Universal Structure of Jet Space

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Operators of total differentiation \( D \), Cartan forms \( \omega \) and infinitesimal symmetries \( P \) constitute the structure of infinite jet space \( J_{n,m} \). We describe these notions compactly for the space \( J_{1,1} \), though reserve the possibility to pass with the help of multi-indices to general case \( J_{n,m} \). Our aim is to show the universality of this structure. Every time when we differentiate a function \( f \) with respect to the vector field \( X \) on a manifold \( M \) we can determine a map \( \varphi : M \to J_{1,1} \) and connect the triple \((X, s, F)\) with the triple \((D, t, U)\) in \( J_{1,1} \), where \( F \) is the set of derivatives \( f^{(k)} = X^k f \), \( k = 0, 1, 2, \ldots \); \( s \) is canonical parameter of \( X \); \( U \) is the set of fiber coordinates \( u^{(k)} = D^k u \), \( k = 0, 1, 2, \ldots \), and \( t \) is canonical parameter of \( D \). Then all the invariants and symmetries of \( D \) as well as all the covariant tensors including Cartan forms can be transformed from \( J_{1,1} \) onto the manifold \( M \). The structure is universal as final object in the category of triples \((X, s, F)\).

Let \( f : V_n \to V_m \) be a smooth mapping. The infinite jet of the map \( f \) is determined by the coordinates \( t^i, u^\alpha \) of the points \( t \in V_n \) and \( u = f(t) \in V_m \), and by the values of partial derivatives at \( t \):

\[
\begin{align*}
  u_\alpha^i &= \frac{\partial f^\alpha}{\partial u^i}(t), & u_{ij}^\alpha &= \frac{\partial^2 f^\alpha}{\partial u^i \partial u^j}(t), & \ldots, \\
  i, j &= 1, 2, \ldots, & n &= \dim V_n, & \alpha, \beta &= 1, 2, \ldots, & m &= \dim V_m.
\end{align*}
\]

The set of the jets of \( f \) is called jet space \( J_{m,n} \) where the quantities

\[
  t^i, \quad u^\alpha, \quad u_i^\alpha, \quad u_{ij}^\alpha, \quad \ldots
\]

are jet coordinates.

In the space \( J_{1,1} \) we have the coordinates

\[
  t, \quad u, \quad u', \quad u'', \quad \ldots
\]

or briefly \((t, U)\) where \( U \) is the column of elements \( u, u', u'', \ldots \).

In \( J_{1,1} \) one has the natural basis \((\partial / \partial t, \partial U/dt)\) associated with the coordinates (2). Here \( \partial / \partial t \) is the row of elements \( \partial / \partial u, \partial / \partial u', \ldots \) and \( dU \) is the column of elements \( du, du', du'', \ldots \).

Let us introduce the infinite-dimensional unit matrix \( E \) and the shift matrix \( C \) as follows:

\[
  E = \begin{pmatrix}
    1 & 0 & 0 & \cdot \\
    0 & 1 & 0 & \cdot \\
    0 & 0 & 1 & \cdot \\
    \vdots & \vdots & \vdots & \ddots
  \end{pmatrix}, \quad C = \begin{pmatrix}
    0 & 1 & 0 & \cdot \\
    0 & 0 & 1 & \cdot \\
    0 & 0 & 0 & \cdot \\
    \vdots & \vdots & \vdots & \ddots
  \end{pmatrix}
\]

and define in \( J_{1,1} \) the total differentiation operator by formula:

\[
  D = \frac{\partial}{\partial t} + \frac{\partial}{\partial U} U', \quad \text{where} \quad U' = CU.
\]

**Proposition 1.** The operator \( D \) is a linear vector field in the jet space \( J_{1,1} \) and its flow is determined by exponential law (see [5]),

\[
  U' = CU \Rightarrow U_t = e^{\lambda t} U.
\]

The curves \((t, U_t)\) are the trajectories of \( D \).
**Proposition 2.** If the operator $\frac{\partial}{\partial t}$ in the frame $(\frac{\partial}{\partial t}, \frac{\partial}{\partial U})$ is replaced by $D$ then the differentials $dU$ in the coframe $(dt, dU)$ have to be replaced by Cartan forms

$$\omega = dU - U' dt.$$  

(5)

The new basis in the matrix form

$$
\begin{pmatrix}
D & \frac{\partial}{\partial U} \\
\frac{\partial}{\partial t} & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
U' & E
\end{pmatrix}
\cdot
\begin{pmatrix}
dt \\
\omega
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
-U' & E
\end{pmatrix}
\cdot
\begin{pmatrix}
dt \\
dU
\end{pmatrix}.
$$

is called **adapted basis** in $J_{1,1}$. The term “adapted basis” proceeds from the theory of connections (see [4, p. 23]).

