Twisted Product of Fock Spaces

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We discuss the construction of the twisted product of the interacting Fock spaces.

1 Introduction

Recall that the $C^*$-probability space is the pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a $C^*$-algebra and $\phi$ is a state. The self-adjoint element $a \in \mathcal{A}$ is called a self-adjoint quantum random variable. The distribution of $a$ with respect to $\phi$ is a probability measure $\mu$ on $\mathbb{R}$ such that

$$\phi(a^k) = \int_{\mathbb{R}} t^k d\mu(t).$$

Remark 1. If $\mathcal{A}$ is a $\ast$-algebra or $W^*$-algebra then the corresponding spaces are called the $\ast$-algebraic and $W^*$-algebraic quantum probability spaces respectively.

The notion of monotone independence was introduced by N. Muraki in [10].

Definition 1. The family $\{Y_i, i = 1, \ldots, n\}$ of random variables in $(\mathcal{A}, \phi)$ is called anti-monotone independent if the following two conditions hold

1. $Y_i^p Y_j^s \phi = \phi(Y_j^p) Y_i^s$ for any $i > j < k$ and $p, s \in \mathbb{N}$.
2. For any $i_1 < i_2 < \cdots < i_s < j > j_1 > \cdots > j_2 > j_1$

$$\phi(Y_{i_1}^{r_{i_1}} \cdots Y_{i_s}^{r_{i_s}} Y_{j_1}^{r_{j_1}} \cdots Y_{j_t}^{r_{j_t}} Y_{j_1}^{r_{j_1}} \cdots Y_{j_2}^{r_{j_2}}) = \prod_{\nu=1}^{s} \phi(Y_{i_\nu}^{r_{i_\nu}}) \phi(Y_{j_\nu}^{r_{j_\nu}}) \prod_{\omega=1}^{t} \phi(Y_{i_\omega}^{r_{i_\omega}}).$$

The notation $i < j > k$ means $i < j$, $k < j$ and $i > j < k$ means $i > j$, $k > j$.

The notion of boolean independence was studied by many authors, see for example [11].

Definition 2. Let $(\mathcal{B}, \phi)$ be the $\ast$-algebraic probability space. The family of elements $\{X_i, i \in \mathcal{I}\}$ is called boolean independent if one has

$$\phi(X_{i_1}^{r_{i_1}} X_{i_2}^{r_{i_2}} \cdots X_{i_n}^{r_{i_n}}) = \phi(X_{i_1}^{r_{i_1}}) \phi(X_{i_2}^{r_{i_2}}) \cdots \phi(X_{i_n}^{r_{i_n}})$$

for any $i_1 \neq i_2 \neq \cdots \neq i_n \in \mathcal{I}$.

The notion of the interacting Fock space was introduced by L. Accardi, Y. Lu and I. Volovich, see [2]. Let $\mathcal{H}$ be the Hilbert space. Denote by $\mathcal{T}(\mathcal{H})$ the full tensor space over $\mathcal{H}$ and by $\omega$ the vacuum vector. Then construct operators of the creation $a(f)$, $f \in \mathcal{H}$,

$$a(f) f_1 \otimes f_2 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n, \quad n \in \mathbb{N}, \quad a(f) \omega = f.$$

Finally for any $n \in \mathbb{N}$ we supply $\mathcal{H}^\otimes n$ with some scalar product $\langle \cdot | \cdot \rangle_n$ and define the scalar product $\langle \cdot | \cdot \rangle$ on the $\mathcal{T}(\mathcal{H})$ by the rule

$$\mathcal{H}^\otimes n \perp \mathcal{H}^\otimes m, \quad m \neq n, \quad \langle x | y \rangle = \langle x | y \rangle_n, \quad x, y \in \mathcal{H}^\otimes n.$$
The operators $a^*(f), f \in \mathcal{H}$ are called the annihilation operators. The following properties of the annihilation operators are obvious

$$a^*(f)\omega = 0, \quad a^*(f): \mathcal{H}^\otimes n \to \mathcal{H}^\otimes n^{-1}, \quad n \in \mathbb{N}.$$  

**Definition 3.** The system $\mathcal{F} = \left( \mathcal{T}(H), \langle \cdot | \cdot \rangle, a(f), a^*(f), f \in \mathcal{H} \right)$ is called the interacting Fock space.

