On the Algebra of Unharmonic Quantum Oscillator

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In present work we consider $C^*$-algebras $C^*(A_f)$ associated with simple unimodal non-bijective dynamical system $(f, \mathbb{R})$ with special requirements. In the case when $f$ is polynomial, $A_f = \mathbb{C} \langle X, X^* \mid XX^* = f(X^*X) \rangle$ and $C^*(A_f)$ is its enveloping $C^*$-algebra. As typical examples we consider one-parameter family $f_\mu(x) = \mu x (1-x)$ and two-parameter family called Unharmonic Quantum Oscillator $f_{p,q}(x) = 1 + px - qx^2$. The crossed product structure of $C^*(A_f)$ is investigated. As a consequence we describe complete isomorphism invariant in terms of corresponding dynamical systems.

1 Introduction

$C^*$-algebras associated with dynamical systems arise naturally in pure mathematics as well as in applications to physics (see [1] and bibliography for more details) in particular to quantum optics (see [4]). For example Heisenberg algebra generated by operator $X$ such that $XX^* - X^*X = \hbar I$ associated with linear dynamical system $x \rightarrow h - x$ on $\mathbb{R}$, q-CCR algebra also associated with linear dynamics $x \rightarrow h - qx$. More complicated dynamics appear in algebra of Quantum unit Disk (see [5]), more precisely this algebra is associated with one dimensional dynamical system $x \rightarrow \frac{(q+\mu)x+1-a-\mu}{\mu x+1-\mu}$ where $\mu$ and $q$ are parameters of deformation. In this article we are concerned with non-linear deformation of q-CCR which we call algebra of Unharmonic Quantum Oscillator. It is given by generator $X$ obeying the following relation $XX^* = \hbar + pX^*X - q(X^*X)^2$ where $q > 0$, $p > 0$.

The representation theory of $C^*$-algebras given by “dynamical relations” is extensively studied and well known (see [1]). Its connection with many concurrent approaches to associate $C^*$-algebra to a dynamical systems, for example groupoid approach and cross-product by partial actions of a group or semigroup (see [13, 12]) is very intriguing. In the paper we use recent work [3] to establish connection of algebra of unharmonic quantum oscillator with cross product like algebras.

2 Cross-product like structure of $C^*$-algebras associated with dynamical systems

Here we present some recent results on cross-product like structure of $C^*$-algebras associated with dynamical systems developed in [3] which are necessary for the last section of the paper.

Let $A$ be some unital $C^*$-subalgebra of $B(H)$ and $U \in B(H)$ be a partial isometry such that the mapping $A \ni a \mapsto UaU^*$ is an endomorphism of $A$. If in addition pair $A$ and $U$ satisfies $Ua = UaU^*U$ and $U^*aU \in A$ for all $a \in A$ then $A$ is called coefficient algebra for the $C^*$-subalgebra $B$ generated by $A$ and $U$.

Let us fix some notations: $d(x) = UxU^*$, $d_+(x) = U^*xU$. Then the condition that $A$ is an algebra of coefficients for $B$ will be reformulated in the following form

$$Ua = d(a)U^*, \quad a \in A, \quad d : A \rightarrow A, \quad d_+ : A \rightarrow A.$$
If \( \mathcal{A} \) is an algebra of coefficients for \( \mathcal{B} \) then \( \mathcal{B} \) is a uniform closure of the finite combinations of the form
\[
x = U^N a_N + \cdots + U a_1 + a_0 + a_1 U + \cdots + a_N U^N.
\]
(1)

Where \( a_j, a_j \in \mathcal{A} \) and satisfy the following property for all \( k \):
\[
a_k U^k U^* k = a_k, \quad a_{k^*} U^k U^* k = a_{k^*}.
\]

In order to guarantee the very important property of uniqueness of representation in the form (1) one needs to impose the following \((*)\)-property for all \( x \) of the form (1):
\[
||a_0|| \leq ||x||.
\]

\((*)\)

We will need the following central result from [3, 2.13]:

**Theorem 1.** Let \( \mathcal{A}_j \) be an algebra of coefficients for \( \mathcal{B}_j \) generated by \( \mathcal{A}_j \) and \( U_j \) where \( j = 1, 2 \). Assume that for both algebras \((*)\) is satisfied. And assume that a mapping \( \vartheta : \mathcal{A}_1 \to \mathcal{A}_2 \) is an isomorphism such that \( \vartheta \circ d_1 = d_2 \circ \vartheta \). Then the mapping \( \Psi(x) = \vartheta(x) \) for \( x \in \mathcal{A}_1 \) and \( \Psi(U_1) = U_2 \) can be extended to isomorphism of \( C^*\)-algebras \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \).