**Proposition 3.** The derivation formulae valid for the adapted basis (vertical part):

$$
\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U} C, \quad \omega' = C\omega.
$$

(6)

The stroke means Lie derivative with respect to $D$. The frame $\frac{\partial}{\partial U}$ and the coframe $\omega$ are transported by the flow of $D$ according to the law (4):

$$
\left(\frac{\partial}{\partial U}\right]' = -\frac{\partial}{\partial U} C \quad \Rightarrow \quad \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U} e^{-Ct},
$$

$$
\omega' = C\omega \quad \Rightarrow \quad \omega_t = e^{Ct}\omega.
$$

**Proposition 4.** The quantities

$$I = e^{-Ct} U$$

(7)

are the invariants of $D$ because $I' = e^{-Ct}(U' - CU) = 0$. Replacing $U$ by $I$ in the fibers of $J_{1,1}$ we have the invariant basis:

$$
\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial I}; dt, dI\right).
$$

The exponential $e^{-Ct}$ is **integrating matrix** for Cartan forms $\omega$ and the operators $\frac{\partial}{\partial I}$ are **infinitesimal symmetries** of $D$ (infinitesimals after [1]) in the following sense:

$$dI = e^{-Ct}\omega, \quad \frac{\partial}{\partial I} = \frac{\partial}{\partial U} e^{Ct}.$$  

(8)

Infinitesimal symmetries of $D$ are called **Lie vector fields** in $J_{1,1}$.

**Proposition 5.** A vector field $P$ written in three frames of $J_{1,1}$, natural, adapted and invariant as follows

$$P = \frac{\partial}{\partial t}\xi + \frac{\partial}{\partial U}\lambda = D\xi + \frac{\partial}{\partial U}\mu = \frac{\partial}{\partial t}\xi + \frac{\partial}{\partial I}\nu$$

(9)

has for components the entities

$$\xi = Pt, \quad \lambda = PU, \quad \mu = \omega(P), \quad \nu = PI,$$

(10)

with relations

$$\nu = e^{-Ct}\mu, \quad \mu = \lambda - U'\xi.$$  

(11)

The field $P$ is a Lie vector field if and only if one of equivalent conditions is satisfied:

$$\nu' = 0, \quad \mu' = C\mu, \quad \lambda' = C\lambda + U'\xi.$$  

(12)
It is obvious from the Lie derivatives:

\[ P' = D\xi' + \frac{\partial}{\partial U}(\lambda' - C\lambda - U'\xi') = D\xi' + \frac{\partial}{\partial U}(\mu' - C\mu) = D\xi' + \frac{\partial}{\partial U} \nu', \]

\[ L_\nu \omega = (\lambda' - C\lambda - U'\xi')dt + \left( \frac{\partial \lambda}{\partial U} - U' \frac{\partial \xi}{\partial U} \right) \omega = (\mu' - C\mu)dt + \frac{\partial \mu}{\partial U} \omega = e^{Ct} \nu' dt + \frac{\partial \nu}{\partial U} \omega. \]

The most simple condition \( \nu' = 0 \) says that the components \( \nu \) in invariant frame are invariants of \( D \).

The condition \( \mu' = C\mu \) means that each entry of column \( \mu \) is the derivative of preceding entry. Thus all entries of column \( \mu \) in adapted frame are generated by the first entry \( \mu_0 = f \) (generating function, see [1, p. 454]) by means of differentiation:

\[ \mu_k = f^{(k)} = D^k f, \quad k = 0, 1, 2, \ldots. \]

There is an obvious analogy between two equations \( I = e^{-Ct}U \) and \( v = e^{-Ct} \mu \).

