Invariance of Quasilinear Equations of Hyperbolic Type with Respect to Three-Dimensional Lie Algebras

Olena MAGDA

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine
E-mail: magda@imath.kiev.ua

We have completely solved the problem of description of quasi-linear hyperbolic differential equations in two independent variables, that are invariant under three-parameter Lie groups.

The problem of group classification of differential equations is one of the central problems of modern symmetry analysis of differential equations [1]. One of the important classes are hyperbolic equations. The problem of group classification of such equations was discussed by many authors (see for instance [2–6]). In this paper we consider the problem of the group classification of equations of the form:

\[ u_{tt} = u_{xx} + F(t, x, u, u_x), \]  

(1)

where \( u = u(t, x) \) and \( F \) is an arbitrary nonlinear differentiable function, with \( F_{u, u_x} \neq 0 \) is an arbitrary nonlinear smooth function, which depend on variables \( u \) or \( u_x \). We use the following notation \( u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad F_u = \frac{\partial F}{\partial u}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2} \). For the group classification of equation (1) we use the approach proposed in [7]. Here we give three main results (for details, the reader is referred to [8]).

**Theorem 1.** The infinitesimal operator of the symmetry group of the equation (1) has the following form:

\[ X = (\lambda t + \lambda_1) \partial_t + (\lambda x + \lambda_2) \partial_x + (h(x)u + r(t, x)) \partial_u, \]  

(2)

where \( \lambda, \lambda_1, \lambda_2 \) are arbitrary real constants and \( h(x), r(t, x) \) are arbitrary functions which satisfy the condition

\[ r_{tt} - \frac{\partial^2 h}{\partial x^2} u - r_{xx} + (h - 2\lambda) F - (\lambda t + \lambda_1) F_t - (\lambda x + \lambda_2) F_x \]

\[ - (hu + r) F_u - 2u_x \frac{dh}{dx} - u_x (h - \lambda) F_{u_x} - \frac{dh}{dx} u F_{u_x} - r_x F_{u_x} = 0. \]  

(3)

**Theorem 2.** The equivalence group of the equation (1) is given by transformations of the following form:

\[ \bar{t} = \gamma t + \gamma_1, \quad \bar{x} = \epsilon \gamma x + \gamma_2, \quad v = \rho(x)u + \theta(t, x), \]  

(4)

\( \gamma \neq 0, \rho \neq 0, \epsilon = \pm 1. \)

**Theorem 3.** In the class of operators (2), there are no realizations of the algebras \( so(3) \) and \( sl(2, \mathbb{R}) \).

From this theorem we obtain the following:

**Note 1.** In the class of operators (2) there are no realizations of any real semi-simple Lie algebras.
The set of three-dimensional solvable Lie algebras consists of the following two decomposable Lie algebras:

\[ A_{3,1} = A_1 \oplus A_1 \oplus A_1 = 3A_1; \quad A_{3,2} = A_{2,2} \oplus A_1, \quad [e_1, e_2] = e_3, \]

and the following seven of non-decomposable Lie algebras:

\[ A_{3,3} : [e_2, e_3] = e_1; \]
\[ A_{3,4} : [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2; \]
\[ A_{3,5} : [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2; \]
\[ A_{3,6} : [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2; \]
\[ A_{3,7} : [e_1, e_3] = e_1, \quad [e_2, e_3] = qe_2, \quad (0 < |q| < 1); \]
\[ A_{3,8} : [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1; \]
\[ A_{3,9} : [e_1, e_3] = qe_1 - e_2, \quad [e_2, e_3] = e_1 + qe_2, \quad (q > 0). \]

We give the realizations of the algebras \( A_{3,3}, A_{3,4}, A_{3,5}, A_{3,9} \) and the corresponding values of the functions \( F \) in the equation (1). Here we find only equations, which are non-equivalent to equations of the form

\[ u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u \ln |u| + uD(t, x), \]

and which was classified in [8].

