On ∗-Representations of One Deformed Quotient of Affine Temperley–Lieb Algebra

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We consider ∗-algebra generated by orthogonal projections with relations of Temperley–Lieb type. In this article we study all irreducible ∗-representations of this algebra and obtain the set of values of parameters when these representations exist.

1 Introduction

Temperley–Lieb algebras generated by \( n \) projections \( p_1, \ldots, p_n \) with relations

\[
p_ip_j = p_jp_i, \quad |i - j| > 1, \quad p_ip_i\pm1p_i = \tau p_i, \quad \tau \in \mathbb{R},
\]

appeared in [3, 4] in the context of ice-type models but they also play an important role in the analysis of subfactors of \( \text{II}_1 \) factor and in the knot theory (see, e.g., [5–7]). Jones proved that the chain (1) of orthogonal projections in Hilbert space with adding condition involving the trace can be infinite one if \( \tau \in [0; 1/4] \cup \left\{ \frac{1}{4 \cos \frac{n \pi}{2}} \mid n \geq 3 \right\} \).

In the present paper we consider ∗-algebra \( TL_{\vec{\tau},n} \) generated by orthogonal projections \( p_0, \ldots, p_{n-1} \) with relations

\[
p_ip_j = 0, \quad |i - j| > 1, \quad (i, j) \neq (0, n - 1) \quad \text{and} \quad p_ip_{i-1}p_i = \tau_ip_i, \quad p_ip_{i+1}p_i = \tau_{i-1}p_i.
\]

(2)

In [2] we studied such ∗-algebra for \( \tau_i = \tau \). For this more general algebra (2) we have found all irreducible ∗-representations and described the set of values of the parameters when these representations exist.

2 Description of all irreducible ∗-representations of algebra \( TL_{\vec{\tau},n} \), their existence in depending on values of parameter \( \vec{\tau} \)

We study ∗-algebra over complex field generated by \( n \) (\( n \geq 3 \)) orthogonal projections \( p_0, \ldots, p_{n-1} \) with relations of Temperley–Lieb type or orthogonality between any two projections. In other words, \( p_i^2 = p_i^* = p_i \) and any projections \( p_i \) and \( p_j \) fulfil condition \( p_ip_j = 0 \) or for some \( 0 < \tau_{i,j} < 1 \) relations \( p_ip_jp_i = \tau_{i,j}p_i \) and \( p_ip_jp_j = \tau_{i,j}p_j \) are correct. Such algebra can be described by a marked graph \( G \) with \( n \) vertices, where two vertices \( i, j \) are joined by a line marked with \( \tau_{i,j} \) if and only if orthogonal projections \( p_i, p_j \) satisfy relations of Temperley–Lieb type. If \( \vec{\tau} = (\tau_0, \ldots, \tau_{n-1}) \) with \( 0 < \tau_i < 1 \) is fixed vector we may consider ∗-algebra \( TL_{\vec{\tau},n} \) described by a graph (see Fig. 1).

In [1] there were proved that ∗-algebra \( TL_{\vec{\tau},n} \) has only finite-dimensional irreducible ∗-representations, so in the following we consider nontrivial irreducible finite-dimensional ∗-representations of this algebra and name them simply ‘representations’. If \( \pi \) is a ∗-representation of algebra \( TL_{\vec{\tau},n} \) in unitary space \( H \) we write \( P_i \) for \( \pi(p_i) \). Next theorem give a description of ∗-representations of algebra \( TL_{\vec{\tau},n} \).
Theorem 1. Let irreducible \(*\)-representation of algebra $TL_{\bar{\tau},n}$ exists in unitary space $H$. Then we can find the orthonormal basis of $H$ such that in this basis matrices of operators $P_0, \ldots, P_{n-1}$ are as follows:

$$P_0 = \text{diag} (1, 0, \ldots, 0),$$

$$P_i = \begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \sqrt{t_{i-1} - t_{i-1}^2} & 0 & \cdots \\
0 & \cdots & 0 & \sqrt{t_{i-1} - t_{i-1}^2} & 1 - t_{i-1} & 0 & \cdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}, \quad i = 1, \ldots, n - 2,$$

where $t_{i-1} = \frac{\tau_{i-1}}{1 - t_{i-1}^2}$, $t_0 = \tau_0$ and the number of zeroes on the top of diagonal is equal to $i - 1$.