In order to construct an algebra of coefficients we need an additional piece of notations: if \( X \subset B \) then \( E(X) \) will denote the \( C^*\)-algebra generated by \( \{X, d(X), d^2(X), \ldots, d^n(x), \ldots\} \) and analogously \( E_+(X) \) will denote the \( C^*\)-algebra generated by \( \{X, d_+(X), d^2_+(X), \ldots, d^n_+(x), \ldots\} \).

The following theorem (see [3, Theorem 3.11]) gives conditions for existence of algebra of coefficients:

**Theorem 2.** Let \( d : \mathcal{A}_0 \to B(H) \) is a morphism.

1. The following statements are equivalent:
   a) There exists an algebra of coefficients \( \mathcal{A} \supseteq \mathcal{A}_0 \).
   b) \( U^* U \cap \bigcap_{n=0}^{\infty} d^n(\mathcal{A}_0)' \).

2. If the above condition is satisfied then \( E_+(E(\mathcal{A}_0)) \) is the minimal algebra of coefficients containing \( \mathcal{A}_0 \) and \( d \) is an endomorphism of \( E_+(E(\mathcal{A}_0)) \). Moreover, each element \( \beta \in E_+(E(\mathcal{A}_0)) \) can be written as
\[
\beta = a_0 + d_0(a_1) + \cdots + d_0^N(a_N).
\]

\( U^k U^* k \) and \( U^k U^* k \) are the decreasing sequences of commuting projections.

## 3 One-dimensional dynamical systems

For convenience of the reader we repeat the relevant material from [2, 1] without proofs, thus making our exposition self-contained. By the dynamical system we mean a continuous map \( f : \mathbb{R} \to \mathbb{R} \) or \( f : I \to I \), where \( I \subset \mathbb{R} \) is a closed bounded interval. By the orbit of dynamical system \( (f, \mathbb{R}) \) we mean a sequence \( \delta = (x_k)_{k \in P} \), where \( P \) is one of the sets \( \mathbb{Z}, \mathbb{N} \) or \( -\mathbb{N} = \{1, -2, \ldots\} \) such that \( f(x_k) = x_{k+1} \). But sometimes we will consider orbit as the set \( \{x_k | k \in P\} \). The set of all orbits will be denoted by \( \text{Orb}(f) \). For \( x \in \mathbb{R} \) denote by \( \mathcal{O}_+(x) \) the forward orbit, i.e., \( (f^k(x))_{k \geq 0} \). For every orbit \( \delta \in \text{Orb}(f) \) define \( \omega(\delta) \) be the set of accumulation points of forward half-orbit and \( \alpha(\delta) \) be the set of accumulation points of backward half-orbit.

By the positive orbit of a dynamical system \( (f(), \mathbb{R}) \) we mean a sequence \( \omega = (x_k)_{k \in \mathbb{Z}} \) such that \( f(x_k) = x_{k+1} \) and \( x_k > 0 \) for all integer \( k \). Unilateral positive orbit is a sequence \( \omega = (x_k)_{k \in \mathbb{N}} \) (Fock-orbit) such that \( x_1 = 0 \) and \( f(x_k) = x_{k+1}, x_k > 0 \) for \( k > 1 \) or \( \omega = (x_k)_{k \in \mathbb{N}} \) (anti-Fock-orbit) such that \( x_{-1} = 0 \) and \( f(x_k) = x_{k+1}, x_k > 0 \) for \( k < -1 \). Define \( \text{Orb}_+(f) \)
be the set of all positive non-cyclic orbits. Note that \( \omega(\delta) = \emptyset \) for any anti-Fock orbit \( \delta \) and \( \alpha(\delta_1) = \emptyset \) for the Fock orbit \( \delta_1 \).

Cycle \( \beta = \{\beta_1, \ldots, \beta_m\} \) is called attractive if there is a neighborhood \( U \) of \( \beta \) such that \( f(U) \subseteq U \) and \( \bigcap_{i>0} f^i(U) = \beta \).

Point \( x \in \mathbb{R} \) is called non-wandering if for every its neighborhood \( U \) there exists a positive integer \( m \) such that \( f^m(U) \cap U \neq \emptyset \).

Since we will consider only bounded from above functions \( f \) and positive orbits we can always consider our dynamical system defined on a closed interval \([0, \sup f]\).

In this article we will deal with simple dynamical system, which possesses one of the equivalent properties listed in the following theorem (see [2, Theorem 3.14]):

**Theorem 3.** Let \( (f(), I) \) be continuous dynamical system, \( I \subset \mathbb{R} \) is closed bounded interval. The following conditions are equivalent:

1. For every \( x \in I \) \( \omega(x) = \omega(O_+(x)) \) is cycle.
2. \( \text{Per}(f) \) is closed.
3. Every non-wandering point is periodic.

\( f \) is called partially monotone if \( I \) decomposes into a finite union of sub intervals, on which \( f \) is monotone.

For a simple dynamical system \((f, I)\) for some positive integer \( m \) the relation \( \text{Fix}(f^{2m+1}) = \text{Fix}(f^{2m}) \) holds (see [1]).

