Supersymmetry and the MSSM: An Elementary Introduction

Notes of Lectures for Graduate Students in Particle Physics
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Abstract

These notes are an expanded version of a short course of lectures given for graduate students in particle physics at Oxford. The level was intended to be appropriate for students in both experimental and theoretical particle physics. The purpose is to present an elementary and self-contained introduction to SUSY that follows on, relatively straightforwardly, from graduate-level courses in relativistic quantum mechanics and introductory quantum field theory. The notation adopted, at least initially, is one widely used in RQM courses, rather than the 'spinor calculus' (dotted and undotted indices) notation found in most SUSY sources, though the latter is introduced in optional Asides. There is also a strong preference for a 'do-it-yourself' constructive approach, rather than for a top-down formal deductive treatment. The main goal is to provide a practical understanding of how the softly broken MSSM is constructed. Relatively less space is devoted to phenomenology, though simple 'classic' results are covered, including gauge unification, the bound on the mass of the lightest Higgs boson, and sparticle mixing. By the end of the course students (readers) should be provided with access to the contemporary phenomenological literature.

Contents

1 Introduction and Motivation

Supersymmetry (SUSY) - a symmetry relating bosonic and fermionic degrees of freedom - is a remarkable and exciting idea, but its implementation is technically pretty complicated. It can be discouraging to find that after standard courses on, say, the Dirac equation and quantum field theory, one has almost to start afresh and master a new formalism, and moreover one that is not fully standardized. On the other hand, thirty years have passed since the first explorations of SUSY in the early 1970's, without any direct evidence of its relevance to physics having been discovered. The Standard Model (SM) of particle physics (suitably extended to include an adequate neutrino phenomenology) works extremely well. So the hard-nosed seeker after truth may well wonder: Why spend the time learning all this intricate SUSY stuff? Indeed, why speculate at all about how to go 'beyond' the SM, unless or until experiment forces us to? If it's not broken, why try and fix it?

As regards the formalism, most standard sources on SUSY use either the 'dotted and undotted' 2-component spinor notation found in the theory of representations of the Lorentz group, or 4-component Majorana spinors. Neither of these is commonly included in introductory courses on the Dirac equation (though perhaps they should be). But it is of course perfectly possible to present simple aspects of SUSY using a notation which joins smoothly on to standard 4-component Dirac equation courses, and a brute force, 'try-it-and-see' approach to constructing SUSY-invariant theories. That is what I aim to do in these lectures, at least to start with. Somewhat surprisingly, it seems that such an elementary introduction is not available, or at least not in such detail as is given here, which is why these notes have been typed up. I hope that they will help to make the basic nuts and bolts of SUSY accessible to a wider clientele. However, as we go along I shall explain the more compact 'dotted and undotted' notation in optional Asides, and I'll also introduce the powerful superfield formalism; this is partly because the simple-minded approach becomes too cumbersome after a while, and partly because contemporary discussions of the phenomenology of the Minimal Supersymmetric Standard Model (MSSM) make some use this more sophisticated notation.

What of the need to go beyond the Standard Model? Within the SM itself, there is a plausible historical answer to that question. The V-A current-current (four-fermion) theory of weak interactions worked very well for many years, when used at lowest order in perturbation theory. Yet Heisenberg [1] had noted as early as 1939 that problems arose if one tried to compute higher order effects, perturbation theory apparently breaking down completely at the then unimaginably high energy of some 300 GeV (the scale of $G_{\rm F}^{-1/2}$). Later, this became linked to the non-renormalizability of the four-fermion theory, a purely theoretical problem in the years before experiments attained the precision required for sensitivity to electroweak radiative corrections. This perceived disease was alleviated but not cured in the 'Intermediate Vector Boson' model, which envisaged the weak force between two fermions as being mediated by massive vector bosons. The non-renormalizability of such a theory was recognized, but not addressed, by Glashow [2] in his 1961 paper proposing the SU(2)×U(1) structure. Weinberg [3] and Salam [4], in their gauge-theory models, employed the hypothesis of spontaneous symmetry breaking to generate masses for the gauge bosons and the fermions, conjecturing that this form of symmetry breaking would not spoil the renormalizability possessed by the massless (unbroken) theory. When 't Hooft [5] demonstrated this in 1971, the Glashow-Salam-Weinberg theory achieved a theoretical status comparable to that of QED. In due course the precision electroweak experiments spectacularly confirmed the calculated radiative corrections, even yielding a remarkably accurate prediction of the top quark mass, based on its effect as a virtual particle......but note that even this part of the story is not yet over, since we have still not obtained experimental access to the proposed symmetry-breaking (Higgs [6]) sector! If and when we do, it will surely be a remarkable vindication of theoretical pre-occupations dating back to the early 1960's.

It seems fair to conclude that worrying about perceived imperfections of a theory, even a phenomenologically very successful one, can pay off. In the case of the SM, a quite serious imperfection (for many theorists) is the 'hierarchy problem', which we shall discuss in a moment. SUSY can provide a solution to this preceived problem, provided that SUSY partners to known particles have masses no larger than 1-10 TeV (roughly). A lot of work has been done on the phenomenology of SUSY, which has influenced LHC detector design. Once again, it will be extraordinary if, in fact, the world turns out to be this way.

In addition to this kind of motivation for SUSY, there are various other arguments which have been adduced. The rest of this section consists of a brief summary of the main reasons I could find why theorists are keen on SUSY.

1.1 The 'weak scale instability problem' - also known as the 'hierarchy problem'

The electroweak sector of the SM (see for example Aitchison and Hey [12]) contains within it a parameter with the dimensions of energy (i.e. a 'weak scale'), namely the vacuum expectation value of the Higgs field,

$$v \approx 246 \text{ GeV}. \tag{1}$$

This parameter sets the scale, in principle, of all masses in the theory. For example, the mass of the $W^{\pm}$ (neglecting radiative corrections) is given by

$$M_{\rm W} = gv/2 \sim 80\text{GeV}, \tag{2}$$

and the mass of the Higgs boson is

$$M_{\rm H} = v\sqrt{\frac{\lambda}{2}}, \tag{3}$$

where g is the SU(2) gauge coupling constant, and λ is the strength of the Higgs self-interaction in the Higgs potential

$$V = -\mu^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2, \tag{4}$$

where $\lambda > 0$ and $\mu^2 > 0$. Here ϕ is the SU(2) doublet field

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{5}$$

and all fields are understood to be quantum, no 'hat' being used.

Recall now that the *negative* sign of the 'mass^2' term $-\mu^2$ is essential for the spontaneous symmetry breaking mechanism to work. With the sign as in (4), the minimum of V interpreted as a classical potential is at the non-zero value

$$|\phi| = \sqrt{2}\mu/\sqrt{\lambda} \equiv v/\sqrt{2}, \tag{6}$$

where $\mu \equiv \sqrt{\mu^2}$. This classical minimum (equilibrium value) is conventionally interpreted as the expectation value of the quantum field in the quantum vacuum (i.e. the vev), at least at tree level. If '$-\mu^2$' in (4) is replaced by the positive quantity 'μ^2', the classical equilibrium value is at the origin in field space, which would imply $v = 0$ - in which case all particles would be massless. Hence it is vital to preserve the sign, and indeed magnitude, of the coefficient of $\phi^\dagger\phi$ in (4).

The discussion so far has been at tree level (no loops). What happens when we include loops? The SM is renormalizable, which means that finite results are obtained for all higher-order (loop) corrections, even if we extend the virtual momenta in the loop integrals all the way to infinity. But although this certainly implies that the theory is well-defined and calculable up to infinite energies, in practice no-one seriously believes that the SM is really all there is, however high we go in energy. That is to say, in loop integrals of the form

$$\int^\Lambda \mathrm{d}^4k \; f(k, \text{external momenta}) \tag{7}$$

we do not think that the cut-off Λ *should* go to infinity, physically, even though the reormalizability of the theory assures us that no inconsistency will arise if it does. More reasonably, we regard the SM as part of a larger theory which includes as yet unknown 'new physics' at high energy, Λ representing the scale at which this new physics appears, and where the SM must be modified. At the very least, for instance, there surely must be some kind of new physics at the scale when quantum gravity becomes important, which is believed to be indicated by the Planck mass

$$M_{\mathrm{P}} = (G_{\mathrm{N}})^{-1/2} \simeq 1.2 \times 10^{19} \text{ GeV}. \tag{8}$$

If this is indeed the scale of the new physics beyond the SM or, in fact, if there is *any* scale of 'new physics' even several orders of magnitude different from the scale set by v, then we shall see that we meet a problem with the SM, once we go beyond tree level.

The 4-boson self-interaction in (4) generates, at one-loop order, a contribution to the $\phi^\dagger\phi$ term, corresponding to the self-energy diagram of figure 1.1 which is proportional to

$$\lambda \int^\Lambda \mathrm{d}^4k \; \frac{1}{k^2 - M_{\mathrm{H}}^2}. \tag{9}$$

This integral clearly diverges quadratically (there are four powers of k in the numerator, and two in the denominator), and it turns out to be *positive*, producing a correction

$$\sim \lambda\Lambda^2\phi^\dagger\phi \tag{10}$$

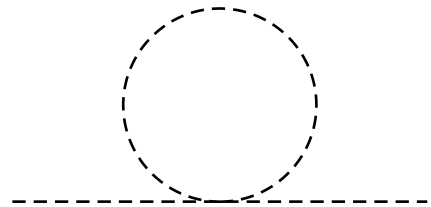

Figure 1: One-loop self-energy graph in ϕ^4 theory.

to the $-\mu^2\phi^\dagger\phi$ term in V. Now we know that the vev v is given in terms of μ by (6), and that its value is fixed phenomenologically by (1). Hence it seems that μ can hardly be much greater than of order a few hundred GeV (or, if it is, λ is much greater than unity - which would imply that we can't do perturbation theory; but since this is all we know how to do, in this problem, we proceed on the assumption that λ had better be in the perturbative regime). On the other hand, if $\Lambda \sim M_{\rm P} \sim 10^{19}$ GeV, the one-loop quantum correction to '$-\mu^2$' is then vastly greater than $\sim (100\ {\rm GeV})^2$, and positive, so that to arrive at a value $\sim -(100\ {\rm Gev})^2$ *after* inclusion of loop corrections would seem to require that we start with an equally huge but negative value of the Lagrangian parameter $-\mu^2$, relying on a remarkable cancellation to get us from $\sim -(10^{19}\ {\rm GeV})^2$ to $\sim -(10^2\ {\rm GeV})^2$.

We stress again, however, that this is *not* a problem if the SM is treated in isolation, with the cut-off Λ going to infinity. There is then no 'second scale' (Λ as well as v), and the Lagrangian parameter $-\mu^2$ can be chosen to depend on the cut-off Λ in just such a way that, when $\Lambda \to \infty$, the final (renormalized) coefficient of $\phi^\dagger\phi$ has the desired value. This is of course what happens to all ordinary mass terms in renormalizable theories.

This 'large cancellation' (or 'fine tuning') problem involving the parameter μ affects not only the mass of the Higgs particle, which is given in terms of μ (combining (3) and (6)) by

$$M_{\rm H} = \sqrt{2}\mu, \tag{11}$$

but also the mass of the W,

$$M_{\rm W} = g\mu/\sqrt{\lambda}, \tag{12}$$

and ultimately all masses in the SM, which derive from v and hence μ.

But wait a minute: haven't we just admitted that something like this *always* happens in mass terms of renormalizable theories? Why are we making a fuss about it now?

Actually, it is a problem which arises in a particularly acute way in theories which involve scalar particles in the Lagrangian - in contrast to theories with only fermions and gauge fields in the Lagrangian, but which are capable of producing scalar particles as some kind of bound states. An example of the latter kind of theory would be QED, for instance. Here the analogue of figure 1.1 would be the one-loop process in which an electron emits and then re-absorbs a photon. This produces a correction δm to the fermion mass m in the Lagrangian, which seems to vary with the cut-off as

$$\delta m \sim \alpha \int^{\Lambda} \frac{\mathrm{d}^4 k}{\not{k} k^2} \sim \alpha\Lambda. \tag{13}$$

(Here we have neglected both the external momentum and the fermion mass, in the fermion propagator, since we are interested in the large k behaviour.) In fact, however, when the calculation is done in detail one finds

$$\delta m \sim \alpha m \ln \Lambda, \tag{14}$$

so that even if $\Lambda \sim 10^{19}$ GeV, we have $\delta m \sim m$ and no unpleasant fine-tuning is necessary after all.

Why does it happen in this case that $\delta m \sim m$? It is because the Lagrangian for QED (and the SM for that matter) has a *symmetry* as the fermion masses go to zero, namely chiral symmetry. This is the symmetry under transformations (on fermion fields) of the form

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi \tag{15}$$

in the U(1) case, or

$$\psi \to \mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2\gamma_5}\psi \tag{16}$$

in the SU(2) case. This symmetry guarantees that all radiative corrections to m, computed in perturbation theory, will vanish as $m \to 0$. Hence δm must be proportional to m, and the dependence on Λ is therefore (from dimensional analysis) only logarithmic.

What about self-energy corrections to the masses of gauge particles? For QED it is of course the (unbroken) gauge symmetry which forces $m_\gamma = 0$, to all orders in perturbation theory. In other words, gauge invariance guarantees that no term of the form

$$m_\gamma^2 A^\mu A_\mu \tag{17}$$

can be radiatively generated in an unbroken gauge theory. On the other hand, the non-zero masses of the W and Z bosons in the SM arise non-perturbatively via spontaneous symmetry breaking, as we have seen - that is, via the Higgs vev v. If v is zero, the W and Z are as massless as the photon. But v is proportional to μ, and so the masses they acquire by symmetry breaking are as sensitive to Λ as M_{H} is.

Can we find a symmetry which would - in a sense similar to chiral symmetry or gauge symmetry - control δm^2 for a scalar particle appearing in a Lagrangian? Well, there are also fermion loop corrections to the $-\mu^2\phi^\dagger\phi$ term, in which a ϕ particle turns into a fermion-antifermion pair, which then annihilates back into a ϕ particle. At zero external momentum, such a contribution behaves as

$$\left(-g_{\mathrm{f}}^2 \int^\Lambda \mathrm{d}^4k \,\mathrm{Tr}\left[\frac{1}{(\not{k} - m_{\mathrm{f}})^2}\right]\right)\phi^\dagger\phi = \left(-4g_{\mathrm{f}}^4 \int^\Lambda \mathrm{d}^4k \,\frac{k^2 + m_{\mathrm{f}}^2}{(k^2 - m_{\mathrm{f}}^2)^2}\right)\phi^\dagger\phi. \tag{18}$$

The sign here is crucial, and comes from the closed fermion loop. The term with the k^2 in the numerator in (18) is quadratically divergent, and of opposite sign to the quadratic divergence (10) due to the Higgs loop. Ignoring numerical factors, these two contributions together have the form

$$(\lambda - g_{\mathrm{f}}^2)\Lambda^2\phi^\dagger\phi. \tag{19}$$

The possibility now arises that *if* for some reason there existed a boson-fermion coupling g_{f} related to the Higgs coupling by

$$g_{\mathrm{f}}^2 = \lambda \tag{20}$$

then this quadratic sensitivity to Λ would not occur. A relation between coupling constants, such as (20), is characteristic of a symmetry - but in this case it must evidently be a symmetry which relates a purely bosonic vertex to a boson-fermion (Yukawa) one. Relations of the form (20) are indeed just what occur in a SUSY theory, as we shall see in section 8. To implement this idea in the context of the (MS)SM, it will be necessary to postulate the existence of new fermionic 'superpartners' of the Higgs field - 'Higgsinos' - as discussed in sections 4 and 12.

But this will by no means deal with all the quadratic divergences present in the $-\mu^2\phi^\dagger\phi$ term. In principle, every SM fermion can play the role of 'f' in (18), since they all have a Yukawa coupling to the Higgs field. To cancel all these quadratic divergences will require the introduction of scalar superpartners for all the SM fermions - that is, an appropriate set of squarks and sleptons (see sections 4 and 12). There are also quadratic divergences associated with the contribution of gauge boson loops to the '$-\mu^2$' term, and these too will have to be cancelled by fermionic superpartners, 'gauginos'. In this way, the outlines of the MSSM are beginning to emerge.

After cancellation of the Λ^2 terms via (20), the next most divergent contributions to the '$-\mu^2$' term grow logarithmically with Λ; essentially, such a 'boson $\leftrightarrow$ fermion' symmetry gives the scalar masses 'protection' from quadratically divergent loop corrections, by virtue of being related by symmetry to the fermion masses, which are protected by chiral symmetry. But even terms logarithmic in the cut-off can be uncomfortably large. Consider a simple 'one Higgs - one new fermion' model. The $\ln\Lambda$ contribution to the '$-\mu^2$' term has the form

$$\sim \lambda(aM_{\rm H}^2 - bm_{\rm f}^2)\ln\Lambda \qquad (21)$$

where a and b are positive numerical factors of order unity. If this model were exactly supersymmetric, we would have $m_{\rm H} = m_{\rm f}$ and (21) would be of order $M_{\rm H}^2$ itself, and no fine tuning would be required. However, no superpartners for the SM particles have been discovered, so SUSY must be a broken symmetry (see section 15), with the masses of the superpartners presumably lying at too high values to be detected yet. In our simple model, this means that $M_{\rm H}^2 \neq m_{\rm f}^2$. But the (mass)^2 difference cannot be too large, or else a (possibly) smaller hierarchy problem threatens. In the realistic case of the (MS)SM, the coefficient of the $\ln\Lambda$ term will involve a sum of (mass)^2 terms for bosonic particles and, with the opposite sign, a sum of (mass)^2 terms for the fermionic particles. As far as the SM particles are concerned, the dominant terms are those from the particles with masses of order the Higgs vev v. We conclude that the new SUSY partners cannot be too much heavier than the scale of v (or $M_{\rm H}$), or we are back to some form of fine tuning. Of course, how much fine-tuning we are prepared to tolerate is a matter of taste.

Thus SUSY *stabilizes* the hierarchy $M_{\rm H,W} \ll M_{\rm P}$, in the sense that radiative corrections will not drag $M_{\rm H,W}$ up to the high scale Λ; and the argument implies that, for the desired stabilization to occur, SUSY should be visible at a scale not too much greater than 1-10 TeV. The origin of this latter scale (that of SUSY-breaking - see section 15.2) is a separate problem.

The reader should not get the impression that SUSY is the only available solution to the hierarchy problem. In fact, there are several others on offer. One, which has been around almost as long as SUSY, is generically called 'technicolour'. It proposes [7] [8] that the Higgs field is not 'elementary' but is analogous to the electron-pair state in the BCS theory of superconductivity, being a bound state of new doublets of massless quarks Q and anti-quarks $\bar{\mathrm{Q}}$ interacting via a new strongly interacting gauge theory, similar to QCD. In this case, the Lagrangian for the Higgs sector is only an effective theory, valid for energies significantly below the scale at which the $\mathrm{Q}-\bar{\mathrm{Q}}$ structure would be revealed - say 1 - 10 TeV. The integral in (9) can then only properly be extended to this scale, certainly not to a hierarchically different scale such as M_{P}. Essentially, new strong interactions enter not too far above the electroweak scale. A relatively recent review is provided by Lane [9]. A quite different possibility is to suggest that the gravitational (or string) scale is actually very much lower than (8) - perhaps even as low as a few TeV [10]. The hierarchy problem then evaporates since the ultaviolet cut-off Λ is not much higher than the weak scale itself. This miracle is worked by appealing to the notion of 'large' hidden extra dimensions, perhaps as large as sub-millimetre scales. This and other related ideas are discussed by Lykken [11], for example. Nevertheless, it is fair to say that SUSY, in the form of the MSSM, is at present the most highly developed framework for guiding and informing explorations of physics 'beyond the SM'.

1.2 Additional positive indications

(a) The precision fits to electroweak data show that M_{H} is less than about 200 GeV, at the 99% confidence level. The 'Minimal Supersymmetric Standard Model' (MSSM) (see section 12), which has two Higgs doublets, predicts (see section 16) that the lightest Higgs particle should be no heavier than about 140 GeV. In the SM, by contrast, we have *no* constraint on M_{H}.[1]

(b) At one-loop order, the inverse gauge couplings $\alpha_1^{-1}(Q^2), \alpha_2^{-1}(Q^2), \alpha_3^{-1}(Q^2)$ of the SM run linearly with $\ln Q^2$. Although α_1^{-1} decreases with Q^2, and α_2^{-1} and α_3^{-1} increase, all three tending to meet at high $Q^2 \sim (10^{16}\ \mathrm{GeV})^2$, they do not in fact meet convincingly in the SM. On the other hand, in the MSSM they do, thus encouraging ideas of unification: see section 13.

[1] Not in quite the same sense (i.e. of a mathematical bound), at any rate. One can certainly say, from (3), that if λ is not much greater than unity, so that perturbation theory has a hope of being applicable, then M_{H} can't be much greater than a few hundred GeV. For more sophisticated versions of this sort of argument, see [12] section 22.10.2.

(c) In any renormalizable theory, the mass parameters in the Lagrangian are also scale-dependent (they 'run'), just as the coupling parameters do. In the MSSM, the evolution of a Higgs $(\text{mass})^2$ parameter from a typical positive value of order v^2 at a scale of the order of 10^{16} GeV, takes it to a negative value of the correct order of magnitude at scales of order 100 GeV, thus providing a possible explanation for the origin of electroweak symmetry breaking, specifically at those much lower scales. Actually, however, this happens because the Yukawa coupling of the top quark is large (being proportional to its mass), and this has a dominant effect on the evolution. You might ask whether, in that case, the same result would be obtained without SUSY.The answer is that it would, but the initial conditions for the evolution are more naturally motivated within a SUSY theory, as discussed in section 15.

1.3 Theoretical considerations

It can certainly be plausibly argued that a dominant theme in twentieth century physics was that of symmetry, the pursuit of which was heuristically very successful. It is natural to ask if our current quantum field theories exploit all the kinds of symmetry which could exist, consistent with Lorentz invariance. Consider the symmetry 'charges' that we are familiar with in the SM, for example an electromagnetic charge of the form

$$Q = e \int \mathrm{d}^3\boldsymbol{x}\ \psi^\dagger \psi \tag{22}$$

or an SU(2) charge (isospin operator) of the form

$$\boldsymbol{T} = g \int \mathrm{d}^3\boldsymbol{x}\ \psi^\dagger (\boldsymbol{\tau}/2) \psi \tag{23}$$

where in (23) ψ is an SU(2) doublet, and in both (22) and (23) ψ is a fermionic field. All such symmetry operators are themselves Lorentz scalars (they carry no uncontracted Lorentz indices of any kind, for example vector or spinor). This implies that when they act on a state of definite spin J, they cannot alter that spin:

$$Q|J\rangle = |\text{ same } J, \text{ possibly different member of symmetry multiplet } \rangle. \tag{24}$$

Need this be the case?

We certainly know of one vector 'charge' - namely, the 4-momentum operators P_μ which generate space-time displacements, and whose eigenvalues are conserved 4-momenta. There are also the angular momentum operators, which belong inside an antisymmetric tensor $M_{\mu\nu}$. Could we, perhaps, have a conserved symmetric tensor charge $Q_{\mu\nu}$? We shall provide a highly simplified version (taken from Ellis [13]) of an argument due to Coleman and Mandula [14] which shows that we cannot. Consider letting such a charge act on a single particle state with 4-momentum p:

$$Q_{\mu\nu}|p\rangle = (\alpha p_\mu p_\nu + \beta g_{\mu\nu})|p\rangle, \tag{25}$$

where the RHS has been written down by 'covariance' arguments (i.e. the most general expression with the indicated tensor transformation character, built from the tensors at our disposal). Now consider a two-particle state $|p^{(1)}, p^{(2)}\rangle$, and assume the $Q_{\mu\nu}$'s are additive, conserved, and act on only one particle at a time, like other known charges. Then

$$Q_{\mu\nu}|p^{(1)}, p^{(2)}\rangle = (\alpha(p^{(1)}_\mu p^{(1)}_\nu + p^{(2)}_\mu p^{(2)}_\nu) + 2\beta g_{\mu\nu})|p^{(1)}, p^{(2)}\rangle. \tag{26}$$

In an elastic scattering process of the form $1 + 2 \to 3 + 4$ we will then need (from conservation of the eigenvalue)

$$p^{(1)}_\mu p^{(1)}_\nu + p^{(2)}_\mu p^{(2)}_\nu = p^{(3)}_\mu p^{(3)}_\nu + p^{(4)}_\mu p^{(4)}_\nu. \tag{27}$$

But we also have 4-momentum conservation:

$$p^{(1)}_\mu + p^{(2)}_\mu = p^{(3)}_\mu + p^{(4)}_\mu \tag{28}$$

The only common solution to (27) and (28) is

$$p^{(1)}_\mu = p^{(3)}_\mu, p^{(2)}_\mu = p^{(4)}_\mu, \text{ or } p^{(1)}_\mu = p^{(4)}_\mu, p^{(2)}_\mu = p^{(3)}_\mu, \tag{29}$$

which means that only forward or backward scattering can occur - which is obviously unacceptable.

