On Four Orthogonal Projections that Satisfy the Linear Relation
\( \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = I, \alpha_i > 0 \)

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In the article we investigate the sets of orthogonal projections which satisfy the linear relation
\[
\sum_{i=1}^{n} \alpha_i P_i = I, \quad \alpha_i > 0, \text{ up to unitary equivalence.}
\]
A problem of unitary classification of four projections that satisfy the linear relation \( \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = I, \alpha_i > 0 \) is considered in [1–4]. We present a new method for solving this problem that is based on functors of Coxeter, which are analogous to those introduced in [5].

Let \( \mathfrak{P}_{n,\vec{\alpha}} = \mathbb{C}(p_1,p_2,\ldots,p_n | p_i^2 = p_i, \sum_{i=1}^{n} \alpha_i p_i = e) \) be a *-algebra, where the vector \( \vec{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_n), \quad \alpha_i > 0, \quad i = 1,\ldots,n; \quad A = \sum_{i=1}^{n} \alpha_i \). We study its representations, up to unitary equivalence, in the category of Hilbert spaces. Define \( \Sigma_n \) as a set of \( \vec{\alpha} \) such that the category of representations \( \text{Rep} \mathfrak{P}_{n,\vec{\alpha}} \) is not empty.

1. Let us consider some properties of \( \mathfrak{P}_{n,\vec{\alpha}} \).

**Lemma 1.** If \( \vec{\alpha} \in \Sigma_n \) then \( A \geq 1 \).

**Proof.** Let \( \pi \) be a representation of the algebra \( \mathfrak{P}_{n,\vec{\alpha}} \):
\[
\sum_{i=1}^{n} \alpha_i \pi (p_i) = I \text{ then } \sum_{i=1}^{n} \alpha_i (I - \pi(p_i)) = (A - 1)I. \]
Since the operator at the left hand-side is positive then \( A \geq 1 \).

**Lemma 2.** If \( A = 1 \) then \( \vec{\alpha} \in \Sigma_n \) and the algebra \( \mathfrak{P}_{n,\vec{\alpha}} \) has (up to unitary equivalence) only one irreducible representation \( \pi : \pi(p_i) = 1 \).

**Proof.** If \( A = 1 \) then \( \sum_{i=1}^{n} \alpha_i (I - \pi(p_i)) = 0 \) and for all \( i = 1,\ldots,n \): \( \pi(p_i) = I \).

**Definition 1.** The algebra \( \mathfrak{P}_{n,\vec{\alpha}} \) and the vector \( \vec{\alpha} \) are called reduced if there exists such a number \( i_0 \) that for all representations \( \pi \) of the algebra we have \( \pi(p_{i_0}) = 0 \) or there exists a number \( j_0 \) that for all representations \( \pi \) of the algebra we have \( \pi(p_{j_0}) = I \).

**Remark 1.** In the case of mapping of a reduced algebra to its enveloping \( C^* \)-algebra the elements \( p_{i_0} \) and \( p_{j_0} - e \) belong to the *-radical, and the corresponding \( C^* \)-algebra will be generated by less than \( n \) linear connected projections.

**Lemma 3.** If \( \vec{\alpha} \in \Sigma_n : \exists \alpha_{i_0} > 1 \) then for all representations \( \pi \) of the algebra \( \mathfrak{P}_{n,\vec{\alpha}} \): \( \pi(p_{i_0}) = 0 \), e.g. the algebra \( \mathfrak{P}_{n,\vec{\alpha}} \) is reduced.

**Proof.** Take an arbitrary representation \( \pi \) of the algebra \( \mathfrak{P}_{n,\vec{\alpha}} \) then \( \sum_{i \neq i_0} \alpha_i \pi(p_i) = I - \alpha_{i_0} \pi(p_{i_0}). \)
The operator at the left-hand side is positive. But the operator at the right-hand side is positive when \( \pi(p_{i_0}) = 0 \) only.
Lemma 4. If \( \vec{\alpha} \in \Sigma_n \) and the algebra \( \Psi_{n,\vec{\alpha}} \) is not reduced then \( A \leq n \).

Proof. If \( A > n \), then there exists a number \( i_0 : \alpha_{i_0} > 1 \) and according to the Lemma 3 the algebra \( \Psi_{n,\vec{\alpha}} \) will be reduced.

Let \( \Sigma^1_n = \Sigma_n \cap (0,1)^n \) e.g. \( \Sigma^1_n \) consists of such points \( \vec{\alpha} \in \Sigma_n \) that \( 0 < \alpha_i < 1 \).

