Jacobson Generators of (Quantum) $sl(n + 1|m)$.

Related Statistics

T.D. PALEV $^\dagger$, N.I. STOILOVA $^\ddagger$ and J. VAN der JEUGT $^§$

$^\dagger$ Institute for Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria
E-mail: tpalev@inrne.bas.bg

$^\ddagger$ Mathematical Physics Group, Department of Physics, Technical University of Clausthal, Leibnizstrasse 10, D-38678 Clausthal-Zellerfeld, Germany
E-mail: ptns@pt.tu-clausthal.de

$^§$ Department of Applied Mathematics and Computer Science, University of Ghent, Krijgslaan 281-S9, B-9000 Gent, Belgium
E-mail: Joris.VanderJeugt@rug.ac.be

A description of the quantum superalgebra $U_q[sl(n + 1|m)]$ and hence (at $q = 1$) of the special linear superalgebra $sl(n + 1|m)$ via a new set of generators, called Jacobson generators, is given. It provides an alternative to the canonical description of $U_q[sl(n + 1|m)]$ in terms of Chevalley generators. The Jacobson generators satisfy three linear supercommutation relations and define $U_q[sl(n + 1|m)]$ as a deformed Lie supertriple system. Fock representations are constructed and the action of the Jacobson generators on the Fock basis is written down. The Jacobson generators and the Fock representations allow for an interpretation in terms of quantum statistics, and the properties of the underlying statistics are shortly discussed.

1 Introduction

The Lie superalgebra $sl(n + 1|m)$ is one of the basic classical simple Lie superalgebras in Kac’s classification [1]. It can be considered as the superanalogue of the special linear Lie algebra $sl(n + 1)$. The quantum superalgebra $U_q[sl(n + 1|m)]$ is a Hopf superalgebra deformation of the universal enveloping superalgebra $U[sl(n + 1|m)]$ of $sl(n + 1|m)$.

Usually, $U_q[sl(n + 1|m)]$ is defined by its Chevalley generators $e_i$, $f_i$, $h_i$, $i = 1, \ldots, n + m$, subject to the Cartan–Kac relations and the Serre relations [2, 3, 4]. Beside these defining relations, also the other Hopf superalgebra maps (comultiplication, co-unit and antipode) are part of the definition. In the present talk, however, we do not use these other Hopf superalgebra maps; so we shall concentrate on $U_q[sl(n + 1|m)]$ as an associative superalgebra.

The definition in terms of Chevalley generators has the advantage that the comultiplication, co-unit and antipode are easy to give. Furthermore, certain representations can be constructed explicitly (e.g. for the essentially typical representations a Gelfand–Zetlin basis exist for which the action of the Chevalley generators is known [5]). Having certain physical applications in mind, however, it is sometimes more useful to work with a different set of generators for $U_q[sl(n + 1|m)]$.

The different set of generators for $U_q[sl(n + 1|m)]$ given here are the Jacobson generators (JGs) (denoted by $a^+_i$, $a^-_i$ and $H_i$, with $i = 1, \ldots, n + m$). For the case of $sl(n + 1)$, such generators were originally introduced by Jacobson [6, 7]. The use of Jacobson generators has a number of advantages.

First of all, in certain applications it is necessary to have a complete basis of $U_q[sl(n + 1|m)]$ (following from the Poincaré–Birkhoff–Witt theorem). Such a basis is given in terms of the
Cartan–Weyl elements. Although it is possible to express all Cartan–Weyl elements in terms of the Chevalley generators, such expressions soon become rather unmanageable. In terms of the Jacobson generators, the description of all Cartan–Weyl elements is very easy.

Secondly, the Jacobson generators allow for the definition of a simple class of representations, the Fock representations of $U_q[sl(n+1|m)]$. In these representations, the Jacobson generators $a_i^+$ and $a_i^-$ share certain properties with ordinary creation and annihilation operators.

A disadvantage of the Jacobson generators compared to the Chevalley generators is that the explicit expressions for the other Hopf (super)algebra maps (comultiplication, co-unit and antipode) become very lengthy and complicated.