The most complicated condition \( \lambda' = C\lambda + U'\xi' \) is principal for the calculation of symmetries in natural basis (see [1, p. 244], [2, p. 110], [3, p. 55]).

**Remark 1.** In \( J_{1,1} \) the invariants \( I = e^{-Ct}U \) are described as follows:

\[ I_k = \sum_{\ell=0}^{\infty} u^{(k+\ell)} \frac{(-t)^\ell}{\ell!}, \quad k = 0, 1, 2, \ldots. \]

The operators \( \frac{\partial}{\partial I} \) are basic Lie vector fields with generating functions \( 1, t, t^2, \ldots \) respectively, that is

\[ \frac{\partial}{\partial I_0} = \frac{\partial}{\partial u}, \]
\[ \frac{\partial}{\partial I_1} = t \frac{\partial}{\partial u} + \frac{\partial}{\partial u'}, \]
\[ \frac{\partial}{\partial I_2} = t^2 \frac{\partial}{2 \partial u} + t \frac{\partial}{\partial u'} + \frac{\partial}{\partial u''}. \]

**Remark 2.** In \( J_{n,m} \) instead of \( D \) we have a system of \( n \) operators \( D_i, i = 1, 2, \ldots, n, \) and instead of 1-dimensional trajectories we have \( n \)-dimensional orbits of the additive group \( \mathbb{R}^n \). For example, in the space \( J_{2,1} \) there are the 2-dimensional time \( t = (t_1, t_2) \) and two operators

\[ D_1 = \frac{\partial}{\partial t_1} + u_1 \frac{\partial}{\partial u} + u_{11} \frac{\partial}{\partial u_1} + u_{12} \frac{\partial}{\partial u_2} + \cdots, \]
\[ D_2 = \frac{\partial}{\partial t_2} + u_2 \frac{\partial}{\partial u} + u_{12} \frac{\partial}{\partial u_1} + u_{22} \frac{\partial}{\partial u_2} + \cdots. \]

Herewith 2-dimensional orbits of \( \mathbb{R}^2 \) are determined by the series

\[ u_t = u + u_{11} t_1 + u_{21} t_2 + \frac{1}{2} \left[ u_{11}(t_1)^2 + 2u_{12} t_1 t_2 + u_{22}(t_2)^2 \right] + \cdots \]

and its partial derivatives of all orders with respect to \( t_1 \) and \( t_2 \).

**Remark 3.** In \( J_{2,1} \) the Lie field \( P \) with the generating function \( f \) can be represented in adapted and natural basis as follows:

\[ P = \xi^1 D_1 + \xi^2 D_2 + f \frac{\partial}{\partial u} + f_1 \frac{\partial}{\partial u_1} + f_2 \frac{\partial}{\partial u_2} + \cdots. \]
\[ = \xi^1 \frac{\partial}{\partial t_1} + \xi^2 \frac{\partial}{\partial t_2} + (f + u_1 \xi^1 + u_2 \xi^2) \frac{\partial}{\partial u} + (f_1 + u_{11} \xi^1 + u_{12} \xi^2) \frac{\partial}{\partial u_1} \\
+ (f_2 + u_{12} \xi^1 + u_{22} \xi^2) \frac{\partial}{\partial u_2} + \cdots, \]

where \( f_i = D_if, \ i = 1, 2 \). The components \( \lambda_k = f_k + u_{ki} \xi^i, \ k = 0, 1, 2, \ldots \) are consistent with the relation \( \lambda = \mu + U' \xi \).

**Theorem 1.** Any smooth vector field \( X \) without singularities on a manifold \( M \) can be connected with the total differentiation operator \( D \) in the jet space \( J_{1,1} \), i.e. there exists a smooth map \( \varphi : M \rightarrow J_{1,1} \) such that the vector field \( X \) is \( \varphi \)-connected with the operator \( D \).

**Proof.** Let \( s \) be the canonical parameter of the vector field \( X \), herewith \( Xs = 1 \). Take a smooth function \( f \) and calculate its derivatives with respect to \( X \), \( f^{(k)} = X^k f, \ k = 1, 2, \ldots \). Let \( F \) be the infinite column of elements \( f, \ f', \ f'', \ldots \) and let us define the mapping \( \varphi \) by the relations

\[ t \circ \varphi = s, \quad U \circ \varphi = F. \tag{13} \]

At some step \( n \) we get the conditions

\[ \Theta = df \wedge df' \wedge df'' \wedge \cdots \wedge df^{(n-1)} \neq 0 \quad \text{and} \quad \Theta \wedge df^{(n)} = 0. \tag{14} \]

There are two possible cases: a) \( n = \dim M \), or b) \( n < \dim M \).