Our aim is to describe the set \( \Sigma^A_n \) (\( 1 \leq A < n \)) and the set of representations of corresponding algebras. There are reduced and nonreduced ones among such class of algebras.

We define functors \( S \) and \( T \) (analogy with [5]), which act on the set of categories \( \text{Rep} \Psi_{n,\vec{\alpha}} \). They are equivalences of categories (if \( \text{Rep} \Psi_{n,\vec{\alpha}} \) is not empty, then \( S(\text{Rep} \Psi_{n,\vec{\alpha}}) \) or \( T(\text{Rep} \Psi_{n,\vec{\alpha}}) \) is not empty and they are equivalent).

Let us define the functor \( T \) (functor of hyperbolic reflection).

Let \( \alpha \in \Sigma_n, A > 1, \pi \in \text{Rep} \Psi_{n,\vec{\alpha}} \), then \( \sum_{i=1}^n \alpha_i \pi(p_i) = I \) and \( \sum_{i=1}^n \alpha_i(I - \pi(p_i)) = (A - 1)I \) or \( \sum_{i=1}^n \frac{\alpha_i(I - \pi(p_i))}{I} = I \). Define \( T(\pi)(p_i) = I - \pi(p_i) \). Thus, we obtain the functor

\[
T : \text{Rep} \Psi_{n,\alpha_1,\alpha_2,\ldots,\alpha_n} \to \text{Rep} \Psi_{n,\frac{\alpha_1}{1-\alpha_1},\frac{\alpha_2}{1-\alpha_2},\ldots,\frac{\alpha_n}{1-\alpha_n}}
\]

which is defined when \( A > 1 \).

It is easy to check that this functor is equivalence of categories (the corresponding algebras are isomorphic).

Let us define the functor \( S \) (functor of linear reflection).

Let \( \vec{\alpha} \in \Sigma^1_n \), \( \sum_{i=1}^n \alpha_i \pi(p_i) = I \) and \( \pi \) be a representation of the algebra \( \Psi_{n,\vec{\alpha}} \) in the Hilbert space \( H_0 \). Since \( \pi(p_i) \) is a projection then \( \pi(p_i) = \Gamma_i \Gamma_i^* \), where \( \Gamma_i \) is the natural isometry of the space \( H_i = \text{Im} \pi(p_i) \) to \( H_0 \).

Let \( H = H_1 \oplus H_2 \oplus \cdots \oplus H_n \). Define the linear operator \( \Gamma : H \to H_0 \) that is given by the matrix

\[
\Gamma = \left( \sqrt{\alpha_1} \Gamma_1 \quad \sqrt{\alpha_2} \Gamma_2 \quad \cdots \quad \sqrt{\alpha_n} \Gamma_n \right).
\]

Since \( \Gamma \Gamma^* = \sum_{i=1}^n \alpha_i \Gamma_i \Gamma_i^* = \sum_{i=1}^n \alpha_i \pi(p_i) = I_{H_0} \), \( \Gamma^* \) is a partial isometry from \( H_0 \) to \( H \). Let \( \hat{H}_0 = (\text{Im} \Gamma^*)^\perp \) and \( \Delta^* \) is the natural isometry from \( \hat{H}_0 \) to \( H \) then \( U^* = (\Gamma^*, \Delta^*) \) be a unitary operator from \( \hat{H}_0 \oplus H_0 \) to \( H \). As \( H = H_1 \oplus H_2 \oplus \cdots \oplus H_n \), the operators \( \Delta \) and \( U \) have the Peirce decomposition

\[
\Delta = \left( \sqrt{1 - \alpha_1} \Delta_1 \quad \sqrt{1 - \alpha_2} \Delta_2 \quad \cdots \quad \sqrt{1 - \alpha_n} \Delta_n \right),
\]
\[
U = \left( \sqrt{\alpha_1} \Gamma_1 \quad \sqrt{\alpha_2} \Gamma_2 \quad \cdots \quad \sqrt{\alpha_n} \Gamma_n \right).
\]

Since \( U \) is a unitary operator and \( \Gamma_i^* \Gamma_i = I_{H_i} \), it is easy to obtain that \( \Delta_i^* \Delta_i = I_{H_i} \) and \( \Delta_i \Delta_i^* = Q_i \) are orthoprojections in the space \( H_0 \). From \( \Delta \Delta^* = I_{\hat{H}_0} \) (\( \Delta \) is an isometry) it follows that

\[
\sum_{i=1}^n (1 - \alpha_i) \Delta_i \Delta_i^* = I_{\hat{H}_0}, \quad \sum_{i=1}^n (1 - \alpha_i) Q_i = I_{\hat{H}_0}.
\]