In Section 2 we define the Jacobson generators of $U_q[sl(n+1|m)]$ as a special subset of the Cartan–Weyl elements. The description of all Cartan–Weyl elements in terms of the Jacobson generators becomes very simple. In order to apply these results (e.g. in representations) one must have a list of all (super)commutation relations between these Cartan–Weyl elements; in terms of Jacobson generators, this means one has to determine certain triple relations. These are given in Theorem 2. In Section 3 we define Fock representations for $U_q[sl(n+1|m)]$, related to the Jacobson generators. The Fock representations are labeled by a number $p$; when $p$ is a nonnegative integer, the Fock representation is finite-dimensional. These representations are further analyzed. Following conditions required in a physical context, it is determined when these Fock representations are unitary, see Theorem 4. In that case, an orthonormal basis of the Fock space is given, together with the action of the Jacobson generators on these basis elements. Finally, in Section 4 the Jacobson generators are interpreted as operators creating or annihilating a “particle”, and the underlying quantum statistics is discussed.

## 2 Jacobson generators of $U_q[sl(n+1|m)]$

The Hopf superalgebra $U_q[sl(n+1|m)]$ is defined in the sense of Drinfeld [8], as a topologically free $\mathbb{C}[[h]]$ module. As a superalgebra, $U_q[sl(n+1|m)]$ is usually defined by means of its Chevalley generators, subject to the Cartan–Kac relations and the Serre relations [2, 3, 4]. Here, we present an alternative description of $U_q[sl(n+1|m)]$ in terms of the so-called Jacobson generators. The definition of JGs can be best presented in the framework of a set of Cartan–Weyl elements $e_{ij}$, $i, j = 0, \ldots, n + m$ of $U_q[gl(n+1|m)]$ [9]. The elements $e_{ij}$ are the $q$-analogues of the defining basis of $gl(n+1|m)$; their grading is given by $\deg(e_{ij}) = \theta_{ij} = \theta_i + \theta_j$, where

$$
\theta_i = \begin{cases} 
0 & \text{if } i = 0, \ldots, n, \\
1 & \text{if } i = n + 1, \ldots, n + m.
\end{cases}
$$

We shall refer to $e_{ij}$ as a positive root vector (resp. negative root vector) if $i < j$ (resp. $i > j$).

For the formulation of the Poincaré–Birkhoff–Witt theorem, it is necessary to fix a total order for the set of elements $e_{ij}$. Among the positive root vectors, this order is given by

$$
e_{ij} < e_{kl}, \quad \text{if } i < k \text{ or } i = k \text{ and } j < l;
$$

for the negative root vectors $e_{ij}$ one takes the same rule (1), and total order is fixed by choosing positive root vectors $< negative root vectors $< e_{ii}$.

The difference between $U_q[sl(n+1|m)]$ and $U_q[gl(n+1|m)]$ is in the elements of the Cartan subalgebra. For $U_q[gl(n+1|m)]$ the Cartan subalgebra is generated by $e_{ii}$ ($i = 0, \ldots, n + m$). For $U_q[sl(n+1|m)]$ the Cartan subalgebra is generated by the elements $H_i$, with

$$H_i = e_{00} - (-1)^{\theta_i} e_{ii}, \quad i = 1, \ldots, n + m.\tag{2}$$
We will use also the elements $L_i$ and $\bar{L}_i$, where
\[
L_i = q^{H_i}, \quad \bar{L}_i = q^{-H_i}, \quad i = 1, \ldots, n + m.
\]