The operators $X(f) = a(f) + a^*(f), f \in \mathcal{H}$ are called the field operators and the operator $N$, defined by the rule

$$N\omega = 0, \quad N(f_1 \otimes \cdots \otimes f_k) = kf_1 \otimes \cdots \otimes f_k, \quad k \in \mathbb{N}$$

is called the number operator. The vector state associated with vacuum vector $\omega$ is called Fock state. If $\dim \mathcal{H} = 1$ then the corresponding interacting Fock space is called a one-mode interacting Fock space.

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The role of the one-mode interacting Fock spaces in quantum probability was clarified by L. Accardi and M. Bożejko in [1]. Namely, it was shown that any self-adjoint quantum variable can be realized in the form $a + a^* + f(N)$, where $a, a^*, N$ are the creation, annihilation and the number operators acting on some one-mode interacting Fock space.

Let us discuss some type of general central limit theorem of the type considered by R. Speicher and W. von Waldenfels, see [15].

**Theorem 1.** Consider the $\ast$-algebra $\mathcal{A}$ and state $\phi$ and the sequence of elements $a_i, a_i^* \in \mathcal{A}, i \in \mathbb{N}$. Denote by $b_{2i-1} := a_i, b_{2i} := a_i^*$ and $X_i := a_i + a_i^*$. Put $S_N$ to be

$$S_N = \frac{X_1 + \cdots + X_N}{\sqrt{N}}.$$  

Suppose that the following assumptions are satisfied.

(i) For any odd $n \in \mathbb{N}$ one has $\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) = 0$.

(ii) For even $n$ the mixed moment $\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) \neq 0$ only if $(\sigma(1), \ldots, \sigma(n))$ is the permutation (with replications) of the collection

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k, k \leq n/2\}.$$  

(iii) Let $(\sigma(1), \ldots, \sigma(n))$ be the permutation of the collection

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k, k \leq n/2\},$$

where $i_1 < i_2 < \cdots < i_k$. Then for any $\{j_1 < j_2 < \cdots < j_k\}$ construct $(\bar{\sigma}(1), \cdots, \bar{\sigma}(n))$ by the rule

$$\bar{\sigma}(i) = 2j_s, \quad if \quad \sigma(i) = 2i_s, \quad \bar{\sigma}(i) = 2j_s - 1, \quad if \quad \sigma(i) = 2i_s - 1.$$  

Obviously the relation $\sigma \sim \bar{\sigma}$ is equivalence. Then if $\sigma$ and $\bar{\sigma}$ are equivalent permutations with property $\exists \sigma^{-1}(i) = 1$, we have the equality

$$\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) = \phi(b_{\bar{\sigma}(1)} \cdots b_{\bar{\sigma}(n)}).$$
Finally, where we denote by $\sigma$ a unique equivalence consisting of $\sigma_N$, $\phi$ the equality 

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We use the standard arguments, see, for example [15]. For any even $m$

$$ \text{Obviously, for any } m \in M \text{ is uniquely determined by the ordered collection } 1 \leq i_1 < \cdots < i_k \leq n. \text{ Then}$$

$$ \phi(S_N^n) = N^{-\frac{n}{2}} \sum_{k=1}^{\frac{n}{2}} \sum_{m \in M_k} \sum_{\sigma \in m} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)})$$

$$ = N^{-\frac{n}{2}} \sum_{k=\frac{n}{2}}^n \sum_{m \in M_k} \sum_{\sigma \in m} \phi(b_{\sigma(m)(1)} \cdots b_{\sigma(m)(n)}) + N^{-\frac{n}{2}} \sum_{m \in M_{n/2}} C_{n/2}^n \phi(b_{\sigma(m)(1)} \cdots b_{\sigma(m)(n)}).$$

Since the number of summands is finite and independent on $N$ and

$$ \lim_{N \to \infty} N^{-\frac{n}{2}} C_n^k = 0, \quad k < \frac{n}{2}, \quad \lim_{N \to \infty} N^{-\frac{n}{2}} C_{n/2}^n = \frac{1}{(n/2)!}$$

and

$$ \left| N^{-\frac{n}{2}} \sum_{k=\frac{n}{2}}^n \sum_{m \in M_k} \sum_{\sigma \in m} \phi(b_{\sigma(m)(1)} \cdots b_{\sigma(m)(n)}) \right| \leq N^{-\frac{n}{2}} \sum_{k=\frac{n}{2}}^n \sum_{m \in M_k} \sum_{\sigma \in m} |\phi(b_{\sigma(m)(1)} \cdots b_{\sigma(m)(n)})|$$

$$ \leq C_n \sum_{k=\frac{n}{2}}^n \sum_{m \in M_k} N^{-\frac{n}{2}} C_{n/2}^k \to 0, \quad N \to \infty.$$ 