Define \( S : \pi \to \hat{\pi} \), where \( \hat{\pi}(p_i) = Q_i \). From the condition \( \sum_{i=1}^n (1 - \alpha_i) Q_i = I \) we have

\[
\hat{\pi} \in \text{Ob} \text{Rep} \Psi_{n,1-\alpha_1,1-\alpha_2,\ldots,1-\alpha_n}.
\]

One can see (in analogy with [5]), that the functor

\[
S : \text{Rep} \Psi_{n,\alpha_1,\alpha_2,\ldots,\alpha_n} \to \text{Rep} \Psi_{n,1-\alpha_1,1-\alpha_2,\ldots,1-\alpha_n},
\]

where \( 0 < \alpha_i < 1 \) (therefore, \( 0 < A < n \)), is an equivalence of categories.
Let \( \pi \) be a representation of the algebra \( \mathfrak{P}_{n,\vec{\alpha}} \) in a finite-dimensional space \( H \). We shall call the vector \((d; d_1, d_2, \ldots, d_n)\), where \( d = \dim H, d_i = \dim \text{Im} \pi(p_i) \), the generalized dimension of the representation \( \pi \).

The functors \( T \) and \( S \) induce actions on the set of vectors \( \vec{\alpha} \), on sums of their coordinates \( A \) and on generalized dimensions of representations of algebras \( \mathfrak{P}_{n,\vec{\alpha}} \).

It is easy to show that

\[
T(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left( \frac{\alpha_1}{A-1}, \frac{\alpha_2}{A-1}, \ldots, \frac{\alpha_n}{A-1} \right), \quad T(A) = \frac{A}{A-1},
\]

\[
T(d; d_1, d_2, \ldots, d_n) = (d; d - d_1, d - d_2, \ldots, d - d_n),
\]

\[
S(\alpha_1, \alpha_2, \ldots, \alpha_n) = (1 - \alpha_1, 1 - \alpha_2, \ldots, 1 - \alpha_n), \quad S(A) = n - A,
\]

\[
S(d; d_1, d_2, \ldots, d_n) = \left( \sum_{i=1}^{n} d_i - d; d_1, d_2, \ldots, d_n \right).
\]

Define the functors of Coxeter as \( \Phi^+ = TS \) and \( \Phi^- = ST \). \( \Phi^+ \) is defined when \( A < n - 1 \), \( \vec{\alpha} \in \Sigma^1_n \). \( \Phi^- \) is defined when \( A > 1 \), \( T(\vec{\alpha}) \in (0, 1)^n \). Since \( T^2 = I_d, S^2 = I_d \), then \( \Phi^+ \Phi^- = I_d \) and \( \Phi^- \Phi^+ = I_d \).

Let \( \Phi^{+(k)} = \Phi^+ \Phi^{+(k-1)} \).

**Lemma 5.** \( \lim_{k \to \infty} \Phi^{+(k)} \left( 1 + \frac{1}{n-2} \right) = \frac{n - \sqrt{n^2 - 4n}}{2} \) and intervals

\[
\left[ 1, 1 + \frac{1}{n-2} \right], \quad \left[ 1 + \frac{1}{n-2}, \Phi^+ \left( 1 + \frac{1}{n-2} \right) \right], \ldots,
\]

\[
\left[ \Phi^{+(k-1)} \left( 1 + \frac{1}{n-2} \right), \Phi^{+(k)} \left( 1 + \frac{1}{n-2} \right) \right], \ldots
\]

do not intersect and cover the interval \( \left[ 1, \frac{n - \sqrt{n^2 - 4n}}{2} \right] \).

**Proof.** It is easy to show that \( \Phi^+(1) = 1 + \frac{1}{n-2} \) and the sequence \( \Phi^{+(k)} \left( 1 + \frac{1}{n-2} \right) \) is increasing. Since it is bounded by 2, the limit \( a \) of the sequence exists and it is a fixed point of the map \( \Phi^+(A) = 1 + \frac{1}{n-A-1} = a \) (taking into account that \( a < 2 \)) we obtain \( a = \frac{n - \sqrt{n^2 - 4n}}{2} \).

**Lemma 6.** \( \vec{\alpha} \in \Sigma^1_n, 0 < A \leq \frac{n}{2} \), if and only if \( T(\vec{\alpha}) \in \Sigma^1_n \) and \( \frac{n}{2} \leq T(A) < n \).