The Cartan–Weyl elements of $U_q[sl(n+1|m)]$ are now given by $\{H_i; i = 1, \ldots, n + m\} \cup \{e_{ij}; i \neq j = 0, \ldots, n + m\}$. The complete set of supercommutation relations between these Cartan–Weyl elements is given by
\[
[H_i, H_j] = 0; \quad [H_i, e_{jk}] = (\delta_{0j} - \delta_{0k} - (-1)^{\theta_i}(\delta_{ij} - \delta_{ik}))e_{jk};
\]
for two positive root vectors $e_{ij} < e_{kl}$:
\[
[e_{ij}, e_{kl}]_{q(-1)^{\theta_i}i} = \delta_{jk}e_{il} + (q - q^{-1})(-1)^{\theta_i}(i > j > k)(-1)^{\theta_i}e_{kj}e_{il};
\]
for two negative root vectors $e_{ij} > e_{kl}$:
\[
[e_{ij}, e_{kl}]_{q(-1)^{\theta_i}i} = \delta_{jk}e_{il} - (q - q^{-1})(-1)^{\theta_i}(i > j > l)(-1)^{\theta_i}e_{kj}e_{il};
\]
and finally for a positive root vector $e_{ij}$ and a negative root vector $e_{kl}$:
\[
[e_{ij}, e_{kl}] = \frac{\delta_{ij}\delta_{jk}}{q - q^{-1}}
\]
\[
\cdot L_j(-1)^{\delta_{kl}}L_i(-1)^{\delta_{ik}}L_i(-1)^{\delta_{kl}}
\]
\[
+ \left[ (q - q^{-1}) \theta(j > k > i > l)(-1)^{\theta_i}e_{kj}e_{il} - \delta_{ij}\theta(j > k)(-1)^{\theta_i}e_{kj} + \delta_{jk}\theta(i > l)e_{il} \right] L_i\bar{L}_k
\]
\[
+ L_j\bar{L}_i(-q - q^{-1}) \theta(k > j > l > i)(-1)^{\theta_i}e_{il}e_{kj} - \delta_{ij}\theta(k > j)(-1)^{\theta_i}e_{kj} + \delta_{jk}\theta(l > i)e_{il}
\]
where
\[
[a, b]_x = ab - xba, \quad \{a, b\}_x = ab + xba, \quad [a, b]_x = ab - (-1)^{\deg(a)\deg(b)}xab,
\]
\[
\theta(i_1 > i_2 > \ldots > i_r) = \left\{ \begin{array}{ll}
1, & \text{if } i_1 > i_2 > \ldots > i_r, \\
0, & \text{otherwise.}
\end{array} \right.
\]

Define the Jacobson generators of $U_q[sl(n+1|m)]$ to be the following Cartan–Weyl vectors:
\[
a_i^- = e_{0i}, \quad a_i^+ = e_{0i}, \quad H_i, \quad i = 1, \ldots, n + m.
\]

Then from (8) one obtains:
\[
[a_i^-, a_j^+] = -(-1)^{\theta_i}L_i e_{ji}, \quad (i < j); \quad [a_i^-, a_j^+] = -(-1)^{\theta_i}e_{ji}L_j, \quad (i > j).
\]

In terms of the JGs the definition of $U_q[sl(n+1|m)]$ reads

**Theorem 1.** $U_q[sl(n+1|m)]$ is a unital associative algebra with generators $\{H_i, a_i^\pm\}_{i=1,\ldots,n+m}$ and relations
\[
[H_i, H_j] = 0; \quad [H_i, a_j^\pm] = \mp(1 + (-1)^{\theta_i}\delta_{ij})a_j^\pm,
\]
\[
[a_i^-, a_j^+] = \frac{L_i - L_j}{q - q^{-1}}, \quad L_i = q^{H_i}, \quad L_i = L_i^{-1} = q^{-H_i}, \quad q = q^{-1},
\]
\[
[[a_i^\eta, a_{i+1}^-], a_k^\eta]_{q^{(1+(-1)^{\theta_i}\delta_{ik})}} = \eta^{\theta_k}\delta_{k,i+1}L_k\xi_L a_i^\eta,
\]
\[
[a_1^\xi, a_2^\eta] = 0, \quad [a_1^\xi, a_1^\eta] = 0, \quad \xi, \eta = \pm \text{ or } \pm 1.
\]
The set of relations (11) is the minimal one defining the algebra \( U_q[sl(n+1|m)] \). This description of \( U_q[sl(n+1|m)] \) (resp. \( sl(n+1|m) \)) is somewhat similar to the Lie triple system description of Lie algebras, initiated by Jacobson [6, 7] and generalized to Lie superalgebras by Okubo [10]. Therefore we have defined \( U_q[sl(n+1|m)] \) (resp. \( sl(n+1|m) \)) as a (deformed) Lie supertriple system.