The general message here is that there seems to be no room for further conserved operators with non-trivial Lorentz transformation character (i.e. not Lorentz scalars). The existing such operators P_μ and $M_{\mu\nu}$ do allow proper scattering processes to occur, but imposing any more conservation laws over-restricts the possible configurations. Such was the conclusion of the Coleman-Mandula theorem [14]. But in fact their argument turns out not to exclude 'charges' which transform under Lorentz transformations as *spinors*: that is to say, things transforming like a fermionic field ψ. We may denote such a charge by Q_a, the subscript a indicating the spinor component (we will see that we'll be dealing with 2-component spinors, rather than 4-component ones, for the most part). For such a charge, equation (24) will clearly not hold; rather,

$$Q_a|J\rangle = |J \pm 1/2\rangle. \tag{30}$$

Such an operator will not contribute to a matrix element for a two-particle $\rightarrow$ two-particle elastic scattering process (in which the particle spins remain the same), and consequently the above kind of 'no-go' argument can't get started.

The question then arises: is it possible to include such spinorial operators in a consistent algebraic scheme, along with the known conserved operators P_μ and $M_{\mu\nu}$? The affirmative answer was first given by Gol'fand and Likhtman [15], and the most general such 'supersymmetry algebra' was obtained by Haag, Lopuszanski and Sohnius [16]. By 'algebra' here we mean (as usual) the set of commutation relations among the 'charges' - which, we recall, are also the generators of the appropriate symmetry transformations. The SU(2) algebra of the angular momentum operators, which are generators of rotations, is a familiar example. The essential new feature here, however, is that the charges which have a spinor character will have *anticommutation* relations among themselves, rather than commutation relations. So such algebras involve some commutation relations and some anticommutation relations.

What will such algebras look like? Since our generic spinorial charge Q_a is a symmetry operator, it must commute with the Hamiltonian of the system, whatever it is:

$$[Q_a, H] = 0, \tag{31}$$

and so must the anticommutator of two different components:

$$[\{Q_a, Q_b\}, H] = 0. \tag{32}$$

As noted above, the spinorial Q's have two components, so as a and b vary the symmetric object $\{Q_a, Q_b\} = Q_aQ_b + Q_bQ_a$ has three independent components, and we suspect that it must transform as a spin-1 object (just like the symmetric combinations of two spin-1/2 wavefunctions). However, as usual in a relativistic theory, this spin-1 object should be described by a 4-vector, not a 3-vector. Further, this 4-vector is conserved, from (32). There is only one such conserved 4-vector operator (from the Coleman-Mandula theorem), namely P_μ. So the Q_a's must satisfy an algebra of the form, roughly,

$$\{Q_a, Q_b\} \sim P_\mu. \tag{33}$$

Clearly (33) is sloppy: the indices on each side don't balance. With more than a little hindsight, we might think of absorbing the '$_\mu$' by multiplying by γ^μ, the γ-matrix itself conveniently having two matrix indices which might correspond to a, b. This is in fact more or less right, as we shall see in section 5, but the precise details are finicky.

Accepting that (33) captures the essence of the matter, we can now begin to see what a radical idea supersymmetry really is. Equation (33) says, roughly speaking, that if you do two SUSY transformations generated by the Q's, one after the other, you get the energy-momentum operator. Or, to put it even more strikingly (but quite equivalently), you get the space-time translation operator, i.e. a derivative. Turning it around, the SUSY spinorial Q's are like square roots of 4-momentum, or square roots of derivatives! It is rather like going one better than the Dirac equation, which can be viewed as providing the square root of the Klein-Gordon equation: how would we take the square root of the Dirac equation?

It is worth pausing to take this in properly. Four-dimensional derivatives are firmly locked to our notions of a four-dimensional space-time. In now entertaining the possibility that we can take square roots of them, we are effectively extending our concept of space-time itself - just as, when the square root of -1 is introduced, we enlarge the real axis to the complex (Argand) plane. That is to say, if we take seriously an algebra involving both P_μ and the Q's, we shall have to say that the space-time co-ordinates are being extended to include further degrees of freedom, which are acted on by the Q's, and that these degrees of freedom are connected to the standard ones by means of transformations generated by the Q's. These further degrees of freedom are, in fact, fermionic. So we may say that SUSY invites us to contemplate 'fermionic dimensions', and enlarge space-time to 'superspace'.

For some reason this doesn't seem to be the thing usually most emphasized about SUSY. Rather, people talk much more about the fact that SUSY implies (if an exact symmetry) degenerate multiplets of bosons and fermions. Of course, that aspect is certainly true, and phenomenologically important, but the fermionic enlargement of space-time is arguably a more striking concept.

One final remark on motivations: if you believe in String Theory (and it still seems to be the only game in town that may provide a consistent quantum theory of gravity), then the phenomenologically most attractive versions incorporate supersymmetry, some trace of which might remain in the theories which effectively describe physics at presently accessible energies.

2 Spinors

Let's begin by recalling in outline how symmetries, such as SU(2), are described in quantum field theory (see for example chapter 12 of [12]). The Lagrangian involves a set of fields ψ_r - they could be bosons or fermions - and it is taken to be invariant under an infinitesimal transformation on the fields of the form

$$\delta_\epsilon \psi_r = -\mathrm{i}\epsilon\lambda_{rs}\psi_s, \tag{34}$$

where a summation is understood on the repeated index s, the λ_{rs} are certain constant coefficients (for instance, the elements of the Pauli matrices), and ϵ is an infinitesimal parameter. Supersymmetry transformations will look something like this, but they will transform bosonic fields into fermionic ones, for example

$$\delta_\xi \phi \sim \xi\psi \tag{35}$$

where ϕ is a bosonic (say spin-0) field, ψ is a fermionic (say spin-1/2) one, and ξ is an infinitesimal parameter (the alert reader will figure out that ξ too has to be a spinor). In due course we shall spell out the details of the '$\sim$' here, but one thing should already be clear at this stage: the number of (field) degrees of freedom, as between the bosonic ϕ fields and the fermionic ψ fields, had better be the same in an equation of the form (35), just as the number of fields $r = 1, 2, \ldots N$ on the LHS of (34) is the same as the number $s = 1, 2 \ldots N$ on the RHS. We can't have some fields being 'left out'. Now the simplest kind of bosonic field is of course a neutral scalar field, which has only one component, which is real: $\phi = \phi^\dagger$ (see [17] chapter 5). On the other hand, there is *no* fermionic field with just one component: being a spinor, it has at least two. So that means that we must consider, at the very least, a two-degree-of-freedom bosonic field, to go with the spinor field, and that takes us to a complex (charged) scalar field (see chapter 7 of [17]).[2]

[2] We have been a bit slipshod here, eliding 'components' with 'degrees of freedom'. In fact, each component of a two-component spinor is complex, so there are 4 degrees of

But what kind of a two-component fermionic field do we 'match' the complex scalar field with? When we learn the Dirac equation, pretty well the first result we arrive at is that fermion wavefunctions, or fields, have *four* components, not two. The simplest SUSY theory, however, involves a complex scalar field and a two-component fermionic field. The Dirac field actually uses *two* two-component fields, which is not the simplest case. Our first, and absolutely inescapable, job is therefore to 'deconstruct' the Dirac field, and understand the nature of the two different two-component fields which together constitute it.

This difference has to do with the different ways the two 'halves' of the 4-component Dirac field transform under Lorentz transformations. Understanding how this works, in detail, is vital to being able to write down SUSY transformations which are consistent with Lorentz invariance. For example, the LHS of (35) refers to a scalar (spin-0) field ϕ; admittedly it's complex, but that just means that it has a real part and an imaginary part, both of which have spin-0. So it is an invariant under Lorentz transformations. On the RHS, however, we have the 2-component spinor (spin-1/2) field ψ, which is certainly not invariant under Lorentz transformations. But the parameter ξ is also a 2-component spinor, in fact, and so we shall have to understand how to put ξ and ψ together properly so as to form a Lorentz invariant, in order to be consistent with the Lorentz transformation character of the LHS. While we may be familiar with how to do this sort of thing for 4-component Dirac spinors, we need to learn the corresponding tricks for 2-component ones. The rest of this lecture is therefore devoted to this groundwork.

2.1 Spinors and Lorentz Transformations

In many branches of theoretical physics, *specific notation* has been invented. There are many reasons for this, including the following (all of which of course assume that the notation has been perfectly mastered): it makes the formulae more compact and less of a drudgery to write out (take a look at Maxwell's original paper on Electromagnetism, written in 1864 before the invention of

freedom in a two-component spinor. If the spinor is assumed to be 'on-shell' - i.e. obeying the appropriate equation of motion - then the number of degrees of freedom reduces to 2, the same as a complex scalar field. But generally in field theory we need to go 'off-shell', so that to match the 4 degrees of freedom in a two-component spinor we shall actually need more bosonic degrees of freedom than just the 2 in a complex scalar field. We shall ignore this complication in our first foray into SUSY, in Section 3, but return to it in Section 7.

vector calculus); it can guarantee, essentially as an automatic consequence of writing a well-formed equation, that it incorporates some desired properties (for example, 4-vectors in Special Relativity, the use of which automatically incorporates the requirement of Lorentz covariance); and a well-conceived notation lends itself to advantageous steps in manipulations (for example, taking dot or cross products in equations involving vectors). Supersymmetry is no exception, there being plenty of specific notation available for things like spinors. The only problem is, that it has not yet been standardized. This is very off-putting to beginners in the subject, because almost all the introductory articles or books they pick up will use notation which is, to a greater or lesser extent, special to that source, making comparisons very frustrating. By contrast, we shall in these lectures not make much use of special SUSY notation. Rather, we shall aim to use a notation with which we assume the reader is familiar - namely, that used in standard relativistic quantum mechanics courses which deal with the Dirac equation. The advantage of this strategy is that the student doesn't, as a first task, have to learn a quite tricky new notation, and can gain access to the subject directly on the basis of standard courses. There will eventually be a price to pay, in the cumbersome nature of some expressions and manipulations, which could be streamlined using professional SUSY notation. And after all, students have, in the end, got to be able to read SUSY formulae when written in the (quasi-)standard notation. So as we progress we'll introduce more specific notation.

We begin with the Dirac equation in momentum space, which we write as

$$E\Psi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\Psi \tag{36}$$

where of course we are taking $c = \hbar = 1$. We shall choose the particular reprsentation

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{37}$$

which implies that

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{38}$$

This is one of the standard representations of the Dirac matrices (see for example [17] page 91, and [12] pages 31-2, and particularly [12] Appendix M, section M.6). It is the one which is commonly used in the 'small mass' or 'high energy' limit, since the (large) momentum term is then (block) diagonal. As usual, $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the 2×2 Pauli matrices.

We write

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}. \tag{39}$$

The Dirac equation is then

$$\begin{aligned} (E - \boldsymbol{\sigma} \cdot \boldsymbol{p})\psi &= m\chi & (40) \\ (E + \boldsymbol{\sigma} \cdot \boldsymbol{p})\chi &= m\psi. & (41) \end{aligned}$$

Notice that as $m \to 0$, (40) becomes $\boldsymbol{\sigma} \cdot \boldsymbol{p}\psi_0 = E\psi_0$, and $E \to |\boldsymbol{p}|$, so the zero mass limit of (40) is

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|)\psi_0 = \psi_0, \tag{42}$$

which means that ψ_0 is an eigenstate of the helicity operator $\boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|$ with eigenvalue +1 ('positive helicity'). Similarly, the zero-mass limit of (41) shows that χ_0 has negative helicity.

For $m \neq 0$, ψ and χ of (40) and (41) are plainly not helicity eigenstates: indeed the mass term (in this representation) 'mixes' them. But, as we shall see shortly, it is these two-component objects, ψ and χ, that have well-defined Lorentz transformation properties, and they are the two-component spinors we shall be dealing with.

Although not helicity eigenstates, ψ and χ are eigenstates of γ_5, in the sense that

$$\gamma_5 \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 \begin{pmatrix} 0 \\ \chi \end{pmatrix} = -\begin{pmatrix} 0 \\ \chi \end{pmatrix}. \tag{43}$$

These two γ_5-eigenstates can be constructed from the original Ψ by using the projection operators P_{R} and P_{L} defined by

$$P_{\mathrm{R}} = \left(\frac{1+\gamma_5}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{44}$$

and

$$P_{\mathrm{L}} = \left(\frac{1-\gamma_5}{2}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{45}$$

Then

$$P_{\mathrm{R}}\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad P_{\mathrm{L}}\Psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \tag{46}$$

It is easy to check that $P_{\rm R}P_{\rm L} = 0$, $P_{\rm R}^2 = P_{\rm L}^2 = 1$. The eigenvalue of γ_5 is called 'chirality'; ψ has chirality +1, and χ has chirality -1. In an unfortunate terminolgy, but one now too late to change, '+' chirality is denoted by 'R' (i.e right-handed) and '-' chirality by 'L' (i.e. left-handed), despite the fact that (as noted above) ψ and χ are *not* helicity eigenstates when $m \neq 0$. Anyway, a 'ψ' type 2-component spinor is often written as $\psi_{\rm R}$, and a 'χ' type one as $\chi_{\rm L}$. For the moment, we shall not use these R and L subscripts, but shall understand that anything called ψ is an R state, and a χ is an L state.

Now, we said above that ψ and χ had well-defined Lorentz transformation character. Let's recall how this goes (see [12] Appendix M, section M.6). There are basically two kinds of transformation: rotations and 'boosts' (i.e. pure velocity transformations). It is sufficient to consider *infinitesimal* transformations, which we can specify by their action on a 4-vector, for example the energy-momentum 4-vector $(E, \boldsymbol{p})$. Under an infinitesimal 3-dimensional rotation,

$$E \to E' = E, \quad \boldsymbol{p} \to \boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{\epsilon} \times \boldsymbol{p} \tag{47}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$ are three infinitesimal parameters specifying the infinitesimal rotation; and under a velocity transformation

$$E \to E' = E - \boldsymbol{\eta} \cdot \boldsymbol{p}, \quad \boldsymbol{p} \to \boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{\eta} E \tag{48}$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ are three infinitesimal velocities. Under the Lorentz transformations thus defined, ψ and χ transform as follows (see equations (M.94) and (M.98) of [12], where however the top two components are called 'ϕ' rather than 'ψ'):

$$\psi \to \psi' = (1 + {\rm i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}/2 - \boldsymbol{\eta} \cdot \boldsymbol{\sigma}/2)\psi \tag{49}$$

and

$$\chi \to \chi' = (1 + {\rm i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}/2 + \boldsymbol{\eta} \cdot \boldsymbol{\sigma}/2)\chi. \tag{50}$$

Equations (49) and (50) are extremely important equations for us. They tell us how to construct the spinors ψ' and χ' for the rotated and boosted frame, in terms of the original spinors ψ and χ. That is to say, the ψ' and χ' specified by (49) and (50) satisfy the 'primed' analogues of (40) and (41), namely

$$(E' - \boldsymbol{\sigma} \cdot \boldsymbol{p}')\psi' = m\chi' \tag{51}$$

$$(E' + \boldsymbol{\sigma} \cdot \boldsymbol{p}')\chi' = m\psi'. \tag{52}$$

Let's pause to check this statement in a special case, that of a pure boost. Define $V_{\boldsymbol{\eta}} = (1 - \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2)$. Then since $\boldsymbol{\eta}$ is infinitesimal, $V_{\boldsymbol{\eta}}^{-1} = (1 + \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2)$. Now take (40), multiply from the left by $V_{\boldsymbol{\eta}}^{-1}$, and insert the unit matrix $V_{\boldsymbol{\eta}}^{-1}V_{\boldsymbol{\eta}}$ as indicated:

$$[V_{\boldsymbol{\eta}}^{-1}(E - \boldsymbol{\sigma}\cdot\boldsymbol{p})V_{\boldsymbol{\eta}}^{-1}]V_{\boldsymbol{\eta}}\psi = mV_{\boldsymbol{\eta}}^{-1}\chi. \tag{53}$$

If (49) is right, we have $\psi' = V_{\boldsymbol{\eta}}\psi$, and if (50) is right we have $\chi' = V_{\boldsymbol{\eta}}^{-1}\chi$, in this pure boost case. So to establish the complete consistency between (49), (50) and (51), we need to show that

$$V_{\boldsymbol{\eta}}^{-1}(E - \boldsymbol{\sigma}\cdot\boldsymbol{p})V_{\boldsymbol{\eta}}^{-1} = (E' - \boldsymbol{\sigma}\cdot\boldsymbol{p}'), \tag{54}$$

that is

$$(1 + \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2)(E - \boldsymbol{\sigma}\cdot\boldsymbol{p})(1 + \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2) = (E - \boldsymbol{\eta}\cdot\boldsymbol{p}) - \boldsymbol{\sigma}\cdot(\boldsymbol{p} - E\boldsymbol{\eta}) \tag{55}$$

to first order in $\boldsymbol{\eta}$, since the RHS of (55) is just $E' - \boldsymbol{\sigma}\cdot\boldsymbol{p}'$ from (48).

Exercise Verify (55).

Returning now to equations (49) and (50), we note that ψ and χ actually behave the same under rotations (they have spin-1/2!), but differently under boosts. The interesting fact is that there are *two* kinds of two-component spinors, distinguished by their different transformation character under boosts. Both are used in the Dirac 4-component spinor. In SUSY, however, one works with the 2-component objects ψ and χ which (as we saw above) may also be labelled by 'R' and 'L' respectively.

Before proceeding, we note another important feature of (49) and (50). Let V be the transformation matrix appearing in (49):

$$V = (1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2). \tag{56}$$

Then

$$V^{-1} = (1 - \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}/2 + \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2) \tag{57}$$

since we merely have to reverse the sense of the infinitesimal parameters, while

$$V^{\dagger} = (1 - \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2) \tag{58}$$

using the Hermiticity of the $\boldsymbol{\sigma}$'s. So

$$V^{\dagger^{-1}} = V^{-1^{\dagger}} = (1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}/2 + \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2), \tag{59}$$

which is the matrix appearing in (50). Hence we may write, compactly,

$$\psi' = V\psi, \quad \chi' = V^{\dagger^{-1}}\chi = V^{-1^{\dagger}}\chi. \tag{60}$$

2.2 Constructing invariants and 4-vectors out of 2-component spinors

Let's start by recalling some things which should be familiar from a Dirac equation course. From the 4-component Dirac spinor we can form a Lorentz invariant

$$\bar{\Psi}\Psi = \Psi^\dagger\beta\Psi, \tag{61}$$

and a 4-vector

$$\bar{\Psi}\gamma^\mu\Psi = \Psi^\dagger\beta(\beta, \beta\boldsymbol{\alpha})\Psi = \Psi^\dagger(1, \boldsymbol{\alpha})\Psi. \tag{62}$$

In terms of our 2-component objects ψ and χ (61) becomes

$$\text{Lorentz invariant} \qquad (\psi^\dagger \chi^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \psi^\dagger\chi + \chi^\dagger\psi. \tag{63}$$

Using (60) it is easy to verify that the RHS of (63) is invariant. Indeed, perhaps more interestingly, each part of it is:

$$\psi^\dagger\chi \to \psi^{\dagger\prime}\chi' = \psi V^\dagger V^{\dagger^{-1}}\chi = \psi^\dagger\chi, \tag{64}$$

and similarly for $\chi^\dagger\psi$. Again, (62) becomes

$$\begin{aligned} 4-\text{vector} \quad (\psi^\dagger\chi^\dagger)\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}\right]\begin{pmatrix} \psi \\ \chi \end{pmatrix} &= (\psi^\dagger\psi + \chi^\dagger\chi, \psi^\dagger\boldsymbol{\sigma}\psi - \chi^\dagger\boldsymbol{\sigma}\chi) \\ &\equiv \psi^\dagger\sigma^\mu\psi + \chi^\dagger\bar{\sigma}^\mu\chi, \end{aligned} \tag{65}$$

where we have introduced the important quantities

$$\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \qquad \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}). \tag{66}$$

As with the Lorentz invariant, it is actually the case that each of $\psi^\dagger\sigma^\mu\psi$ and $\chi^\dagger\bar{\sigma}^\mu\chi$ transforms, separately, as a 4-vector.

Exercise Verify that last statement.

In this 'σ^μ, $\bar{\sigma}^\mu$' notation, the Dirac equation (40) and (41) becomes

$$\sigma^\mu p_\mu \psi = m\chi \tag{67}$$

$$\bar{\sigma}^\mu p_\mu \chi = m\psi. \tag{68}$$

So we can read off the useful news that '$\sigma^\mu p_\mu$' converts a ψ-type object to a χ-type one, and $\bar{\sigma}^\mu p_\mu$ converts a χ to a ψ - or, in slightly more proper language, the Lorentz transformation character of $\sigma^\mu p_\mu \psi$ is the same as that of χ, and the LT character of $\bar{\sigma}^\mu p_\mu \chi$ is the same as that of ψ.

Lastly in this re-play of Dirac stuff, the Dirac Lagrangian can be written in terms of ψ and χ:

$$\bar{\Psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\Psi = \psi^\dagger \mathrm{i}\sigma^\mu \partial_\mu \psi + \chi^\dagger \mathrm{i}\bar{\sigma}^\mu \partial_\mu \chi - m(\psi^\dagger \chi + \chi^\dagger \psi). \tag{69}$$

Note how $\bar{\sigma}^\mu$ belongs with χ, and σ^μ with ψ.

An interesting point may have occurred to the reader here: it is possible to form 4-vectors using only ψ's or only χ's (see the most recent Exercise), but the invariants introduced so far ($\psi^\dagger \chi$ and $\chi^\dagger \psi$) make use of both. So we might ask: *can we make an invariant out of just χ- type spinors, for instance?* This is an important technicality as far as SUSY is concerned, and it is at this point that we part company with what is usually contained in standard Dirac courses.

Another way of putting our question is this: is it possible to construct a spinor from the components of, say, χ, which has the transformation character of a ψ? (and of course vice versa). If it is, then we can use it, with χ-type spinors, in place of ψ-type spinors when making invariants. The answer is that it is possible. Consider how the complex conjugate of χ, denoted by χ^*, transforms under Lorentz transformations. We have

$$\chi' = (1 + \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}/2 + \boldsymbol{\eta} \cdot \boldsymbol{\sigma}/2)\chi. \tag{70}$$

Taking the complex conjugate gives

$$\chi^{*\prime} = (1 - \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}^*/2 + \boldsymbol{\eta} \cdot \boldsymbol{\sigma}^*/2)\chi^*. \tag{71}$$

Now observe that $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$, $\sigma_3^* = \sigma_3$, and that $\sigma_2\sigma_3 = -\sigma_3\sigma_2$ and $\sigma_1\sigma_2 = -\sigma_2\sigma_1$. It follows that

$$\begin{aligned} \sigma_2\chi^{*\prime} &= \sigma_2(1 - \mathrm{i}\boldsymbol{\epsilon} \cdot (\sigma_1, -\sigma_2, \sigma_3)/2 + \boldsymbol{\eta} \cdot (\sigma_1, -\sigma_2, \sigma_3)/2)\chi^* & (72) \\ &= (1 + \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}/2 - \boldsymbol{\eta} \cdot \boldsymbol{\sigma}/2)\sigma_2\chi^* & (73) \\ &= V\sigma_2\chi^*, & (74) \end{aligned}$$

referring to (56) for the definition of V, which is precisely the matrix by which ψ transforms.

