On the Deformations of Dorfman’s and Sokolov’s Operators

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We deform the Dorfman’s and Sokolov’s Hamiltonian operators by the quasi-Miura transformation coming from the topological field theory and investigate the deformed operators.

1 Introduction

The Dorfman’s and Sokolov’s Hamiltonian operators are defined respectively as \( J = D 1 \frac{1}{v_x} D 1 v_x D \) (1) and \( S = v_x D^{-1} v_x \) (2) which are Hamiltonian operators (or \( J^{-1} = D^{-1} v_x D^{-1} v_x D^{-1} \) and \( S^{-1} = \frac{1}{v_x} D^{-1} \frac{1}{v_x} \) are symplectic operators). The Dorfman’s operator \( J \) (or \( J^{-1} \)) and the Sokolov’s operator \( S \) are related to integrable equations as follows.

- The Riemann hierarchy

\[
\begin{align*}
v_{tn} &= v^n v_x = S \delta H_n = \frac{1}{(n+1)(2n+1)} K \delta H_{n+1} = \frac{1}{(n+1)(n+2)} D \delta H_{n+2} \\
&= \frac{1}{(n+1)(n+2)(n+3)(n+4)} J \delta H_{n+4},
\end{align*}
\]

where

\[
K = Dv + vD, \quad H_n = \int v^n dx, \quad n = 1, 2, 3, \ldots,
\]

and \( \delta \) is the variational derivative. When \( n = 1 \), it is called the Riemann equation or dispersionless KdV equation. We notice that it seems that the Riemann hierarchy (3) is a quartet-Hamiltonian system. But one can show that \( S \) and \( J \) is not compatible, i.e., \( S + \lambda J \) are not a Hamiltonian operator for any \( \lambda \neq 0 \) (see below).

- The Schwarzian KdV equation \([10,13]\)

\[
v_t = v_{xxx} - \frac{3}{2} \frac{v_x^2}{v_x} = v_x \{ v, x \} = S \delta H_1 = J^{-1} \delta H_2,
\]

where \( \{ v, x \} \) is the Schwartz derivative and

\[
H_1 = \frac{1}{2} \int (v_x^2 v_x^2) dx, \quad H_2 = \frac{1}{2} \int \left( -v_x^{-2} v_x^2 + \frac{3}{4} v_x^{-4} v_x^4 \right) dx.
\]

**Remark 1.** It is not difficult to verify that \( J^{-1} \) is also a Hamiltonian operator and, then, \( J \) is also a symplectic operator; however, \( S^{-1} = \frac{1}{v_x} D^{-1} \frac{1}{v_x} \) is not a Hamiltonian operator and, then, \( S \) is not a symplectic operator.
Next, to deform the operators $J$ and $S$, we use the free energy in topological field theory of the famous KdV equation

\[ u_t = uu_x + \frac{\epsilon^2}{12} u_{xxx} \]  

(5)

to construct the quasi-Miura transformation as follows. The free energy $F$ of KdV equation (5) in TFT has the form ($F_0 = \frac{1}{6} v^3$)

\[ F = \frac{1}{6} v^3 + \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g (v, v_x, v_{xx}, \ldots, v^{(3g-2)}). \]

Let

\[ \Delta F = \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g (v, v_x, v_{xx}, \ldots, v^{(3g-2)}) = F_1 (v, v_x) + \epsilon^2 F_2 (v, v_x, v_{xx}, v_{xxx}) + \epsilon^4 F_3 (v, v_x, v_{xx}, v_{xxx}, \ldots, v^{(7)}) + \ldots. \]

The $\Delta F$ will satisfy the loop equation [4, p. 151]

\[ \sum_{r \geq 0} \frac{\partial \Delta F}{\partial \nu^{(r)}} \frac{1}{\nu - \lambda} + \sum_{r \geq 1} \frac{\partial \Delta F}{\partial \nu^{(r)}} \sum_{k=1}^{r} \left( \begin{array}{c} r \\ k \end{array} \right) \frac{\partial^{k-1} 1}{\sqrt{\nu - \lambda}} \frac{1}{\sqrt{\nu - \lambda}} = \frac{1}{16 \lambda^2} - \frac{1}{16 (\nu - \lambda)^2} - \frac{\kappa_0}{\lambda^2} \]

\[ + \frac{\epsilon^2}{2} \sum_{k, l \geq 0} \left[ \frac{\partial^2 \Delta F}{\partial \nu^{(k)} \nu^{(l)}} + \frac{\partial \Delta F \partial \Delta F}{\partial \nu^{(k)} \partial \nu^{(l)}} \right] \frac{\partial^{k+1} 1}{\sqrt{\nu - \lambda}} \frac{1}{\sqrt{\nu - \lambda}} - \frac{\epsilon^2}{16} \sum_{k \geq 0} \frac{\partial \Delta F}{\partial \nu^{(k)}} \frac{\partial^{k+2} 1}{(\nu - \lambda)^2}. \]