In order to be able to reorder the Cartan–Weyl elements, which appear when computing the transformations of the Fock spaces, it is convenient to write down all triple relations between the JGs (which certainly follow from the relations (11)).

**Theorem 2.** A set of Cartan–Weyl elements of \( U_q[sl(n+1|m)] \) is given by \( H_i, a_i^+, [a_i^+, a_j^-] \) \((i \neq j = 1, \ldots, n+m)\). A complete set of supercommutation relations between these elements is given by:

\[
[H_i, H_j] = 0; \quad [H_i, a_j^+] = \mp(1 + (-1)\theta_{ij})a_j^+; \\
[a_i^-, a_j^+] = \frac{L_i - \bar{L}_i}{q - q^{-1}}, \\
[a_i^+, a_j^-] = 0 \quad (i < j); \quad (a_i^\pm)^2 = 0 \quad (i = n+1, \ldots, n+m); \\
[[a_i^+, a_j^-], a_k^-]_{q^{(1+(-1)^\theta_{jk})}} = \eta^q \delta_{jk} L_k^{-\xi_0} a_i^+ + (-1)^{\theta_k} \varepsilon(j,k,i)(q - \bar{q}) [a_i^+, a_j^-] a_k^- \\
= \eta^q \delta_{jk} L_k^{-\xi_0} a_i^+ + (-1)^{\theta_k} \varepsilon(j,k,i) q^\xi(q - \bar{q}) a_i^+[a_k^+, a_j^-],
\]

where \((j-i)\xi > 0, \xi, \eta = \pm \) and

\[
\varepsilon(j,k,i) = \begin{cases} 
1, & \text{if } j > k > i; \\
-1, & \text{if } j < k < i; \\
0, & \text{otherwise,}
\end{cases}
\]

and we have used the notation \( \bar{q} = q^{-1} \).

### 3 Fock representations

We construct the Fock modules using the induced module procedure. \( G = U_q[sl(n+1|m)] \), with Cartan–Weyl elements \( H_i, a_i^\pm \) and \([a_i^+, a_j^-] \) \((i \neq j = 1, \ldots, n+m)\), has a subalgebra \( A = U_q[gl(n|m)] \) with Cartan–Weyl elements \( H_i \) and \([a_i^+, a_j^-] \) \((i \neq j = 1, \ldots, n+m)\). Define a trivial one-dimensional \( A \) module as follows:

\[
[a_i^-, a_j^+]|0\rangle = 0, \quad (i \neq j = 1, \ldots, n+m), \\
H_i|0\rangle = p|0\rangle,
\]

where \( p \) is any complex number. Let \( P \) be the (associative) subalgebra of \( G = U_q[sl(n+1|m)] \) generated by the elements of \( A \) and \( \{a_i^-; i = 1, \ldots, n+m\} \). The one-dimensional module \( \mathbb{C}|0\rangle \) can be extended to a one-dimensional \( P \) module by requiring:

\[
a_i^-|0\rangle = 0, \quad i = 1, \ldots, n+m.
\]

Now the \( G \) module \( \mathcal{W}_p \) is defined as

\[
\mathcal{W}_p = \text{Ind}^G_A \mathbb{C}|0\rangle.
\]

Clearly \( \mathcal{W}_p \) is freely generated by the generators \( a_i^+ \) \((i = 1, \ldots, n+m)\) acting on \( |0\rangle \). Therefore a basis for \( \mathcal{W}_p \) is given by

\[
|p; r_1, r_2, \ldots, r_{n+m}\rangle \equiv (a_1^+)^{r_1}(a_2^+)^{r_2} \cdots (a_n^+)^{r_n}(a_{n+1}^+)^{r_{n+1}}(a_{n+2}^+)^{r_{n+2}} \cdots (a_{n+m}^+)^{r_{n+m}}|0\rangle,
\]

where \( r_i \in \mathbb{Z}_+ \) for \( i = 1, \ldots, n \) and \( r_i \in \{0, 1\} \) for \( i = n+1, \ldots, n+m \).
Theorem 3. The transformation of the basis (19) of $\tilde{W}_p$ under the action of the JGs reads:

\[
H_i|p; r_1, r_2, \ldots, r_{n+m}\rangle = \left( p - (-1)^i r_i - \sum_{j=1}^{n+m} r_j \right) |p; r_1, r_2, \ldots, r_{n+m}\rangle,
\]