We have therefore established the important result that

$$\sigma_2\chi^* \text{ transforms like a } \psi. \tag{75}$$

So let's at once introduce 'the ψ-like thing constructed from χ' via the definition

$$\psi_\chi \equiv \mathrm{i}\sigma_2\chi^*, \tag{76}$$

where the i has been put in for convenience (remember σ_2 involves i's). Then we are guaranteed that

$$\psi^\dagger_{\chi^{(1)}}\chi^{(2)} = \left(\mathrm{i}\sigma_2\chi^{(1)*}\right)^{*\mathrm{T}}\chi^{(2)} = \left(\mathrm{i}\sigma_2\chi^{(1)}\right)^{\mathrm{T}}\chi^{(2)} = \chi^{(1)\mathrm{T}}(-\mathrm{i}\sigma_2)\chi^{(2)} \tag{77}$$

where $^{\mathrm{T}}$ denotes transpose, is Lorentz invariant, for any two χ-like things $\chi^{(1)}, \chi^{(2)}$, just as $\psi^\dagger\chi$ was. (Equally, so is $\chi^{(2)\dagger}\psi_{\chi^{(1)}}$.) Equation (77) is important, because it tells us *how to form the Lorentz invariant scalar product of two* χ's. This is the kind of product that we will need in SUSY transformations of the form (35).

In particular, $\psi^\dagger_\chi\chi$ is Lorentz invariant, where the χ's are the same. This quantity is

$$(\mathrm{i}\sigma_2\chi^*)^{*\mathrm{T}}\chi = (\mathrm{i}\sigma_2\chi)^{\mathrm{T}}\chi = \chi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi. \tag{78}$$

Let's write it out in detail. We have

$$\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{79}$$

so that

$$\mathrm{i}\sigma_2\chi = \begin{pmatrix} \chi_2 \\ -\chi_1 \end{pmatrix}, \quad \text{and} \quad (\mathrm{i}\sigma_2\chi)^{\mathrm{T}}\chi = \chi_2\chi_1 - \chi_1\chi_2. \tag{80}$$

But now this seems like something of an anti-climax! It vanishes, doesn't it? Well, yes if χ_1 and χ_2 are ordinary functions, but not if they are *anticommuting* quantities, as appear in (quantized) fermionic fields. So certainly this is a satisfactory invariant in terms of two-component quantized fields, or in terms of Grassmann numbers (see Appendix O of [12]).

Notational Aside (1). It looks as if it's going to get pretty tedious keeping track of which two-component spinor is a χ-type one and which is ψ-type one, by writing things like $\chi^{(1)}, \chi^{(2)}, \ldots, \psi^{(1)}, \psi^{(2)}, \ldots$, all the time, and (even worse) things like $\psi^\dagger_{\chi^{(1)}}\chi^{(2)}$. A first step in the direction of a more powerful notation is to agree that the components of χ-type spinors have *lower indices*, as in (79). That is, anything written with lower indices is a χ-type spinor. So then we are free to name them how we please: $\chi_a, \xi_a, \ldots$, even ψ_a.

We can also streamline the cumbersome notation '$\psi_{\chi^{(1)}}{}^\dagger\chi^{(2)}$'. The point here is that this notation was - at this stage - introduced in order to construct invariants out of just χ-type things. But (77) tells us how to do this, in terms of the two χ's involved: you take one of them, say $\chi^{(1)}$, and form $\mathrm{i}\sigma_2\chi^{(1)}$. Then you take the matrix dot product (in the sense of '$u^{\mathrm{T}}v$') of this quantity and the second χ-type spinor. So, starting from a χ with lower indices, χ_a, let's define a χ with *upper indices* via (see equation (80))

$$\begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} \equiv \mathrm{i}\sigma_2\chi = \begin{pmatrix} \chi_2 \\ -\chi_1 \end{pmatrix}, \tag{81}$$

that is,

$$\chi^1 \equiv \chi_2, \quad \chi^2 \equiv -\chi_1. \tag{82}$$

Suppose now that ξ is a second χ-type spinor, and

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{83}$$

Then we know that $(\mathrm{i}\sigma_2\chi)^{\mathrm{T}}\xi$ is a Lorentz invariant, and this is just

$$(\chi^1\chi^2)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \chi^1\xi_1 + \chi^2\xi_2 = \chi^a\xi_a, \tag{84}$$

where a runs over the values 1 and 2. Equation (84) is a compact notation for this scalar product: it is a 'spinor dot product', analogous to the 'upstairs-downstairs' dot-products of Special Relativity, like $A^\mu B_\mu$. We can shorten the notation even further, indeed, to $\chi\cdot\xi$, or even to $\chi\xi$ if we know what we are doing. Note that if the components of χ and ξ commute, then it doesn't matter whether we write this invariant as $\chi\cdot\xi = \chi^1\xi_1 + \chi^2\xi_2$ or as $\xi_1\chi^1 + \xi_2\chi^2$. But if they are anticommuting these will differ by a sign, and we need a convention as to which we take to be the 'positive' dot product. It is as in (84), which is remembered as 'summed-over χ-type indices appear diagonally downwards, top left to bottom right'.

The 4-D Lorewntz-invariant dot product $A^\mu B_\nu$ of Special Relativity can also be written as $g^{\mu\nu}A_\nu B_\mu$, where $g^{\mu\nu}$ is the *metric tensor* of SR with components (in one common convention!) $g^{00} = +1, g^{11} = g^{22} = g^{33} = -1$, all others vanishing (see Appendix D of [12]). In a similar way we can introduce a metric tensor ϵ^{ab} for forming the Lorentz-invariant spinor dot product of two two-component L-type spinors, by writing

$$\chi^a = \epsilon^{ab}\chi_b \tag{85}$$

(always summing on repeated indices, of course), so that

$$\chi^a\xi_a = \epsilon^{ab}\chi_b\xi_a. \tag{86}$$

For (85) to be consistent with (82), we require

$$\epsilon^{12} = +1, \epsilon^{21} = -1, \epsilon^{11} = \epsilon^{22} = 0. \tag{87}$$

Clearly ϵ^{ab}, regarded as a 2×2 matrix, is nothing but the matrix $\mathrm{i}\sigma_2$ of (79). We shall, however, continue to use the explicit '$\mathrm{i}\sigma_2$' notation for the most part, rather than the 'ϵ^{ab}' notation.

We can also introduce ϵ_{ab} via

$$\chi_a = \epsilon_{ab}\chi^b, \tag{88}$$

which is consistent with (82) if

$$\epsilon_{12} = -1, \epsilon_{21} = +1, \epsilon_{11} = \epsilon_{22} = 0. \tag{89}$$

Finally, you can verify that

$$\epsilon_{ab}\epsilon^{bc} = \delta^c_a, \tag{90}$$

as one would expect. It is important to note that these 'ϵ' metrics are *antisymmetric* under the interchange of the two indices a and b, whereas the SR metric $g^{\mu\nu}$ is symmetric under $\mu \leftrightarrow \nu$.

Exercise (a) What is $\xi \cdot \chi$ in terms of $\chi \cdot \xi$ (assuming the components anticommute)? (b) What is $\chi_a \xi^a$ in terms of $\chi^a \xi_a$? Do these both by brute force via components, and by using the ϵ dot product.

Given that χ transforms by $V^{-1\dagger}$ of (59), it is interesting to ask: how does the 'raised-index' version, $\mathrm{i}\sigma_2\chi$, transform?

Exercise Show that $\mathrm{i}\sigma_2\chi$ transforms by V^*.

We can use the result of this Exercise to verify once more the invariance of $(\mathrm{i}\sigma_2\chi)^{\mathrm{T}}\xi$: $(\mathrm{i}\sigma_2\chi)^{\mathrm{T}}\xi \to (\mathrm{i}\sigma_2\chi)'^{\mathrm{T}}\xi' = (\mathrm{i}\sigma_2\chi)^{\mathrm{T}}(V^*)^{\mathrm{T}}V^{-1\dagger}\xi$. But $(V^*)^{\mathrm{T}} = V^\dagger$, and so the invariance is established.

We can therefore summarize the state of play so far by saying that a downstairs χ-type spinor transforms by $V^{-1\dagger}$, while an upstairs χ-type spinor transforms by V^*.

It is natural to ask: what about ψ^*? Performing manipulations analogous to those in (71), (72)-(74), you can verify that

$$\sigma_2\psi^* \quad \text{transforms like} \quad \chi. \tag{91}$$

This licenses us to introduce a χ-type object constructed from a ψ, which we define by

$$\chi_\psi \equiv -\mathrm{i}\sigma_2\psi^*. \tag{92}$$

Then for any two ψ's $\psi^{(1)}, \psi^{(2)}$ say, we know that

$$(-\mathrm{i}\sigma_2\psi^{(1)*})^{*\mathrm{T}}\psi^{(2)} = (-\mathrm{i}\sigma_2\psi^{(1)})^{\mathrm{T}}\psi^{(2)} = \psi^{(1)\mathrm{T}}\mathrm{i}\sigma_2\psi^{(2)} \tag{93}$$

is an invariant. In particular, for the same ψ, the quantity

$$(-\mathrm{i}\sigma_2\psi)^{\mathrm{T}}\psi \tag{94}$$

is an invariant.

Notational Aside (2). Clearly we want an 'index' notation for ψ-type spinors. The general convention is that they are given 'dotted indices' i.e. we write things like $\psi^{\dot{a}}$. By convention, also, we decide that our ψ-type thing has an *upstairs* index, just as it was a convention that our χ-type thing had a downstairs index. Equation (93) tells us how to form scalar products out of two ψ-like things, $\psi^{(1)}$ and $\psi^{(2)}$, and invites us to define downstairs-indexed quantities

$$\begin{pmatrix} \psi_{\dot{1}} \\ \psi_{\dot{2}} \end{pmatrix} \equiv -\mathrm{i}\sigma_2\psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi^{\dot{1}} \\ \psi^{\dot{2}} \end{pmatrix} \tag{95}$$

so that

$$\psi_{\dot{1}} \equiv -\psi^{\dot{2}}, \quad \psi_{\dot{2}} \equiv \psi^{\dot{1}}. \tag{96}$$

Then if ζ ('zeta') is a second ψ-type spinor, and

$$\zeta = \begin{pmatrix} \zeta^{\dot{1}} \\ \zeta^{\dot{2}} \end{pmatrix}, \tag{97}$$

we know that $(-\mathrm{i}\sigma_2\psi)^{\mathrm{T}}\zeta$ is a Lorentz invariant, which is

$$(\psi_{\dot{1}}\psi_{\dot{2}}) \begin{pmatrix} \zeta^{\dot{1}} \\ \zeta^{\dot{2}} \end{pmatrix} = \psi_{\dot{1}}\zeta^{\dot{1}} + \psi_{\dot{2}}\zeta^{\dot{2}} = \psi_{\dot{a}}\zeta^{\dot{a}}, \tag{98}$$

where $\dot{a}$ runs over the values 1,2. Notice that with all these conventions, the 'positive' scalar product has been defined so that the summed-over dotted indices appear diagonally upwards, bottom left to top right.

We can introduce a metric tensor notation for the Lorentz-invariant scalar product of two two-component R-type (dotted) spinors, too. We write

$$\psi_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\psi^{\dot{b}} \tag{99}$$

where, to be consistent with (96), we need

$$\epsilon_{\dot{1}\dot{2}} = -1, \epsilon_{\dot{2}\dot{1}} = +1, \epsilon_{\dot{1}\dot{1}} = \epsilon_{\dot{2}\dot{2}} = 0. \tag{100}$$

Then

$$\psi_{\dot{a}}\zeta^{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\psi^{\dot{b}}\zeta^{\dot{a}}. \tag{101}$$

We can also define

$$\epsilon^{\dot{1}\dot{2}} = +1, \epsilon^{\dot{2}\dot{1}} = -1, \epsilon^{\dot{1}\dot{1}} = \epsilon^{\dot{2}\dot{2}} = 0, \tag{102}$$

with

$$\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{b}\dot{c}} = \delta_{\dot{b}}^{\dot{c}}. \tag{103}$$

Again, the ϵs with dotted indices are antisymmetric under interchange of their indices.

We could of course think of shortening (98) further to $\psi \cdot \zeta$ or $\psi\zeta$, but without the dotted indices to tell us, we wouldn't in general know whether such expressions referred to what we have been calling ψ- or χ-type spinors. So it is common to find people using a '$^{-}$' notation for ψ-type spinors. Then (98) would be just $\bar{\psi}\bar{\zeta}$.

As in the previous Aside, we can ask how (in terms of V) the downstairs dotted spinor $-\mathrm{i}\sigma_2\psi$ transforms.

Exercise Show that $-\mathrm{i}\sigma_2\psi$ transforms by $V^{-1\mathrm{T}}$, and hence verify once again that $(-\mathrm{i}\sigma_2\psi)^{\mathrm{T}}\zeta$ is invariant.

So altogether we have arrived at four types of two-component spinor: upstairs and downstairs χ-type, which transform by V^* and $V^{-1\dagger}$ respectively; and upstairs and downstairs ψ-type which transform by V and $V^{-1\mathrm{T}}$ respectively. The essential point is that invariants are formed by taking the matrix dot product between one quantity transforming by M say, and another transforming by $M^{-1\mathrm{T}}$.

In the notation of this and the previous Aside, then, the familiar Dirac 4-component spinor (39) would be written as

$$\Psi = \begin{pmatrix} \psi^{\dot{a}} \\ \chi_a \end{pmatrix}. \tag{104}$$

The conventions of different authors typically do not agree here. As far as I can tell, the notation I am using is the same as that of Shifman [18], see his equation (68) on page 335. Other authors, for example Bailin and Love [19], use a choice for the Dirac matrices which is different from (37) and (38), and which has the effect of interchanging the position, in Ψ, of the L (undotted) and R (dotted) parts - which, furthermore, they call 'ψ' and 'χ' respectively, the opposite way round from us - so that for them

$$\Psi_{\mathrm{D}} = \begin{pmatrix} \psi_a \\ \chi^{\dot{a}} \end{pmatrix}. \tag{105}$$

Bailin and Love also employ the '$^-$ ' notation, so that

$$\Psi_{\mathrm{BL}} = \begin{pmatrix} \psi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix}. \tag{106}$$

Note particularly, however, that this 'bar' has nothing to do with the 'bar' used in 4-component Dirac theory, as in (61). Also, BL's ϵ symbols, and hence their spinor scalar products, have the opposite sign from ours.

2.3 Majorana fermions

We stated in (92) that $\chi_\psi \equiv \mathrm{i}\sigma_2\psi^*$ transforms like a χ-type object. It follows that it should be perfectly consistent with Dirac theory to assemble ψ and χ_ψ into a 4-component object:

$$\Psi_{\mathrm{M}}^{\psi} = \begin{pmatrix} \psi \\ -\mathrm{i}\sigma_2\psi^* \end{pmatrix}. \tag{107}$$

This must behave under Lorentz transformations just like an 'ordinary' Dirac 4-component object Ψ built from a ψ and a χ. But Ψ_{M}^{ψ} of (107) has *fewer degrees of freedom* than an ordinary Dirac 4-component spinor Ψ, since it is fully determined by the 2-component object ψ. In a Dirac spinor Ψ involving a ψ and a χ, as in (39), there are two 2-component spinors, each of which is specified by 4 real quantities (each has two complex components), making 8 in all. In $\Psi_{\rm M}^{\psi}$, by contrast, there are only 4 real quantities, contained in the single spinor ψ.

What this means physically becomes clearer when we consider the operation of *charge conjugation*. On a Dirac 4-component spinor, this is defined by

$$\Psi_{\mathbf{C}} = C_0 \Psi^* \tag{108}$$

where[3]

$$C_0 = -\mathrm{i}\gamma^2 = \begin{pmatrix} 0 & \mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & 0 \end{pmatrix}. \tag{109}$$

Then

$$\Psi_{\rm M,\mathbf{C}}^{\psi} = \begin{pmatrix} 0 & \mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \psi^* \\ -\mathrm{i}\sigma_2\psi \end{pmatrix} = \begin{pmatrix} \psi \\ -\mathrm{i}\sigma_2\psi^* \end{pmatrix} = \Psi_{\rm M}^{\psi}. \tag{110}$$

So $\Psi_{\rm M}^{\psi}$ describes a spin-1/2 particle which is even under charge-conjugation - that is, it is its own antiparticle. Such a particle is called a Majorana fermion.

This charge-self-conjugate property is clearly the physical reason for the difference in the number of degrees of freedom in $\Psi_{\rm M}^{\psi}$ as compared with Ψ of (36). There are 4 physically distinguishable modes in a Dirac field, for example $\mathrm{e}_{\rm L}^-, \mathrm{e}_{\rm R}^-, \mathrm{e}_{\rm L}^+, \mathrm{e}_{\rm R}^+$. But in a Majorana field one there are only two, the antiparticle being the same as the particle; for example $\nu_{\rm L}, \nu_{\rm R}$ - supposing, as is possible (see [12] section 20.6), that neutrinos are Majorana particles.

We could also construct

$$\Psi_{\rm M}^{\chi} = \begin{pmatrix} \mathrm{i}\sigma_2\chi^* \\ \chi \end{pmatrix} \tag{111}$$

which also satisfies

$$\Psi_{\rm M,\mathbf{C}}^{\chi} = \Psi_{\rm M}^{\chi}. \tag{112}$$

Clearly a formalism using χ's only is equivalent to one using $\Psi_{\rm M}^{\chi}$'s only, and one using ψ's is equivalent to one using $\Psi_{\rm M}^{\psi}$'s. A mass term of the form

[3]This choice of C_0 has the opposite sign from the one in equation (20.63) of [12] page 290; the present choice is more in conformity with SUSY conventions. We are sticking to the convention that the indices of the γ- matrices as defined in (38) appear upstairs; no significance should be attached to the position of the indices of the σ-matrices - it is common to write them downstairs.

'$\bar{\Psi}\Psi$' would now be, for instance,

$$\bar{\Psi}^{\chi}_{\rm M}\Psi^{\chi}_{\rm M} = ((\mathrm{i}\sigma_2\chi^*)^\dagger\chi^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i}\sigma_2\chi^* \\ \chi \end{pmatrix} = \chi^{\rm T}(-\mathrm{i}\sigma_2)\chi + \chi^\dagger(\mathrm{i}\sigma_2)\chi^*. \quad (113)$$

The first term on the RHS of the last equality in (113) we have seen before in (78); the second is also a possible Lorentz invariant formed from χ's.[4]

Similarly, a mass term made from $\Psi^{\psi}_{\rm M}$ would be

$$\bar{\Psi}^{\psi}_{\rm M}\Psi^{\psi}_{\rm M} = (\psi^\dagger(-\mathrm{i}\sigma_2\psi^*)^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ -\mathrm{i}\sigma_2\psi^* \end{pmatrix} = \psi^\dagger(-\mathrm{i}\sigma_2)\psi^* + \psi^{\rm T}(\mathrm{i}\sigma_2)\psi. \quad (114)$$

Again, we have seen the second term on the RHS of the last equality in (114) before, and the first is also a Lorentz invariant formed from ψ (from (92), it transforms as a '$\psi^\dagger\chi$' object, which we know from (64) is invariant). Note that all the terms in (113) and (114) would vanish if the field components did not anticommute.

We can similarly consider the Lorentz-invariant product of two different Majorana spinors $\Psi_{\rm 1M}$ and $\Psi_{\rm 2M}$, namely

$$\bar{\Psi}_{\rm 1M}\Psi_{\rm 2M} = \Psi^\dagger_{\rm 1M}\beta\Psi_{\rm 2M}. \quad (115)$$

But equations (108) and (110) tell us that

$$\Psi_{\rm 1M} = -\mathrm{i}\gamma^2\Psi^*_{\rm 1M}, \quad (116)$$

and hence

$$\Psi^\dagger_{\rm 1M} = \Psi^{\rm T}_{\rm 1M}(-\mathrm{i}\gamma^2) \quad (117)$$

using $\gamma^{2\dagger} = -\gamma^2$. It follows that

$$\Psi^\dagger_{\rm 1M}\beta\Psi_{\rm 2M} = \Psi^{\rm T}_{\rm 1M}(-\mathrm{i}\gamma^2\beta)\Psi_{\rm 2M}. \quad (118)$$

The matrix

$$-\mathrm{i}\gamma^2\beta = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix} \quad (119)$$

therefore acts as a metric in forming the dot product of the two $\Psi_{\rm M}$'s. It is easy to check that (118) is the same as (113) when $\Psi_{\rm 1M} = \Psi_{\rm 2M} = \Psi^{\chi}_{\rm M}$, and the same as (114) when $\Psi_{\rm 1M} = \Psi_{\rm 2M} = \Psi^{\psi}_{\rm M}$.

[4] Here's a useful check on why. We know from (76) that $\mathrm{i}\sigma_2\chi^*$ transforms under Lorentz transformations like a ψ-type thing, which is to say it transforms by the matrix V of (56). And we also know from (60) that χ transforms by the matrix $V^{-1\dagger}$. Hence $\chi^\dagger(\mathrm{i}\sigma_2)\chi^* \to \chi^\dagger V^{-1}V(\mathrm{i}\sigma_2)\chi^* = \chi^\dagger(\mathrm{i}\sigma_2)\chi^*$.

2.4 Dirac fermions using χ- (or L-) type spinors only

We noted at the beginning of Section 2 that the simplest SUSY theory (which is just around the corner now) involves a complex scalar field and a two-component spinor field. This is in fact the archetype of SUSY models leading to the MSSM (Minimal Supersymmetric [version of the] Standard Model). By convention, one uses χ-type spinors, i.e. (see section 2.1) L-type spinors, no doubt because the V-A structure of the electroweak sector of the SM distinguishes the L parts of the fields, and one might as well give them a privileged status. But of course there are the R parts as well. In a SUSY context, it is very convenient to be able to use only one kind of spinors, which in the MSSM is (for the reason just outlined) going to be L-type ones - but in that case how are we going to deal with the R parts of the SM fields?

Consider for example the electron field which we write as

$$\Psi^{(\mathrm{e}^-)} = \begin{pmatrix} \psi_{\mathrm{R}}^{(\mathrm{e}^-)} \\ \chi_{\mathrm{L}}^{(\mathrm{e}^-)} \end{pmatrix}. \tag{120}$$

Instead of using the *right-handed electron* field in the top 2 components, we can just as well use the *charge conjugate of the left-handed positron* field. That is, instead of (120) we choose to write

$$\Psi^{(\mathrm{e})} = \begin{pmatrix} \mathrm{i}\sigma_2\chi_{\mathrm{L}}^{(\mathrm{e}^+)*} \\ \chi_{\mathrm{L}}^{(\mathrm{e}^-)} \end{pmatrix}. \tag{121}$$

A commonly used notation is to write

$$\chi_{\mathrm{L}}^{(\mathrm{e}^+)\mathrm{c}} \equiv \mathrm{i}\sigma_2\chi_{\mathrm{L}}^{(\mathrm{e}^+)*}, \tag{122}$$

or, more compactly, $e_{\mathrm{L}}^{+\mathrm{c}}$, accompanying the L-type electron field e_{L}^{-}.