The class of such dynamical system is denoted by \( \mathcal{F}_{2m} \). Let us note that when \( \text{Per}(f) \) is closed [2, Theorem 3.12] implies that the length of every cycle is a power of 2 and They’re no homoclinical orbits (i.e. orbit \( \delta \) such that \( \alpha(\delta) = \omega(\delta) \) is a cycle).

We will need the following lemma (see [10]):

**Lemma 1.** Let \( (f, \mathbb{R}) \) be dynamical system with bounded from above \( f \) such that \( f(), [0, \sup f]) \) is simple d.s. And let the set of periodic points which are not the points of attractive cycles, i.e. the set \( [0, \sup f] \cap \text{Per}(f) \) \( \cup \beta \) is attractive cycle \( \beta \) be finite then for every orbit \( \delta \in \text{Orb}_+(f) \) the \( \alpha \)-boundary \( \alpha(\delta) \) is cycle, which is not attractive.

## 4 Simple unimodal mapping

**Definition 1.** Let \( f \in C^0(I, I) \) where \( I = [0, 1] \). Then \( f \) is called unimodal mapping if it satisfies the following conditions:

1. \( f(0) = f(1) = 0 \).
2. There is unique extreme point \( c \in \text{int} I \) and \( f \) is monotonously increasing on \([0, c]\) and is monotonously decreasing on \([c, 1]\).

**Definition 2.** Let dynamical system \((f, I)\) be as in theorem 4 with minimal possible \( n \).

1. Then \( B_0 = s_0 \) and \( B_1 = s_1 \) are two one-dimensional cycles. \( B_{2k} = \{\beta_1, \beta_2, \ldots, \beta_{2k}\} \) denote the unique cycle of period \( 2^k \) where \( \beta_i < \beta_j \) whereas \( i < j \). \( B_{2k}^- = B_{2k}^+ \cup B_{2k}^- \) where \( B_{2k}^- = \{\beta_1, \ldots, \beta_{2k-1}\} \) and \( B_{2k}^- = \{\beta_{2k-1}, \ldots, \beta_{2k}\} \). Denote by \( B_{2k}(f^2) \) the cycle of period \( 2^k \) of dynamical system \((f^2, I_2)\).

2. We will say that orbit \( \delta = (x_k)_{k \in \mathbb{Z}} \) is glued to point \( \beta_i \) of cycle \( B_{2k} \) if there exists integer \( k_0 \) such that \( x_{k_0} = \beta_i \) and \( x_k \notin B_{2k} \) for all \( k < k_0 \). An orbit is glued to cycle if it is glued to some point of this cycle.

3. We will say that an orbit is degenerate if it is glued to a cycle of period less then \( 2^n \).

**Definition 3.** Let \( \beta_i \in B_{2^m} \) and denote \( D_{B_{2k}}^\beta \) \( = \{ \delta \in P_{B_{2k}} | \delta \text{ is glued to } \beta_i \} \). Denote by \( D_{B_{2k}}^\beta (f^2) \) the set \( D_{B_{2k}}^\beta (f^2) \) where \( j = i - 2^{m-1} \) and \( \beta_j \in B_{2^{k-1}}(f^2) \).
In the following theorem from [11] an analog of measurable section for dynamical system has been constructed.

**Theorem 4.** Let $(f, I)$ be $\mathcal{F}_{2^n}$ dynamical system with unimodal mapping $f$, which has only two
fix points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \leq n$ there is only one cycle of
period $2^m$ which is repellent for $m < n$ and attractive for $m = n$. Define $P_B = \{ \delta \mid \delta \in \text{Orb}_+(f), \alpha(\delta) = B \}$ for every cycle $B$ of period $m \leq n$. Then
1. $\text{Orb}_+(f) = \bigcup_B P_B$, where union is taken over all repellent cycles.
2. for each $B$ there is $I_B = [t_1, t_2]$ and one-to-one mapping $\phi : I_B \to P_B$ such that $t \in \phi(t)$
for every $t \in I_B$. Moreover $I_B$ can be chosen to lie in arbitrary neighborhood of $B$.
3. $I_B \cap I_B' = \emptyset$ for $B_1 \neq B_2$.

**Corollary 1.** Mapping $\cup \beta I_\beta \ni t \to \delta_t$ constructed in the proof is bijective correspondence
between $n$-copies of $[0,1)$ and the set of non-cyclic positive orbits.

## 5 Enveloping $C^*$-algebra

By $C^*(A_f)$ we mean a $C^*$-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by $X$ with
sub-norm $\|b\| = \sup_{x} \|\pi(b)x\|$ where supremum is taken over all $\pi \in \text{Rep}(\mathcal{F}(X, X^*))$ such that
$\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure.