(20)

\[
a_i^- |p; r_1, r_2, \ldots, r_{n+m}\rangle = (-1)^{\theta_1 r_1 + \theta_2 r_2 + \cdots + \theta_{n-1} r_{n-1}} q^{r_1 + \cdots + r_{n-1}} [r_i] \left[ p - \sum_{j=1}^{n+m} r_j + 1 \right]
\times |p; r_1, r_2, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{n+m}\rangle,
\]

(21)

\[
a_i^+ |p; r_1, r_2, \ldots, r_{n+m}\rangle = (-1)^{\theta_1 r_1 + \theta_2 r_2 + \cdots + \theta_{n-1} r_{n-1}} q^{r_1 + \cdots + r_{n-1}} (1 - \theta_i r_i)
\times |p; r_1, r_2, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_{n+m}\rangle,
\]

(22)

where $i = 1, \ldots, n+m$.

Proof. We sketch the proof. Equation (20) is an immediate consequence of $[H_i, a_i^+] = -(1 + (-1)^i \delta_{ij}) a_j^+$, which is one of the last relations in (12). Also the action of $a_i^+$ on the basis vectors is easy: (22) follows directly from (14). The proof of (21) follows from the following relations [11]:

\[ [A, B_1 B_2 \cdots B_{i-1} B_i B_{i+1} \cdots B_{j}]_{q^{b_1 + b_2 + \cdots + b_{j-1}}} = \sum_{i=1}^{j} q^{b_1 + b_2 + \cdots + b_{i-1}} (-1)^{\alpha(b_i + \cdots + b_{j-1})} B_1 B_2 \cdots B_{i-1} \left[ A, B_i \right]_{q^{b_i}} B_{i+1} \cdots B_{j}, \]

where $\alpha = \deg(A)$ and $\beta_i = \deg(B_i)$;

\[ [a_i^-, (a_j^+)^r] = \begin{cases} \frac{q^{2r}-1}{q^2-1} (a_j^+)^{r-1} [a_i^-, a_j^+] & \text{when } i < j, \\ \frac{q^{2r}-1}{q^2-1} (a_i^+)^{r-1} [a_i^-, a_j^+] & \text{when } i > j; \end{cases} \]

(23)

\[ [a_i^-, (a_j^+)^r] = \frac{(a_i^+)^{r-1}}{q - q^r} \left( \frac{q^{2r}-1}{q^2-1} L_i - \frac{q^{2r}-1}{q^2-1} L_j \right); \]

(24)

\[ \left[ [a_i^-, a_j^+] (a_k^+)^r]_{q^r} = \begin{cases} (-1)^{\theta_j} \frac{q^{2r}-1}{q^2-1} L_i (a_j^+)^{r-1}, & i > j, \\ (-1)^{\theta_j} (q^{2r}-1) (a_j^+) (a_k^+)^{r-1} [a_i^-, a_k^+], & i > k > j, \\ \left[ (a_i^+, a_j^+) (a_k^+)^r \cdots (a_{n+m}^+)^r \right]_{q^{n+m}} [0] = (-1)^{\theta_i + \cdots + \theta_{n-1} r_{n-1}} q^{2r_1 + \cdots + 2r_{n-1} + r_i} (a_i^+)^{r-1} (a_i^+)^{r-1} \cdots (a_i^+)^{r-1} (a_{i+1}^+)^{r-1} \cdots (a_{n+m}^+)^{r-1} \cdots [0], & i > 1. \end{cases} \]

(25)

\[ [a_i^-, a_j^+] (a_k^+)^r \cdots (a_{n+m}^+)^r \cdots [0] \]

\[ \sum_{i=1}^{n+m} r_i > p. \]