Our previous work guarantees, of course, that the Lorentz transformation character of (121) is OK. In terms of the choice (121), a mass term for a (non-Majorana!) Dirac fermion is (omitting now the 'L' subscripts from the χ's)

$$\begin{aligned}\bar{\Psi}^{(\mathrm{e})}\Psi^{(\mathrm{e})} = \Psi^{(\mathrm{e})\dagger}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Psi^{(\mathrm{e})} &= \left((\mathrm{i}\sigma_2\chi^{(\mathrm{e}^+)})^{\mathrm{T}}\chi^{(\mathrm{e}^-)\dagger}\right)\begin{pmatrix} \chi^{(\mathrm{e}^-)} \\ \mathrm{i}\sigma_2\chi^{(\mathrm{e}^+)*} \end{pmatrix} \\ &= \chi^{(\mathrm{e}^+)}\cdot\chi^{(\mathrm{e}^-)} + \chi^{(\mathrm{e}^-)\dagger}\mathrm{i}\sigma_2\chi^{(\mathrm{e}^+)*}. \end{aligned} \tag{123}$$

In the first term on the RHS of (123) we have used the quick 'dot' notation for two χ-type spinors introduced in Aside (1); see Notational Aside (3) for a similar treatment of the second term. So the 'Dirac' mass has here been re-written wholly in terms of two L-type spinors, one associated with the e^- mode, the other with the e^+ mode.

Notational Aside (3) Readers who have patiently ploughed through Asides (1) and (2) may be beginning to think we have now got altogether too many different kinds of spinor in play. We previously agreed that we'd identified four kinds of spinor: χ_a and χ^a transforming by $V^{-1\dagger}$ and V^* respectively, and $\psi^{\dot{a}}$ and $\psi_{\dot{a}}$ transforming by V and $V^{-1\mathrm{T}}$. Surely χ_a^* can't be yet another kind? Indeed, since χ_a transforms by $V^{-1\dagger}$, it follows that χ_a^* transforms by the complex conjugate of this, which is $V^{-1\mathrm{T}}$. But this is exactly how a '$\psi_{\dot{a}}$' (or a '$\bar{\psi}_{\dot{a}}$', using the bar notation for the dotted spinor) transforms. So it is consistent to *define*

$$\bar{\chi}_{\dot{a}} \equiv \chi_a^* \tag{124}$$

and then raise the lower dotted index with the matrix $i\sigma_2$, using the inverse of (95) (remember, once we have the dotted indices, or the bar, to tell us what kind of spinor it is, we no longer care what letter we use!). Then the second term of (123) becomes

$$\chi^{(e^-)*\mathrm{T}} i\sigma_2 \chi^{(e^+)*} = \bar{\chi}_{\dot{a}}^{(e^-)} \bar{\chi}^{(e^+)\dot{a}} = \bar{\chi}^{(e^-)} \cdot \bar{\chi}^{(e^+)}. \tag{125}$$

Exercise (a) What is $\bar{\chi} \cdot \bar{\xi}$ in terms of $\bar{\xi} \cdot \bar{\chi}$ (assuming the components anticommute)? (b) What is $\bar{\chi}_{\dot{a}}\bar{\xi}^{\dot{a}}$ in terms of $\bar{\chi}^{\dot{a}}\bar{\xi}_{\dot{a}}$? Do these by components and by using ϵ symbols.

Now, at last, we are ready to take our first steps in SUSY.

3 A Simple Supersymmetric Lagrangian

In this section we'll look at one of the simplest supersymmetric theories, one involving just two free fields: a complex spin-0 field ϕ and an L-type spinor field χ , both massless. The Lagrangian (density) for this system is

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi. \tag{126}$$

The ϕ part is familiar from introductory qft courses: the χ bit is just the appropriate part of the Dirac Lagrangian (69). The equation of motion for ϕ is of course $\Box\phi = 0$, while that for χ is $i\bar{\sigma}^\mu \partial_\mu \chi = 0$ (compare (68)). We are going to try and find, by 'brute force', transformations in which the change in ϕ is proportional to χ (as in (35)), and the change in χ is proportional to ϕ, such that $\mathcal{L}$ is invariant.[5]

[5]Actually we shan't succeed: instead, we have to settle for the Action to be invariant, which means that $\mathcal{L}$ changes by a total derivative; it turns out that this has to do with the 'mis-match' in the number of degrees of freedom (off-shell) in ϕ and χ.

As a preliminary, it is useful to get the *dimensions* of everything straight. The Action is the integral of the density $\mathcal{L}$ over all 4-dimensional space, and is dimensionless in units $\hbar = c = 1$. In this system, there is only one independent dimension left, which we take to be that of mass (or energy), M (see Appendix B of [17]). Length has the same dimension as time (because $c = 1$), and both have the dimension of M^{-1} (because $\hbar = 1$). It follows that, for the Action to be dimensionless, $\mathcal{L}$ has dimension M^4. Since the gradients have dimension M, we can then read off the dimensions of ϕ and χ (denoted by $[\phi]$ and $[\chi]$):

$$[\phi] = \mathrm{M} \quad [\chi] = \mathrm{M}^{3/2}. \tag{127}$$

Now, what are the SUSY transformations linking ϕ and χ? Several considerations can guide us to, if not the answer, then at least a good guess. Consider the change in ϕ, $\delta_\xi \phi$, first where ξ is a constant (x-independent) parameter. This has the form (already stated in (35))

$$\text{` change in } \phi = \text{parameter } \xi \times \text{ other field } \chi\text{'}. \tag{128}$$

On the LHS, we have a spin-0 field, which is invariant under Lorentz transformations. So we must construct a Lorentz invariant out of χ and the parameter ξ. One simple way to do this is to declare that ξ is also a χ- (or L-) type spinor, and use the invariant product (77). This gives

$$\delta_\xi \phi = \xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi, \tag{129}$$

or in the notation of Aside (1)

$$\delta_\xi \phi = \xi^a \chi_a = \xi \cdot \chi. \tag{130}$$

It is worth pausing to note some things about the parameter ξ. First, we repeat that it is a spinor. It doesn't depend on x, but it is not an invariant under Lorentz transformations: it transforms as a χ-type spinor, i.e. by $V^{-1\dagger}$. It has two components, of course, each of which is complex - hence 4 real numbers in all. These specify the transformation (129). Secondly, although ξ doesn't depend on x, and isn't a field (operator) in that sense, we shall assume that its components *anticommute* with the components of spinor fields - that is, we assume they are Grassmann numbers (see [12] Appendix O). Lastly, since $[\phi] = \mathrm{M}$ and $[\chi] = \mathrm{M}^{3/2}$, to make the dimensions balance on both sides of (129) we need to assign the dimension

$$[\xi] = \mathrm{M}^{-1/2} \tag{131}$$

to ξ.

Now let's think what the corresponding $\delta_\xi \chi$ might be. This has to be something like

$$\delta_\xi \chi \sim \text{product of } \xi \text{ and } \phi. \tag{132}$$

Now, on the LHS of (132) we have a quantity with dimensions $\mathrm{M}^{3/2}$, while on the RHS the algebraic product of ξ and ϕ has dimensions $\mathrm{M}^{-1/2+1} = \mathrm{M}^{1/2}$. Hence we need to introduce something with dimensions M^1 on the RHS. In this massless theory, there is only one possibility - the gradient operator ∂_μ, or more conveniently the momentum operator $\mathrm{i}\partial_\mu$. But now we have a 'loose' index μ on the RHS! The LHS is a spinor, and there is a spinor (ξ) also on the RHS, so we should probably get rid of the μ index altogether, by contracting it. We try

$$\delta_\xi \chi = (\mathrm{i}\sigma^\mu \partial_\mu \phi)\ \xi \tag{133}$$

where σ^μ is given in (66). Note that the 2×2 matrices in σ^μ act on the 2-component column ξ to give, correctly, a 2-component column to match the LHS. But although both sides of (133) are 2-component column vectors, the RHS does not transform as a χ-type spinor. If we look back at (67) and (68), we see that the combination $\sigma^\mu \partial_\mu$ acting on a ψ transforms as a χ (and $\bar{\sigma}^\mu \partial_\mu$ on a χ transforms as a ψ). This suggests that we should let the $\sigma^\mu \partial_\mu \phi$ in (133) multiply a ψ-like thing, not a ξ, in order to get something transforming as a χ. But we know how to manufacture a ψ-like thing out of ξ! We just take (see (76)) $\mathrm{i}\sigma_2 \chi^*$. We therefore arrive at the guess

$$\delta_\xi \chi_a = A[\mathrm{i}\sigma^\mu(\mathrm{i}\sigma_2 \xi^*)]_a \partial_\mu \phi \tag{134}$$

where A is some constant to be determined from the condition that $\mathcal{L}$ is invariant under (129) and (134), and we have indicated the χ-type spinor index on both sides. Note that '$\partial_\mu \phi$' has no matrix structure and has been moved to the end.

Equations (129) and (134) give the proposed SUSY transformations for ϕ and χ, but both are complex fields and we need to be clear what the corresponding transformations are for their Hermitian conjugates $\phi^\dagger$ and $\chi^\dagger$. There are some notational concerns here which we shall not put in small print. First, remember that ϕ and χ are quantum fields, even though we are not explicitly putting hats on them; on the other hand, ξ is not a field (it's x-independent). In the discussion of Lorentz transformations of spinors in Section 2, we used the symbol * to denote complex conjugation, it being tacitly understood that we were dealing with wave functions rather than field operators. But consider the (quantum) field ϕ with a mode expansion

$$\phi = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\,[a(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + b^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}]. \tag{135}$$

Here the operator $a(k)$ destroys (say) a particle with 4-momentum k, and $b^\dagger(k)$ creates an anti-particle of 4-momentum k, while $\exp[\pm \mathrm{i}k\cdot x]$ are of course ordinary wavefunctions. For (135) the simple complex conjugation * is not appropriate, since '$a^*(k)$' is not defined; instead, we want '$a^\dagger(k)$'. So instead of 'ϕ^*' we deal with $\phi^\dagger$, which is defined in terms of (135) by (a) taking the complex conjugate of the wavefunction parts and (b) taking the dagger of the mode operators. This gives

$$\phi^\dagger = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega}}\,[a^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x} + b(k)\mathrm{e}^{-\mathrm{i}k\cdot x}], \tag{136}$$

the conventional definition of the Hermitian conjugate of (135).

For spinor fields like χ, on the other hand, the situation is slightly more complicated, since now in the analogue of (135) the scalar (spin-0) wavefunctions $\exp[\pm \mathrm{i}k\cdot x]$ will be replaced by (free-particle) 2-component spinors. Thus, symbolically, the first (upper) component of the quantum field χ will have the form

$$\chi_1 \sim \text{mode operator} \times \text{first component of free-particle spinor of } \chi\text{-type} \tag{137}$$

where we are of course using the 'downstairs, undotted' notation for the components of χ. In the same way as (136) we then define

$$\chi_1^\dagger \sim (\text{mode operator})^\dagger \times (\text{ first component of free-particle spinor})^*. \tag{138}$$

With this in hand, let's consider the Hermitian conjugate of (129), that is $\delta_\xi\phi^\dagger$. Written out in terms of components (129) is

$$\delta_\xi\phi = (\xi_1\xi_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = -\xi_1\chi_2 + \xi_2\chi_1. \tag{139}$$

We want to take the 'dagger' of this - but we are now faced with a decision about how to take the dagger of products of (anticommuting) spinor components, like $\xi_1\chi_2$. In the case of two matrices A and B, we know that

$(AB)^\dagger = B^\dagger A^\dagger$. By analogy, we shall *define* the dagger to reverse the order of the spinors:

$$\delta_\xi \phi^\dagger = -\chi_2^\dagger \xi_1^* + \chi_1^\dagger \xi_2^*; \tag{140}$$

ξ isn't a quantum field and the '*' notation is OK for it. But (140) can be written in more compact form:

$$\begin{aligned} \delta_\xi \phi^\dagger &= \chi_1^\dagger \xi_2^* - \chi_2^\dagger \xi_1^* \\ &= (\chi_1^\dagger \chi_2^\dagger) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} \\ &= \chi^\dagger (\mathrm{i}\sigma_2) \xi^*, \end{aligned} \tag{141}$$

where in the last line the † symbol, as applied to the two-component spinor field χ, is understood in a matrix sense as well: that is

$$\chi^\dagger = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}^\dagger = (\chi_1^\dagger \chi_2^\dagger). \tag{142}$$

Equation (141) is a satisfactory outcome of these rather fiddly considerations because (a) we have seen exactly this spinor structure before, in (123), and we are assured its Lorentz transformation character is OK, and (b) it is nicely consistent with 'naively' taking the dagger of (129), treating it like a matrix product. In particular, the RHS of the last line of (141) can be written in the notation of Aside (3) as $\bar{\chi} \cdot \bar{\xi}$ or equally, using the Exercise in Aside (3), as $\bar{\xi} \cdot \bar{\chi}$. Referring to (130) we therefore note the useful result

$$(\xi \cdot \chi)^\dagger = (\chi \cdot \xi)^\dagger = \bar{\xi} \cdot \bar{\chi} = \bar{\chi} \cdot \bar{\xi}. \tag{143}$$

In the same way, therefore, we can take the dagger of (134) to obtain

$$\delta_\xi \chi^\dagger = A \partial_\mu \phi^\dagger \xi^{\mathrm{T}} \mathrm{i}\sigma_2 \mathrm{i}\sigma^\mu, \tag{144}$$

where for later convenience we have here moved the $\partial_\mu \phi^\dagger$ to the front, and we have taken A to be real (which will be sufficient, as we'll see). We are now ready to see if we can choose A so as to make $\mathcal{L}$ invariant under (129), (134), (141) and (144).

We have

$$\begin{aligned} \delta_\xi \mathcal{L} &= \partial_\mu (\delta_\xi \phi^\dagger) \partial^\mu \phi + \partial_\mu \phi^\dagger \partial^\mu (\delta_\xi \phi) + (\delta_\xi \chi^\dagger) \mathrm{i}\bar{\sigma}^\mu \partial_\mu \chi + \chi^\dagger \mathrm{i}\bar{\sigma}^\mu \partial (\delta_\xi \chi) \\ &= \partial_\mu (\chi^\dagger \mathrm{i}\sigma_2 \xi^*) \partial^\mu \phi + \partial_\mu \phi^\dagger \partial^\mu (\xi^{\mathrm{T}} (-\mathrm{i}\sigma_2) \chi) \end{aligned}$$

$$+A(\partial_\mu\phi^\dagger\xi^{\mathrm{T}}\mathrm{i}\sigma_2\mathrm{i}\sigma^\mu)\mathrm{i}\bar{\sigma}^\nu\partial_\nu\chi + A\chi^\dagger\mathrm{i}\bar{\sigma}^\nu\partial_\nu(\mathrm{i}\sigma^\mu\mathrm{i}\sigma_2\xi^*)\partial_\mu\phi. \tag{145}$$

Inspection of (145) shows that there are two types of term, one involving the parameters ξ^* and the other the parameters ξ^{T}. Consider the term involving $A\xi^*$. In it there appears the combination (pulling ∂_μ through the constant ξ^*)

$$\bar{\sigma}^\nu\partial_\nu\sigma^\mu\partial_\mu = (\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla})(\partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) = \partial_0^2 - \boldsymbol{\nabla}^2 = \partial_\mu\partial^\mu. \tag{146}$$

We can therefore combine this and the other term in ξ^* from (145) to give

$$\delta_\xi\mathcal{L}|_{\xi^*} = \partial_\mu\chi^\dagger\mathrm{i}\sigma_2\xi^*\partial^\mu\phi - \mathrm{i}A\chi^\dagger\partial_\mu\partial^\mu\sigma_2\xi^*\phi. \tag{147}$$

This represents a change in $\mathcal{L}$ under our transformations, so it seems we have not succeeded in finding an invariance (or symmetry), since we cannot hope to cancel this change against the term involving ξ^{T}, which involves quite independent parameters. However, we must remember that the Action is the space-time integral of $\mathcal{L}$, and this will be invariant if we can arrange for the change in $\mathcal{L}$ to be a *total derivative* (assuming as usual that the expression obtained by integrating it vanishes at the boundaries of space-time). Since ξ does not depend on x, we can indeed write (147) as a total derivative

$$\delta_\xi\mathcal{L}|_{\xi^*} = \partial_\mu(\chi^\dagger\mathrm{i}\sigma_2\xi^*\partial^\mu\phi) \tag{148}$$

provided that

$$A = -1. \tag{149}$$

Similarly, if $A = -1$ the terms in ξ^{T} combine to give

$$\delta_\xi\mathcal{L}|_{\xi^{\mathrm{T}}} = \partial_\mu\phi^\dagger\partial^\mu(\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi) + \partial_\mu\phi^\dagger\xi^{\mathrm{T}}\mathrm{i}\sigma_2\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi. \tag{150}$$

The second term in (150) we can write as

$$\partial_\mu(\phi^\dagger\xi^{\mathrm{T}}\mathrm{i}\sigma_2\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi) + \phi^\dagger\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi \tag{151}$$

$$= \partial_\mu(\phi^\dagger\xi^{\mathrm{T}}\mathrm{i}\sigma_2\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi) + \phi^\dagger\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\partial_\mu\partial^\mu\chi. \tag{152}$$

The second term of (152) and the first term of (150) now combine to give the total derivative

$$\partial_\mu(\phi^\dagger\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\partial^\mu\chi), \tag{153}$$

so that finally

$$\delta_\xi\mathcal{L}|_{\xi^{\mathrm{T}}} = \partial_\mu(\phi^\dagger\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\partial^\mu\chi) + \partial_\mu(\phi^\dagger\xi^{\mathrm{T}}\mathrm{i}\sigma_2\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi), \tag{154}$$

which is also a total derivative. In summary, we have shown that under (129), (134), (141) and (144), with $A = -1$, $\mathcal{L}$ changes by a total derivative:

$$\delta_\xi \mathcal{L} = \partial_\mu(\chi^\dagger \mathrm{i}\sigma_2 \xi^* \partial^\mu \phi + \phi^\dagger \xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\partial^\mu \chi + \phi^\dagger \xi^{\mathrm{T}} \mathrm{i}\sigma_2 \sigma^\mu \bar{\sigma}^\nu \partial_\nu \chi) \tag{155}$$

and the Action is therefore invariant: we have a SUSY theory, in this sense. As we shall see in Section 6, the pair (ϕ, spin-0) and (χ, L-type spin-1/2) constitute a *chiral supermultiplet* in SUSY.

Exercise Show that (155) can also be written as

$$\delta_\xi \mathcal{L} = \partial_\mu(\chi^\dagger \mathrm{i}\sigma_2 \xi^* \partial^\mu \phi + \xi^{\mathrm{T}} \mathrm{i}\sigma_2 \sigma^\nu \bar{\sigma}^\mu \chi \partial_\nu \phi^\dagger + \xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi \partial^\mu \phi^\dagger). \tag{156}$$

The reader may well feel that it's been pretty heavy going, considering especially the simplicity, triviality almost, of the Lagrangian (126). A more professional notation would have been more efficient, of course, but there is a lot to be said for doing it the most explicit and straightforward way, first time through. As we proceed, we shall speed up the notation. In fact, interactions don't constitute an order of magnitude increase in labour, and the manipulations gone through in this simple example are quite representative.

4 A First Glance at the MSSM

Before ploughing on with more formal work, let's consider how the SUSY idea might relate to particle physics. All we have so far, of course, is 1 complex scalar field and one L-type fermion field, and they aren't even interacting. All the same, let's see how we might apply it to physics. One important point to realise is that SUSY transformations do not act on the $\mathrm{SU(3)_c}$, $\mathrm{SU(2)_L}$ or $\mathrm{U(1)_{em}}$ degrees of freedom. Consider for example the left-handed lepton fields, e.g. the electron one e_{L}. This is in an $\mathrm{SU(2)_L}$ doublet, the partner field being ν_{eL}:

$$\begin{pmatrix} \nu_{\mathrm{eL}} \\ e_{\mathrm{L}} \end{pmatrix}. \tag{157}$$

These need to be partnered, in a SUSY theory, by spin-0 bosons forming another $\mathrm{SU(2)_L}$ doublet, presumably. Indeed there is such a doublet in the SM, the Higgs doublet

$$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{158}$$

or its charge-conjugate doublet

$$\begin{pmatrix} \bar{\phi}^0 \\ \phi^- \end{pmatrix}. \tag{159}$$

But these Higgs doublets don't carry lepton number (which we shall assume to be conserved), and we can't have some particles in a symmetry (SUSY) multiplet carrying a conserved quantum number, and others not. So we seem to need *new* particles to go with our doublet (157):

$$\begin{pmatrix} \nu_{\rm eL} \\ e_{\rm L} \end{pmatrix} \text{ partnered by } \begin{pmatrix} \tilde{\nu}_{\rm eL} \\ \tilde{e}_{\rm L} \end{pmatrix} \tag{160}$$

where '$\tilde{\nu}$' is a scalar partner for the neutrino ('sneutrino'), and '$\tilde{e}$' is a scalar partner for the electron ('selectron'). Similarly, we'd have smuons and staus, and their sneutrinos. These are all in chiral supermultiplets, and $\mathrm{SU}(2)_{\rm L}$ doublets.

What about quarks? They are a triplet of the $\mathrm{SU}(3)_{\rm c}$ colour gauge group, and no other SM particles are colour triplets. So we will need new (scalar) partners for the quarks too, called squarks, which are colour triplets, and also in chiral supermultiplets.

The electroweak interactions of both leptons and quarks are 'chiral', which means that the 'L' parts of the fields interact differently from the 'R' parts. The L parts belong to $\mathrm{SU}(2)_{\rm L}$ doublets, as above, while the R parts are $\mathrm{SU}(2)_{\rm L}$ singlets. So we need to arrange for scalar partners for the L and R parts separately: for example $(e_{\rm R}, \tilde{e}_{\rm R})$, $(u_{\rm R}, \tilde{u}_{\rm R})$, $(d_{\rm R}, \tilde{d}_{\rm R})$, etc; and

$$\begin{pmatrix} u_{\rm L} \\ d_{\rm L} \end{pmatrix}, \begin{pmatrix} \tilde{u}_{\rm L} \\ \tilde{d}_{\rm L} \end{pmatrix} \tag{161}$$

and so on.[6]

We haven't yet learned about SUSY for spin-1 fields, but we shall see in Section 10 that there is a *vector* (or *gauge*) supermultiplet, which associates a massless vector field (which has two on-shell degrees of freedom) with an L fermion, the latter being called generically a 'gaugino'. Once again, the gauge group quantum numbers for the gauginos have to be the same as for

[6] As noted in Section 2.4, the 'particle R-parts' will actually be represented by the charge conjugates of the 'antiparticle L-parts'.

the gauge bosons - i.e. we need a colour octet of 'gluinos' to supersymmetrize QCD, plus an $\mathrm{SU(2)_L}$ triplet of -inos and a $\mathrm{U(1)_{em}}$ -ino for the SUSY electroweak theory. After $\mathrm{SU(2)_L}$ symmetry-breaking (a la Higgs) we'll have three fermionic partners for the $\mathrm{W}^{\pm}, \mathrm{Z}^0$, namely $\tilde{\mathrm{W}}^{\pm}, \tilde{Z}^0$, and the photino $\tilde{\gamma}$.

Finally, the Higgs sector: we haven't been able to partner the Higgs doublets with any known fermion, so they need their own 'higgsinos', i.e. fermionic analogues forming chiral supermultiplets. In fact, a crucial consequence of making the SM supersymmetric, in the MSSM, is - as we shall see in Section 8 - that *two separate* Higgs doublets are required: whereas in the SM itself the doublet (159) can be satisfactorily represented as the charge conjugate of the doublet (158), this is not possible in the SUSY version. So we need

$$H_{\mathrm{u}}: \quad \begin{pmatrix} H_{\mathrm{u}}^{+} \\ H_{\mathrm{u}}^{0} \end{pmatrix}, \quad \begin{pmatrix} \tilde{H}_{\mathrm{u}}^{+} \\ \tilde{H}_{\mathrm{u}}^{0} \end{pmatrix} \tag{162}$$

and

$$H_{\mathrm{d}}: \quad \begin{pmatrix} H_{\mathrm{d}}^{0} \\ H_{\mathrm{d}}^{-} \end{pmatrix}, \quad \begin{pmatrix} \tilde{H}_{\mathrm{d}}^{0} \\ \tilde{H}_{\mathrm{d}}^{-} \end{pmatrix}. \tag{163}$$

The chiral and gauge supermultiplets introduced here constitute the 'minimal' extension of the SM field content which is required to make it supersymmetric. The full theory, including supersymmetric interactions, is called the minimal supersymmetric standard model (MSSM). It has been around for over 20 years: early reviews are given in [20] and [21]; a more recent and very helpful 'supersymmetry primer' was provided by Martin [22], to which we shall make quite frequent reference in what follows. A recent and very comprehensive review may be found in [23].