Then we can determine $F_1, F_2, F_3, \ldots$ recursively by substituting $\Delta F$ into equation (6). For $F_1$, one obtains

\[ \frac{1}{v - \lambda} \frac{\partial F_1}{\partial \nu} = \frac{3}{2} \frac{v_x}{(v - \lambda)^2} \frac{\partial F_1}{\partial v_x} = \frac{1}{16 \lambda^2} - \frac{1}{16 (v - \lambda)^2} - \frac{\kappa_0}{\lambda^2}. \]

From this, we have

\[ \kappa_0 = \frac{1}{16}, \quad F_1 = \frac{1}{24} \log v_x. \]

For the next terms $F_2 (v, v_x, v_{xx}, v_{xxx})$, it can be similarly computed and the result is

\[ F_2 = \frac{v_{xxx}}{1152 v_x^2} - \frac{7 v_{xxx} v_{xxxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4}. \]

Now, one can define the quasi-Miura transformation as

\[ u = v + \epsilon^2 (\Delta F)_{xx} = v + \epsilon^2 (F_1)_{xx} + \epsilon^4 (F_2)_{xx} + \ldots \]

\[ = v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + \epsilon^4 \left( \frac{v_{xxx}}{1152 v_x^2} - \frac{7 v_{xxx} v_{xxxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4} \right)_{xx} + \ldots. \]  

(7)
One remarks that Miura-type transformation means the coefficients of $\epsilon$ are homogeneous polynomials in the derivatives $v_x, v_{xx}, \ldots, v^{(m)}$ [4, p. 37], [5] and “quasi” means the ones of $\epsilon$ are quasi-homogeneous rational functions in the derivatives, too [4, p. 109] (see also [12]).

The truncated quasi-Miura transformation

$$u = v + \sum_{n=1}^{g} \epsilon^{2n} \left[ F_n \left( v; v_x, v_{xx}, \ldots, v^{(3g-2)} \right) \right]_{xx}$$  \hspace{1cm} (8)

has the basic property [4, p. 117] that it reduces the Magri–Poisson pencil [6] of KdV equation (5)

$$\{u(x), u(y)\}_\lambda = [u(x) - \lambda]D\delta(x - y) + \frac{1}{2} u_x(x)\delta(x - y) + \frac{\epsilon^2}{8} D^3\delta(x - y)$$  \hspace{1cm} (9)

to the Poisson pencil of the Riemann hierarchy (3):

$$\{v(x), v(y)\}_\lambda = [v(x) - \lambda]D\delta(x - y) + \frac{1}{2} v_x(x)\delta(x - y) + O(\epsilon^{2g+2}).$$  \hspace{1cm} (10)

One can also say that the truncated quasi-Miura transformation (8) deforms the KdV equation (5) to the Riemann equation $v_t = vv_x$ up to $O(\epsilon^{2g+2})$.

**Remark 2.** A simple calculation shows that, under the transformation $u = \epsilon^2 \{m, x\}$, the KdV equation (5) is transformed into the Schwarzian KdV equation

$$m_t = \frac{\epsilon^2}{12} m_x \{m, x\} = \frac{\epsilon^2}{12} \left( m_{xxx} - \frac{3}{2} \frac{m_x^2}{m_x} \right).$$

Furthermore, after a direct calculation, one can see that the Magri Poisson bracket

$$K(\epsilon) = \{u(x), u(y)\} = u(x)D\delta(x - y) + \frac{1}{2} u_x(x)\delta(x - y) + \frac{\epsilon^2}{8} D^3\delta(x - y)$$  \hspace{1cm} (11)

is transformed into the Dorfman’s symplectic operator $J^{-1} (m = v)$

$$\{m(x), m(y)\} = -\frac{\epsilon^2}{8} D^{-1} m_x D^{-1} m_x D^{-1} \delta(x - y).$$