Thus

$$ \lim_{N \to \infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{m \in M_{n/2}} \phi(b_{\sigma(m)(1)} \cdots b_{\sigma(m)(n)}).$$

Evidently, the canonical representatives $\sigma_m$ of classes $m \in M_{n/2}$ are the elements $\sigma_m \in S_n$, where we denote by $S_n$ the group of permutations on $n$ symbols; please do not confuse with $S_N$. Finally

$$ \lim_{N \to \infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{\sigma \in S_n} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}).$$

The equality $\phi(S_N^n) = 0$ for odd $n$ is evident.
Example 1. Consider the family of centered identically distributed elements \(\{a_i, a_i^*, \ i \in \mathbb{N}\}\) satisfying the following conditions

1. \(a_i^*a_j = qa_ja_i^*, \ a_ja_i = qa_ia_j, \ i < j, \ q \neq 0.\)
2. If \(\sigma(1) < \cdots < \sigma(k),\) one has
   \[
   \phi(y_{\sigma(1)} \cdots y_{\sigma(k)}) = \phi(y_{\sigma(1)}) \cdots \phi(y_{\sigma(k)}),
   \]
   where \(y_s \in \{a_s, a_s^*, a_s^*a_s, a_s^*a_s^*\}.\)

Then the assumptions of the central limit theorem hold. Moreover, if we suppose that operators \(a_i, a_i^*, \ i \in \mathbb{N}\) are the creation and annihilation operators acting on the interacting Fock space and \(\phi\) is the Fock state, then we additionally have the property \(\phi(b_{\sigma(1)} \cdots b_{\sigma(2m)}) \neq 0,\) where \(\sigma \in S_{2k},\) only if \(\sigma^{-1}(2i) < \sigma^{-1}(2i-1)\) for any \(i = 1, \ldots, k.\)

In particular, if \(a_i, a_i^*\) are the creation and annihilation operators on the twisted product of copies of the one-mode interacting Fock space, see next Section, and \(\phi\) is the Fock state, the conditions above are satisfied.

Example 2. Suppose that \(X_i, \ i \in \mathbb{N}\) are anti-monotone independent centered symmetric identically distributed random variables with variance 1. Then one can realize them as the field operators acting on the monotone product of the one-mode interacting Fock spaces, i.e. suppose that \(X_i = a_i + a_i^*,\) \(a_i\) the creation and \(a_i^*\) the annihilation operators and \(\phi\) is the Fock state. Let us find the measure given by the central limit theorem. To do it we note that \(\phi(b_{\sigma(1)} \cdots b_{\sigma(2m)})\) is either 1 or 0. Let us find the number of the non-zero summands in the sum

\[
\sum_{\sigma \in S_{2m}} \phi(b_{\sigma(1)} \cdots b_{\sigma(2m)}) \tag{1}
\]

Firstly note that \(\phi(Y_1 a_1 Y_2 a_1^* Y_3) = 0\) if \(Y_2 \neq 1,\) here

\[
Y_1 = \prod_{j < \sigma^{-1}(2i-1)} b_{\sigma(j)}, \quad Y_2 = \prod_{\sigma^{-1}(2i-1) < j < \sigma^{-1}(2i)} b_{\sigma(j)}, \quad Y_3 = \prod_{\sigma^{-1}(2i) < j} b_{\sigma(j)}.
\]

Further, by definition of the anti-monotone independence if \(Y_2 \neq 1\) we have \(\phi(Y_1 a_1^* Y_2 a_1 Y_3) = \phi(a_1^*) \phi(Y_1 Y_2 a_1^* Y_3) = 0,\) here

\[
Y_1 = \prod_{j < \sigma^{-1}(2)} b_{\sigma(j)}, \quad Y_2 = \prod_{\sigma^{-1}(2) < j < \sigma^{-1}(1)} b_{\sigma(j)}, \quad Y_3 = \prod_{\sigma^{-1}(2) < j} b_{\sigma(j)}.
\]

Hence if \(\phi(b_{\sigma(1)} \cdots \sigma(n)) \neq 0\) we have \(Y_2 = 1\) and

\[
\phi(b_{\sigma(1)} \cdots \sigma(n)) = \phi(Y_1 a_1^* Y_3) = \phi(a_1^* a_1) \phi(Y_1 Y_3),
\]

where \(Y_1 Y_3\) is any product of \(a_2, a_2^*, \ldots, a_m, a_m^*\) where each term appears only once. Let \(k_m\) be the number of the non-zero summands in (1), then arguments presented above imply the following recurrent formula

\[
k_m = (2m - 1)k_{m-1}.
\]