We'll return to the MSSM in Section 12. For the moment, we should simply note that (a) none of the 'superpartners' has yet been seen experimentally, in particular they certainly cannot have the same mass as their SM partner states (as would normally be expected for a symmetry multiplet), so that (b) SUSY - as applied in the MSSM - must be broken somehow. We'll include a brief discussion of SUSY breaking in section 15, but a more detailed treatment is well beyond the scope of these lectures.

5 Towards a Supersymmetry Algebra

A fundamental aspect of any symmetry (other than a U(1) symmetry) is the *algebra* associated with the *symmetry generators* - see for example Appendix

M of [12]. For example, the generators T_i of SU(2) satisfy the commutation relations

$$[T_i, T_j] = \mathrm{i}\epsilon_{ijk}T_k \tag{164}$$

where i, j and k run over the values 1, 2 and 3, and where the repeated index k is summed over; ϵ_{ijk} is the totally antisymmetric symbol such that $\epsilon_{123} = +1$, $\epsilon_{213} = -1$, etc. The commutation relations summarized in (164) constitute the 'SU(2) alegra', and it is of course exactly that of the angular momentum operators in quantum mechanics, in units $\hbar = 1$. Readers will be familiar with the way in which the whole theory of angular momentum in quantum mechanics can be developed just from these commutation relations. In the same way, in order to proceed in a reasonably systematic way with SUSY, we must know what the SUSY algebra is. In Section 1.3, we introduced the idea of generators of SUSY transformations, Q_a, and their associated algebra - which now involves anticommutation relations - was roughly indicated in (33). The purpose of this section is to find the actual SUSY algebra, by a 'brute force' method once again, making use of what we have learned in Section 2.

5.1 One way of obtaining the SU(2) algebra

In Section 2, we arrived at recipes for SUSY transformations of spin-0 fields ϕ and $\phi^\dagger$, and spin-1/2 fields χ and $\chi^\dagger$. From these transformations, the algebra of the SUSY generators can be deduced. To understand the method, it is helpful to see it in action in a more familiar context, namely that of SU(2), as we now discuss. Readers should skip this subsection if they've seen it all before.

Consider an SU(2) doublet of fields

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \tag{165}$$

where u and d have equal mass, and identical interactions, so that the Lagrangian is invariant under (infinitesimal) transformations of the form (see for example equation (12.95) of [12])

$$q \rightarrow q' = (1 - \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2)q \equiv q + \delta_{\boldsymbol{\epsilon}}q \tag{166}$$

where

$$\delta_{\boldsymbol{\epsilon}}q = -\mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2\ q. \tag{167}$$

Here, as usual, the three matrices $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the same as the Pauli $\boldsymbol{\sigma}$ matrices, and $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$ are three real infinitesimal parameters specifying the transformation. The transformed fields q' satisfy the same anticommutations relations as the original fields q, so that q' and q are related by a unitary transformation

$$q' = UqU^\dagger. \tag{168}$$

For infinitesimal transformations, U has the general form

$$U_{\text{infl}} = (1 + \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{T}) \tag{169}$$

where

$$\boldsymbol{T} = (T_1, T_2, T_3) \tag{170}$$

are the *generators* of infinitesimal SU(2) transformations; the unitarity of U implies that the $\boldsymbol{T}$'s are Hermitian. For infinitesimal transformations, therefore, we have (from (168) and (169))

$$\begin{aligned} q' &= (1 + \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{T})q(1 - \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{T}) \\ &= q + \mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{T}q - \mathrm{i}\boldsymbol{\epsilon} \cdot q\boldsymbol{T} \quad \text{to first order in } \boldsymbol{\epsilon} \\ &= q + \mathrm{i}\boldsymbol{\epsilon} \cdot [\boldsymbol{T}, q] \end{aligned} \tag{171}$$

Hence from (166) and (167) we deduce (see equation (12.100) of [12])

$$\delta_{\boldsymbol{\epsilon}} q = \mathrm{i}\boldsymbol{\epsilon} \cdot [\boldsymbol{T}, q] = -\mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{\tau}/2 \; q, \tag{172}$$

It is important to realise that the $\boldsymbol{T}$'s are themselves quantum field operators, constructed from the fields of the Lagrangian; for example in this simple case they would be

$$\boldsymbol{T} = \int q^\dagger(\boldsymbol{\tau}/2)q \; \mathrm{d}^3\boldsymbol{x} \tag{173}$$

as explained for example in section 12.3 of [12].

Given an explicit formula for the generators, such as (173), we can proceed to calculate the commutation relations of the $\boldsymbol{T}$'s, knowing how the q's anticommute. The answer is that the $\boldsymbol{T}$'s obey the relations (164). However, there is another way to get these commutation relations, just by considering the small changes in the fields, as given by (172). Consider two such transformations

$$\delta_{\epsilon_1} q = \mathrm{i}\epsilon_1[T_1, q] = -\mathrm{i}\epsilon_1(\tau_1/2)q \tag{174}$$

and

$$\delta_{\epsilon_2} q = \mathrm{i}\epsilon_2 [T_2, q] = -\mathrm{i}\epsilon_2 (\tau_2/2) q. \qquad (175)$$

We shall calculate the *difference* $(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})q$ two different ways: first via the second equality in (174) and (175), and then via the first equalities. Equating the two results will lead us to the algebra (164).

First, then, we use the second equality of (174) and (175) to obtain

$$\begin{aligned} \delta_{\epsilon_1}\delta_{\epsilon_2} q &= \delta_{\epsilon_1}\{-\mathrm{i}\epsilon_2(\tau_2/2\} q \\ &= -\mathrm{i}\epsilon_2(\tau_2/2)\delta_{\epsilon_1} q \\ &= -\mathrm{i}\epsilon_2(\tau_2/2). - \mathrm{i}\epsilon_1(\tau_1/2) q \\ &= -(1/4)\epsilon_1\epsilon_2\tau_2\tau_1 q. \end{aligned} \qquad (176)$$

Note that in the last line we have changed the order of the ϵ parameters as we are free to do since they are ordinary numbers, but we cannot alter the order of the τ's since they are matrices which don't commute. Similarly,

$$\begin{aligned} \delta_{\epsilon_2}\delta_{\epsilon_1} q &= \delta_{\epsilon_2}\{-\mathrm{i}\epsilon_1(\tau_1/2\} q \\ &= -\mathrm{i}\epsilon_1(\tau_1/2)\delta_{\epsilon_2} q \\ &= -(1/4)\epsilon_1\epsilon_2\tau_1\tau_2 q. \end{aligned} \qquad (177)$$

Hence

$$\begin{aligned} (\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1}) q &= \epsilon_1\epsilon_2[\tau_1/2, \tau_2/2] q \\ &= \epsilon_1\epsilon_2 \mathrm{i}(\tau_3/2) q \\ &= -\mathrm{i}\epsilon_1\epsilon_2[T_3, q] \end{aligned} \qquad (178)$$

where the second line follows from the fact that the τ's, as matrices, satisfy the algebra (164), and the third line results from the '$_3$' analogue of (174) and (175).

Now we calculate $(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})q$ using the first equality of (174) and (175). We have

$$\begin{aligned} \delta_{\epsilon_1}\delta_{\epsilon_2} q &= \delta_{\epsilon_1}\{\mathrm{i}\epsilon_2[T_2, q]\} \\ &= \mathrm{i}\epsilon_2\delta_{\epsilon_1}\{[T_2, q]\} \\ &= \mathrm{i}\epsilon_1\mathrm{i}\epsilon_2[T_1, [T_2, q]]. \end{aligned} \qquad (179)$$

Similarly,

$$\delta_{\epsilon_2}\delta_{\epsilon_1} q = \mathrm{i}\epsilon_1\mathrm{i}\epsilon_2[T_2, [T_1, q]]. \qquad (180)$$

Hence

$$(\delta_{\epsilon_1}\delta_{\epsilon_2}-\delta_{\epsilon_2}\delta_{\epsilon_1})q=-\epsilon_1\epsilon_2\{[T_1,[T_2,q]]-[T_2,[T_1,q]]\}. \qquad (181)$$

Now we can rearrange the RHS of this equation by using the identity (which is easily checked by multiplying it all out)

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0. \qquad (182)$$

We first write

$$[T_2,[T_1,q]]=-[T_2,[q,T_1]] \qquad (183)$$

so that the two double commutators in (181) become

$$[T_1,[T_2,q]]-[T_2,[T_1,q]]=[T_1,[T_2,q]]+[T_2,[q,T_1]]=-[q,[T_1,T_2]] \qquad (184)$$

where the last step follows by use of (182). Finally, then, (181) can be written as

$$(\delta_{\epsilon_1}\delta_{\epsilon_2}-\delta_{\epsilon_2}\delta_{\epsilon_1})q=-\epsilon_1\epsilon_2[[T_1,T_2],q] \qquad (185)$$

which can be compared with (178). We deduce

$$[T_1,T_2]=\mathrm{i}T_3 \qquad (186)$$

exactly as stated in (164).

This is the method we shall use to find the SUSY algebra, at least as far as it concerns the transformations for scalar and spinor fields found in Section 2.

5.2 Supersymmetry generators ('charges') and their algebra

In order to apply the preceding method, we need the SUSY analogue of (172). Equations (129) and (134) (with $A=-1$) provide us with the analogue of the second equality in (172), for $\delta_\xi\phi$ and for $\delta_\xi\chi$; what about the first? We want to write something like

$$\delta_\xi\phi\sim\mathrm{i}[\xi Q,\phi]=\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi. \qquad (187)$$

where Q is a SUSY generator. In the first (tentative) equality in (187), we must remember that ξ is a χ-type spinor quantity, and so it is clear that Q must be a spinor quantity also, or else one side of the equality would be bosonic and the other fermionic. In fact, since ϕ is a Lorentz scalar, we must combine ξ and Q into a Lorentz invariant. Let us suppose that Q transforms as a χ-type spinor also: then we know that $\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)Q$ is Lorentz invariant. So we shall write

$$\delta_\xi \phi = \mathrm{i}[\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)Q, \phi] = \xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi \tag{188}$$

or in the faster notation of Aside (1)

$$\delta_\xi \phi = \mathrm{i}[\xi \cdot Q, \phi] = \xi \cdot \chi. \tag{189}$$

We are going to calculate $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi$, so (since $\delta\phi \sim \chi$) we shall need (134) as well. This involves ξ^*, so to get the complete analogue of '$\mathrm{i}\boldsymbol{\epsilon} \cdot \boldsymbol{T}$' we shall need to extend '$\mathrm{i}\xi \cdot Q$' to

$$\mathrm{i}(\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)Q + \xi^{\dagger}(\mathrm{i}\sigma_2)Q^*) = \mathrm{i}(\xi \cdot Q + \bar{\xi} \cdot \bar{Q}). \tag{190}$$

We first calculate $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi$ using (129) and (134) (with $A = -1$):

$$\begin{aligned} (\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi &= \delta_\eta(\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2\chi) - (\eta \leftrightarrow \xi) \\ &= \xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\mathrm{i}\sigma^\mu(-\mathrm{i}\sigma_2)\eta^*\partial_\mu\phi - (\eta \leftrightarrow \xi) \\ &= (\xi^{\mathrm{T}} c\sigma^\mu c\eta^* - \eta^{\mathrm{T}} c\sigma^\mu c\xi^*)\mathrm{i}\partial_\mu\phi, \end{aligned} \tag{191}$$

where (none too soon) we have introduced the notation

$$c \equiv -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{192}$$

(191) is sometimes written more compactly by using

$$c\sigma^\mu c = -\bar{\sigma}^{\mu\mathrm{T}} \tag{193}$$

(see (66) for the definition of σ^μ and $\bar{\sigma}^\mu$). Now $\xi^{\mathrm{T}}\bar{\sigma}^{\mu\mathrm{T}}\eta^*$ is a single quantity (row vector times matrix times column vector) so it must equal its formal transpose, apart from a minus sign due to interchanging the order of anti-commuting variables.[7] Hence

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi = (\eta^\dagger \bar{\sigma}^\mu \xi - \xi^\dagger \bar{\sigma}^\mu \eta)\mathrm{i}\partial_\mu\phi. \tag{194}$$

[7]Check this statement by looking at $(\eta^{\mathrm{T}}(-\mathrm{i}\sigma_2)\xi)^{\mathrm{T}}$, for instance.

On the other hand, we also have

$$\delta_\xi \phi = \mathrm{i}[\xi \cdot Q + \bar{\xi} \cdot \bar{Q}, \phi] \tag{195}$$

and so

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\phi = -\{[\eta \cdot Q + \bar{\eta} \cdot \bar{Q}, [\xi \cdot Q + \bar{\xi} \cdot \bar{Q}, \phi]] - [\xi \cdot Q + \bar{\xi} \cdot \bar{Q}, [\eta \cdot Q + \bar{\eta} \cdot \bar{Q}, \phi]]\}. \tag{196}$$

Just as in (184), the RHS of (196) can be rearranged using (182) and we obtain

$$\begin{aligned} [[\eta \cdot Q + \bar{\eta} \cdot \bar{Q}, \xi \cdot Q + \bar{\xi} \cdot \bar{Q}], \phi] &= (\eta^{\mathrm{T}} c \sigma^\mu c \xi^* - \xi^{\mathrm{T}} c \sigma^\mu c \eta^*) \mathrm{i} \partial_\mu \phi \\ &= -(\eta^{\mathrm{T}} c \sigma^\mu c \xi^* - \xi^{\mathrm{T}} c \sigma^\mu c \eta^*)[P_\mu, \phi] \end{aligned} \tag{197}$$

where in the last step we have introduced the 4-momentum operator P_μ, which is also the generator of translations, such that

$$[P_\mu, \phi] = -\mathrm{i}\partial_\mu \phi \tag{198}$$

(we shall recall the proof of this equation in section 9 - see (316)).

It is *tempting* now to conclude that, just as in going from (178) and (185) to (186), we can infer from (197) the result

$$[\eta \cdot Q + \bar{\eta} \cdot \bar{Q}, \xi \cdot Q + \bar{\xi} \cdot \bar{Q}] = -(\eta^{\mathrm{T}} c \sigma^\mu c \xi^* - \xi^{\mathrm{T}} c \sigma^\mu c \eta^*) P_\mu. \tag{199}$$

But we have, so far, only established the RHS of (197) by considering the difference $\delta_\eta \delta_\xi - \delta_\xi \delta_\eta$ acting on ϕ (see (191)). Is it also true that

$$(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\chi = (\xi^{\mathrm{T}} c \sigma^\mu c \eta^* - \eta^{\mathrm{T}} c \sigma^\mu c \xi^*) \mathrm{i} \partial_\mu \chi \ ? \tag{200}$$

Unfortunately, the answer to this is *no*, as we shall see in Section 7, where we shall also learn how to repair the situation. For the moment, we proceed on the basis of (199).

In order to obtain, finally, the (anti)commutation relations of the Q's from (199), we need to get rid of the parameters η and ξ on both sides. First of all, we note that since the RHS of (199) has no terms in $\eta \ldots \xi$ or $\eta^* \ldots \xi^*$ we can deduce

$$[\eta \cdot Q, \xi \cdot Q] = [\bar{\eta} \cdot \bar{Q}, \bar{\xi} \cdot \bar{Q}] = 0. \tag{201}$$

The first commutator is

$$
\begin{aligned}
0 = (\eta^1 Q_1 + \eta^2 Q_2)(\xi^1 Q_1 + \xi^2 Q_2) - (\xi^1 Q_1 + \xi^2 Q_2)(\eta^1 Q_1 + \eta^2 Q_2) \\
= -\eta^1\xi^1(2Q_1Q_1) - \eta^1\xi^2(Q_1Q_2 + Q_2Q_1) \\
-\eta^2\xi^1(Q_2Q_1 + Q_1Q_2) - \eta^2\xi^2(2Q_2Q_2),
\end{aligned}
\tag{202}
$$

remembering that all quantities anticommute. Since all these combinations of parameters are independent, we can deduce

$$
\{Q_a, Q_b\} = 0, \tag{203}
$$

and similarly

$$
\{Q_a^*, Q_b^*\} = 0. \tag{204}
$$

Notice how, when the anti-commuting quantities ξ and η are 'stripped away' from the Q and $\bar{Q}$, the commutators in (201) become *anti*-commutators in (203) and (204).

Now let's look at the $[\eta \cdot Q, \bar{\xi} \cdot \bar{Q}]$ term in (199). Writing everything out long-hand, we have

$$
\bar{\xi} \cdot \bar{Q} = \xi^\dagger \mathrm{i}\sigma_2 Q^* = \xi_1^* Q_2^* - \xi_2^* Q_1^* \tag{205}
$$

and

$$
\eta \cdot Q = -\eta_1 Q_2 + \eta_2 Q_1. \tag{206}
$$

So

$$
\begin{aligned}
[\eta \cdot Q, \bar{\xi} \cdot \bar{Q}] = \eta_1\xi_1^*(Q_2Q_2^* + Q_2^*Q_2) - \eta_1\xi_2^*(Q_2Q_1^* + Q_1^*Q_2) \\
-\eta_2\xi_1^*(Q_1Q_2^* + Q_2^*Q_1) + \eta_2\xi_2^*(Q_1Q_1^* + Q_1^*Q_1).
\end{aligned}
\tag{207}
$$

Meanwhile, the RHS of (199) is

$$
\begin{aligned}
-(\eta_1 \eta_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma^\mu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} P_\mu \\
= -(\eta_2 - \eta_1)\sigma^\mu \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} P_\mu \\
= [\eta_2\xi_2^*(\sigma^\mu)_{11} - \eta_2\xi_1^*(\sigma^\mu)_{12} - \eta_1\xi_2^*(\sigma^\mu)_{21} + \eta_1\xi_1^*(\sigma^\mu)_{22}]\ P_\mu,
\end{aligned}
\tag{208}
$$

where the subscripts on the matrices σ^μ denote the particular element of the matrix, as usual. Comparing (207) and (208) we deduce

$$
\{Q_a, Q_b^*\} = (\sigma^\mu)_{ab} P_\mu. \tag{209}
$$

We have been writing Q^* throughout, like ξ^* and η^*, but the Q's are quantum field operators and so (in accord with the discussion in section 3) we should more properly write (209) as

$$\{Q_a, Q_b^\dagger\} = (\sigma^\mu)_{ab} P_\mu. \tag{210}$$

Once again, the commutator in (207) has led to an anticommutator in (210).

Equation (210) is the main result of this section, and is a most important equation; it provides the 'proper' version of (33). Although we have derived it by our customary brute force methods as applied to a particular (and very simple) case, it must be emphasized that equation (210) is indeed the correct SUSY algebra (up to normalization conventions[8]). Equation (210) shows (to repeat what was said in Section 1) that the SUSY generators are directly connected to the energy-momentum operator, which is the generator of space-time displacements. So it is justified to regard SUSY as some kind of extension of space-time symmetry. We shall see further aspects of this in section 9.

We note finally that the commutator of two P's is zero (translations commute), and that the commutator of a Q and a P also vanishes, since the Q's are independent of x. So all the commutation relations between Q's, $Q^\dagger$'s, and P's are now defined, and they involve only these quantities; we say that 'the supertranslation algebra is closed'.

Appendix to Section 5: The Supersymmetry Current

In the case of ordinary symmetries, the invariance of a Lagrangian under a transformation of the fields (characterised by certain parameters) implies the existence of a 4-vector j^μ (the 'symmetry current'), which is conserved: $\partial_\mu j^\mu = 0$. The generator of the symmetry is the 'charge' associated with this current, namely the spatial integral of j^0. An expression for j^μ is easily found (see for example [12] section 12.3.1). Suppose the Lagrangian $\mathcal{L}$ is invariant under the transformation

$$\phi_r \to \phi_r + \delta\phi_r \tag{211}$$

[8]Many authors normalize the SUSY charges differently, so that they get a '2' on the RHS. For completeness, we take the opportunity of this footnote to mention that more general SUSY algebras also exist, in which the single generator Q_a is replaced by N generators $Q_a^A (A = 1, 2, ...N)$. Equation (210) is then replaced by $\{Q_a^A, Q_b^{B\dagger}\} = \delta^{AB}(\sigma^\mu)_{ab} P_\mu$. The more significant change occurs in the commutator (203), which becomes $\{Q_a^A, Q_b^B\} = \epsilon_{ab} Z^{AB}$, where $\epsilon_{12} = -1, \epsilon_{21} = +1, \epsilon_{11} = \epsilon_{22} = 0$ and the 'central charge' Z^{AB} is antisymmetric under $A \leftrightarrow B$. See footnote 9 for why only the $N = 1$ case seems to have any immediate physical relevance.

where 'ϕ_r' stands generically for any field in $\mathcal{L}$, having several components labelled by r. Then

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}\partial^\mu(\delta\phi_r) + \text{hermitian conjugate.} \tag{212}$$

But the equation of motion for ϕ_r is

$$\frac{\partial\mathcal{L}}{\partial\phi_r} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\right). \tag{213}$$

Using (213) in (212) yields

$$\partial_\mu j^\mu = 0 \tag{214}$$

where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r + \text{hermitian conjugate.} \tag{215}$$

For example, consider the Lagrangian

$$\mathcal{L} = \bar{q}(\mathrm{i}\not{\partial} - m)q \tag{216}$$

where

$$q = \begin{pmatrix} u \\ d \end{pmatrix}. \tag{217}$$

This is invariant under the SU(2) transformation (167), which is characterised by three independent infinitesimal parameters, so there are three independent symmetries, three currents, and three generators (or charges). Consider for instance a transformation involving ϵ_1 alone. Then

$$\delta q = -\mathrm{i}\epsilon_1(\tau_1/2)q, \tag{218}$$

while from (216) we have

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu q)} = \bar{q}\mathrm{i}\gamma^\mu. \tag{219}$$

Hence from (215) and (218) we obtain the corresponding current as

$$\epsilon_1\bar{q}\gamma^\mu(\tau_1/2)q. \tag{220}$$

Clearly the constant factor ϵ_1 is irrelevant and can be dropped. Repeating the same steps for transformations associated with ϵ_2 and ϵ_3 we deduce the existence of the *isospin currents*

$$\boldsymbol{j}^\mu = \bar{q}\gamma^\mu(\boldsymbol{\tau}/2)q \tag{221}$$

and charges (generators)

$$\boldsymbol{T} = \int q^\dagger(\boldsymbol{\tau}/2)q \; \mathrm{d}^3\boldsymbol{x} \tag{222}$$

just as stated in (173).

We can apply the same procedure to find the *supersymmetry curent* associated with the supersymmetry exhibited by the simple model considered in section 3. However, there is an important difference between this example and the SU(2) model just considered: in the latter, the Lagrangian is indeed invariant under the transformation (166), but in the SUSY case we were only able to ensure that the Action was invariant, the Lagrangian changing by a total derivative, as given in (155) or (156). In this case, the '0' on the LHS of (212) must be replaced by $\partial_\mu K^\mu$ say, where K^μ is the expression in brackets in (155) or (156).

Furthermore, since the SUSY charges are spinors Q_a, we anticipate that the associated currents carry a spinor index too, so we write them as J^μ_a, where a is a spinor index. These will be associated with transformations characterised by the usual spinor parameters ξ. Similarly, there will be the Hermitian conjugate currents associated with the parameters ξ^*.