**Case a)** \( n = \dim M \). Let the functions \( f, f', f'', \ldots, f^{(n-1)} \) be the coordinates on \( M \) and let us represent the field \( X \) as follows:

\[ X = f' \frac{\partial}{\partial f} + f'' \frac{\partial}{\partial f'} + \cdots + f^{(n)} \frac{\partial}{\partial f^{(n-1)}}. \]

The Jacobian matrix of \( \varphi \) relate the components of \( X \) to the components of \( D \) (the subscripts mean the partial derivatives):

\[
\begin{pmatrix}
  s_1 & \cdots & s_n \\
  1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  f^{(n)} & \cdots & f^{(n)}
\end{pmatrix}
\begin{pmatrix}
  f' \\
  \vdots \\
  f^{(n)}
\end{pmatrix}
= 
\begin{pmatrix}
  s' \\
  f' \\
  \vdots \\
  f^{(n)}
\end{pmatrix}
= 
\begin{pmatrix}
  1 \\
  u' \\
  \vdots \\
  u^{(n+1)}
\end{pmatrix} \circ \varphi.
\]

The rank of the Jacobian matrix is equal to \( n \) and \( \varphi \) is an immersion of \( M \) into \( J_{1,1} \). The triple \( (X, s, F) \) on the manifold \( M \) is \( \varphi \)-connected with the triple \( (D, t, U) \) in the jet space \( J_{1,1} \).

**Case b)** \( n < N = \dim M \). It follows from \( \Theta \wedge df^{(n)} = 0 \) that \( df^{(n)} \) is a linear combination of \( df, df', df'', \ldots, df^{(n-1)} \),

\[ df^{(n)} = \sum_{i=1}^{n} \alpha_i df^{(n-i)} \quad \text{and} \quad L_X \Theta = \alpha_1 \Theta. \]

The functions \( f, f', f'', \ldots, f^{(n-1)} \) determine a submersion \( \pi : M \rightarrow W \). The vector field \( X \) transports the fibers of \( \pi \) into the fibers of the same bundle and because of this the field \( X \) can be projected on the \( n \)-dimensional manifold \( W \). In the coordinates \( v^{(i)} \),

\[ v^{(i)} \circ \pi = f^{(i)}, \quad i = 0, 1, 2, \ldots, n-1, \]
the projection of $X$ is a vector field

$$T\pi X = v' \frac{\partial}{\partial u} + v'' \frac{\partial}{\partial v'} + \cdots + v^{(n-1)} \frac{\partial}{\partial v^{(n-2)}} + f^{(n)} \frac{\partial}{\partial v^{(n-1)}}$$

which can be connected by a map $\tilde{\varphi} : W \longrightarrow J_{1,1}$ with the operator $D$. Then the vector field $X$ is $\varphi$-connected with $D$, where $\varphi = \tilde{\varphi} \circ \pi$.

**General case.** How to make the correspondence between a system of $n$ vector fields $Y_i$ on a manifold $M$ with the operators of total differentiation $D_i$ in the jet space $J_{n,m}$? Let $u^\alpha$ be the coordinates on $M$, $w^i$ the canonical parameters of $Y_i$, $Y_i w^j = \delta^j_i$, and $y_i^\alpha$ the natural components of the fields $Y_i$. The operators

$$X_i = \frac{\partial}{\partial w^i} + Y_i = \frac{\partial}{\partial w^i} + y_i^\alpha \frac{\partial}{\partial w^\alpha}$$

determine a $n$-dimensional distribution in the “space-time” $R^n \times M$ with the coordinates $(u^i, u^\alpha)$, $i = 1, 2, \ldots, n$; $\alpha = n + 1, \ldots, n + m$; $m = \text{dim } M$. This is a particular case of connection in the fiber space, see [4], where the operators

$$X_i = \frac{\partial}{\partial w^i} + \Gamma_i^\alpha \frac{\partial}{\partial w^\alpha} \quad (15)$$