**Proof.** Obviously, the map \( S \) sets one-to-one correspondence between points of \( \Sigma^1_n \) with the sum \( A < n \) and points \( \Sigma^1_n \) with the sum \( n - A \).

**Lemma 7.** If \( n - 1 < A < n \) then \( \vec{\alpha} \notin \Sigma^1_n \).

**Proof.** If \( n - 1 < A < n \) then \( 0 < S(A) < 1 \), whence, by the Lemma 1, \( S(\vec{\alpha}) \notin \Sigma_n \) and it means that \( \vec{\alpha} \notin \Sigma^1_n \).

**Lemma 8.** If \( \vec{\alpha} \in \Sigma_n, A \neq 1 \) and \( \mathfrak{P}_{n,\vec{\alpha}} \) is not reduced then \( \frac{\alpha_1}{A-1} \leq 1 \) and \( A \geq \frac{n}{n-1} \).

**Proof.** If there exists a number \( i_0 \) that \( \frac{\alpha_{i_0}}{A-1} > 1 \), then the algebra \( \mathfrak{P}_{n,T(\vec{\alpha})} \) will be reduced. Take any representation \( \pi \) of the algebra \( \mathfrak{P}_{n,\vec{\alpha}} \). Denote \( \hat{\pi} \) as the correspondent representation of the algebra \( \mathfrak{P}_{n,T(\vec{\alpha})} \) then by the lemma 3 \( \hat{\pi}(p_{i_0}) = 0 \), so \( \pi(p_{i_0}) = I \) and \( \mathfrak{P}_{n,\vec{\alpha}} \) is reduced.

If for all \( i : \frac{\alpha_i}{A-1} \leq 1 \) then \( \frac{A}{A-1} \leq n \) and from here \( A \geq \frac{n}{n-1} \).

2. Now we describe \( \Sigma^1_n \), when \( n = 3 \) and \( n = 4 \).

**Lemma 9.** Let \( \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \Sigma_3 \). Then for some subset \( J \subseteq \{1,2,3\} : \sum_{i \in J} \alpha_i = 1 \) or \( \alpha_1 + \alpha_2 + \alpha_3 = 2 \). To every pointed subset \( J \), there corresponds a unique one-dimensional irreducible representation \( \pi \) : \( \pi(p_i) = 1, \ i \in J \), and \( \pi(p_i) = 0, \ i \notin J \). If \( \alpha_1 + \alpha_2 + \alpha_3 = 2 \) then, furthermore, the algebra has a unique, up to unitary equivalence, irreducible two-dimensional representation.
Lemma 11. If \( \vec{\alpha} \in \Sigma_4, 0 < A < 2, \) is reduced then the following condition, which we will call the R-condition, is satisfied: \( \exists J \subset \{1, 2, 3, 4\} : \sum_{i \in J} \alpha_i = 1 \) or \( \exists \alpha_{i_0} : 2 - A = \alpha_{i_0}. \)

Proof. There are two possible cases.

1) Let \( \pi(p_{i_0}) = 0 \) then \( \sum_{i \neq i_0} \alpha_i \pi(p_i) = I. \) Let \( \vec{\alpha}' \) be obtained from \( \vec{\alpha} \) by omitting the coordinate \( \alpha_{i_0}. \) Obviously, \( \vec{\alpha}' \in \Sigma_3. \) So \( \sum_i \alpha_i = 1, \) for some subset \( J \subset \{1, 2, 3, 4\} \setminus \{i_0\}, \) (if \( \sum_i \alpha_i = 2, \) then \( A > 2 \)).

2) If for all \( \pi : \pi(p_{i_0}) = I \) then \( \sum_{i \neq i_0} \alpha_i \pi(p_i) = (1 - \alpha_{i_0})I. \) The operator at the left hand-side is positive. From here \( \alpha_{i_0} \leq 1. \) If \( \alpha_{i_0} = 1, \) then the R-condition is satisfied, else \( \sum_{i \neq i_0} \frac{\alpha_i}{1 - \alpha_{i_0}} \pi(p_i) = I. \)

From the previous lemma we have either: \( a \) \( \sum_{i \in J} \frac{\alpha_i}{1 - \alpha_{i_0}} = 1, \) for some subset \( J \subset \{1, 2, 3, 4\} \setminus \{i_0\}, \) hence \( \sum \alpha_i + \alpha_4 = 1 \) or \( b \) \( \frac{\alpha_1}{1 - \alpha_{i_0}} + \frac{\alpha_2}{1 - \alpha_{i_0}} + \frac{\alpha_3}{1 - \alpha_{i_0}} = 2, \alpha_1 + \alpha_2 + \alpha_3 = 2(1 - \alpha_4) \) and \( 2 - A = \alpha_4. \)

Note, that if \( \vec{\alpha} \) satisfies R-condition then \( \vec{\alpha} \) is not necessary reduced.