Altogether, then, we can write (forming Lorentz invariants in the now familiar way)

$$\begin{aligned}\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)J^\mu + \xi^\dagger \mathrm{i}\sigma_2 J^{\mu\dagger} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \delta\phi^\dagger\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)}\delta\chi - K^\mu. \\ &= \partial^\mu\phi^\dagger\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi + \chi^\dagger\mathrm{i}\sigma_2\xi^*\partial^\mu\phi + \chi^\dagger\mathrm{i}\bar{\sigma}^\mu(-\mathrm{i}\sigma^\nu)\mathrm{i}\sigma_2\xi^*\partial_\nu\phi \\ &\quad -(\chi^\dagger\mathrm{i}\sigma_2\xi^*\partial^\mu\phi + \xi^{\mathrm{T}}\mathrm{i}\sigma_2\sigma^\nu\bar{\sigma}^\mu\chi\partial_\nu\phi^\dagger + \xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi\partial^\mu\phi^\dagger) \\ &= \chi^\dagger\bar{\sigma}^\mu\sigma^\nu\mathrm{i}\sigma_2\xi^*\partial_\nu\phi + \xi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\sigma^\nu\bar{\sigma}^\mu\chi\partial_\nu\phi^\dagger, \end{aligned} \tag{223}$$

whence we read off the SUSY current as

$$J^\mu = \sigma^\nu\bar{\sigma}^\mu\chi\partial_\nu\phi^\dagger. \tag{224}$$

As expected, this current has two spinorial components, and it contains an unpaired fermionic operator χ.

6 Supermultiplets

We proceed to extract some physical consequences of (210). In a theory which is supersymmetric, the operators Q - being generators of the symmetry - will commute with the Hamiltonian H:

$$[Q_a, H] = [Q_a^\dagger, H] = 0. \tag{225}$$

So acting on one state of mass M the Q's will create another state also of mass M, but since they are spinor operators this other state will not have the same spin j as the first. In fact, we know that under rotations (compare equation (172) for the case of isospin rotations, and equations (315) and (317) below for spatial translations)

$$\delta Q = -(\mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}/2)Q = \mathrm{i}\boldsymbol{\epsilon}\cdot[\boldsymbol{J}, Q], \tag{226}$$

where the J's are the generators of rotations (i.e. angular momentum operators). For example, for a rotation about the 3-axis,

$$-\frac{1}{2}\sigma_3 Q = [J_3, Q], \tag{227}$$

which implies that

$$[J_3, Q_1] = -\frac{1}{2}Q_1, \quad [J_3, Q_2] = \frac{1}{2}Q_2. \tag{228}$$

It follows that if $|jm\rangle$ is a spin-j state with $J_3 = m$, then

$$(J_3Q_1 - Q_1J_3)|jm\rangle = -\frac{1}{2}Q_1|jm\rangle, \tag{229}$$

whence

$$J_3(Q_1|jm\rangle) = (m - \frac{1}{2})Q_1|jm\rangle, \tag{230}$$

showing that $Q_1|jm\rangle$ has $J_3 = m - \frac{1}{2}$ - that is, Q_1 lowers the M-value by $\frac{1}{2}$ (like an annihilation operator for a 'u'-state). Similarly, Q_2 raises it by $\frac{1}{2}$ (like an annihilation operator for a 'd'-state). Likewise, since

$$[J_3, Q_1^\dagger] = \frac{1}{2}Q_1^\dagger, \tag{231}$$

we find that $Q_1^\dagger$ raises the m-value by $\frac{1}{2}$; and by the same token $Q_2^\dagger$ lowers it by $\frac{1}{2}$.

We now want to find the nature of the states which are 'connected' to each other by the application of the operators Q_a and $Q_a^\dagger$ - that is, the analogue of the $(2j+1)$-fold multiplet structure familiar in angular momentum theory. Our states will be labelled as $|p, \lambda\rangle$, where we take the 4-momentum to be $p = (E, 0, 0, E)$ (since the fields are massless), and where λ is a helicity label. Let's choose $|p, \lambda\rangle$ such that

$$Q_a^\dagger|p, \lambda\rangle = 0 \quad \text{for } a\text{=1,2}. \tag{232}$$

Note that this is always possible: for if we started with a state $|p, \lambda\rangle'$ which did not satisfy this condition, then we could choose instead the state $Q_a^\dagger|p, \lambda\rangle'$ which does, because $Q_a^\dagger Q_a^\dagger = 0$ (which follows either from (204) with $a = b$ or simply from the fact that the Q's are fermionic operators). There are then only two states 'connected' to $|p, \lambda\rangle$, namely $Q_1|p, \lambda\rangle$ and $Q_2|p, \lambda\rangle$. The first of these is not an acceptable state since its norm is zero. This follows by considering the SUSY algebra (210) with $a = b = 1$:

$$Q_1^\dagger Q_1 + Q_1 Q_1^\dagger = (\sigma^\mu)_{11} P_\mu. \tag{233}$$

The only components of σ^μ to have a non-vanishing '11' entry are $(\sigma^0)_{11} = 1$ and $(\sigma^3)_{11} = 1$, so we have

$$Q_1^\dagger Q_1 + Q_1 Q_1^\dagger = P_0 + P_3 = P^0 - P^3. \tag{234}$$

Hence, taking the expectation value in the state $|p, \lambda\rangle$, we find

$$\langle p, \lambda | Q_1^\dagger Q_1 + Q_1 Q_1^\dagger | p, \lambda\rangle = 0 \tag{235}$$

since the eigenvalue of $P^0 - P^3$ vanishes in this state. But also (by choice) $Q_1^\dagger|p, \lambda\rangle = 0$, from which we deduce

$$\langle p, \lambda | Q_1^\dagger Q_1 | p, \lambda\rangle = 0, \tag{236}$$

which shows that the norm of $Q_1|p, \lambda\rangle$ is zero, as claimed.

This leaves just one connected state, namely

$$Q_2|p, \lambda\rangle. \tag{237}$$

We know that Q_2 raises λ by 1/2, and hence

$$Q_2|p, \lambda\rangle \propto |p, \lambda + \frac{1}{2}\rangle. \tag{238}$$

We expect that the application of $Q_2^\dagger$ to this second state will take us back to the one we started from, and this is correct:

$$Q_2^\dagger|p, \lambda + \frac{1}{2}\rangle \propto Q_2^\dagger Q_2|p, \lambda\rangle \propto (2E - Q_2 Q_2^\dagger)|p, \lambda\rangle \propto |p, \lambda\rangle, \tag{239}$$

where we have used (210) with $a = b = 2$. So there are just two distinct states, degenerate in mass, and linked by the operators Q_2 and $Q_2^\dagger$. Equations (238) and (239) are suitable for the case $\lambda = -1/2$ (L-type); clearly a separate, but analogous, choice may be made for the case $\lambda = +1/2$ (R-type).[9]

[9]In $N = 2$ SUSY (see footnote 8) the corresponding supermultiplet contains 4 states: $\lambda = +1/2, \lambda = -1/2$ and two states with $\lambda = 0$. The problem phenomenologically with this is that the R ($\lambda = +1/2$) and L ($\lambda = -1/2$) states must transform in the same way under any gauge symmetry. (Similar remarks hold for all $N \geq 1$ supermultiplets.) But we know that the $SU(2)_L$ gauge symmetry of the SM treats the L and R components of quark and lepton fields differently. So if we want to make a SUSY extension of the SM, it had better be the simple $N = 1$ SUSY, where we are free to treat the supermultiplet ($\lambda = -1/2, \lambda = 0$) differently from the supermultiplet ($\lambda = 0, \lambda = +1/2$). See [19] section 1.6.

In terms of particle states, a *supermultiplet* contains just two types of particles, differing by a 1/2-unit of helicity. The free SUSY theory of Section 3 is an example of a *left chiral supermultiplet* containing a complex scalar field and a single 2-component fermion of L-type. Later, we shall learn how to include interactions (Section 8). We shall also need to consider the *vector, or gauge supermultiplet*, which contains a gauge field and a two-component spinor (see Section 11). Finally there is the *gravity supermultiplet* (which we shall not present), containing a spin-2 graviton field and a spin-3/2 gravitino field.

We must now take up an issue raised after (197).

7 A Snag, and the Need for a Significant Complication

In Section 5.2 we arrived at the SUSY algebra by calculating the difference $\delta_\eta\delta_\xi - \delta_\xi\delta_\eta$ two different ways. We explicitly evaluated this difference as applied to ϕ, but in deducing the operator relation (199), it is crucial that a consistent result be obtained when $\delta_\eta\delta_\xi - \delta_\xi\delta_\eta$ is applied to χ. In fact, as noted after (200), it is not, as we now show. This will necessitate a significant modification of the SUSY transformations given so far, in order to bring about this desired consistency.

Consider first $\delta_\eta\delta_\xi\chi_a$, where we are indicating the spinor component explicitly:

$$\begin{aligned}\delta_\eta\delta_\xi\chi_a &= \delta_\eta(-\mathrm{i}\sigma^\mu(\mathrm{i}\sigma_2\xi^*))_a\partial_\mu\phi \\ &= (\mathrm{i}\sigma^\mu(-\mathrm{i}\sigma_2\xi^*))_a\partial_\mu\delta_\eta\phi \\ &= (\mathrm{i}\sigma^\mu(-\mathrm{i}\sigma_2\xi^*))_a(\eta^{\mathrm{T}}(-\mathrm{i}\sigma_2)\partial_\mu\chi). \end{aligned} \tag{240}$$

There is an important identity involving products of three spinors, which we can use to simplify (240). The identity reads, for any three spinors λ, ζ and ρ,

$$\lambda_a(\zeta^{\mathrm{T}}(-\mathrm{i}\sigma_2)\rho) + \zeta_a(\rho^{\mathrm{T}}(-\mathrm{i}\sigma_2)\lambda) + \rho_a(\lambda^{\mathrm{T}}(-\mathrm{i}\sigma_2)\zeta) = 0, \tag{241}$$

or in the faster notation

$$\lambda_a(\zeta\cdot\rho) + \zeta_a(\rho\cdot\lambda) + \rho_a(\lambda\cdot\zeta) = 0. \tag{242}$$

Exercise Check the identity (241).

We take, in (241),

$$\lambda_a = (\sigma^\mu(-\mathrm{i}\sigma_2)\xi^*)_a, \quad \zeta_a = \eta_a, \quad \rho_a = \partial_\mu\chi_a. \tag{243}$$

The RHS of (240) is then equal to

$$-\mathrm{i}\{\eta_a\partial_\mu\chi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\sigma^\mu(-\mathrm{i}\sigma_2)\xi^* + \partial_\mu\chi_a(\sigma^\mu(-\mathrm{i}\sigma_2\xi^*))^{\mathrm{T}}(-\mathrm{i}\sigma_2)\eta.\} \tag{244}$$

But we know from (193) that the first term in (244) can be written as

$$\mathrm{i}\eta_a(\partial_\mu\chi^{\mathrm{T}}\bar{\sigma}^{\mu\mathrm{T}}\xi^*) = -\mathrm{i}\eta_a(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi), \tag{245}$$

where to reach the second equality in (245) we have taken the formal transpose of the quantity in brackets, remembering the sign change from re-ordering the spinors. As regards the second term in (244), we again take the transpose of the quantity multiplying $\partial_\mu\chi_a$, so that it becomes

$$-\mathrm{i}\partial_\mu\chi_a(-\eta^{\mathrm{T}}\mathrm{i}\sigma_2\sigma^\mu(-\mathrm{i}\sigma_2)\xi^* = -\mathrm{i}\eta^{\mathrm{T}}c\sigma^\mu c\xi^*\partial_\mu\chi_a. \tag{246}$$

After these manipulations, then, we have arrived at

$$\delta_\eta\delta_\xi\chi_a = -\mathrm{i}\eta_a(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi) - \mathrm{i}\eta^{\mathrm{T}}c\sigma^\mu c\xi^*\partial_\mu\chi_a, \tag{247}$$

and so

$$\begin{aligned}(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\chi_a = {} & (\xi^{\mathrm{T}}c\sigma^\mu c\eta^* - \eta^{\mathrm{T}}c\sigma^\mu c\xi^*)\mathrm{i}\partial_\mu\chi_a \\ & +\mathrm{i}\xi_a(\eta^\dagger\bar{\sigma}^\mu\partial_\mu\chi) - \mathrm{i}\eta_a(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi).\end{aligned} \tag{248}$$

We now see the difficulty: the first term on the RHS of (248) is indeed exactly the same as (191) with ϕ replaced by χ, as hoped for in (200), *but there are in addition two unwanted terms.*

The two unwanted terms vanish when the equation of motion $\bar{\sigma}^\mu\partial_\mu\chi = 0$ is satisfied (for a massless field) - i.e. 'on-shell'. But this is not good enough - we want a symmetry that applies for the internal (off-shell) lines in Feynman graphs, as well as for the on-shell external lines. Actually, we should not be too surprised that our naive SUSY of Section 5.2 has failed off-shell, for a reason that has already been touched upon: the numbers of degrees of freedom in ϕ and χ don't match up properly, the former having two (one complex field) and the latter four (two complex components). This suggests that we need to introduce another two degrees of freedom to supplement the two in ϕ - say a second scalar field F. We do this in the 'cheapest' possible way (provided it works), which is simply to add a term $F^\dagger F$ to the Lagrangian (126), so that F has no kinetic term, and therefore no propagator:

$$\mathcal{L}_F = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger \mathrm{i} \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F. \tag{249}$$

The strategy now is to invent a SUSY transformation for the *auxiliary field* F, and the existing fields ϕ and χ, such that (a) $\mathcal{L}_F$ is invariant, at least up to a total derivative, and (b) the unwanted terms in $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\chi$ are removed.

We note that F has dimension M^2, suggesting that $\delta_\xi F$ should probably be of the form

$$\delta_\xi F \sim \xi \partial_\mu \chi, \tag{250}$$

which is consistent dimensionally. But as usual we need to ensure Lorentz covariance, and in this case that means that the RHS of (250) must be a Lorentz invariant. We know that $\bar{\sigma}^\mu \partial_\mu \chi$ transforms as a 'ψ'-type spinor (see (68)), and we know that an object of the form '$\xi^\dagger \psi$ is Lorentz invariant (see (63)). So we try

$$\delta_\xi F = -\mathrm{i} \xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \tag{251}$$

and correspondingly

$$\delta_\xi F^\dagger = \mathrm{i} \partial_\mu \chi^\dagger \bar{\sigma}^\mu \xi. \tag{252}$$

The fact that these changes vanish if the equation of motion for χ is imposed (the on-shell condition) suggests that they might be capable of cancelling the unwanted terms in (248). Note also that, since ξ is independent of x, the changes in F and $F^\dagger$ are total derivatives: this will be important later (see the end of section 9.3).

We must first ensure that the enlarged Lagrangian (249) - or at least the corresponding Action - remains SUSY-invariant. Under these changes, the $F^\dagger F$ term in (249) changes by

$$(\delta_\xi F^\dagger) F + F^\dagger (\delta_\xi F) = (\mathrm{i} \partial_\mu \chi^\dagger \bar{\sigma}^\mu \xi) F - F^\dagger (\mathrm{i} \xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi). \tag{253}$$

These terms have a structure very similar to the χ term in (249), which changes by

$$\delta_\xi (\chi^\dagger \mathrm{i} \bar{\sigma}^\mu \partial_\mu \chi) = (\delta_\xi \chi^\dagger) \mathrm{i} \bar{\sigma}^\mu \partial_\mu \chi + \chi^\dagger \mathrm{i} \bar{\sigma}^\mu \partial_\mu (\delta_\xi \chi). \tag{254}$$

We see that if we choose

$$\delta_\xi \chi_a^\dagger = \text{previous change} + F^\dagger \xi_a^\dagger. \tag{255}$$

then the $F^\dagger$ part of the first term in (254) cancels the second term in (253). As regards the second term in (254), we write it as

$$\begin{aligned} \chi^\dagger \mathrm{i}\bar{\sigma}^\mu \partial_\mu \delta_\xi \chi &= \partial_\mu(\chi^\dagger \mathrm{i}\bar{\sigma}^\mu \delta_\xi \chi) - \partial_\mu \chi^\dagger \mathrm{i}\bar{\sigma}^\mu \delta_\xi \chi \\ &= \partial_\mu(\chi^\dagger \mathrm{i}\bar{\sigma}^\mu \xi F) - \partial_\mu \chi^\dagger \mathrm{i}\bar{\sigma}^\mu \xi F \end{aligned} \tag{256}$$

where we have used the dagger of (255), namely

$$\delta_\xi \chi_a = \text{previous change} + \xi_a F. \tag{257}$$

The first term of (256) is a total derivative, leaving the Action invariant, while the second cancels the first term in (253). This has been achieved by allowing χ to 'mix', under SUSY transformations, with the auxiliary field F, while the transformation of ϕ is unaltered.

Let us now re-calculate $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\chi$, including the new terms involving the auxiliary field F. Since the transformation of ϕ is unaltered, $\delta_\eta \delta_\xi \chi$ will be the same as before, in (247), together with an extra term

$$\delta_\eta(\xi_a F) = -\mathrm{i}\xi_a(\eta^\dagger \bar{\sigma}^\mu \partial_\mu \chi). \tag{258}$$

So $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)\chi$ will be as before, in (248), together with the extra terms

$$\mathrm{i}\eta_a(\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi) - \mathrm{i}\xi_a(\eta^\dagger \bar{\sigma}^\mu \partial_\mu \chi). \tag{259}$$

These extra terms precisely cancel the unwanted terms in (248), as required. Similar results hold for the action of $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)$ on ϕ and on F, and so with this enlarged structure including F we can indeed claim that (199) holds as an operator relation, being true when acting on any field of the theory.

8 Interactions: the Wess-Zumino model

The Lagrangian (249) describes a free (left) chiral supermultiplet, with a massless spin-0 field ϕ, a massless L-type spinor field χ, and a non-propagating field F. As we saw in Section 4, we have to put the quarks, leptons and Higgs bosons of the SM, labelled by gauge and flavour degrees of freedom, into chiral supermultiplets, partnered by the appropriate s-particle. So we shall generalize (249) to

$$\mathcal{L}_{\text{free WZ}} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \chi_i^\dagger \mathrm{i}\bar{\sigma}^\mu \partial_\mu \chi_i + F_i^\dagger F_i \tag{260}$$

where the summed-over index i runs over internal degrees of freedom (e.g. flavour, and eventually gauge - see section 10), and is not to be confused (in the case of χ_i) with the spinor component index. The corresponding Action is invariant under the SUSY transformations

$$\delta_\xi \phi_i = \xi \cdot \chi_i, \quad \delta_\xi \chi_i = -\mathrm{i}\sigma^\mu \mathrm{i}\sigma_2 \xi^* \partial_\mu \phi_i + \xi F_i, \quad \delta_\xi F_i = -\mathrm{i}\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i, \tag{261}$$

together with their hermitian conjugates.The obvious next step is to introduce interactions in such a way as to preserve SUSY - that is, invariance of the Lagrangian (or the Action) under the transformations (261). This was first done (for this type of theory, in four dimensions) by Wess and Zumino [24], so the resulting model is called the Wess-Zumino model. We shall largely follow the account given by [22], section 3.2.

We shall impose the important condition that the interactions should be renormalizable. This means that the mass dimension of all interaction terms must not be greater than 4 - or, equivalently, that the coupling constants in the interaction terms should be dimensionless, or have positive dimension (see [17] section 11.8). The most general possible set of renormalizable interactions among the fields ϕ, χ and F is, in fact, rather simple:

$$\mathcal{L}_{\text{int}} = W_i(\phi, \phi^\dagger) F_i - \frac{1}{2} W_{ij}(\phi, \phi^\dagger) \chi_i \cdot \chi_j + \text{hermitian conjugate} \tag{262}$$

where there is a sum on i and on j. Here W_i and W_{ij} are - for the moment - arbitrary functions of the bosonic fields; we shall see that they are actually related, and have a simple form. There is no term in the ϕ_i's alone, because under the transformation (261) this would become some function of the ϕ_i's multiplied by $\delta_\xi \phi_i = \xi \cdot \chi_i$ or $\delta_\xi \phi_i^\dagger = \bar{\xi} \cdot \bar{\chi}$; but these terms do not include any derivatives ∂_μ, or F_i or $F_i^\dagger$ fields, and it is clear by inspection of (261) that they couldn't be cancelled by the transformation of any other term.

As regards W_i and W_{ij}, we first note that since F_i has dimension 2, W_i cannot depend on χ_i, which has dimension 3/2, nor on any power of F_i other than the first, which is already included in (260). Indeed, W_i can involve no higher powers of ϕ_i and $\phi_i^\dagger$ than the second. Similarly, since $\chi_i \cdot \chi_j$ has dimension 3, W_{ij} can only depend on ϕ_i and $\phi_i^\dagger$, and contain no powers higher than the first. Also, since $\chi_i \cdot \chi_j = \chi_j \cdot \chi_i$ (see the first Exercise in Aside (1)), W_{ij} must be symmetric in the indices i and j.

Since we know that the Action for the 'free' part (260) is invariant under (261), we consider now only the change in $\mathcal{L}_{\text{int}}$ under (261), namely $\delta_\xi \mathcal{L}_{\text{int}}$.

First, consider the part involving four spinors, which is

$$-\frac{1}{2}\frac{\partial W_{ij}}{\partial \phi_k}(\xi\cdot\chi_k)(\chi_i\cdot\chi_j)-\frac{1}{2}\frac{\partial W_{ij}}{\partial \phi_k^\dagger}(\bar{\xi}\cdot\bar{\chi}_k)(\chi_i\cdot\chi_j)+\text{hermitian conjugate.} \quad (263)$$

Neither of these terms can be cancelled by the variation of any other term. However, the first term will vanish provided that

$$\frac{\partial W_{ij}}{\partial \phi_k} \text{ is symmetric in } i, j \text{ and } k. \quad (264)$$

The reason is that the identity (241) (with $\lambda \to \chi_k, \zeta \to \chi_i, \rho \to \chi_j$) implies

$$(\xi\cdot\chi_k)(\chi_i\cdot\chi_j)+(\xi\cdot\chi_i)(\chi_j\cdot\chi_k)+(\xi\cdot\chi_j)(\chi_k\cdot\chi_i)=0, \quad (265)$$

from which it follows that if (264) is true, then the first term in (263) will vanish identically. However, there is no corresponding identity for the 4-spinor product in the second term of (263). The only way to get rid of this second term, and thus preserve SUSY for such interactions, is to say that W_{ij} cannot depend on $\phi_k^\dagger$, only on ϕ_k.[10] Thus we now know that W_{ij} must have the form

$$W_{ij} = M_{ij} + y_{ijk}\phi_k \quad (266)$$

where the matrix M_{ij} (which has the dimensions - and significance - of a mass) is symmetric in i and j, and where the 'Yukawa couplings' y_{ijk} are symmetric in i, j and k. It is convenient to write (266) as

$$W_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \quad (267)$$

which is automatically symmetric in i and j, and where[11] (bearing in mind the symmetry properties of W_{ij})

$$W = \frac{1}{2}M_{ij}\phi_i\phi_j + \frac{1}{6}y_{ijk}\phi_i\phi_j\phi_k. \quad (268)$$

Exercise Justify (268).

[10] This is a point of great importance for the MSSM: the SM uses both the Higgs field ϕ and its charge conjugate, which is related to $\phi^\dagger$, but in the MSSM we shall need to have two separate ϕ's.

[11] A linear term of the form $A_l\phi_l$ could be added to (268), consistently with (267) and (266). This is relevant to one model of SUSY breaking - see Section 14.