Now, a natural question arises: under the truncated quasi-Miura transformation (8), are the deformed Dorfman’s operator $J(\epsilon)$ and Sokolov’s operator $S(\epsilon)$ still Hamiltonian operators up to $O(\epsilon^{2g+2})$? For simplicity, we consider only the case $g = 1$, i.e.,

$$u = v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + O(\epsilon^4)$$  \hspace{1cm} (12)

or

$$v = u - \frac{\epsilon^2}{24} (\log u_x)_{xx} + O(\epsilon^4).$$  \hspace{1cm} (13)

The answer is true for the Dorfman’s operator $J(\epsilon)$ but it is false for the Sokolov’s operator $S(\epsilon)$. It is the purpose of this article.
2 Deformations under quasi-Miura transformation

In the new “u-coordinate”, \( J \) and \( S \) will be given by the operators

\[
J(\epsilon) = M^* D \frac{1}{u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx}} D \frac{1}{u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx}} DM + O(\epsilon^4),
\]

(14)

\[
S(\epsilon) = M^* \left( u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) D^{-1} \left( u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) M + O(\epsilon^4),
\]

(15)

where

\[
M = 1 - \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2, \quad M^* = 1 + \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D,
\]

(16)

\( M^* \) being the adjoint operator of \( M \). Then we have the following

**Theorem 1.** 1. \( J(\epsilon) \) is a Hamiltonian operator up to \( O(\epsilon^4) \). 2. \( S(\epsilon) \) is not a Hamiltonian operator up to \( O(\epsilon^4) \).

**Proof.** 1. The fact that \( J(\epsilon) \) is a skew-adjoint (or \( J^*(\epsilon) = -J(\epsilon) \)) differential operator (up to \( O(\epsilon^4) \)) follows immediately from (14). Rather than prove the Poisson form [7] of the Jacobi identity for \( J(\epsilon) \), it is simpler to prove that the symplectic two-form

\[
\Omega_J(\epsilon) = \int \{ du \wedge J(\epsilon)^{-1} du \} dx + O(\epsilon^4)
\]

is closed [8,9]: \( d\Omega_J(\epsilon) = O(\epsilon^4) \).

A simple calculation can yield

\[
J(\epsilon)^{-1} = \left( 1 + \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2 \right) D^{-1} \left( u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) D^{-1}
\]

\[
\times \left( u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) D^{-1} \left( 1 - \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D \right)
\]

\[
= \left( D^{-1} u_x - \frac{\epsilon^2}{24} D^{-1} (\log u_x)_{xxx} + \frac{\epsilon^2}{24} D \frac{1}{u_x} D u_x \right) D^{-1}
\]

\[
\times \left( u_x D^{-1} - \frac{\epsilon^2}{24} (\log u_x)_{xxx} D^{-1} - \frac{\epsilon^2}{24} u_x D \frac{1}{u_x} D \right) + O(\epsilon^4)
\]

\[
= D^{-1} u_x D^{-1} u_x D^{-1} + \frac{\epsilon^2}{24} \left[ D \frac{1}{u_x} D u_x D^{-1} u_x D^{-1} (\log u_x)_{xxx} D^{-1} u_x D^{-1}
\right.
\]

\[
- D^{-1} u_x D^{-1} u_x D \frac{1}{u_x} D - D^{-1} u_x D^{-1} (\log u_x)_{xxx} D^{-1} \left] + O(\epsilon^4) \right.
\]

\[
= D^{-1} u_x D^{-1} u_x D^{-1}
\]

\[
+ \frac{\epsilon^2}{24} \left[ D u_x D^{-1} - D^{-1} u_x D + (\log u_x)_{xxx} u_x D^{-1} + D^{-1} (\log u_x)_{xxx} u_x \right] + O(\epsilon^4)
\]

Let \( \psi \) denote the potential function for \( u \), i.e., \( u = \psi_x \). Thus, formally,

\[
D_x^{-1}(du) = d\psi
\]

and hence, after a series of integration by parts, one has

\[
\Omega_J(\epsilon) = \int \left\{ \left[ D^{-1} d \left( \frac{\psi_x^2}{2} \right) \right] \wedge d \left( \frac{\psi_x^2}{2} \right) - \psi_x d\psi \wedge d \left( \frac{\psi_x^2}{2} \right) \right\} dx + O(\epsilon^4)
\]

\[
+ \frac{\epsilon^2}{24} \left[ 2\psi_{xx} d\psi \wedge d\psi_{xx} + 2\psi_{xxx} d\psi_x \wedge d\psi \right] dx + O(\epsilon^4).
\]
So
\[
\begin{align*}
d\Omega_J(\epsilon) &= \int \left\{ 0 + \frac{\epsilon^2}{12} [d\psi_{xxx} \wedge d\psi_x \wedge d\psi] \right\} dx + O(\epsilon^4) \\
&= \frac{\epsilon^2}{12} \int \{ (d\psi_{xx} \wedge d\psi_x \wedge d\psi) \} dx + O(\epsilon^4) = O(\epsilon^4). 
\end{align*}
\]
This completes the proof of (1).

