“Leonard Pairs” in Classical Mechanics

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We propose a concept of Leonard duality in classical mechanics. It is shown that Leonard duality leads to non-linear relations of the AW-type with respect to Poisson brackets.

1 Introduction

Let $F, G, \ldots$ be classical dynamical variables (DV) that can be represented as differentiable functions of the canonical finite-dimensional variables $q_i, p_i, i = 1, 2, \ldots, N$.

The Poisson brackets (PB) $\{F, G\}$ are defined as [1]

$$\{F, G\} = \sum_{i=1}^{N} \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i}.$$ 

The PB satisfies fundamental properties [1]

(i) PB is a linear function in both $F$ and $G$;
(ii) PB is anti-symmetric $\{F, G\} = -\{G, F\}$;
(iii) PB satisfies the Leibnitz rule $\{F_1 F_2, G\} = F_1 \{F_2, G\} + F_2 \{F_1, G\}$;
(iv) for any dynamical variables $F, G, H$ PB satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$ 

PB are important in classical mechanics because they determine time dynamics: if the DV $H$ is a Hamiltonian of the system, then for any DV $G$ one has Poisson equation

$$\dot{G} = \{G, H\}.$$ 

In particular, the DV $F$ is called an integral if it has zero PB with the Hamiltonian $\{F, H\} = 0$. In this case $F$ does not depend on $t$.

In many problems of the classical mechanics DV form elegant algebraic structures which are closed with respect to PB.

The Poisson structures with non-linear PB were discussed in [9] and [6]. Sklyanin introduced [9] the so-called quadratic Poisson algebra consisting of 4 DV $S_0, S_1, S_2, S_3$ such that PB $\{S_i, S_k\}$ is expressed as a quadratic function of the generators $S_i$. The Sklyanin algebra appears quite naturally from theory of algebraic structures related to the Yang–Baxter equation in mathematical physics. Sklyanin also proposed to study general non-linear Poisson structures. Assume that there exists $N$ dynamical variables $F_i, i = 1, 2, \ldots, N$ such that PB of these variables are closed in frames of the non-linear relations

$$\{F_i, F_k\} = \Phi_{ik}(F_1, \ldots, F_N), \quad i, k = 1, 2, \ldots, N,$$

where $\Phi_{ik}(F_1, \ldots, F_N)$ are (nonlinear, in general) functions of $N$ variables.

Several interesting examples of such non-linear Poisson structures are described in [6].
In [4] another example of such non-linear Poisson algebra was proposed. This example is connected with the property of “mutual integrability” and leads to the so-called classical AW-relations, where abbreviation AW means “Askey–Wilson algebra”. Indeed, as was shown in [11] the operator (i.e. non-commutative) version of AW-relations has a natural representation in terms of generic Askey–Wilson polynomials, introduced in [2] (see also [7]).

In the present work we show that the property of “Leonard pairs” proposed in [5, 10] for matrices can be naturally generalized to the case of classical mechanics. For details and proofs of corresponding statements see [12].

Recall that two $N \times N$ matrices $X$, $Y$ form the Leonard pair if there exists invertible matrices $S$ and $T$ such that the matrix $S^{-1}XS$ is diagonal whereas the matrix $S^{-1}YS$ is irreducible tri-diagonal and similarly, the matrix $T^{-1}YT$ is diagonal whereas the matrix $T^{-1}XT$ is irreducible tri-diagonal. We will call such the property “mutual tri-diagonality”. Leonard showed [8] that the eigenvalue problem for a Leonard pair $X$, $Y$ leads to the $q$-Racah polynomials (for definition see, e.g. [7]).

Terwilliger showed [5, 10] that a Leonard pair $X$, $Y$ satisfies a certain algebraic relations with respect to commutators. In turn, the Terwilliger relations follow from to the so-called relations of the AW-algebra studied in [11] and [4].

We say that $X$ and $Y$ form a classical Leonard pair (CLP) whenever $X$, $Y$ are of the form

$$X = \phi(x), \quad Y = A_1(x) \exp(p) + A_2(x) \exp(-p) + A_3(x)$$

and

$$Y = \psi(q), \quad X = B_1(q) \exp(P) + B_2(q) \exp(-P) + B_3(q),$$

where $\phi(x)$, $A_i(x)$, $\psi(x)$, $B_i(x)$ are some functions such that at least $A_1$ or $A_2$ are non-zero, and $x$, $p$, $q$, $P$ are some canonical variables such that

$$\{x, p\} = 1, \quad \{Q, P\} = 1.$$

Note that the concept of the Leonard pair is closely related with the so-called “bispectrality problem” [3]. We thus arrive also at the classical analogue of the bispectrality problem.

Assuming that $X$, $Y$ are algebraically independent we can show that the following algebraic relations hold

$$\{Z, X\} = -1/2 F_Y(X, Y) = -\left( Y \left( \alpha_1 X^2 + \alpha_3 X + \alpha_5 \right) + \left( \alpha_2 X^2 + \alpha_6 X + \alpha_8 \right) \right)/2$$

and

$$\{Y, Z\} = -1/2 F_X(Y, X) = -\left( X \left( \alpha_1 Y^2 + \alpha_2 Y + \alpha_4 \right) + \left( \alpha_3 Y^2 + \alpha_6 Y + \alpha_7 \right) \right)/2.$$

This is classical version of the AW-algebra introduced in [4] (i.e. Askey–Wilson algebra).

We mention also a remarkable property of the classical AW-algebra [4]. Assume that $X$ is chosen as Hamiltonian: $H = X$. Then we have $Y = \{Y, H\} = -Z$. Hence $Y^2 = F(H, Y) + \alpha_9$ quadratic in $Y$. Hence $Y(t)$ is elementary function in the time $t$. This means that

$$Y(t) = G_1(H) \exp(\omega(H)t) + G_2(H) \exp(-\omega(H)t) + G_3(H)$$

or

$$Y(t) = G_1(H)t^2 + G_2(H)t + G_3(H),$$

where $G_i(H)$, $\omega(H)$ are some functions in the Hamiltonian $H$. Due to obvious symmetry between $X$, $Y$, the same property holds if one chooses $Y$ as Hamiltonian: $H = Y$. In this case $X(t)$
behaves as elementary function in the time \( t \). This property was called “mutual integrability” in \([4]\). It can be considered classical analogues of the property of “mutual tri-diagonality” \([10, 5]\) in the “quantum” case.

It is interesting to note that when dynamics of the system is described by elliptic functions it is possible to generalize AW-algebra obtaining algebra with cubic non-linearity. We announce here the following result: Euler and Lagrange tops (which are known to be integrated in terms of Jacobi elliptic function) have the same symmetry Poisson algebra with cubic non-linearity \([13]\).

\[\begin{align*}
[1] \text{Arnold V.I., Mathematical methods of classical mechanics, Moscow, Nauka, 1989 (in Russian).} \\
[5] \text{Ito T., Tanabe K. and Terwilliger P., Some algebra related to \( P \)- and \( Q \)-polynomial association schemes, Preprint, 1999.} \\
[10] \text{Terwilliger P., Two linear transformations each tridiagonal with respect to an eigenbasis of the other, Preprint, 1999.} \\
\end{align*}\]