Lemma 11. If \( \vec{\alpha} \in \Sigma_4 \setminus \Sigma_4^1 \) then \( T(\vec{\alpha}) \) satisfies R-condition.

Proof. From the condition \( \vec{\alpha} \in \Sigma_4 \setminus \Sigma_4^1, \) we obtain \( \alpha_{i_0} \geq 1 \) for some \( i_0. \) Suppose \( \alpha_{i_0} > 1, \) \( \pi \in \text{Rep} \, \mathfrak{P}_4, T(\alpha) \) then, by the Lemma 3, \( T(\pi)(p_{i_0}) = 0. \) From here \( \pi(p_{i_0}) = I, \) so \( \vec{\alpha} \) is reduced.

Assume \( \alpha_{i_0} = 1. \) From \( T(\vec{\alpha}) = \left( \frac{\alpha_1}{A - 1}, \frac{\alpha_2}{A - 1}, \frac{\alpha_3}{A - 1}, \frac{\alpha_4}{A - 1} \right) = \left( \sum_{i \neq i_0} \frac{\alpha_i}{A - 1}, \sum_{i \neq i_0} \frac{\alpha_i}{A - 1}, \sum_{i \neq i_0} \frac{\alpha_i}{A - 1}, \sum_{i \neq i_0} \frac{\alpha_i}{A - 1} \right), \) the sum \( \sum_{j \neq i_0} \left( \frac{\alpha_j}{\sum_{i \neq i_0} \alpha_i} \right) = 1, \) so \( T(\vec{\alpha}) \) satisfies R-condition.

From Lemmas 2, 3, 8, 10, it follows

Lemma 12. If \( 1 \leq A < 1 + \frac{1}{n-2} \big|_{n=4} = \frac{3}{2} \) then \( \vec{\alpha} \) satisfy R-condition.

Using the lemmas proved above, we obtain:

Theorem 1. Let \( \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), 0 < \alpha_i < 1, A = \sum_{i=1}^{4} \alpha_i, \Sigma_4^1 \) be the set of such \( \vec{\alpha} \) that the algebra \( \mathfrak{P}_{4,\vec{\alpha}} \) has a nonzero representation.

1) Dimensions of all irreducible representations of the algebra \( \mathfrak{P}_{4,\vec{\alpha}} \) are finite.

2) If \( A = 1 \) then \( \vec{\alpha} \in \Sigma_4^1 \) and the corresponding algebra \( \mathfrak{P}_{4,\vec{\alpha}} \) has a unique irreducible representation \( \pi, \) which is a one-dimensional representation and \( \pi(p_i) = 1. \)

3) If \( A = 2 \) then \( \vec{\alpha} \in \Sigma_4^1 \) and all irreducible representations have dimension one or two (their description see in [4]).

4) The functor \( S \) is equivalence of categories of representations of “symmetry” algebras \( \mathfrak{P}_{4,\{1\},\{2,3,4\}}, \mathfrak{P}_{4,\{1\},\{2,3,4\}} \) and \( \mathfrak{P}_{4,\{1\},\{2,3,4\}} \) and \( \mathfrak{P}_{4,\{1\},\{2,3,4\}}, \) \( \vec{\alpha} \in \Sigma_4^1, \) with the center of symmetry \( A = 2. \)

5) Every point \( \vec{\alpha} \in \Sigma_4^1, 1 < A < 2, \) or satisfies R-condition or \( \Phi^{-1}(\alpha) \) belongs to \( \Sigma_4^1. \)

6) \( \vec{\alpha} \in \Sigma_4^1, 1 < A < 2 \) if and only if \( \Phi^{-1}(\vec{\alpha}) \) satisfies R-condition for some \( k. \) The number \( k \) is bounded by \( N : \Phi^{-1}(\vec{\alpha}) \in [1, \frac{3}{2}] \). The functor \( \Phi^{-1} \) is equivalence of categories of representations of algebras \( \mathfrak{P}_{n,\vec{\alpha}} \) and reduced algebra \( \mathfrak{P}_{n,\Phi^{-1}(\vec{\alpha})}. \)

The theorem allows us to reduce the solution of the problem about belonging of a point \( \vec{\alpha} \) to \( \Sigma_4^1 \) to verifying R-condition for some another point.
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