As shown in (see [1]) there is a bijective correspondence between representations of $C^*$-
algebras

$$A = C^*(X, X^* \mid XX^* = f(X^*X))$$

with certain orbits of dynamical systems $(f, \mathbb{R})$. In particular, if $f$ partially monotone continuous
map and $(f, \mathbb{R})$ is $\mathcal{F}_{2^m}$ dynamical system. Then every positive non-cyclic orbit $\omega(x_k)_{k \in \mathbb{Z}}$
corresponds to an irreducible representation $\pi_{\omega}$ in Hilbert space $l_2(\mathbb{Z})$ given by the formulæ:
$U e_k = e_{k-1}$, $C e_k = \sqrt{x_k} e_k$ for $k \in \mathbb{Z}$ and $X = UC$ is a polar decomposition. For the
Fock and anti-Fock representations the similar formulæ hold with the exception that space is
$l_2(N)(l_2(-N))$ and $U e_1 = 0$ for the Fock representation. To cyclic positive orbit $\omega = (x_k)_{k \in \mathbb{N}}$ of
length $m$ there corresponds a family of $m$-dimensional irreducible representation $\pi_{\omega, \phi}$ in Hilbert
space $l_2\{1, \ldots, m\}$ given by the formulæ: $U e_0 = e^{i\phi} e_{m-1}$, $U e_k = e_{k-1}$ $C e_k = \sqrt{x_k} e_k$
for $k = 1, \ldots, m$; $0 \leq \phi \leq 2\pi$ and $X = UC$

Let $f$ be bounded from above Hermitian polynomial (hence $f$ is always partially monotone
and continuous). Let $A_f = \mathbb{C}(X, X^* \mid XX^* = f(X^*X))$ be $*$-algebra given by generators and
relations which has at least one representation. Let $C = \sup f$. Then for any representation $\pi$ of
$*$-algebra $A_f$ we have $\|X\| \leq \sqrt{C}$. Thus there is (exists) enveloping $C^*$-algebra, which we
denote by $C^*(A_f)$. Let us note that by Theorem 3.3 [2] for $f \in C^1(I, I)$ simplicity of dynamical
system is equivalent to $(f, I) \in \mathcal{F}_{2^m}$ for some integer $m$.

## 6 Description of the dual space of $C^*(A_f)$

Let $A$ be $C^*$-algebra by its spectrum (sometimes called dual space), denoted by $\hat{A}$ we understand
the set of unitary equivalence classes in the set $\text{Irr}(A)$ of irreducible representations of $A$ with
the Jacobson topology (see [7, Chapter 3] about several equivalent definitions). The closure of the set $S \subseteq A$ is $[S] = \{ \pi \in \hat{A} \mid \text{Ker} \pi \supseteq \bigcap_{\rho \in S} \text{Ker} \rho \}$ or equivalently $[S] = \{ \pi \in \hat{A} \mid \text{for all } y \in A \|\pi(y)\| \leq \sup_{\rho \in S} \|\rho(y)\| \}$ obviously it is enough to verify last inequality only for elements of a dense subspace of $A$.

In the following and consequent theorems, in case $f$ is not a polynomial, by $C^*(A_f)$ we
mean a $C^*$-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by $X$ with prenorm $\|b\| = \sup_{x} \|\pi(b)x\|$
sup_\pi \| \pi(b) \| \text{ where supremum is taken over all } \pi \in \text{Rep}(\mathcal{F}(X,X*)) \text{ such that } \pi(XX^*) = f(\pi(X^*X)) \text{ by standard factorization and completion procedure. This } C^*\text{-algebra has obvious universal properties similar to those in case of polynomial map } f. \text{ Theorem 4 describes the set } \text{Orb}_+(f) \text{ but in order to describe spectrum of } C^*(A_f) \text{ we need finer description. The reason is that some orbit } \delta \in P_0 \text{ may be eventually periodic, hence } \omega(\delta) \text{ could be a cycle of length } 2^m \text{ for } m < n \text{ and so } C^*(\pi_\delta) \text{ would not be isomorphic to } C^*(\pi_\gamma) \text{ for non-eventually periodic ("generic") orbit } \gamma. 

Let us give some definitions.

**Definition 4.** 1. Let \( \delta = (x_k)_{k \in \mathbb{Z}} \in P_{B_2}^k \) where \( k \geq 0 \) be such that \( x_0 \in I_2 \) then \( r(\delta) = (x_{2k})_{k \in \mathbb{Z}} \) is an orbit of \((f^2, I_2)\). If \( \delta = (y_k)_{k \in \mathbb{Z}} \in \text{Orb}_+(f^2, I_2) \) then \( r^{-1}(\delta) \) where \( (r^{-1}(\delta))_{2k} = y_k, (r^{-1}(\delta))_{2k+1} = f(y_k) \) is an orbit of \((f, I)\), moreover \( r^{-1} \) is inverse to \( r \).