Next, consider those parts of $\delta_\xi \mathcal{L}_{\rm int}$ which contain one derivative ∂_μ. These are (recall $c = -{\rm i}\sigma_2$)

$$W_i(-{\rm i}\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i) - \frac{1}{2} W_{ij} \{\chi_i^{\rm T} c {\rm i} \sigma^\mu c \xi^*\} \partial_\mu \phi_j + \frac{1}{2} W_{ij} \xi^\dagger c {\rm i} \sigma^{{\rm T}\mu} \partial_\mu \phi_i c \chi_j. \quad (269)$$

Consider the expression in curly brackets, $\{\chi_i^{\rm T} \ldots \xi^*\}$. Since this is a single quantity (after evaluating the matrix products), it is equal to its transpose, which is

$$-\xi^\dagger c {\rm i} \sigma^{\mu{\rm T}} c \chi_i = \xi^\dagger {\rm i} \bar{\sigma}^\mu \chi_i \quad (270)$$

where the first minus sign comes from interchanging two fermionic quantities, and the second equality uses the result $c\sigma^{\mu{\rm T}} c = -\bar{\sigma}^\mu$ (c.f. (193)). So the second term in (269) is

$$-\frac{1}{2} W_{ij} {\rm i} \xi^\dagger \bar{\sigma}^\mu \chi_i \partial_\mu \phi_j, \quad (271)$$

and the third term is

$$\frac{1}{2} W_{ij} \xi^\dagger c {\rm i} \sigma^{\mu{\rm T}} c \chi_j \partial_\mu \phi_i = -\frac{1}{2} W_{ij} {\rm i} \xi^\dagger \bar{\sigma}^\mu \chi_j \partial_\mu \phi_i. \quad (272)$$

So these two terms add to give

$$-W_{ij} {\rm i} \xi^\dagger \bar{\sigma}^\mu \chi_i \partial_\mu \phi_j = -{\rm i} \xi^\dagger \bar{\sigma}^\mu \chi_i \partial_\mu \left(\frac{\partial W}{\partial \phi_i} \right), \quad (273)$$

where in the second equality we have used

$$\partial_\mu \left(\frac{\partial W}{\partial \phi_i} \right) = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j = W_{ij} \partial_\mu \phi_j. \quad (274)$$

Altogether, then, (269) has become

$$-{\rm i} W_i \xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i - {\rm i} \xi^\dagger \bar{\sigma}^\mu \chi_i \partial_\mu \left(\frac{\partial W}{\partial \phi_i} \right). \quad (275)$$

This variation cannot be cancelled by anything else, and our only chance of saving SUSY is to have it equal a total derivative (giving an invariant Action, as usual). The condition for (275) to be a total derivative is that W_i should have the form

$$W_i = \frac{\partial W}{\partial \phi_i}, \quad (276)$$

in which case (275) becomes

$$\partial_\mu \{ \frac{\partial W}{\partial \phi_i} (-\mathrm{i} \xi^\dagger \bar{\sigma}^\mu \chi_j) \}. \tag{277}$$

Referring to (268), we see that the condition (276) implies

$$W_i = M_{ij}\phi_j + \frac{1}{2} y_{ijk}\phi_j\phi_k \tag{278}$$

together with a possible constant term A_i (see footnote 10).

Exercise Verify that the remaining terms in $\delta_\xi \mathcal{L}$ do cancel.

In summary, we have found conditions on W_i and W_{ij} (namely equations(276) and (267) with W given by (268)) such that the interactions (262) give an Action which is invariant under the SUSY transformations (261). Consider now the part of the complete Lagrangian (including (260)) containing F_i and $F_i^\dagger$, which is just $F_i F_i^\dagger + W_i F_i + W_i^\dagger F_i^\dagger$. Since this contains no gradients, the Euler-Lagrange equations for F_i and $F_i^\dagger$ are simply

$$\frac{\partial \mathcal{L}}{\partial F_i} = 0, \text{ or } F_i^\dagger + W_i = 0. \tag{279}$$

Hence $F_i = -W_i^\dagger$, and similarly $F_i^\dagger = -W_i$. These relations, coming from the E-L equations, involve (again) no derivatives, and hence the canonical commutation relations will not be affected, and it is permissible to replace F_i and $F_i^\dagger$ in the Lagrangian by these values determined from the E-L equations. This results in the complete (Wess-Zumino [24]) Lagrangian now having the form

$$\mathcal{L}_{\mathrm{WZ}} = \mathcal{L}_{\text{free WZ}} - |W_i|^2 - \frac{1}{2}\{W_{ij}\chi_i \cdot \chi_j + \mathrm{h.c}\} \tag{280}$$

where 'h.c.' means hermitian conjugate.

It is worth spending a little time looking in more detail at the model of (280). For simplicity we shall discuss just one supermultiplet, dropping the indices i and j. First, consider the terms which are quadratic in the fields ϕ and χ, which correspond to kinetic and mass terms (rather than interactions proper). This will give us an opportunity to learn about mass terms for two-component spinors. The quadratic terms for a single supermultiplet are

$$\mathcal{L}_{\mathrm{WZ,quad}} = \partial_\mu\phi^\dagger\partial^\mu\phi + \chi^\dagger \mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi - MM^*\phi^\dagger\phi - \frac{1}{2}M\chi^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi - \frac{1}{2}M^*\chi^\dagger(\mathrm{i}\sigma_2)\chi^{\dagger\mathrm{T}} \tag{281}$$

where we have reverted to the explicit forms of the spinor products. In (281), $\chi^\dagger$ is as given in (142), while evidently

$$\chi^{\dagger \mathrm{T}} = \begin{pmatrix} \chi_1^\dagger \\ \chi_2^\dagger \end{pmatrix} \tag{282}$$

where '1' and '2', of course, label the spinor components. The E-L equation for $\phi^\dagger$ is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^\dagger} = 0, \tag{283}$$

which leads immediately to

$$\partial_\mu \partial^\mu \phi + |M|^2 \phi = 0, \tag{284}$$

which is just the standard free Klein-Gordon equation for a spinless field of mass $|M|$.

In considering the analogous E-L equation for (say) $\chi^\dagger$, we need to take care in evaluating (functional) derivatives of $\mathcal{L}$ with respect to fields such as χ or $\chi^\dagger$ which anticommute. Consider the term $-(1/2)M\chi\cdot\chi$ in (281), which is

$$-\frac{1}{2}M(\chi_1\chi_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = -\frac{1}{2}M(-\chi_1\chi_2+\chi_2\chi_1) = -M\chi_2\chi_1 = +M\chi_1\chi_2. \tag{285}$$

We define

$$\frac{\partial}{\partial \chi_1}(\chi_1\chi_2) = \chi_2, \tag{286}$$

and then necessarily

$$\frac{\partial}{\partial \chi_2}(\chi_1\chi_2) = -\chi_1. \tag{287}$$

Hence

$$\frac{\partial}{\partial \chi_1}\{-\frac{1}{2}M\chi\cdot\chi\} = M\chi_2, \tag{288}$$

and

$$\frac{\partial}{\partial \chi_2}\{-\frac{1}{2}M\chi\cdot\chi\} = -M\chi_1. \tag{289}$$

Equations (288) and (289) can be combined as

$$\frac{\partial}{\partial \chi_a}\{-\frac{1}{2}M\chi\cdot\chi\} = M(\mathrm{i}\sigma_2\chi)_a. \tag{290}$$

Exercise Show similarly that

$$\frac{\partial}{\partial \chi^{\dagger}_{\dot{a}}}\{-\frac{1}{2}M^* \chi^{\dagger}\mathrm{i}\sigma_2\chi^{\dagger\mathrm{T}}\} = M^*(-\mathrm{i}\sigma_2\chi^{\dagger\mathrm{T}})_{\dot{a}}. \tag{291}$$

We are now ready to consider the E-L equation for $\chi^{\dagger}$, which is

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\chi^{\dagger}_{\dot{a}})}\right) - \frac{\partial\mathcal{L}}{\partial\chi^{\dagger}_{\dot{a}}} = 0. \tag{292}$$

Using just the quadratic parts (281) this yields

$$\mathrm{i}\bar{\sigma}^{\mu}\partial_{\mu}\chi = M^*\mathrm{i}\sigma_2\chi^{\dagger\mathrm{T}}. \tag{293}$$

From Notational Aside (3) in section 2.4 we know that χ transforms by $V^{-1\dagger}$, and hence $\chi^{\dagger\mathrm{T}}$ transforms by $V^{-1\mathrm{T}}$, which is the same as a 'lower dotted' spinor of type $\psi_{\dot{a}}$. The lower dotted index is raised by the matrix $\mathrm{i}\sigma_2$. Hence the RHS of (293) transforms like a $\psi^{\dot{a}}$ spinor, and this is consistent with the LHS, by (68).

Exercise Similarly, show that

$$\mathrm{i}\sigma^{\mu}\partial_{\mu}(\mathrm{i}\sigma_2\chi^{\dagger\mathrm{T}}) = M\chi. \tag{294}$$

It follows from (293) and (294) that

$$\begin{aligned}\mathrm{i}\sigma^{\mu}\partial_{\mu}(\mathrm{i}\bar{\sigma}^{\nu}\partial_{\nu}\chi) &= \mathrm{i}\sigma^{\mu}\partial_{\mu}(M^*\mathrm{i}\sigma_2\chi^{\dagger\mathrm{T}}) \\ &= |M|^2\chi. \end{aligned}\tag{295}$$

So, using (146) on the LHS we have simply

$$\partial_{\mu}\partial^{\mu}\chi + |M|^2\chi = 0, \tag{296}$$

which shows that the χ field also has mass $|M|$. So we have verified that the quadratic parts (281) describe a free spin-0 and spin-1/2 field which are degenerate, both having mass $|M|$. It is perhaps worth pointing out that, although we started (for simplicity) with massless fields, we now see that it is perfectly possible to have massive supersymmetric theories, the bosonic and fermionic superpartners having (of course) the same mass.

Aside: Majorana version This seems as good a moment as any to say a few words about a possible alternative formalism, in which one uses 4-component Majorana spinor fields (see section 2.3) rather than the 2-component L- or R-spinor fields we have been using up till now (and will continue to use). First recall from section 2.3 that the number of degrees of freedom is the same, because the Majorana spinor $\Psi_{\rm M}^{\chi}$ of equation (111), for example, is constructed explicitly from χ; so we expect that it must be possible to re-write everything involving χ's in terms of $\Psi_{\rm M}^{\chi}$'s. To check this, consider first the fermion mass term in (281). We may take M to be real, by absorbing any phase into the undefined phase of χ. It then follows at once from equation (113) that

$$-\frac{1}{2}M[\chi^{\rm T}(-{\rm i}\sigma_2)\chi + \chi^{\dagger}({\rm i}\sigma_2)\chi^{\dagger {\rm T}}] = -\frac{1}{2}M\bar{\Psi}_{\rm M}^{\chi}\Psi_{\rm M}^{\chi}. \tag{297}$$

The kinetic term is a little more involved. We have

$$\begin{aligned}\Psi_{\rm M}^{\chi\dagger}\beta{\rm i}\gamma^{\mu}\partial_{\mu}\Psi_{\rm m}^{\chi} &= (\chi^{\rm T}(-{\rm i}\sigma_2)\ \ \chi^{\dagger})\begin{pmatrix} {\rm i}\sigma^{\mu}\partial_{\mu} & 0 \\ 0 & {\rm i}\bar{\sigma}^{\mu}\partial_{\mu}\end{pmatrix}\begin{pmatrix} {\rm i}\sigma_2\chi^{\dagger {\rm T}} \\ \chi \end{pmatrix} \\ &= [\chi^{\rm T}(-{\rm i}\sigma_2){\rm i}\sigma^{\mu}\partial_{\mu}{\rm i}\sigma_2\chi^{\dagger {\rm T}}] + \chi^{\dagger}{\rm i}\bar{\sigma}^{\mu}\partial_{\mu}\chi. \end{aligned} \tag{298}$$

The first term can be maniplulated in the now familiar way, by taking its transpose and using (193); one finds that it is just equal to the second term, and so

$$\chi^{\dagger}{\rm i}\bar{\sigma}^{\mu}\partial_{\mu}\chi = \frac{1}{2}\bar{\Psi}_{\rm M}^{\chi}{\rm i}\gamma^{\mu}\partial_{\mu}\Psi_{\rm M}^{\chi}. \tag{299}$$

Thus the kinetic term and the mass terms have been re-written in terms of the Majorana spinor $\Psi_{\rm M}^{\chi}$; a factor of '1/2' appears in the Majorana version of both the kinetic and the mass terms. The other terms can be treated similarly.

One might wonder about a Majorana version of the SUSY algebra. Just as we can construct a 4-component Majorana spinor from a χ (or of course a ψ), so we can make a 4-component Majorana spinor charge $Q_{\rm M}$ from our L-type spinor charge Q, by setting (c.f. (111))

$$Q_{\rm M} = \begin{pmatrix} {\rm i}\sigma_2 Q^{\dagger {\rm T}} \\ Q \end{pmatrix} = \begin{pmatrix} Q_2^{\dagger} \\ -Q_1^{\dagger} \\ Q_1 \\ Q_2 \end{pmatrix}. \tag{300}$$

Let's call the components of this $Q_{{\rm M}\alpha}$, so that $Q_{\rm M1} = Q_2^{\dagger}$, $Q_{\rm M2} = -Q_1^{\dagger}$, etc. It is not completely obvious (to me, at any rate) what the anticommutation relations of the $Q_{{\rm M}\alpha}$'s ought to be (given those of the Q_a's and $Q_a^{\dagger}$'s), but the answer turns out to be

$$\{Q_{{\rm M}\alpha}, Q_{{\rm M}\beta}\} = (\gamma^{\mu}({\rm i}\gamma^2\gamma^0))_{\alpha\beta}P_{\mu}, \tag{301}$$

as can be checked with the help of (203), (204), (210) and (300). Note that '$-{\rm i}\gamma^2\gamma^0$' is the 'metric' we met in section 2.3. The anticommutator (301) can be re-written rather more suggestively as

$$\{Q_{{\rm M}\alpha}, \bar{Q}_{{\rm M}\beta}\} = (\gamma^{\mu})_{\alpha\beta}P_{\mu} \tag{302}$$

where (compare (118))

$$\bar{Q}_{{\rm M}\beta} = (Q_{\rm M}^{\rm T}(-{\rm i}\gamma^2\gamma^0))_{\beta} = (Q_{\rm M}^{\dagger}\gamma^0)_{\beta}. \tag{303}$$

Next, let's consider briefly the interaction terms in (280), again just for the case of one chiral superfield. These terms are

$$-|M\phi + \frac{1}{2}y\phi^2|^2 - \frac{1}{2}\{(M + y\phi)\chi \cdot \chi + \text{h.c.}\}. \tag{304}$$

In addition to the quadratic parts $|M|^2\phi^\dagger\phi$ and $-(1/2)M\chi \cdot \chi + \text{h.c.}$ which we have just discussed, (304) contains three true interactions, namely

(i) a 'cubic' interaction among the ϕ fields,

$$-\frac{1}{2}(My^*\phi\phi^{\dagger 2} + M^*y\phi^2\phi^\dagger); \tag{305}$$

(ii) a 'quartic' interaction among the ϕ fields,

$$-\frac{1}{4}|y|^2\phi^2\phi^{\dagger 2}; \tag{306}$$

(iii) a Yukawa-type coupling between the ϕ and χ fields,

$$-\frac{1}{2}\{y\phi\chi \cdot \chi + \text{h.c.}\}. \tag{307}$$

It is noteworthy that the same coupling parameter y enters into the cubic and quartic bosonic interactions (305) and (306), as well as the Yukawa-like fermion-boson interaction (307). In particular, the quartic coupling constant appearing in (306) is equal to the square of the Yukawa coupling in (307). This is exactly the relationship noted in (20), as being required for a cancellation between the bosonic self-energy graph of figure 1, and the fermion-antifermion loop contribution to this self energy (see Peskin [25] section 3.3 for details of the calculation in the massless case).

The cancellation of radiative (loop) corrections in models of this type is actually a more general phenomenon: the only non-vanishing radiative corrections to the interaction terms (including masses) are field rescalings ([26], [27]).

Thus far in these lectures we have adopted (pretty much) a 'brute force', or 'do-it-yourself' approach, retreating quite often to explicit matrix expressions, and arriving at SUSY-invariant Lagrangians by direct construction. We might well wonder whether there is not a more general procedure which would somehow automatically generate SUSY-invariant interactions. Such a procedure is indeed available within the superfield approach, to which we now turn. This formalism has other advantages too. First, it gives us more insight into SUSY transformations, and their linkage with space-time translations. Second, the appearance of the auxiliary field F is better motivated. And finally, and in practice rather importantly, the superfield notation is widely used in discussions of the MSSM.

9 Superfields

9.1 SUSY transformations on fields

By way of a warm-up exercise, let's recall some things about space-time translations. A translation of coordinates takes the form

$$x'^{\mu} = x^{\mu} + a^{\mu} \tag{308}$$

where a^{μ} is a constant 4-vector. In the unprimed coordinate frame, observers use states $|\alpha\rangle, |\beta\rangle, \ldots$, and deal with amplitudes of the form $\langle\beta|\phi(x)|\alpha\rangle$, where $\phi(x)$ is scalar field. In the primed frame, observers evaluate ϕ at x', and use states $|\alpha\rangle' = U|\alpha\rangle, \ldots$, where U is unitary, in such a way that their matrix elements (and hence transition probabilities) are equal to those calculated in the unprimed frame:

$$\langle\beta|U^{-1}\phi(x')U|\alpha\rangle = \langle\beta|\phi(x)|\alpha\rangle. \tag{309}$$

Since this has to be true for all pairs of states, we can deduce

$$U^{-1}\phi(x')U = \phi(x) \tag{310}$$

or

$$U\phi(x)U^{-1} = \phi(x') = \phi(x+a). \tag{311}$$

For an infinitesimal translation, $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$, we may write

$$U = 1 + \mathrm{i}\epsilon_{\mu}P^{\mu} \tag{312}$$

where the four operators P^{μ} are the *generators* of this transformation (c.f. (169)); (311) then becomes

$$\begin{aligned}(1+\mathrm{i}\epsilon_{\mu}P^{\mu})\phi(x)(1-\mathrm{i}\epsilon_{\mu}P^{\mu}) &= \phi(x^{\mu}+\epsilon^{\mu}) \\ &= \phi(x^{\mu}) + \epsilon^{\mu}\frac{\partial\phi}{\partial x^{\mu}};\end{aligned} \tag{313}$$

that is,

$$\phi(x) + \delta\phi(x) = \phi(x) + \epsilon^\mu \partial_\mu \phi(x), \tag{314}$$

where (c.f. (172))

$$\delta\phi(x) = \mathrm{i}\epsilon_\mu [P^\mu, \phi(x)] = \epsilon_\mu \partial^\mu \phi(x). \tag{315}$$

We therefore obtain the fundamental relation

$$\mathrm{i}[P^\mu, \phi(x)] = \partial^\mu \phi(x). \tag{316}$$

In (316) the P^μ are constructed from field operators - for example P^0 is the Hamiltonian, which is the spatial integral of the appropriate Hamiltonian density - and the canonical commutation relations of the fields must be consistent with (316). We used (316) in section 5.2 - see (198).

But we can also look at (315) another way: we can say

$$\delta\phi = \epsilon^\mu \partial_\mu \phi = (1 - \mathrm{i}\epsilon^\mu \hat{P}_\mu)\phi, \tag{317}$$

where $\hat{P}_\mu$ is a *differential operator* acting on the *argument* of ϕ. Clearly $\hat{P}^\mu = \mathrm{i}\partial^\mu$ as usual.

We are now going to carry out analogous steps using SUSY transformations. This will entail enlarging the space of coordinates x^μ on which the fields can depend to include also *fermionic* degrees of freedom - specifically, spinor degrees of freedom θ and θ^*. Fields which depend on these spinorial degrees of freedom as well as on x are called *superfields*, and the extended space of x^μ, θ and θ^* is called *superspace*. Just as the operators P^μ generate (via the unitary operator U of (311)) a shift in the space-time argument of ϕ, so we expect to be able to construct analogous unitary operators from Q and $Q^\dagger$ which should similarly effect shifts in the spinorial arguments of the field. Actually, we shall see that the matter is rather more interesting than that, because a shift will also be induced in the space-time argument x; this is actually to be expected, given the link between the SUSY generators and the space-time translation generators P^μ embodied in the SUSY algebra (210). Having constructed these operators and seen what shifts they induce, we shall then look at the analogue of (317), and arrive at a differential operator representation of the SUSY generators, say $\hat{Q}$ and $\hat{Q}^\dagger$, the differentials in this case being with respect to the spinor degrees of freedom of superspace (i.e. θ and θ^*). We can close the circle by checking that the generators $\hat{Q}$ and $\hat{Q}^\dagger$ defined this way do indeed satisfy the SUSY algebra (210) (this step being analogous to checking that the angular momentum operators $\hat{\boldsymbol{L}} = -\mathrm{i}\boldsymbol{x} \times \boldsymbol{\nabla}$ obey the SU(2) algebra).

The basic idea is simple. We may write (311) as

$$\mathrm{e}^{\mathrm{i}x\cdot P}\phi(0)\mathrm{e}^{-\mathrm{i}x\cdot P} = \phi(x). \tag{318}$$

In analogy to this, let's consider a 'U' for a SUSY transformation which has the form

$$U(x,\theta,\theta^*) = \mathrm{e}^{\mathrm{i}x\cdot P}\mathrm{e}^{\mathrm{i}\theta\cdot Q}\mathrm{e}^{\mathrm{i}\bar{\theta}\cdot\bar{Q}}. \tag{319}$$

Here Q and Q^* (or $Q^{\dagger\mathrm{T}}$) are the (spinorial) SUSY generators met in section 5.2, and θ and θ^* are spinor degrees of freedom associated with these SUSY 'translations'. Note that, as usual,

$$\theta\cdot Q \equiv \theta^{\mathrm{T}}(-\mathrm{i}\sigma_2)Q, \tag{320}$$

and

$$\bar{\theta}\cdot\bar{Q} \equiv \theta^{\dagger}(\mathrm{i}\sigma_2)Q^{\dagger\mathrm{T}}. \tag{321}$$

When the field $\phi(0)$ is transformed via '$U(x,\theta,\theta^*)\phi(0)U^{-1}(x,\theta,\theta^*)$', we expect to obtain a ϕ which is a function of x, but also now of the 'fermionic coordinates' θ and θ^*, so we shall write it as Φ, a superfield:

$$U(x,\theta,\theta^*)\Phi(0)U^{-1}(x,\theta,\theta^*) = \Phi(x,\theta,\theta^*). \tag{322}$$

Now consider the product of two ordinary spatial translation operators:

$$\mathrm{e}^{\mathrm{i}x\cdot P}\mathrm{e}^{\mathrm{i}a\cdot P} = \mathrm{e}^{\mathrm{i}(x+a)\cdot P}, \tag{323}$$

since all the components of P commute. We say that this product of translation operators 'induces the transformation $x \to x + a$ in parameter (coordinate) space'. We are going to generalize this by multiplying two U's of the form (319) together, and asking: *what transformations are induced in the space-time coordinates, and in the spinorial degrees of freedom*?