form in the coordinates $(u^i, u^\alpha)$ an adapted frame of the horizontal distribution $\Delta_h$, with the components

$$\Gamma_i^\alpha = \Gamma_i^\alpha (w^j, u^\beta).$$

In our case we have $\Gamma_i^\alpha = y_i^\alpha (w^j)$. Let us immerse the operators $X_i$ in the space $J_{n,m}$ with the help of the map $\varphi : M \longrightarrow J_{n,m}$ supposing

$$t^i \circ \varphi = u^i, \quad u^\alpha \circ \varphi = u^\alpha, \quad u_i^\alpha \circ \varphi = \Gamma_i^\alpha, \quad u_{ij}^\alpha \circ \varphi = X_i (X_j \Gamma_i^\alpha), \quad \ldots \quad (16)$$

The operators $X_i$ and the vector fields $Y_i$ are $\varphi$-connected with the operators $D_i$.

As corollaries we have the next Propositions.

**Proposition 6.** If the vector field $X$ is $\varphi$-connected with the operator $D$ then for any function $I$ in $J_{1,1}$ the derivatives $X(I \circ \varphi)$ and $DI$ are $\varphi$-connected, i.e. $X(I \circ \varphi) = (DI) \circ \varphi$. From this it follows that $DI = 0 \Longrightarrow X(I \circ \varphi) = 0$ and all the invariants of $D$ can be transported on the manifold $M$ in the invariants of the vector field $X$. In particular the invariants $I = e^{-Ct} U$ are transported from $J_{1,1}$ on $M$ in the invariants $I \circ \varphi = e^{-Ct} F$.

**Proposition 7.** If the vector field $X$ is $\varphi$-connected with the operator $D$ then all the covariant tensors can be transported from $J_{1,1}$ on the manifold $M$. For example, the Cartan forms $\omega = dU - U'dt$ can be transported in the forms $\omega \circ T \varphi = dF - F'dt$, where $F' = XF$. The sequence of Lie derivatives with respect to $D$ (Cartan forms)

$$\omega_0 = du - u'dt, \quad \omega'_0 = du' - u''dt, \quad \omega''_0 = du'' - u'''dt, \quad \ldots$$

induces the sequence of Lie derivatives with respect to $X$:

$$\omega_0 \circ T \varphi = df - f'ds, \quad \omega'_0 \circ T \varphi = df' - f''ds, \quad \omega''_0 \circ T \varphi = df'' - f'''ds, \quad \ldots$$
Proposition 8. In the general case (16) the Cartan forms in $J_{n,m}$
\[ \omega^\alpha = du^\alpha - u^\alpha dt^i, \quad \omega_i^\alpha = du_i^\alpha - u_i^\alpha dt^j, \quad \ldots \]
induce on the manifold $\mathbb{R}^n \times M$ the sequence of 1-forms
\[ \theta^\alpha = \omega^\alpha \circ T \varphi = du^\alpha - \Gamma_i^\alpha d\varphi, \quad \theta_i^\alpha = \omega_i^\alpha \circ T \varphi = d\Gamma_i^\alpha - X_i^\alpha \varphi d\varphi, \quad \ldots \]
The horizontal distribution $\Delta_h$ is the annihilator of the forms $\theta^\alpha$, i.e. $\theta^\alpha(X_i) = 0$. The forms $\theta_i^\alpha$ imply the appearance of two important objects:
\[ K_{ij}^\alpha = X_i^\alpha \Gamma_j^\alpha, \quad \text{object of curvature}, \]
\[ \Gamma_{ij}^\alpha = -\partial_j \Gamma_i^\alpha, \quad \text{object of connection}. \]
Namely, because $d\Gamma_i^\alpha = X_j \Gamma_i^\alpha d\varphi^j + \partial_j \Gamma_i^\alpha d\theta^j$ and $X_j \Gamma_i^\alpha = X_i^\alpha \Gamma_j^\alpha - X_i^\alpha \Gamma_j^\alpha$ we have
\[ \theta_i^\alpha = -K_{ij}^\alpha d\varphi^j - \Gamma_{ij}^\alpha d\theta^j. \]
For the linear connection the quantities $\Gamma_i^\alpha$ are linear functions on the fibers: $\Gamma_i^\alpha = -\Gamma_{ij}^\beta u^\beta_i$, and we have $K_{ij}^\alpha = -K_{ij}^\beta u^\beta$, where $K_{ij}^\beta = \partial_i \Gamma_j^\beta + \Gamma_i^\alpha \Gamma_j^\beta$ (see [4, p. 26]).
Extending the linear connection onto the tangent bundle $TM \rightarrow M$ we get the affine connection on the manifold $M$ in the classical sense.