2. Let \( x \in [0, M] \) define \( \mu_-(x) = (y_k)_{k \in \mathbb{Z}} \in P_{B_0} \) where \( y_k = f^k(x) \) for \( k \geq 0 \) and \( y_{-k} = f^{1-1}(y_{-k+1}) \) for \( k > 0 \). If \( x \in [s_1, M] \) define \( \mu_+(x) = (y_k)_{k \in \mathbb{Z}} \in P_{B_0} \) where \( y_k = f^k(x) \) for \( k \geq 0 \) and \( y_{-1} = f_1^{-1}(x) \) and \( y_{-k} = f^{-1}(y_{-k+1}) \) for \( k > 1 \).

Denote \( R_{B_{2k}} = P_{B_{2k}} \setminus \cup_{k < m < n} D_{B_{2m}} \).

Let \( H \) be Hilbert space with orthonormal basis \( (e_k)_{k \in \mathbb{Z}} \). Let \( U \) be unitary operator defined by \( U e_k = e_{k+1} \). For every orbit \( \delta = (x_k)_{k \in \mathbb{Z}} \in \text{Orb}_+(f) \) there is repellent cycle \( B \) such that \( \delta \in P_B \) further on we will always assume that \( x_0 \in I_B \). Let us define operator \( C_\delta \) via the rule \( C_\delta e_k = x_k e_k \). Let \( Z \) denote the set of non-periodic orbit. Define \( (\Psi(X))(\delta) = U \sqrt{C_\delta} \) and extend it to \( C^*(A_f) \). We have presentation \( \Psi : C^*(A_f) \rightarrow B(H)^Z \text{ of elements of enveloping algebra as a operator-valid functions on } Z. \) Later on we will see that if \( Z \) endowed with topology induced from dual space, \( \hat{C}^*(A_f) \), and \( R \) is a subspace of non-degenerate orbits then for all \( y \in C^*(A_f) \) \( \Psi(y) \) is continuous on \( R \) in norm topology on \( B(H) \) and continuous on \( Z \) in strong topology.

In the following theorem we denote by \( [X] \) the closure of \( X \) in the topology of \( \hat{C}^*(A_f) \) where subset \( X \subset \text{Orb}_+(f) \) is identified with the corresponding set of irreducible representations. If \( Y \subset \mathbb{R} \) then \( \overline{Y} \) denote closure in topology of \( \mathbb{R} \). The set of cyclic orbits is \( \text{Per}(f) / \sim \text{ where } x \sim y \text{ iff } x \text{ and } y \text{ belong to the same orbit. The following theorem from [11] gives the complete description of the dual space.}

**Theorem 5.** Let dynamical system \( (f, I) \) be as in Theorem 4 with minimal possible \( n \). The dual space (spectrum) of \( C^*(A_f) \) is homeomorphic to \( \text{Orb}_+(f) \sqcup_\theta \text{ (Per}(f)/\sim \times \mathbb{T}) \) where \( \theta : \text{Per}(f)/\sim \times \mathbb{T} \rightarrow \text{Orb}_+(f) \) via the rule \( \theta((x, \phi)) \) is the cyclic orbit containing \( x \). Topology on \( \text{Orb}_+(f) \) is given by the following family of closed sets \( \Sigma = \{ \gamma \in \text{Orb}_+(f) | \gamma \in \bigcup_{k \in \mathbb{Z}} \delta \}; \Sigma \subset \text{Orb}_+(f) \) and the space \( \text{Orb}_+(f) \bigcup_\theta \text{ (Per}(f)/\sim \times \mathbb{T}) \text{ is factor-set of disjoint union } \text{Orb}_+(f) \bigcup_\theta \text{(Per}(f)/\sim \times \mathbb{T}) \text{ with equivalence relation which identifies cyclic orbit containing } x \text{ with } \{x, 1\} \in \text{Per}(f)/\sim \times \mathbb{T} \text{ where topology is given by the following family of closed sets: } S \text{ is closed in } \text{Per}(f)/\sim \times \mathbb{T} \text{ and } \Sigma \cup \theta^{-1}(\Sigma) \text{ where } \Sigma \subset \text{Orb}_+(f) \text{. Moreover it is determined up to homeomorphism by integer } n. \)

**Corollary 2.** Representation \( \oplus_{\delta \in R_{B_0}} \pi_\delta \) of \( C^*(A_f) \) is faithful. Let \( \delta \in R_{B_0} \). Then \( X = \{x | x \in \delta \} \) is a compact subspace in \( \mathbb{R} \) and \( R_{B_0} \) is locally compact space. \( C^*(A_f) \) is a \( C^* \)-subalgebra in the \( C^*-\text{algebra } C(R_{B_0}, \mathbb{Z} \times \psi C(X)) \) of all continuous mappings from \( R_{B_0} \) to cross-product \( C^*-\text{algebra } \mathbb{Z} \times \psi C(X) \).

