilde{A}) = F_{ij}(A_1, \ldots, A_m) \mid i, j = 1, \ldots, m\}, G : H^m \to H^m \). Let \( \tilde{F}(\Lambda) = \{F_{ij}(\Lambda) = F_{ij}(\lambda_1, \ldots, \lambda_m) \mid i, j = 1, \ldots, m\} \) be a continuous matrix function \( \tilde{F}(\Lambda) : U(S(\tilde{A})) \to \mathbb{C}^{n \times n} \), where \( U \) is a neighborhood of \( S \) and \( \tilde{F}(\tilde{A}) \) is a function on family of self-adjoint commuting bounded operators. Denote \( \Delta(G, \lambda) = \Delta(\tilde{F}(\Lambda), \lambda) := \text{det}(\{F_{ij}(\tilde{A}) - \delta^2_i \lambda\}_{i,j=1}^n) \), where \( \delta^2_i \) is a Kroneker symbol and \( \Delta(\Lambda, \lambda) := (F_{ij}(\Lambda) - \delta^2_i \lambda), \) where \( \Lambda = (\lambda_1, \ldots, \lambda_m) \in S(\tilde{A}) \).

Theorem 1. A spectrum of the operator \( G \) is equal to \( \sigma(G) = \{\lambda \in \mathbb{C} \mid \exists \Lambda \in S : \Delta(\Lambda, \lambda) = 0\} \).

Proof. \((\Rightarrow)\) Let \( \forall \lambda \in S(\tilde{A}) : \Delta(\Lambda, \lambda) \neq 0 \), then existence of operator \( (G - \lambda I)^{-1} \) means existence of a solution of the system of operator equations \( \sum_{j=1}^n F_{ij}(\tilde{A})x_j = y_i, i = 1, \ldots, m \), where \( x_j, y_i \in H \).

The sufficient condition for it is existence of the operator \( (\Delta(\tilde{F}(\tilde{A}), \lambda))^{-1} \).
It follows that \( \exists (G - \lambda I)^{-1} = B \Rightarrow \lambda \in \rho(G) \), where \( \rho \) is a resolvent set.

\((\Leftarrow)\) Let \( \exists A \in S(A) : \Delta(F(A, \lambda)) = 0 \), then \( \exists f \), where \( f \) is the eigenvector of operator \( F(A) \).

Denote \( \forall z \geq 1 : 
\)
\[
0 \neq f^{(z)} := E \left( \left( \lambda_1 - \frac{1}{z}, \lambda_1 + \frac{1}{z} \right) \times \cdots \times \left( \lambda_m - \frac{1}{z}, \lambda_m + \frac{1}{z} \right) \right) f
\]
(notice that \( E \) is a projector) and \( y^{(z)} := \frac{f^{(z)}}{\|f^{(z)}\|} \). It follows that
\[
(G - \lambda I)y^{(z)} = (\{F_{ij}(A) - \lambda_{ij}\}_{i,j=1,m})y^{(z)} - \Delta(A, \lambda)y^{(z)}.
\]

The condition \( \Delta(\lambda, \lambda)y^{(z)} = 0 \) is clearly fulfilled, and
\[
\|\Delta_{ij}(\tilde{A}) - \lambda_{ij}\|y^{(z)}_j = \int_{\mathbb{R}} |\alpha_{ij} - \lambda_{ij}|^2 d(E(\alpha_{ij})y^{(z)}_j, y^{(z)}_j) \to 0, \quad z \to 0.
\]

And finally \( \|G - \lambda I\|y^{(z)} \rightharpoonup 0, \quad z \to 0, \quad \|y^{(z)}\| = 1 \Rightarrow \lambda \in \sigma(G). \)

The following proposition will be useful in examples.

**Proposition 1.** Let \( A^* = A \subset L(H) \) be a self-adjoint bounded operator. Let \( \tilde{A} = \{f_1(A), \ldots, f_n(A)\} \), where \( \{f_i|i = \overline{1,m}\} \subset C(U(\sigma(A)), \mathbb{R}) \), \( U \) is a neighborhood of \( \sigma(A) \), then
\[
S(\{f_1(A), f_2(A), \ldots, f_m(A)\}) = \{(f_1(\lambda), f_2(\lambda), \ldots, f_m(\lambda))|\lambda \in \sigma(A)\}.
\]

**Proof.** There is \( E_i(\alpha) = \int_{\sigma(A)} \chi(f_i(\lambda))dE \), where \( \chi(A) \) is the characteristic function, \( i = \overline{1,m} \).

Then denote a decomposition \( \times, \sigma_i = \text{Supp} \tilde{E} \cup \Theta \), where the common measure of any opening set with \( \Theta \) is equal to zero, and the common measure of any opening set is not equal to zero if this set contains a point from \( \text{Supp} \tilde{E} \). Since \( \int E(\alpha_1, \alpha_2 \cdots \alpha_m) = \int \chi_{\alpha_1}(\lambda) \cdots \chi_{\alpha_m}(\lambda)dE(\lambda) = 0 \) and \( f(\sigma(A)) = \sigma(f(A)) \), then proposition is proved.