\[ A_{3,3}^1 = \langle u \partial_u, \partial_x, m \partial_t + xu \partial_u \rangle, \quad m \neq 0: \quad F = -u^{-1}u_x^2 + u \tilde{G}(\omega), \quad \omega = t - mu_x u^{-1}; \]
\[ A_{3,3}^2 = \langle \partial_u, \partial_x, m \partial_t + x \partial_u \rangle, \quad m \neq 0: \quad F = \tilde{G}(\omega), \quad \omega = mu_x - t; \]
\[ A_{3,3}^3 = \langle \partial_u, \partial_t, \partial_x + t \partial_u \rangle: \quad F = \tilde{G}(u_x); \]
\[ A_{3,3}^4 = \langle u \partial_u, \partial_t + k \partial_x, \partial_x \rangle, \quad m \in \mathbb{R}: \]
\[ F = -u^{-1}u_x^2 + u \tilde{G}(\omega), \quad \omega = x - kt + mk^{-1}u_x; \]
\[ A_{3,3}^5 = \langle e^{mt} \partial_u, \partial_t, \partial_x + (mu + xe^{mt}) \partial_u \rangle, \quad (m > 0): \]
\[ F = m^2 u + e^{mt} \tilde{G}(\omega), \quad \omega = e^{-mt}u_x - t; \]
\[ A_{3,3}^6 = \langle \partial_u, \partial_t, t \partial_u \rangle: \quad F = \tilde{G}(x, u_x); \]
\[ A_{3,3}^7 = \langle u \partial_u, \partial_t - \beta^{-1}xu \partial_u, \partial_t + \beta \partial_x \rangle, \quad \beta > 0: \]
\[ F = -u^{-1}u_x^2 + u \tilde{G}(\omega), \quad \omega = x - \beta t - \beta^2 u_x u^{-1}; \]
\[ A_{3,3}^8 = \langle u \partial_u, \partial_t - xu \partial_u, \partial_x \rangle: \quad F = t^2 u + 2tu_x + u \tilde{G}(\omega), \quad \omega = t + u_x u^{-1}; \]
\[ A_{3,3}^9 = \langle e^{kt} \partial_u, \partial_t + ku \partial_u, \beta \partial_x + te^{kt} \partial_u \rangle, \quad \beta > 0, \quad k > 0: \]
\[ F = k^2 u + \frac{2ku}{\beta} e^{kt} + e^{kt} \tilde{G}(\omega), \quad \omega = e^{-kt}u_x; \]
\[ A_{3,3}^{10} = \langle |t|^{1/2} \partial_u, -|t|^{1/2} \ln |t| \partial_u, t \partial_t + x \partial_x + \frac{1}{2} u \partial_u \rangle: \]
\[ F = -u^{-1}t^{-2} + u_x^2 \tilde{G}(\omega, v), \quad \omega = tx^{-1}, \quad v = xu_x^2; \]
\[ A_{3,3}^{11} = \langle \partial_u, -t \partial_u, \partial_t + k \partial_x \rangle, \quad k > 0: \quad F = \tilde{G}(\omega, u_x), \quad \omega = x - kt; \]
\[ A_{3,1}^1 = \langle \eta^{m-1} \partial_\eta, \partial_t + k \partial_x, t \partial_t + x \partial_x + (mu + \eta^{m-1}) \partial_\eta \rangle, \]
\[ \eta = x - kt, \quad k \geq 0, \quad m \in \mathbb{R}, \quad m \neq 2 : \]
\[ F = (k^2 - 1)(m-1)(m-2)\eta^{-2} u - \frac{2k(1-m)}{2m-4} \eta^{m-2} + \eta^{2-m} \tilde{G}(\omega), \]
\[ \omega = (1-m)u + \eta u \eta \eta^{3m-4}, \]
\[ A_{3,1}^2 = \langle e^{ktx^{-1}} \partial_\eta, \partial_t + kx^{-1} u \partial_\eta, t \partial_t + x \partial_x + (u + te^{ktx^{-1}}) \partial_\eta \rangle, \quad k \neq 0 : \]
\[ F = u \left( k^2 t^2 x^{-4} - 2ktx^3 - k^2 x^2 \right) + 2ktu x^{-2} + e^{ktx^{-1}} \left( 2k \ln |x| x^{-1} + x^{-1} \tilde{G}(\omega) \right), \]
\[ \omega = e^{-ktx^{-1}} \left( u x + ktx^{-2} \right); \]
\[ A_{3,1}^3 = \langle kx^{-1} u \partial_\eta, \partial_t - k \ln |x| x^{-1} u \partial_\eta, t \partial_t + x \partial_x \rangle, \quad k > 0 : \]
\[ F = k^2 t^2 u x^{-4} - 3ktux^3 + 2ktux^{-2} + 2ktux^{-3} \ln |u| \]
\[ - 2ux^{-2} \ln |u| + 2ux^{-1} \ln |u| + x^{-2} u \ln^2 |u| + ux^{-2} \tilde{G}(\omega), \]