Note that in case \( n = 0 \) there systems that are not orbit-equivalent but as we saw they have isomorphic enveloping \( C^* \)-algebras.

Before considering examples let us make some remarks

**Remark 1.** Then class of \( \mathcal{F}_{2^n} \) unimodal dynamical systems with negative Swartzian, i.e. \( \mathcal{F}_{2^n} \cap SU \) gives examples of \( \text{d.s. satisfying conditions imposed in this section.} \)
Now let us consider some examples. Consider the family of quadratic maps \( f_\mu(x) = \mu x(1-x) \) where \( \mu > 0 \). It is known that \( f_\mu \in SU \). Then for \( \mu < \mu^* \approx 3.57 \) dynamical system \((f_\mu, \mathbb{R})\) is \( F_2 \) for some \( n \). Denote by \( \mu_n > 0 \) the greatest value for which \((f_\mu, \mathbb{R})\) is \( F_2 \). Then for \( 0 < \mu \leq \mu_1 \) dynamical system \((f_\mu, \mathbb{R})\) has two stable points and has no other cycles, for \( \mu_1 < \mu \leq \mu_2 \) dynamical system \((f_\mu, \mathbb{R})\) has two stable points and one cycle of period 2, for \( \mu_n < \mu \leq \mu_{n+1} \) dynamical system \((f_\mu, \mathbb{R})\) has two stable points and one repellent cycle of period \( 2^m \) for \( m < n + 1 \) one attractive cycle of period \( 2^{n+1} \) and no other cycles. The following proposition is a consequence of more general result from [14].

**Proposition 1.** There is \( \nu_{n+1} : \mu_n < \nu_{n+1} \leq \mu_{n+1} \) such that dynamical systems \((f_{\mu_1}, I)\) and \((f_{\mu_2}, I)\) are conjugate iff \( \mu_1 \) and \( \mu_2 \) belongs to the same set \((\mu_n, \nu_{n+1})\) or \( (\nu_{n+1}, \mu_{n+1})\).

7 Cross-product structure of \( C^*(A_f) \)

Let \( X \) be a compact topological Hausdorff space and \((f, X)\) a continuous dynamical system. Let \( \Xi = \{ \delta \subseteq X \} \) be minimal invariant subset such that for all \( x \in \delta \) exists \( y \in \delta : f(y) = x \). For each non eventually periodic orbit \( \delta \in \Xi \) for all \( x \in \delta \) exists unique \( y \in \delta \) such that \( f(y) = x \). Denote such \( y \) by \( f_\delta^{-1}(x) \). And for arbitrary integer \( l > 0 \) define \( f_\delta^{(l)}(x) \) recursively as \( f_\delta^{(l)}(x) = f_\delta^{(l-1)}(f_\delta^{-1}(x)) \). If \( \delta \) is eventually periodic then \( f_\delta^{(l)}(x) \) is defined as such an \( y \in \delta \) that \( f(y) = x \) and \( y \) is non-cyclic. Define topological space \( \Omega = \{ (\delta, y) \mid \delta \in \Xi, y \in \delta \} \).

The family of sets of the form \( B_{n,U}(\delta, y) = \{ (\tau, z) \in \Omega \mid f_\delta^n(z) \in U \} \) where \( n \in \mathbb{Z}, U \) is an open neighborhood of \( f_\delta^n(y) \) constitute a basis of open neighborhoods of \( (\delta, y) \in \Omega \).

In terms of convergent sequences the topology on \( \Omega \) could be defined as follows \((\delta_k, y_k)\) converges to \((\delta, y)\) if and only if for any integer \( n \in \mathbb{Z} \)

\[
\lim_{k \to +\infty} f_\delta^{(n)}(y_k) = f_\delta^{(n)}(y).
\]

**Proposition 2.** \( \Omega \) is a compact Hausdorff space. And mapping \( \sigma : \Omega \to \Omega, \sigma((\delta, y)) = (\delta, f(y)) \) is a homeomorphism of \( \Omega \).

**Theorem 6.** Let \((f, I)\) be \( F_{2^n} \) and \((g, I)\) be \( F_{2^k} \) dynamical systems with unimodal mapping \( f \), which has only two fixed points \( s_0 = 0, 0 < s_1 < 1 \), and assume that for every \( m \leq n \) \((m \leq k \) for \( g \) correspondingly) there is only one cycle of period \( 2^m \) which is repellent for \( m < n \) \((m < k \) for \( g \) correspondingly) and attractive for \( m = n \) \((m = k \) for \( g \) correspondingly). Then \( \Omega(f) \) is homeomorphic to \( \Omega(g) \) if and only if \( m = n \). In this case dynamical systems \((\Omega(f), \sigma)\) and \((\Omega(g), \sigma)\) are conjugated.