2. The skew-adjoint property of the deformed Sokolov’s operator $S(\epsilon)$ (15) is obvious. To see whether $S(\epsilon)$ is Hamiltonian operator or not, we must check whether $S(\epsilon)$ satisfies the Jacobi identity up to $O(\epsilon^4)$. Following [7, 8], we introduce the arbitrary basis of tangent vector Θ, which is then conveniently manipulated according to the rules of exterior calculus. The Jacobi identity is given by the compact expression

\[
P(\epsilon) \wedge I = O(\epsilon^4) \quad \text{(mod. div.)},
\]
where $P(\epsilon) = S(\epsilon) \Theta$, $I = \frac{1}{2} \Theta \wedge P(\epsilon)$ and $\delta$ denotes the variational derivative. The vanishing of the tri-vector (17) modulo a divergence is equivalent to the satisfaction of the Jacobi identity.

After a tedious calculation, one can obtain

\[
S(\epsilon) = M^* \left( u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) D^{-1} \left( u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) M + O(\epsilon^4)
\]
\[
= u_x D^{-1} u_x + \frac{\epsilon^2}{24} \left[ D^2 u_x - D^2 (\log u_x)_x D^{-1} u_x - (\log u_x)_{xxx} D^{-1} u_x - u_x D^2 \right. \\
+ u_x D^{-1} (\log u_x)_x D^2 - u_x D^{-1} (\log u_x)_{xxx} + O(\epsilon^4)
\]
\[
= u_x D^{-1} u_x + \frac{\epsilon^2}{24} \left[ D^2 u_x - u_x D^2 + (\log u_x)_x D u_x + u_x D (\log u_x)_x \right] + O(\epsilon^4)
\]
\[
= u_x D^{-1} u_x + \frac{\epsilon^2}{12} [D u_{xx} + u_{xx} D] + O(\epsilon^4).
\]

So

\[
P(\epsilon) = S(\epsilon) \Theta = u_x D^{-1} (u_x \Theta) + \frac{\epsilon^2}{12} [2u_{xx} \Theta_x + u_{xxx} \Theta] + O(\epsilon^4).
\]

Hence

\[
I = \frac{1}{2} \Theta \wedge P(\epsilon) = \frac{1}{2} u_x \Theta \wedge D^{-1} (u_x \Theta) + \frac{\epsilon^2}{12} u_{xx} \Theta \wedge \Theta_x + O(\epsilon^4)
\]

and then

\[
\delta I = -\frac{1}{2} [\Theta \wedge D^{-1} (u_x \Theta)]_x - \frac{1}{2} u_x \Theta \wedge D^{-1} (\Theta_x) + \frac{\epsilon^2}{12} [\Theta \wedge \Theta_x]_{xx} + O(\epsilon^4)
\]
\[
= \frac{1}{2} \Theta_x \wedge D^{-1} (u_x \Theta) + \frac{\epsilon^2}{12} [\Theta \wedge \Theta_x]_{xx} + O(\epsilon^4).
\]
Finally,

\[ P(\epsilon) \wedge \delta I = \left\{ u_x D^{-1}(u_x \Theta) + \frac{\epsilon^2}{12} 2u_{xx} \Theta_x + u_{xxx} \Theta \right\} \]

\[ \wedge \left\{ -\frac{1}{2} \Theta_x \wedge D^{-1}(u_x \Theta) + \frac{\epsilon^2}{12} [\Theta \wedge \Theta_x]_{xx} \right\} + O(\epsilon^4) \]

\[ = 0 + \frac{\epsilon^2}{12} \left\{ -\frac{1}{2} u_{xxx} \Theta \wedge \Theta_x \wedge D^{-1}(u_x \Theta) + u_{xxx} D^{-1}(u_x \Theta) \wedge \Theta \wedge \Theta_x + 3u_{xx} u_x \Theta \wedge \Theta \wedge \Theta_x + u_x^2 \Theta_x \wedge \Theta \wedge \Theta_x \right\} + O(\epsilon^4) \]

\[ = 0 + \frac{\epsilon^2}{24} u_{xxx} \Theta \wedge \Theta_x \wedge D^{-1}(u_x \Theta), \]

which can be easily checked that it cannot be expressed as a total divergence. So \( S(\epsilon) \) cannot satisfy the Jacobi identity and therefore \( S(\epsilon) \) is not a Hamiltonian operator. This completes the proof of (2). \[ \blacksquare \]