$$P_{n-1} = \begin{pmatrix}
\tau_{n-1} & b_1 & b_{n-3} & \lambda & \mu \\
b_1 & b_1^2 & b_1b_{n-3} & b_1\lambda & b_1\mu \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
b_{n-3} & b_{n-3}b_{n-3} & b_{n-3}^2 & b_{n-3}\lambda & b_{n-3}\mu \\
\bar{\lambda} & \bar{\lambda} & \bar{\lambda} & \bar{\lambda} & \bar{\lambda} \\
\mu & \mu & \mu & \mu & \mu 
\end{pmatrix},$$

where $b_i = (-1)^i \tau_{n-1} \prod_{j=0}^{i-1} \frac{t_j}{\sqrt{t_j - t_j^2}}$. Entry $\lambda \in \mathbb{C}$ that ‘number’ the representations is such that

$$\left| t_{n-3}b_{n-3} + \lambda \sqrt{t_{n-3} - t_{n-3}^2} \right|^2 = \tau_{n-2}\tau_{n-1}t_{n-3}$$

and $\mu = \sqrt{\tau_{n-1} - \tau_{n-1}^2 - \sum_{j=1}^{n-3} b_j^2 - |\lambda|^2}$.

Remark 1. If parameter $\bar{\tau}$ is such that $t_{n-3} = 1$ the matrix of operator $P_{n-1}$ differs from the one pointed out in the Theorem 1, more precisely, first $n - 2$-nd rows and columns are the same but $n - 1$-st (or even $n - 1$-st and $n$-th) row and column are absent, $b_{n-3}$ satisfies additional condition $b_{n-3}^2 = \tau_{n-2}\tau_{n-1}$ and $\mu^2 = \tau_{n-1} - \tau_{n-1}^2 - \sum_{i=1}^{n-3} b_i^2$, \left(\tau_{n-1} - \tau_{n-1}^2 - \sum_{i=1}^{n-3} b_i^2 = 0\right).
Remark 2. The rank of all orthogonal projections $P_i$ is 1 and dimension of irreducible $*$-representation may be equal to $n$, $n-1$, or to $n-2$.

Remark 3. If parameter $\vec{\tau}$ is fixed then different permissible $\lambda$’s define inequivalent irreducible $*$-representations. So, we may say that each irreducible $*$-representation of algebra $TL_{\vec{\tau},n}$ is given by the number $\lambda$.

Now our goal is to produce the set of values of parameter $\vec{\tau}$ for that the $*$-representations exist. Let $F_i^{(k)}$, $i \geq 0$, $0 \leq k \leq n-1$ be the collection of numbers given by recurrent formulas

$$F_0^{(k)} = F_1^{(k)} = 1, \quad F_{i+2}^{(k)} = F_{i+1}^{(k)} - \tau_i F_i^{(k)}.$$

Proposition 1. The irreducible $*$-representations of algebra $TL_{\vec{\tau},n}$ exist if and only if one of following two cases takes place:

1) $F_i^{(0)} > 0$, $i = 2, \ldots, n-1$ and at least one of the following inequalities is true:

$$\left| (-1)^n \sqrt{\tau_0 \cdots \tau_{n-3} \tau_{n-1}} \pm \sqrt{\tau_{n-2} F_{n-2}^{(0)}} \right| \leq \sqrt{1 - \tau_{n-1}} \frac{F_{n-2}^{(0)} - \tau_0 \tau_{n-1} F_{n-4}^{(2)}}{F_{n-1}^{(0)}},$$

2) $F_i^{(0)} > 0$, $i = 2, \ldots, n-2$, $F_{n-1}^{(0)} = 0$, $F_{n-2}^{(0)} = \sqrt{\frac{\tau_0 \cdots \tau_{n-3} \tau_{n-1}}{\tau_{n-2}}}$ and

$$1 - \tau_{n-1} - \tau_0 \tau_{n-1} \frac{F_{n-4}^{(2)}}{F_{n-2}^{(0)}} \geq 0.$$

Note that for $n = 3$ the expressions in the proposition 1 will be correct if $P_{-1}^{(2)} := 0$.

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