**Proof.** We give an internal description of \( \Omega \) from which this theorem follows. Consider in more detail the case \( n = 1 \). The general case could be dealt with by induction in \( n \) which is a common situation for \( F_{2^n} \) dynamical systems (see [11]). In this case \( \Xi = I_0 \cup I_1 \cup \{ 0, s_1, \beta_1, \beta_2 \} \) where \( 0, s_1 \) are fixed points and \( B_2 = \{ \beta_1, \beta_2 \} \) is a cycle of period 2. \((I_0\) and \( I_1\) are \( I_{B_0}\) and \( I_{B_1}\) correspondingly in our previous notations). Then \( \Omega \) could be identified with \((I_0 \times \mathbb{Z}) \cup(I_1 \times \mathbb{Z}) \cup \{ 0, s_1, \beta_1, \beta_2 \} \). We use the homeomorphism \( \Phi : \alpha B \to \mathbb{R} \) to identify the subspace of orbits \( \Omega \) with \( \alpha \)-boundary \( B \) with semi-interval \( I_B \). Then point \((x, m) \in I_B \times \mathbb{Z}\) is identified with \((\Phi(x), f_\Phi^{(m)}(x)) \in \Omega \). Periodic points \( 0, s_1, \beta_1, \beta_2 \) correspond to \((0, 0), (0, 0), (B_2, \beta_1), (B_2, \beta_2) \) respectively. In order to distinguish point \((x, m) \in I_0 \times \mathbb{Z}\) from the point \((x, m) \in I_1 \times \mathbb{Z}\) we will write subscript \((x, m)_0\) and \((x, m)_1\) respectively. From the definition of the topology on \( \Omega \) one can easily obtain the following *homogeneity* condition: sequence \((x_k, m_k)\) converges to \((x, 0)\) if and only if \((x_k, m_k - m)\) converge to \((x, 0)\). We will need some notation, definitions and facts from the proof of Theorem 3 [11]. Let \( I_0 = (a, b) \) and \( I_1 = (t_1, t_2) \). Then there is
a family of semi-intervals $J^k_{B_0}$ where $k \in \mathbb{Z}$ such that $\bigcup_{k \geq 1} J^k_{B_0} = [b_1, b)$ and $\bigcup_{k \leq 1} J^k_{B_0} = (a, a_1]$ for some $a_1 < b_1$ and such that $f^{2k} : J^k_{B_0} \to I_1$ is a homeomorphism for each $k \in \mathbb{Z}$. Denote $J^\infty_{B_0} = (a_1, b_1)$. Further on we will identify $J^k_{B_0}$ with $(0, 1]$ by means of a homeomorphism $\phi_k : J^k_{B_0} \to (0, 1]$ and $\phi_{\infty} : J^\infty_{B_0} \to (0, 1)$. Thus each point $(x, m)_0$ will be coded by a triple $(y, k, m) \in (0, 1] \times (\mathbb{Z} \cup \{\infty\}) \times \mathbb{Z}$ where $x \in J^k_{B_0}$ and $y = \phi_k(x)$. Then using nice descriptions of $\mathcal{F}_{2^n}$ dynamics (see Theorems 1, 2, 3 [11]) the convergent sequences in $\Omega$ could be completely described, for example $(x_k, m)_1$ converges to $(x, t)_1$ if and only if $x_k$ converge to $x < 1$ and $m = t$ or $x_k$ converge to 1 and $x = 0$, $t = m + 1$; $(x_m, k, 2 + 2k) \to (x, 0)_1$ if $k \to \infty$, $x_m \to x$; $(x, 2m)_i \to (B_2, B_2)$ whenever $m \to +\infty$ and $i = 0, 1$; $(x, m)_0 \to (B_0, 0)$ and $(x, m)_1 \to (B_1, s_1)$ whenever $m \to -\infty$. This description of $\Omega$ does not depend on concrete function $f$ but only on the integer $n$. The rest of the claims are consequences of this fact. ■

Let us proceed with the description of the $C^* (A_f)$. In the case if $f$ is unimodal $C^* (A_f)$ is generated by unitary $U$ and Hermitian $C$ such that $UC^2 U^* = f (C^2)$. Let $A_0$ be unital $C^*$-algebra generated by $C$. Obviously, $A_0 \subset C^* (A_f)$ and letter is generated by $A_0$ and $U$. Since each irreducible representation $\pi$ of $C^* (A_f)$ is associated to some orbit $\delta \in \text{Orb} (f)$ and $\pi (C^2)$ is the diagonal operator with points of $\delta$ on its main diagonal the universal representation $\pi_u$ of $C^* (A_f)$ acts on $H = \oplus_{\delta \in \text{Orb} (f)} l^2 (\mathbb{Z})$ and $\pi_u (C^2)$ is the multiplication operator $\pi_u (C^2) \xi (\delta) = \delta (\xi) x \delta (\xi) \in l^2 (\mathbb{Z})$ and $\pi_u (U) \xi (\delta) = \xi (f (\delta))$, $\pi_u (U^*) \xi (\delta) = \xi (f^{-1} (\delta))$. Then $A = E_u (E (A_0))$ is a commutative algebra of diagonal operators generated by $\pi_u (C^2)$ and $\pi_u (U)$ and thus is isomorphic to some $C (Y)$ where $Y$ is a space of multiplicative linear functionals on $A$. From the above description of $A$ follows that a multiplicative linear functional $\rho : A \to \mathbb{C}$ is of the form $p (g) = (g (\delta))_k$ where $g \in A$ (considered as a diagonal operator in $H$), $\delta \in \text{Orb} (f)$ and $k \in \mathbb{Z}$. Hence $A$ can be identified with $\Omega$. One can check that the weak topology on $A$ coincides with the topology on $\Omega$ under this identification. Thus $A$ is an algebra of coefficients for $C^* (A_f)$. Consider an element