Indeed, we have \(2m - 1\) different positions for \(a_m^* a_1\) in our permutation. Evidently, \(k_1 = 1,\) hence

\[
k_m = \lim_{N \to \infty} \frac{1}{m!} \phi(S_N^{2m}) = \frac{(2m - 1)!!}{m!} = \frac{C_m^{2m}}{2^m},
\]

and these moments correspond to the arcsin distribution with density

\[
d\mu(x) = \frac{1}{\pi} \chi(-\sqrt{2}, \sqrt{2}) \frac{dx}{\sqrt{2 - x^2}}.
\]

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Further define the creation operators \( \tilde{a}_i \) Then consider the Hilbert space \( T \otimes \cdots \otimes \phi \) where \( Y \) and so, \( \varepsilon_i \in \{1, -1\} \) and \( a_s^{(1)} = a_s, a_s^{(-1)} = a_s^* \). Hence, as in the monotone case, we have \( \phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) \neq 0 \) if and only if
\[
b_{\sigma(1)} \cdots b_{\sigma(2n)} = Y_1 a_1^* a_1 Y_3
\]
and
\[
\phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) = \phi(a_1^* a_1)\phi(Y_1)\phi(Y_3),
\]
where \( Y_1 \) is the word obtained by some permutation of the word \( a_1^* a_{i_1} \cdots a_{i_k}^* a_{i_k}, i_1 < i_2 < \cdots < i_k \in \{2, \ldots, n\} \) and analogously for \( Y_3 \). The arguments presented above shows that \( \phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) \neq 0 \) if and only if
\[
b_{\sigma(1)} \cdots b_{\sigma(2n)} = a_{\pi(1)}^* a_{\pi(1)} \cdots a_{\pi(n)}^* a_{\pi(n)} = 1,
\]
where \( \pi \in S_n \) is any permutation. Hence we have
\[
\lim_{N \to \infty} \phi(S_n^{2n}) = \frac{1}{n!} n! = 1
\]
so, \( m_{2n-1} = 0 \) and \( m_{2n} = 1 - \) the moments of the discrete measure concentrated on \( \{-1, 1\} \).

### 2 Twisted product

In this Section we discuss the construction of the twisted product Fock space. This is the special kind of the interacting Fock space.

Let \( \mathcal{I} \) be totally ordered set. Consider the collection of the one-mode interacting Fock spaces
\[
\{(\mathcal{T}(H_i), a_i, a_i^* \mid i \in \mathcal{I}\}.
\]
Let \( \Omega_i \in \mathcal{T}(H_i) \) be vacuum vector, consider the orthonormal system
\[
\{e_i^{(n)} \mid n \in \mathbb{Z}_+\},
\]
such that \( e_i^{(0)} := \Omega_i \) and \( a_i e_i^{(n)} = a_i^{(n)} e_{n+1}, n \in \mathbb{Z}_+ \). Denote by \( \phi_i \) the Fock state on \( \mathcal{T}(H_i) \). Then consider the Hilbert space \( \mathcal{T} \) with orthonormal basis
\[
\Omega, \ e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_k)}_{i_k}, \ i_1 < \cdots < i_k, \ k \in \mathbb{N}, \ i_s \in \mathcal{I}, \ n_s \in \mathbb{N}, \ s = 1, \ldots, k.
\]
Further define the creation operators \( \tilde{a}_j, j \in \mathcal{I} \)
\[
\tilde{a}_j \Omega = \alpha_j^{(0)} e^{(1)}_j,
\]
\[
\tilde{a}_j e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_k)}_{i_k} = \mu^j e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_k)}_{i_k} e_j, \ j > i_k,
\]
\[
\tilde{a}_j e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_s)}_{i_s} e^{(n_{s+1})}_{i_{s+1}} \otimes \cdots \otimes e^{(n_k)}_{i_k} = \mu^{n_1 + \cdots + n_s} \alpha_j^{(n_1)} e^{(n_{s+1})}_{i_{s+1}} \otimes \cdots \otimes e^{(n_k)}_{i_k}, \ i_s < j < i_{s+1},
\]
\[
\tilde{a}_j e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_s)}_{i_s} e^{(n_{s+1})}_{i_{s+1}} \otimes \cdots \otimes e^{(n_k)}_{i_k} = \mu^{n_1 + \cdots + n_s} \alpha_j^{(n_s)} e^{(n_{s+1})}_{i_{s+1}} \otimes \cdots \otimes e^{(n_k)}_{i_k}, \ j = i_s,
\]
\[
\tilde{a}_j e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_k)}_{i_k} = \alpha_j^{(0)} e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n_k)}_{i_k}, \ j < i_1.
\]
For the adjoint (annihilation) operators $\tilde{a}_j^*$ one has the following