Corollary 1. The $U_q[sl(n+1|m)]$ module $\tilde{W}_p$ has an invariant submodule $V_p$ with basis vectors

The action of the elements $H_i$ and $a_i^\pm (i = 1, \ldots, n+m)$ on the basis vectors of $\tilde{W}_p$, determined in Theorem 3, imply that $\tilde{W}_p$ has an invariant submodule when $p$ is a nonnegative integer. From now on we shall assume that $p \in \mathbb{Z}_+$. Then we have

Corollary 1. The $U_q[sl(n+1|m)]$ module $\tilde{W}_p$ has an invariant submodule $V_p$ with basis vectors

\[ |p; r_1, r_2, \ldots, r_{n+m}\rangle, \quad \text{with } \sum_{i=1}^{n+m} r_i > p. \]
The quotient module $W_p = W_p/V_p$ is an irreducible representation for $U_q[sl(n+1|m)]$. The basis vectors of $W_p$ are given by (the representatives of)

$$|p; r_1, r_2, \ldots, r_{n+m} \rangle, \quad \text{with} \quad \sum_{i=1}^{n+m} r_i \leq p.$$  \hspace{1cm} (29)

Now we select a class of Fock modules important for physical applications. These are the ones for which the standard Fock metric is positive definite, and for which the representatives of $a_i^\pm$ and $H_i$ ($i = 1, \ldots, n + m$) satisfy the Hermiticity conditions:

$$(a_i^+)^\dagger = a_i^-, \quad (a_i^-)^\dagger = a_i^+, \quad (H_i)^\dagger = H_i.$$  \hspace{1cm} (30)

For the Fock representation $W_p$, we can define a Hermitian form $(,)$ by requiring

$$(|0\rangle, |0\rangle) = \langle 0|0 \rangle = 1,$$  \hspace{1cm} (31)

and by postulating that the Hermiticity conditions (30) should be satisfied, i.e.

$$(a_i^\pm v, w) = (v, a_i^\mp w), \quad \forall v, w \in W_p.$$  \hspace{1cm} (32)

Then any two vectors $|p; r_1, r_2, \ldots, r_{n+m} \rangle$ and $|p; r_1', r_2', \ldots, r_{n+m}' \rangle$ with $(r_1, r_2, \ldots, r_{n+m}) \neq (r_1', r_2', \ldots, r_{n+m}')$ are orthogonal and

$$\langle |p; r_1, r_2, \ldots, r_{n+m} \rangle, |p; r_1, r_2, \ldots, r_{n+m} \rangle \rangle = \frac{[p]!}{[p-R]!} \prod_{i=1}^{n+m} [r_i]! = \frac{[p]!}{[p-R]!} \prod_{i=1}^{n} [r_i]!,$$  \hspace{1cm} (33)

where $R = r_1 + r_2 + \cdots + r_{n+m}$. The straightforward computations show that Hermiticity conditions hold if $q$ is a phase, i.e.

$$q = e^{i\phi} \quad (-\pi < \phi < \pi).$$  \hspace{1cm} (34)

Let us now further investigate when the Hermitian form $(,)$ is an inner product. This means that for every $(r_1, \ldots, r_{n+m})$ with $0 \leq R \leq p$, the value in (33) should be positive. In particular, this implies that all the numbers

$[p], \ [p-1], \ [p-2], \ldots, [2], \ [1]$ should be positive. However, since $q = e^{i\phi}$ is a phase, we have

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{\sin(k\phi)}{\sin(\phi)}.$$  