\[ \omega = xu u x^{-1} + \ln |u| + ktx^{-1}; \]
\[ A_{3,1}^4 = \left( |t|^\frac{1}{2} \partial_\eta, -|t|^\frac{1}{2} \ln |t| \partial_\eta, t \partial_t + x \partial_x + \frac{3}{2} u \partial_\eta \right) : \]
\[ F = -\frac{u}{4} t^2 + u x^{-1} \tilde{G}(\omega, v), \quad \omega = tx^{-1}, \quad v = x^{-1} u^2; \]
\[ A_{3,2}^1 = \langle \partial_\eta, -t \partial_\eta, \partial_t + k \partial_x + u \partial_\eta, k > 0 : \]
\[ F = u \tilde{G}(\omega, v), \quad \omega = x - kt, \quad v = \ln |u| - t; \]
\[ A_{3,2}^2 = \langle \eta^{m-1} \partial_\eta, \partial_t + k \partial_x, t \partial_t + x \partial_x + mu \partial_\eta, \eta = x - kt, \quad k > 0, \quad m \in \mathbb{R} : \]
\[ F = (k^2 - 1)(m-1)(m-2)\eta^{-2} + \eta^{m-2} \tilde{G}(\omega), \quad \omega = ((1-m)u + ux \eta) \eta^{-m}; \]
\[ A_{3,2}^3 = \langle \partial_\eta, |t|^{m-1} \partial_\eta, t \partial_t + x \partial_x + mu \partial_\eta, m \in \mathbb{R} : \]
\[ F = (2u - 3mu - m^2 u) t^2 + t^{m-2} \tilde{G}(\omega), \quad \omega = u x t^{m-1}; \]
\[ A_{3,2}^4 = \langle \partial_t, \partial_x, t \partial_t + x \partial_x \rangle : \quad F = u^2 \tilde{G}(u); \]
\[ A_{3,2}^5 = \langle \partial_t, \partial_x, t \partial_t + x \partial_x + mu \partial_\eta, \quad m \neq 0, 1, 2 \rangle : \quad F = |u|^{-\frac{m-2}{m-1}} \tilde{G}(\omega), \quad \omega = u x^{-1} |u|^{-\frac{m-1}{m-1}}; \]
\[ A_{3,2}^6 = \langle \partial_t, \partial_x, t \partial_t + x \partial_x + \partial_\eta, \quad m \neq 0 \rangle : \quad F = e^{-2u} \tilde{G}(\omega), \quad \omega = e^u u x; \]
\[ A_{3,2}^7 = \langle \partial_t, x^{-1} u \partial_\eta, t \partial_t + x \partial_x \rangle, \quad k \neq 0 : \]
\[ F = 2ux^{-1} \ln |u| + u \ln^2 |u| x^{-2} - 2u \ln |u| x^{-2} + x^{-2} u \tilde{G}(\omega), \quad \omega = u x u x^{-1} + \ln |u|; \]
\[ A_{3,2}^8 = \langle \partial_t + kx^{-1} u \partial_\eta, e^{ktx^{-1}} \partial_\eta, t \partial_t + x \partial_x + u \partial_\eta \rangle, \quad k \in \mathbb{R} : \]
\[ F = u k \left( k t^2 - 2x t + x^2 \right) x^{-4} + 2ktu x^{-2} + e^{ktx^{-1}} x^{-1} \tilde{G}(\omega), \]
\[ \omega = e^{-ktx^{-1}} \left( u x + ktx^{-2} \right); \]
\[ A_{3,2}^9 = \langle \sin(t) \partial_\eta, \cos(t) \partial_\eta, \partial_t + k \partial_x + q \partial_\eta, k \geq 0, \quad q > 0 : \]
\[ F = -u + u \tilde{G}(\eta, v), \quad \eta = x - kt, \quad v = e^{-qu} u x; \]
\[ A_{3,3}^1 = \left( \frac{\ln |t|}{2(k - q)} \right) \partial_\eta, \left( \frac{\ln |t|}{2(k - q)} \right) \partial_\eta, 2(k - q) (t \partial_t + x \partial_x) + ku \partial_\eta, \quad k \in \mathbb{R}, \quad q > 0, \quad k \neq q \rangle : \]
\[ F = -\frac{(k - q)^2 + 1}{4(k - q)^2} t^2 u + |t|^{\frac{4q-3k}{4(k - q)}} \tilde{G}(\omega, v), \]
\[ \omega = tx^{-1}, \quad v = |t|^{k-2q} |u x|^{2(k-q)}. \]
Acknowledgements

The author is grateful to R.Z. Zhdanov and V.I. Lahno for proposing the problem and for the help in the research. This research was supported by the INTAS, grant number 01/1-243.