Proposition 9. The vertical distribution $\Delta_v$ is integrable because $\Delta_v = \ker T \pi$ and the vector fields (15) are infinitesimals of $\Delta_v$. For any coframe $\theta^i$ of $\Delta_v$ there exists an integrating matrix $B^i_j$ such that $B^i_j \theta^j = du^i$. Then $B^i_j \theta^k(X_j) = \delta^i_j$ is unit matrix and $B^i_j$ is inverse to the matrix $\theta^i(X_j)$.

Let us mention that from (8) we have the same situation $e^{-Ct} \omega(\frac{\partial}{\partial t}) = E$. This generalizes the known property of integrating factor for $n = 1$ (see [1, p. 60]).

Proposition 10. The vector field $P$ represented in natural and adapted frames as follows (see [5, p. 286])
\[ P = \xi^i \frac{\partial}{\partial u^i} + \lambda^\alpha \frac{\partial}{\partial u^\alpha} = \xi^i X_i + \mu^\alpha \frac{\partial}{\partial u^\alpha}, \quad \mu^\alpha = \lambda^\alpha - \Gamma_i^\alpha \xi^i, \]
is an infinitesimal symmetry of horizontal distribution $\Delta_h$ if and only if either
\[ X_i \lambda^\alpha - P \Gamma_i^\alpha - \Gamma_j^\alpha X_i \xi^j = 0 \] (17)
or
\[ X_i \mu^\alpha + \Gamma_i^\alpha \mu^\beta + 2K_{ij}^\alpha \xi^j = 0. \] (18)
For the case
\[ \Gamma_i^\alpha = -\Gamma_j^\beta u^\beta, \quad \mu^\alpha = \mu^\beta u^\beta, \quad \lambda^\alpha = \lambda^\beta u^\beta, \quad \mu^\alpha = \lambda^\beta \Gamma_{ij}^\alpha \xi^j \]
the conditions (17) and (18) are equivalent to
\[ \partial_i \lambda_j^\beta - \lambda_i^\gamma \Gamma_{ij}^\gamma + \Gamma_i^\alpha \lambda_j^\beta + \partial_i \Gamma_j^\alpha \xi^j + \Gamma_{ij}^\alpha X_i \xi^j = 0, \]
\[ \partial_i \mu_j^\beta - \mu_i^\gamma \Gamma_{ij}^\gamma + \Gamma_i^\alpha \mu_j^\beta - 2K_{ij}^\alpha \xi^j = 0. \] (19) (20)
On the tangent bundle $TM \rightarrow M$ we have the correspondence
\[ (u^i, u^\alpha) \sim (u^i, du^i), \quad (\xi^i, \lambda^\alpha) \sim (\xi^i, d\xi^i), \quad \lambda_j^\beta \sim \frac{\partial \xi_j^i}{\partial u^j}, \quad \mu^\alpha \sim \frac{\partial \xi^i}{\partial u^\alpha} + \Gamma_{ij}^k \xi^k \]
and the conditions (19) and (20) define $P$ as an affine collineation (infinitesimal movement in the space of affine connection or Killing’s vector field in Riemannian geometry), see [6, p. 37, formulae (2.30) and (2.31)].
Remark 4. For ODE $y' + p(x)y + q(x) = 0$ we have $\omega = (py + q)dx + dy$ and the condition (18) gives $\mu = e^{-\int pdx}$. The form

$$\frac{\omega}{\mu} = d\left(\frac{y}{\mu}\right) + \frac{q}{\mu}dx$$

is exact and determines the first integral (see [1, p. 160])

$$\frac{y}{\mu} + \int \frac{q}{\mu}dx.$$

Acknowledgements

The paper had supported by ETF grants nr. 4420 and 5281.