**Remark 3.** Using the technics of the last proof, one can show that \( J \) and \( S \) is not compatible. Since \( J \) and \( S \) are Hamiltonian operators, what we are going to do is show that [7,8]

\[ \tilde{Q}(\Theta) \wedge \delta R + Q(\Theta) \wedge \delta \tilde{R} \neq 0 \quad (\text{mod. div.}), \]

where

\[ Q(\Theta) = v_x D^{-1}(v_x \Theta), \quad R = \frac{1}{2} \Theta \wedge Q(\Theta), \]

\[ \tilde{Q}(\Theta) = \left( \frac{1}{v_x} \left( \frac{\Theta_x}{v_x} \right) \right)_x, \quad \tilde{R} = \frac{1}{2} \Theta \wedge \tilde{Q}(\Theta) = -\frac{1}{2v_x^2} \Theta_x \wedge \Theta_{xx}. \]

Then

\[ \delta R = -\frac{1}{2} \left[ \Theta \wedge D^{-1}(v_x \Theta) \right]_x - \frac{1}{2} v_x \Theta \wedge D^{-1}(\Theta_x) = -\frac{1}{2} \Theta_x \wedge D^{-1}(v_x \Theta) \]

and

\[ \delta \tilde{R} = -\left( \frac{1}{v_x^3} \Theta_x \wedge \Theta_{xx} \right)_x. \]

Hence

\[ \tilde{Q}(\Theta) \wedge \delta R + Q(\Theta) \wedge \delta \tilde{R} \]

\[ = \left( \frac{1}{v_x} \left( \frac{\Theta_x}{v_x} \right) \right)_x \wedge \left( -\frac{1}{2} \Theta_x \wedge D^{-1}(v_x \Theta) \right) - v_x D^{-1}(v_x \Theta) \wedge \left( \frac{1}{v_x^3} \Theta_x \wedge \Theta_{xx} \right)_x \]

\[ = \frac{1}{2} \frac{1}{v_x} \left( \frac{\Theta_x}{v_x} \right)_x \wedge \left[ \Theta_{xx} \wedge D^{-1}(v_x \Theta) + v_x \Theta_x \wedge \Theta \right] \]

\[ + \left[ v_{xx} D^{-1}(v_x \Theta) + v_x^2 \Theta \right] \wedge \left( \frac{1}{v_x^3} \Theta_x \wedge \Theta_{xx} \right) \]

\[ = \frac{1}{2v_x} \Theta_{xx} \wedge \Theta_x \wedge \Theta - \frac{v_{xx}}{2v_x^3} \Theta_x \wedge \Theta_{xx} \wedge D^{-1}(v_x \Theta) \]

\[ + \frac{v_{xx}}{v_x^3} D^{-1}(v_x \Theta) \wedge \Theta_x \wedge \Theta_{xx} + \frac{1}{v_x} \Theta \wedge \Theta_x \wedge \Theta_{xx} \]

\[ = \frac{1}{2v_x} \Theta \wedge \Theta_x \wedge \Theta_{xx} + \frac{v_{xx}}{2v_x^3} \Theta_x \wedge \Theta_{xx} \wedge D^{-1}(v_x \Theta) \]

\[ \neq 0 \quad (\text{mod. div.}), \]

as required.
3 Concluding remarks

- That $J(\epsilon)$ is a Hamiltonian operator (up to $O(\epsilon^4)$) is proved in [1]. We give another proof here, which remarkably simplifies the proof given in [1].
- We notice that all the deformed operators $J(\epsilon)$ (14), $D(\epsilon) (\equiv D + O(\epsilon^4))$, $K(\epsilon)$ (11) under the quasi-Miura transformation (7) are Hamiltonian operators (up to $O(\epsilon^4)$). That the deformed Sokolov’s operator $S(\epsilon)$ is not Hamiltonian is a little surprising that means that the Poisson bracket of the Hamiltonians $H_m(u; \epsilon)$, $H_n(u; \epsilon)$ for $S(\epsilon)$

$$\{H_m(u; \epsilon), H_n(u; \epsilon)\}_{S(\epsilon)}$$

will not be $O(\epsilon^4)$ but $O(\epsilon^2)$, i.e., it cannot be a conserved quantity of the Riemann hierarchy (3).

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