The common domain where all functions

$$\frac{\sin(2\phi)}{\sin(\phi)}, \frac{\sin(3\phi)}{\sin(\phi)}, \ldots, \frac{\sin(p\phi)}{\sin(\phi)}$$

are positive is

$$-\frac{\pi}{p} < \phi < \frac{\pi}{p}.$$  

Thus we have

**Theorem 4.** The irreducible Fock module $W_p \ (p \geq 2)$ is unitary if and only if $q$ is a phase, i.e. $q = e^{i\phi}$, with $-\frac{\pi}{p} < \phi < \frac{\pi}{p}$.
Observe that whether \(q\) is a root of unity or not does not have any effect on the irreducibility or unitarity of the Fock module \(W_p\), as long as the conditions of Theorem 4 are satisfied. Indeed, suppose that \(q = e^{i\phi}\) is a root of unity with \(\phi\) a rational multiple of \(\pi\) and \(-\frac{\pi}{p} < \phi < \frac{\pi}{p}\). Then the smallest integer \(N\) for which \(q^N = -1\) is greater than \(p\). As a consequence, the rhs in (33) is never zero. This implies that there are no singular vectors among the weight vectors \(|p; r_1, \ldots, r_{n+m}\rangle\), and thus irreducibility holds.

Under the conditions of Theorem 4, we can define an orthonormal basis of \(W_p\):

\[
|p; r_1, r_2, \ldots, r_{n+m}\rangle = \frac{1}{\sqrt{[p]! [r_1]! \cdots [r_{n+m}]!}} |p; r_1, r_2, \ldots, r_{n+m}\rangle,
\]

where \(0 \leq \sum_{l=1}^{n+m} r_l \leq p\). In the new basis (35) the transformation formulas (20)–(22) read (\(i = 1, \ldots, n+m\)):

\[
H_i |p; r_1, r_2, \ldots, r_{n+m}\rangle = \left(p - (-1)^{\theta_i} r_i - \sum_{j=1}^{n+m} r_j\right) |p; r_1, r_2, \ldots, r_{n+m}\rangle,
\]

\[
a^-_i |p; r_1, r_2, \ldots, r_{n+m}\rangle = (-1)^{\theta_i} q^{r_1 + \cdots + r_{i-1}} \sqrt{[r_i]} \left(p - \sum_{l=1}^{n+m} r_l + 1\right) |p; r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{n+m}\rangle,
\]

\[
a^+_i |p; r_1, r_2, \ldots, r_{n+m}\rangle = (-1)^{\theta_i} q^{r_1 + \cdots + r_{i-1}} \sqrt{[r_i]} \left(p - \sum_{l=1}^{n+m} r_l\right) |p; r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_{n+m}\rangle.
\]

4 Properties of the underlying statistics

In the present section we indicate that each \(U_q[sl(n+1|m)]\) module \(W_p\) can be considered as a state space, where \(a^+_i\) (resp. \(a^-_i\)) can be interpreted as operators creating (resp. annihilating) “particles” with, say, energy \(\varepsilon_i\). To this end consider a “free” Hamiltonian

\[
H = \sum_{i=1}^{n+m} \varepsilon_i e_{ii}.
\]

Then

\[
[H, a^+_i] = \pm \varepsilon_i a^+_i.
\]

This result together with equations (37)–(38) allows one to interpret \(a^+_i\) as an operator creating a particle with energy \(\varepsilon_i\), or more precisely, creating a particle on the \(i\)-th orbital. The operator \(a^-_i\) annihilates a particle with energy \(\varepsilon_i\), or equivalently annihilates a particle on the \(i\)-th orbital. On every orbital \(i\) with \(i = n + 1, \ldots, n+m\) there cannot be more than one particle since \((a^+_i)^2 = 0\) for \(i = n + 1, \ldots, n+m\), whereas such a restriction does not hold for the first \(n\) orbitals. These are Fermi like (resp. Bose like) properties. There is however one essential difference. If the corresponding Fock module is characterized by \(p\) then no more than \(p\) “particles” can be accommodated in the system, \(\sum_{i=1}^{n+m} r_i \leq p\). Hence the number of
particles that can be accommodated on a given orbital, keeping the number of particles on all other orbitals fixed, depends on how many particles have already been accommodated in the system. If \( \sum_{i=1}^{p+n+m} r_i < p \) the particles behave similar to bosons and fermions, but are neither bosons nor fermions since the maximum number of the particles in the system cannot exceed \( p \). This condition together with the restrictions for the orbitals with \( i = n+1, \ldots, n+m \) is the analogue of the Pauli principle for this statistics.

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