$$\tilde{a}_j^*\Omega = 0,$$

$$\tilde{a}_j e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_k)}_{i_k} = 0, \quad j \neq i, \quad s = 1, \ldots, k,$$

$$\tilde{a}_j^* e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_s)}_{i_s} \otimes e^{(n_{s+1})}_{i_{s+1}} \otimes \cdots \otimes e^{(n_k)}_{i_k}$$

$$= (\Omega)^{n_1+\cdots+n_{s-1}} (\tilde{a}_j)^{(n_s-1)} e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_{s-1})}_{i_{s-1}} \otimes e^{(n_{s+1})}_{i_{s+1}} \otimes \cdots \otimes e^{(n_k)}_{i_k}, \quad j = i,$$

where we identify $e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_s)}_{i_s} \otimes e^{(n_{s+1})}_{i_{s+1}} \otimes \cdots \otimes e^{(n_k)}_{i_k}$ with $e^{(n_1)}_{i_1} \otimes \cdots \otimes e^{(n_{s-1})}_{i_{s-1}} \otimes e^{(n_{s+1})}_{i_{s+1}}$.

We call the interacting Fock space $\mathcal{T}$ the twisted product of the one-mode interacting Fock spaces $(\mathcal{T}(\mathcal{H}_i), a_i, a_i^*)$, $i \in \mathcal{I}$ with the twist parameter $\mu \in \mathbb{C}$. Below we denote by $\phi$ the Fock state, i.e. the vector state defined by $\Omega$, on $\mathcal{T}$.

It is easy to verify that the operators $\tilde{a}_i$, $\tilde{a}_j$ and $\tilde{a}_i^*$, $\tilde{a}_j$, $i > j$ satisfy the $\mu$-commutation relations, i.e.

$$\tilde{a}_i \tilde{a}_j = \mu \tilde{a}_j \tilde{a}_i, \quad \tilde{a}_i \tilde{a}_j = \mu \tilde{a}_j \tilde{a}_i.$$

One can verify also that the joint distributions of $a_i$, $a_i^*$ with respect to $\phi_i$ and $\tilde{a}_i$, $\tilde{a}_i^*$ with respect to $\phi$ coincide, i.e. for any non-commutative polynomial $p(x, y)$ one has

$$\phi_i (p(a_i, a_i^*)) = \phi (p(\tilde{a}_i, \tilde{a}_i^*)).$$

Finally note that for $\mu = 1$ one has the usual tensor product and for $\mu = 0$ the monotone product of interacting Fock spaces considered by N. Muraki, see [12].

When we have the finite set $\mathcal{I} = \{1, 2, \ldots, d\}$ the twisted product is just the twisted Fock space constructed by W. Pusz and S.L. Woronowicz, see [13]. In this case the orthonormal basis of $\mathcal{T}$ has the form

$$e^{(n_1)}_1 \otimes \cdots \otimes e^{(n_d)}_d, \quad n_i \in \mathbb{Z}_+, \quad s = 1, \ldots, d,$$

where $\Omega := e^{(0)}_1 \otimes \cdots \otimes e^{(0)}_d$, i.e. $\mathcal{T} = \bigotimes_{i=1}^d \mathcal{T}(\mathcal{H}_i)$. In this case the operators $\tilde{a}_i$, $i = 1, \ldots, d$ can be presented as tensor products

$$\tilde{a}_i = \bigotimes_{j=1}^{i-1} d(\mu) a_i \otimes \bigotimes_{j=i+1}^d 1,$$

where $d(\mu) e^{(n)}_j = \mu^n e^{(n)}_j$, $n \in \mathbb{Z}_+$, $j = 1, \ldots, d$. If we consider the special case $\mu = 0$, we get the realization of the monotone independent non-commutative random variables constructed by U. Franz, see [6]

$$\tilde{a}_i = \bigotimes_{j=1}^{i-1} P_j \otimes a_i \otimes \bigotimes_{j=i+1}^d 1.$$