### 3 Examples

**Example 1.** Let \( H = L_2([-\pi, \pi], dt) \) and \( G = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \), and \( K_j(t) := je^{it}, \quad j = \overline{1,4}, \quad i = \sqrt{-1}, \)

and \( \tilde{A} := \{A_j = \int_{-\pi}^{\pi} K_j(t - \tau)x(\tau)d\tau | j = \overline{1,4}\} \) family of commuting self-adjoint bounded operators.

The common spectrum of \( \tilde{A} \) is equal to
\[
S(\tilde{A}) = \{(0,0,0,0), (2\pi, 4\pi, 6\pi, 8\pi)\} = \{\Lambda_1, \Lambda_2\}.
\]

Let us solve the following equations \( \Delta(A_1, \lambda) = 0, \Delta(A_2, \lambda) = 0 \):
\[
\det \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \quad \text{and} \quad \det \begin{pmatrix} 2\pi - \lambda & 4\pi \\ 6\pi & 8\pi - \lambda \end{pmatrix} = 0 \Rightarrow
\]

\( \Rightarrow \) The spectrum of matrix operator is equal to \( \sigma(G) = \{0, \pi(5 \pm \sqrt{33})\} \).
Example 2. Let $H = L_2([0, 1], dt)$ and $f_1, f_2, f_3, f_4 \in C([0, 1])$. Let $\tilde{A} = \{(A_1x)(t) = f_1(t)x(t), (A_2x)(t) = f_2(t)x(t), (A_3x)(t) = f_3(t)x(t), (A_4x)(t) = f_4(t)x(t)\}$ be a family of self-adjoint commuting operators, then

$$S(A_1, A_2, A_3, A_4) = \{f_1(t), f_2(t), f_3(t), f_4(t) | t \in [0, 1]\},$$

$$G = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} F_{11}(\tilde{A}) & F_{12}(\tilde{A}) \\ F_{21}(\tilde{A}) & F_{22}(\tilde{A}) \end{pmatrix},$$

where

$$F_{11}(t_1, t_2, t_3, t_4) = t_1, F_{12}(t_1, t_2, t_3, t_4) = t_2, F_{21}(t_1, t_2, t_3, t_4) = t_3, F_{22}(t_1, t_2, t_3, t_4) = t_4,$$

$$\sigma(G) = \left\{ \lambda \in \mathbb{C} | \exists t \in [0, 1] : \det \begin{pmatrix} f_1(t) - \lambda & f_2(t) \\ f_3(t) & f_4(t) - \lambda \end{pmatrix} = 0 \right\} \Leftrightarrow$$

$$\Leftrightarrow \sigma(G) = \left\{ f_1(t) + f_4(t) \pm \sqrt{(f_1(t) + f_4(t))^2 + 4(2f_2(t) - f_1(t)f_4(t))} \middle| t \in [0, 1] \right\}.$$

Example 3. Let $(P_1x)(t) = t_1x(t), (P_2y)(t) = 2\pi t_2x(t)$ be two commuting self-adjoint bounded operators, where $x(t) \in L_2([0, 1]^2, dt), t = (t_1, t_2)$. The operator $N = P_1(\cos P_2 + i \sin P_2)$ is a normal operator $(N^*N = NN^*)$. Let us consider the subalgebra $B_V$ of $C(D, \mathbb{C}^{2 \times 2})$ of the form $B_V(D) = \{a \in C(D, \mathbb{C}^{2 \times 2}) : a(z) = V^{-1}(z)a(1)V(z), |f(z)| = 1\}$, where $D = \{z \in \mathbb{C} : |z| \leq 1\}$ is the unit disk and $V(z) = \text{diag} \{z^{-1}, 1\}$. The algebra $B_V(D)$ is an example of a non-trivial $2$-homogeneous $C^*$-algebra (see [5]).

It follows that $B(N) = \{a(N) | a \in B_V(D)\}$ is an example of a non-trivial operator $C^*$-algebra, because the spectrum of operator $N$ is equal to $\sigma(N) = D$. The following conclusion may be drawn

$$\forall a = a(N) \in B_V(N) : a(N) = \begin{pmatrix} a_{11}(N) & a_{12}(N) \\ a_{21}(N) & a_{22}(N) \end{pmatrix} = \begin{pmatrix} a_{11}(f(P_1, P_2)) & a_{12}(f(P_1, P_2)) \\ a_{21}(f(P_1, P_2)) & a_{22}(f(P_1, P_2)) \end{pmatrix},$$

where

$$f(x, y) = x(\cos y + i \sin y),$$

$$S(\{a_{ij}(N), i, j = 1, 2\}) = \{a_{ij}(t_1(\cos 2\pi t_2 + i \sin 2\pi t_2)) | t \in [0, 1]^2\},$$

$$\sigma(a(N)) = \left\{ \frac{a_{11}(\tau) + a_{22}(\tau)}{2} \pm \sqrt{\frac{(a_{11}(\tau) + a_{22}(\tau))^2 - 4(a_{11}(\tau)a_{22}(\tau) - a_{12}(\tau)a_{21}(\tau))}{2}} \middle| \tau = f(t_1, 2\pi t_2), t \in [0, 1]^2 \right\}.$$