On Integrability of Some Nonlinear Model with Variable Separant

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In this paper a new integrable nonlinear Hamiltonian system in (1 + 1)-dimension is introduced. Nontrivial connection with well-known multicomponent nonlinear Schrödinger model is found.

Let us consider a non-linear Hamiltonian system
\[ \psi_t = \{\psi, H\} \]  
(1)
in the Schwarz space of smooth fast decreasing on the \( \pm \infty \) complex value \( l \)-component vector-functions \( \psi = (\psi_1, \ldots, \psi_l)(x), l \in \mathbb{N} \) of the variable \( x \in \mathbb{R} \) with the Hamiltonian
\[ H = \int_{-\infty}^{+\infty} |\psi_x|^2 dx, \]  
(2)
and local brackets of Poisson for dynamic variables \( \psi_m, \psi_n, m, n = 1, l \):
\[ \{\psi_n(x), \bar{\psi}_m(y)\} := i\delta_{mn} (c + |\psi|^2(x))^2 \delta(x-y), \]  
(3)
where \( \delta_{mn} \) is the Kronecker symbol, \( \delta(z) \) is the Dirac function, \( c \in \mathbb{R} \).

System (1) is non-linear evolutionary system of differential equations with variable separant (coefficient at higher derivative) and has the next form:
\[ i\psi_t = - (c + |\psi|^2)^2 \frac{\delta H}{\delta \psi^*} = (c + |\psi|^2)^2 \psi_{xx}, \]  
(4)
where \( \frac{\delta}{\delta \psi^*} \) is the Euler operator of variative derivative over the vector-function \( \psi^* := \bar{\psi}^T \).

**Proposition 1.** Hamiltonian system (1)–(4) is formally integrable (by Lax) and assumes infinitive hierarchy non-trivial local laws of motion.

**Proof.** For simplicity we restrict ourselves with Lax commutative representation discovered by us \([L, M] := LM - ML = 0\) in algebra of integro-differential operators \([1, 2]\) which is equivalent to system (4), where
\[ L = (c + |\psi|^2) D + \psi_x \psi^* - \psi_x D^{-1} \psi^*_x, \]  
(5)
\[ M = i\partial_t - (c + |\psi|^2)^2 D^2 - 2 (c + |\psi|^2) |\psi|^2 x D = i\partial_t - (L^2) > 0, \]  
(6)
and, as consequence of operators commutativity in (5)–(6), known \([1]\) procedure for finding density \( \rho_k \) of first integrals \( H_k := \int_{-\infty}^{+\infty} \rho_k dx \):
\[ \rho_k = \text{Res} \left( L^k \right), \quad k \in \mathbb{Z}. \]  
(7)
Remark 1. Obviously, $k = 1$ corresponds to Hamiltonian $H(2)$, and one of the simplest first integrals ($k = -1$) in the formula (7) has the form:

$$H_{-1} = \int_{-\infty}^{\infty} \frac{|\psi|^2}{c + |\psi|^2} dx, \quad c \in \mathbb{R} \setminus \{0\}.$$  

Remark 2. In the formula (5) integral item $\psi_x D^{-1} \psi^*_x$ is a symbol of skew-Hermitian operator of Volterra $\hat{V}$ with the degenerated kernel $V(x, s) := \frac{\partial \psi(x)}{\partial x} \frac{\partial \psi^*(s)}{\partial s}$

$$\left( \hat{V} f \right)(x) = \frac{1}{2} \left\{ \int_{-\infty}^{x} \sum_{i=1}^{l} \frac{\partial \psi_i(x)}{\partial x} \frac{\partial \psi_i^*(s)}{\partial s} f(s) ds - \int_{x}^{+\infty} \sum_{i=1}^{l} \frac{\partial \psi_i(x)}{\partial x} \frac{\partial \psi_i^*(s)}{\partial s} f(s) ds \right\}.$$  

The symbol $(L^k)_{k>0}$ stands for the differential part without free term (multiplier operator by function) of an integro-differential operator $L^k$.

Proposition 2. The following non-local replacement of variables $(t, x, \psi) \rightarrow (\tau, y, \varphi)$:

$$\tau = t, \quad y'_x = \frac{1}{c + |\psi|^2}, \quad \varphi(\tau, y) = \frac{\psi_y}{c + |\psi|^2} \exp \int_{-\infty}^{y} \frac{\psi_y \psi^*}{c + |\psi|^2} dy$$  \hspace{1cm} (8)

transforms non-linear system (4) into the multicomponent non-linear equation of Schrödinger [3]

$$i\varphi_\tau = \varphi_{xx} + 2|\varphi|^2 \varphi.$$  \hspace{1cm} (9)

Proof. The proof is conducted by direct calculation. We restrict ourselves by the Lax operator (5). Making replacement (8) we get

$$L = \left( c + |\psi|^2 \right) D_x + \psi_x \psi^* - \psi_x D^{-1} \psi^* \rightarrow \tilde{L} = D_y + \frac{\psi_y \psi^*}{c + |\psi|^2} - \frac{\psi_y}{c + |\psi|^2} D^{-1} \psi^*_y,$$

and after trasformation $\tilde{L} \rightarrow \Phi \tilde{L} \Phi^{-1}$ with the function $\Phi = \exp \int_{-\infty}^{y} \frac{\psi_y \psi^*}{c + |\psi|^2} dy$ the operator $L$ to pass into the Lax operator $L_{NS}$ [2, 4, 5] for the model (9):

$$L_{NS} = \Phi \tilde{L} \Phi^{-1} = D_y - \varphi D^{-1} \varphi^*,$$

where the dynamic variable $\varphi = \varphi(\tau, y)$ is defined by substitution (8).  