$$x = U^* a_N + \cdots + U^* a_1 + a_0 + a_1 U + \cdots + a_N U^N.$$ 

Since $\langle x e_k (\delta), e_k (\delta) \rangle = (a_0 e_k (\delta), e_k (\delta))$ we get that $||x|| \le ||a_0||$. Thus property $\ast$ is satisfied also. Applying theorem (isomorphism) we get the following

**Theorem 7.** Let $(f, I)$ be $\mathcal{F}_{2^n}$ and $(g, I)$ be $\mathcal{F}_{2^k}$ dynamical systems with unimodal mapping $f$, which has only two fixed points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \le n$ ($m \le k$ for $g$ correspondingly) there is only one cycle of period $2^m$ which is repelling for $m < n$ ($m < k$ for $g$ correspondingly) and attractive for $m = n$ ($m = k$ for $g$ correspondingly). In particular, if $f \in \mathcal{F}_{2^n} \cap U$ and $g \in \mathcal{F}_{2^k} \cap U$. Then

1. $C^* (A_f) \cong C (\Omega) \times \pi \mathbb{Z}$.
2. $C^* (A_f) \cong C^* (A_g)$ if and only if $n = k$.

Let $f_{p,q} (x) = 1 + px - qx^2$ with $\{p, q\} \subset \mathbb{R}$ and $q > 0$ to provide boundedness. Since when $p < 0$ dynamical system is one-to-one on $\mathbb{R}_+$ (and so all irreducible representations are one-dimensional) we assume that $p > 0$. This dynamical system is conjugated to $f_\mu (x) = \mu x (1 - x)$ where $\mu = 1 + \sqrt{p^2 - 2p + 1 + 4q}$. The values of parameter $\mu$ when bifurcations of cycles of one parametric family $\{f_\mu\}$ occurs are given in [2]. However, conjugacy relation does not preserve positivity, i.e. $\text{Orb} (f_{p,q})$ may not map into $\text{Orb} (f_\mu)$. This two-parameter deformation unlike previously considered $f_\mu$ give rise to Fock and anti-Fock representations. If $(p, q)$ belong to domain $D = \{ (p, q) \mid q < -\frac{1}{4} + \frac{p^2}{4} + \frac{p}{2} + \frac{\sqrt{1 + 2p}}{2} \}$ then for every $x \in [0; \sup f_{p,q}] \mathcal{O}_+ (x) \subset [0; \sup f_{p,q}]$. Thus for such $(p, q)$ algebra $C^* (A_{f_{p,q}})$ has Fock representation and as it easily can be shown has no anti-Fock representations. In the complement of $D$ algebra $C^* (A_{f_{p,q}})$ has anti-Fock representations. In present paper we consider only the case when there is no
anti-Fock representations. Let \((p, q)\) be such that \(\mu_{n-1} < 1 + \sqrt{p^2 - 2p + 1 + 4q} \leq \mu_n\) and \((p, q) \in D = \{(p, q) \mid q < \frac{1}{2} - \frac{p^2}{4} + \frac{1}{2} + \frac{\sqrt{1+2p}}{2}\}\) then \(f_{p,q} \in \mathcal{F}_{2^n}\). Using results cited in Section 1 and above theorem one can prove the following

**Theorem 8.** Let \((p, q) \in D\) and \(\mu_{n-1} < 1 + \sqrt{p^2 - 2p + 1 + 4q} \leq \mu_n\). Let \(C^* (A_{f_{p,q}})'\) is a \(C^*\)-algebra generated by elements of polar decomposition of operator \(X\) in universal representation of \(C^* (A_{f_{p,q}})'\) that is by partial isometry \(U\) and positive operator \(C\) such that \(X = UC\). Then \(U\) is an isometry and \(C^* (A_{f_{p,q}})'\) has an algebra of coefficients extending \(C^* (C)\). The isomorphism class of \(C^* (A_{f_{p,q}})'\) does not depend on \((p, q)\).