Such a product is

$$U(a,\xi,\xi^*)U(x,\theta,\theta^*) = \mathrm{e}^{\mathrm{i}a\cdot P}\mathrm{e}^{\mathrm{i}\xi\cdot Q}\mathrm{e}^{\mathrm{i}\bar{\xi}\cdot\bar{Q}}\mathrm{e}^{\mathrm{i}x\cdot P}\mathrm{e}^{\mathrm{i}\theta\cdot Q}\mathrm{e}^{\mathrm{i}\bar{\theta}\cdot\bar{Q}}. \tag{324}$$

Unlike in (323), it is *not* possible simply to combine all the exponents here, because the operators Q and $Q^{\dagger}$ do not commute - rather, they satisfy the algebra (210). However, as noted in section 5.2, the components of P do commute with those of Q and $Q^\dagger$, so we can freely move the operator $\exp[\mathrm{i}x \cdot P]$ through the operators to the left of it, and combine it with $\exp[\mathrm{i}a \cdot P]$ to yield $\exp[\mathrm{i}(x+a) \cdot P]$, as in (323). The non-trivial part is

$$\mathrm{e}^{\mathrm{i}\xi \cdot Q} \mathrm{e}^{\mathrm{i}\bar{\xi} \cdot \bar{Q}} \mathrm{e}^{\mathrm{i}\theta \cdot Q} \mathrm{e}^{\mathrm{i}\bar{\theta} \cdot \bar{Q}}. \tag{325}$$

To simplify this, we use the Baker-Campbell-Hausdorff identity:

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]+\frac{1}{6}[[A,B],B]+\ldots}. \tag{326}$$

Let's apply (326) to the first two products in (325), taking $A = \mathrm{i}\xi \cdot Q$ and $B = \mathrm{i}\bar{\xi} \cdot \bar{Q}$. We get

$$\mathrm{e}^{\mathrm{i}\xi \cdot Q} \mathrm{e}^{\mathrm{i}\bar{\xi} \cdot \bar{Q}} = \mathrm{e}^{\mathrm{i}\xi \cdot Q + \mathrm{i}\bar{\xi} \cdot \bar{Q} - \frac{1}{2}[\xi \cdot Q, \bar{\xi} \cdot \bar{Q}]+\ldots} \tag{327}$$

Writing out the commutator in detail, we have

$$\begin{aligned}
[\xi \cdot Q, \bar{\xi} \cdot \bar{Q}] &= [\xi^1 Q_1 + \xi^2 Q_2, \xi_1^* Q_2^\dagger - \xi_2^* Q_1^\dagger] \\
&= [\xi^1 Q_1 + \xi^2 Q_2, -\xi^{2*} Q_2^\dagger - \xi^{1*} Q_1^\dagger] \\
&= [\xi^a Q_a, -\xi^{b*} Q_b^\dagger] \\
&= -\xi^a Q_a \xi^{b*} Q_b^\dagger + \xi^{b*} Q_b^\dagger \xi^a Q_a \\
&= \xi^a \xi^{b*} (Q_a Q_b^\dagger + Q_b^\dagger Q_a) \\
&= \xi^a \xi^{b*} (\sigma^\mu)_{ab} P_\mu
\end{aligned} \tag{328}$$

using (210). This means that life is not so bad after all: since P commutes with Q and $Q^\dagger$, there are no more terms in the B-C-H identity to calculate, and we have established the result

$$\mathrm{e}^{\mathrm{i}\xi \cdot Q} \mathrm{e}^{\mathrm{i}\bar{\xi} \cdot \bar{Q}} = \mathrm{e}^{\mathrm{i}A \cdot P} \mathrm{e}^{\mathrm{i}(\xi \cdot Q + \bar{\xi} \cdot \bar{Q})}, \tag{329}$$

where

$$A^\mu = \frac{1}{2} \mathrm{i} \xi^a (\sigma^\mu)_{ab} \xi^{b*}. \tag{330}$$

Note that we have moved the $\exp[\mathrm{i}A \cdot P]$ expression to the front, using the fact that P commutes with Q and $Q^\dagger$.

We pause in the development to comment immediately on (329): under this kind of transformation, the spacetime coordinate acquires an additional shift, namely A^μ, which is built out of the spinor parameters ξ and ξ^*.

Exercise Explain why $\xi^a (\sigma^\mu)_{ab} \xi^{b*}$ is a 4-vector.

Continuing on with the reduction of (325), we consider

$$e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} = e^{iA\cdot P}e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q})}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}}, \tag{331}$$

and apply B-C-H to the second and third terms in the product on the RHS:

$$\begin{aligned} e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q})}e^{i\theta\cdot Q} &= e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q)-\frac{1}{2}[\xi\cdot Q+\bar{\xi}\cdot\bar{Q},\theta\cdot Q]+\ldots} \\ &= e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q)+\frac{1}{2}\theta^a(\sigma^\mu)_{ab}\xi^{b*}P_\mu}, \end{aligned} \tag{332}$$

using (328) and (201). The expression (325) is now

$$e^{-\frac{1}{2}\xi^a(\sigma^\mu)_{ab}\xi^{b*}P_\mu+\frac{1}{2}\theta^a(\sigma^\mu)_{ab}\xi^{b*}P_\mu}e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q)}e^{i\bar{\theta}\cdot\bar{Q}}. \tag{333}$$

We now apply B-C-H 'backwards' to the penultimate factor:

$$e^{i(\xi\cdot Q+\bar{\xi}\cdot\bar{Q}+\theta\cdot Q)} = e^{i(\xi+\theta)\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{\frac{1}{2}[(\xi+\theta)\cdot Q,\bar{\xi}\cdot\bar{Q}]}. \tag{334}$$

Evaluating the commutator as before leads to the final result

$$e^{i\xi\cdot Q}e^{i\bar{\xi}\cdot\bar{Q}}e^{i\theta\cdot Q}e^{i\bar{\theta}\cdot\bar{Q}} = e^{i[-i\theta^a(\sigma^\mu)_{ab}\xi^{b*}P_\mu]}e^{i(\xi+\theta)\cdot Q}e^{i(\bar{\xi}+\bar{\theta})\cdot\bar{Q}} \tag{335}$$

where in the final product we have again used (201) to add the exponents.

Exercise Check (335).

Inspecting (335), we infer that the product $U(a,\xi,\xi^*)U(x,\theta,\theta^*)$ induces the transformations

$$\begin{aligned} 0 &\to \theta \to \theta+\xi \\ 0 &\to \theta^* \to \theta^*+\xi^* \\ 0 &\to x^\mu \to x^\mu + a^\mu - i\theta^a(\sigma^\mu)_{ab}\xi^{b*}. \end{aligned} \tag{336}$$

That is to say,

$$U(a,\xi,\xi^*)U(x,\theta,\theta^*)\Phi(0)U^{-1}(x,\theta,\theta^*)U^{-1}(a,\xi,\xi^*) = U(a,\xi,\xi^*)\Phi(x,\theta,\theta^*)U^{-1}(a,\xi,\xi^*) \tag{337}$$

$$= \Phi(x^\mu + a^\mu - i\theta^a(\sigma^\mu)_{ab}\xi^{b*}, \theta+\xi, \theta^*+\xi^*). \tag{338}$$

We now proceed with the second part of our SUSY extension of ordinary translations, namely the analogue of equation (317).

9.2 A differential operator representation of the SUSY generators

Equation (317) provided us with a differential operator representation of the generators of translations, by considering an infinitesimal displacement (the reader might care to recall similar steps for infinitesimal rotations, which lead to the usual representation of the angular momentum operators as $\hat{\boldsymbol{L}} = -\mathrm{i}\boldsymbol{x} \times \boldsymbol{\nabla}$). Analogous steps applied to (338) will lead to an explicit representation of the SUSY generators as certain differential operators. We will then check that they satisfy the anticommutation relations (210), just as the angular momentum operators satisfy the familiar SU(2) algebra.

We regard (338) as the result of applying the transformation parametrized by a, ξ, ξ^* to the field $\Phi(x, \theta, \theta^*)$. For an infinitesimal such transformation, the change in Φ is

$$\delta\Phi = -\mathrm{i}\theta^a(\sigma^\mu)_{ab}\xi^{b*}\partial_\mu\Phi + \xi^a\frac{\partial\Phi}{\partial\theta^a} + \xi^*_a\frac{\partial\Phi}{\partial\theta^*_a}. \tag{339}$$

Notation check In Notational Aside (1), section 2.2, we stated the convention for summing over undotted labels, which was 'diagonally from top left to bottom right, as in $\xi^a\chi_a$'. For (339) to be consistent with this convention, it should be the case that the derivative $\partial/\partial\theta^a$ behaves as a 'χ_a'-type object. A quick way of seeing that this is likely to be OK is simply to calculate

$$\frac{\partial}{\partial\theta^a}(\theta^b\theta_b) \tag{340}$$

Consider $a = 1$. Now $\theta^b\theta_b = -2\theta^1\theta^2$ and so

$$\frac{\partial}{\partial\theta^1}(\theta^b\theta_b) = -2\theta^2 = 2\theta_1. \tag{341}$$

Similarly,

$$\frac{\partial}{\partial\theta^2}(\theta^b\theta_b) = 2\theta_2, \tag{342}$$

or generally

$$\frac{\partial}{\partial\theta^a}(\theta\cdot\theta) = 2\theta_a, \tag{343}$$

which at least checks the claim in this simple case. Similarly, in Notational Aside (2), we stated the convention for products of dotted indices as $\psi_{\dot{a}}\zeta^{\dot{a}}$, and in Aside (3) we related dotted-index quantities to complex conjugated quantities, via $\bar{\chi}_{\dot{a}} \equiv \chi^*_a$. Consider the last term in (339): since $\xi^*_a \equiv \bar{\xi}_{\dot{a}}$, it should be the case that $\partial/\partial\theta^*_a$ behaves as a '$\bar{\zeta}^{\dot{a}}$'-type (or equivalently as a 'ζ^{a*}') object.

Exercise Check this by considering $\partial/\partial\theta^*_a(\bar{\theta}\cdot\bar{\theta})$.

In analogy with (317), we want to write (339) as

$$\delta\Phi = (-\mathrm{i}\xi \cdot \hat{Q} - \mathrm{i}\bar{\xi} \cdot \bar{\hat{Q}})\Phi = (-\mathrm{i}\xi^a \hat{Q}_a - \mathrm{i}\xi^*_a \hat{Q}^{\dagger a})\Phi. \tag{344}$$

Comparing (339) with (344), it is easy to identify $\hat{Q}_a$ as

$$\hat{Q}_a = \mathrm{i}\frac{\partial}{\partial\theta^a}. \tag{345}$$

There is a similar term in $\hat{Q}^{\dagger a}$, namely

$$\hat{Q}^{\dagger a} = \mathrm{i}\frac{\partial}{\partial\theta^*_a}, \tag{346}$$

and in addition another contribution given by

$$-\mathrm{i}\xi^*_a \hat{Q}^{\dagger a}\Phi = -\mathrm{i}\theta^a(\sigma^\mu)_{ab}\xi^{b*}\partial_\mu\Phi. \tag{347}$$

Our present objective is to verify that these $\hat{Q}$ operators satisfy the SUSY anticommutation relations (210). To do this, we need to deal with the lower-index operators $\hat{Q}^\dagger_a$ rather than $\hat{Q}^{\dagger a}$.

Exercise Check that (346) can be converted to

$$\hat{Q}^\dagger_a = -\mathrm{i}\frac{\partial}{\partial\theta^{a*}}. \tag{348}$$

As regards (347), we use $\xi^*_a \hat{Q}^{\dagger a} = -\xi^{a*}\hat{Q}^\dagger_a$ (see Exercise (b) in Notational Aside (3)) , and $\theta^a\xi^{b*} = -\xi^{b*}\theta^a$, followed by an interchange of the indices a and b to give finally

$$\hat{Q}^\dagger_a = -\mathrm{i}\frac{\partial}{\partial\theta^{a*}} + \theta^b(\sigma^\mu)_{ba}\partial_\mu. \tag{349}$$

It is now a useful **Exercise** to check that the explicit representations (345) and (349) do indeed result in the required relations

$$[\hat{Q}_a, \hat{Q}^\dagger_b] = \mathrm{i}(\sigma^\mu)_{ab}\partial_\mu = (\sigma^\mu)_{ab}\hat{P}_\mu, \tag{350}$$

as well as $[\hat{Q}_a, \hat{Q}_b] = [\hat{Q}^\dagger_a, \hat{Q}^\dagger_b] = 0$. We have therefore produced a representation of the SUSY generators in terms of fermionic parameters, and derivatives with respect to them, which satisfies the SUSY algebra (210).

9.3 Chiral superfields, and their (chiral) component fields

Suppose now that a superfield $\Phi(x,\theta,\theta^*)$ does not in fact depend on θ^*, only on x and θ: $\Phi(x,\theta)$[12]. Consider the expansion of such a Φ in powers of θ. Because of the fermionic nature of the variables θ, which implies that $(\theta_1)^2 = (\theta_2)^2 = 0$, there will only be three terms in the expansion, namely a term independent of θ, a term linear in θ and a term involving $\frac{1}{2}\theta\cdot\theta = -\theta_1\theta_2$:

$$\Phi(x,\theta) = \phi(x) + \theta\cdot\chi(x) + \frac{1}{2}\theta\cdot\theta F(x). \tag{351}$$

This is the most general form of such a superfield (which depends only on x and θ), and it depends on three *component* fields, ϕ, χ and F. We have of course deliberately given these component fields the same names as those in our previous chiral supermultiplet. We shall now verify that the transformation law (344) for the superfield Φ, with $\hat{Q}$ given by (345) and $\hat{Q}^\dagger$ by (349), implies precisely the previous transformations (261) for the component fields ϕ, χ and F, thus justifying this identification.

We have

$$\begin{aligned}
\delta\Phi &= (-\mathrm{i}\xi^a\hat{Q}_a - \mathrm{i}\xi^*_a\hat{Q}^{\dagger a})\Phi = (-\mathrm{i}\xi^a\hat{Q}_a + \mathrm{i}\xi^{a*}\hat{Q}^\dagger_a)\Phi \\
&= (\xi^a\frac{\partial}{\partial\theta_a} + \xi^{a*}\frac{\partial}{\partial\theta^{a*}} + \mathrm{i}\xi^{a*}\theta^b(\sigma^\mu)_{ba}\partial_\mu)[\phi(x) + \theta^c\chi_c + \frac{1}{2}\theta\cdot\theta F] \\
&\equiv \delta_\xi\phi + \theta^a\delta_\xi\chi_a + \frac{1}{2}\theta\cdot\theta\delta_\xi F.
\end{aligned} \tag{352}$$

We evaluate the derivatives in the second line as follows. First, we have

$$\frac{\partial}{\partial\theta^a}[\theta^c\chi_c + \frac{1}{2}\theta\cdot\theta] = \chi_a + \theta_a, \tag{353}$$

using (343), so that the $\xi^a\partial/\partial\theta^a$ term yields

$$\xi^a\chi_a + \theta^a\xi_a F. \tag{354}$$

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[12] Such a superfield is usually called a 'left-chiral superfield' and denoted by Φ_{L}, because (see (351)) it contains only the L-type spinor χ, and not the R-type spinor ψ. By the same token, the transformation (319) could be denoted by U_{L}, while the representations (345) and (349) of the generators $\hat{Q}_a$ and $\hat{Q}^\dagger_a$ could be called the L-representation of these operators. For more on this see the comment in small print soon after (360). We shall only be dealing with left-chiral superfields and we shall therefore omit the L-subscript.

Next, the term in $\partial/\partial\theta^{a*}$ vanishes since Φ doesn't depend on θ^*. The remaining term is

$$\mathrm{i}\xi^{a*}\theta^b(\sigma^\mu)_{ba}\partial_\mu\phi + \mathrm{i}\xi^{a*}\theta^b(\sigma^\mu)_{ba}\theta^c\partial_\mu\chi_c; \tag{355}$$

note that the fermionic nature of θ precludes any cubic term in θ. The first term in (355) can alternatively be written as

$$-\mathrm{i}\theta^b(\sigma^\mu)_{ba}\xi^{a*}\partial_\mu\phi. \tag{356}$$

Referring to (352) we can therefore identify the part independent of θ as

$$\delta_\xi\phi = \xi^a\chi_a, \tag{357}$$

and the part linear in θ as

$$\theta^a\delta_\xi\chi_a = \theta^a(\xi_a F - \mathrm{i}(\sigma^\mu)_{ab}\xi^{b*}\partial_\mu\phi). \tag{358}$$

Since (358) has to be true for all θ we can remove the θ^a throughout, and then (357) and (358) indeed reproduce (261) for the fields ϕ and χ (recall that $(\mathrm{i}\sigma_2\xi^*)_b = \xi^{b*}$).

We are left with the second term of (355), which is bilinear in θ, and which ought to yield $\delta_\xi F$. We manipulate this term as follows. First, we write the general product $\theta^a\theta^b$ in terms of the scalar product $\theta\cdot\theta$ by using the result of this exercise:

Exercise Show that $\theta^a\theta^b = -\frac{1}{2}\epsilon^{ab}\theta\cdot\theta$, where $\epsilon^{12} = 1, \epsilon^{21} = -1, \epsilon^{11} = \epsilon^{22} = 0$.

The second term in (355) is then

$$-\mathrm{i}\xi^{a*}(\sigma^{\mu\mathrm{T}})_{ab}\epsilon^{bc}\partial_\mu\chi_c\frac{1}{2}\theta\cdot\theta. \tag{359}$$

Comparing this with (352) we deduce

$$\delta_\xi F = -\mathrm{i}\xi^{a*}(\sigma^{\mu\mathrm{T}})_{ab}\epsilon^{bc}\partial_\mu\chi_c. \tag{360}$$

Exercise Verify that this is in fact the same as the $\delta_\xi F$ given in (261) (remember that '$\xi^\dagger$' means (ξ_1^*, ξ_2^*), not (ξ^{1*}, ξ^{2*})).

So the chiral superfield $\Phi(x,\theta)$ of (351) contains the component fields ϕ, χ and F transforming correctly under SUSY transformations; we say that the chiral superfield provides a linear representation of the SUSY algebra.

Note that *three* component fields (ϕ, χ and F) are required for this result: here is a more 'deductive' justification for the introduction of the field F.

The thoughtful reader may be troubled by the following thought. Our development has been based on the form (319) for the unitary operator associated with finite SUSY transformations. But we could have started, instead, from

$$U_{\mathrm{red}}(x, \theta, \theta^*) = \mathrm{e}^{\mathrm{i}x\cdot P}\mathrm{e}^{\mathrm{i}[\theta\cdot Q+\bar{\theta}\cdot\bar{Q}]}, \tag{361}$$

and since Q and $Q^\dagger$ don't commute, (361) is not the same as (319). Indeed, (361) might be regarded as more natural - and certainly more in line with the angular momentum case, which also involves non-commuting generators, and where the corresponding unitary operator is $\exp[\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{J}]$. In the case of (361), the induced transformation corresponding to (336) is

$$\begin{aligned} 0 &\rightarrow \theta \rightarrow \theta + \xi \\ 0 &\rightarrow \theta^* \rightarrow \theta^* + \xi^* \\ 0 &\rightarrow x^\mu \rightarrow x^\mu + a^\mu + \frac{1}{2}\mathrm{i}\xi^a(\sigma^\mu)_{ab}\theta^{b*} - \frac{1}{2}\mathrm{i}\theta^a(\sigma^\mu)_{ab}\xi^{b*}. \end{aligned} \tag{362}$$

We can again find differential operators representing the SUSY generators by expanding the change in the field up to first order in ξ and ξ^*, as in (339), and this will lead to different expressions from those given in (345) and (349). However, the new operators will be found to satisy the *same* SUSY algebra (210). We could also imagine using

$$U_{\mathrm{R}}(x, \theta, \theta^*) = \mathrm{e}^{\mathrm{i}x\cdot P}\mathrm{e}^{\mathrm{i}\bar{\theta}\cdot\bar{Q}}\mathrm{e}^{\mathrm{i}\theta\cdot Q}, \tag{363}$$

which is not the same either, and for which the induced transformation is

$$\begin{aligned} 0 &\rightarrow \theta \rightarrow \theta + \xi \\ 0 &\rightarrow \theta^* \rightarrow \theta^* + \xi^* \\ 0 &\rightarrow x^\mu \rightarrow x^\mu + a^\mu + \mathrm{i}\xi^a(\sigma^\mu)_{ab}\theta^{b*}. \end{aligned} \tag{364}$$

Here also yet a third set of (differential operator) generators will be found, but again they'll satisfy the same SUSY algebra (210). The superfields produced by U - or more properly in the present context U_{L} - of (319), U_{red} and U_{R} are called ' left' or 'type I' (our Φ, or more properly Φ_{L}), 'reducible' or 'real' (Φ_{red}), and 'right' or 'type II' (Φ_{R}) superfields respectively. It can be shown that

$$\Phi_{\mathrm{red}}(x, \theta, \theta^*) = \Phi_{\mathrm{L}}(x^\mu - \frac{1}{2}\mathrm{i}\theta^a(\sigma^\mu)_{ab}\theta^{b*}, \theta, \theta^*) = \Phi_{\mathrm{R}}(x^\mu + \frac{1}{2}\mathrm{i}\theta^a(\sigma^\mu)_{ab}\theta^{b*}, \theta, \theta^*). \tag{365}$$

We can expand $\Phi_{\mathrm{red}}(x, \theta, \theta^*)$ as a power series in θ and θ^*, just as we did for $\Phi(x, \theta)$. But such an expansion will contain a lot more terms than (351), and will involve more component fields than ϕ, χ and F. This 'enlarged' superfield will again provide a representation of the SUSY algebra, but it will be a *reducible* one, in the sense that we'd find that we could pick out sets of components that only transformed among themselves - such as those in a chiral supermultiplet, for example. These *irreducible* sets of fields can be selected out from the beginning by applying a suitable constraint. For example, we got straight to the irreducible left chiral supermultiplet by starting with the chiral superfield Φ and requiring it not to depend on θ^*. In general, the constraints which may be applied must commute with the SUSY transformation when expressed in terms of differential operators. See [19] section 2.2.

We close this rather heavily formal section with a most important observation: *the change in the F field, (360), is actually a total derivative, since the parameters ξ are independent of x; it follows that, in general, the 'F-component' of a chiral superfield, in the sense of the expansion (351), will always transform by a total derivative - and will therefore automatically correspond to a SUSY-invariant Action.*

We now consider products of chiral superfields, and show how to exploit the italicized remark so as to obtain SUSY-invariant interactions - in particular, those of the W-Z model introduced in section 8.

9.4 Products of chiral superfields

Let Φ_i be a chiral superfield (understood to be of 'left' type) where, as in section 8, the suffix i labels the gauge and flavour degrees of freedom of the component fields. Φ_i has an expansion of the form (351):

$$\Phi_i(x,\theta) = \phi_i(x) + \theta\cdot\chi_i(x) + \frac{1}{2}\theta\cdot\theta F_i(x). \qquad (366)$$

Consider now the product of two such superfields:

$$\Phi_i\Phi_j = (\phi_i + \theta\cdot\chi_i + \frac{1}{2}\theta\cdot\theta F_i)(\phi_j + \theta\cdot\chi_j + \frac{1}{2}\theta\cdot\theta F_j). \qquad (367)$$

On the RHS there are the following terms:

$$\text{independent of } \theta\text{: } \phi_i\phi_j; \qquad (368)$$

$$\text{linear in } \theta\text{: } \theta\cdot(\chi_i\phi_j + \chi_j\phi_i); \qquad (369)$$

$$\text{bilinear in } \theta\text{: } \frac{1}{2}\theta\cdot\theta(\phi_i F_j + \phi_j F_i) + \theta\cdot\chi_i\ \theta\cdot\chi_j. \qquad (370)$$

In the second term of (370) we use the result given in the Exercise above equation (359) to write it as

$$\begin{aligned}
\theta\cdot\chi_i\ \theta\cdot\chi_j &= \theta^a\chi_{ia}\theta^b\chi_{jb} = -\theta^a\theta^b\chi_{ia}\chi_{jb} \\
&= \frac{1}{2}\epsilon^{ab}\theta\cdot\theta\chi_{ia}\chi_{jb} = \frac{1}{2}\theta\cdot\theta(\chi_{i1}\chi_{j2} - \chi_{i2}\chi_{j1}) \\
&= -\frac{1}{2}\theta\cdot\theta\chi_i\cdot\chi_j. \qquad (371)
\end{aligned}$$

Hence the term in the product (367) which is bilinear in θ is

$$\frac{1}{2}\theta\cdot\theta(\phi_i F_j + \phi_j F_i - \chi_i\cdot\chi_j). \tag{372}$$

Exercise Show that the terms in the product (367) which are cubic and quartic in θ vanish.

Altogether, then, we have shown that if the product (367) is itself expanded in component fields via

$$\Phi_i\Phi_j = \phi_{ij} + \theta\cdot\chi_{ij} + \frac{1}{2}\theta\cdot\theta F_{ij}, \tag{373}$$

then

$$\phi_{ij} = \phi_i\phi_j, \quad \chi_{ij} = \chi_i\phi_j + \phi_j\chi_i, \quad F_{ij} = \phi_i F_j + \phi_j F_i - \chi_i\cdot\chi_j. \tag{374}$$

Suppose now that we introduce a quantity W_{quad} defined by

$$W_{\text{quad}} = \frac{1}{2}M_{ij}\Phi_i\Phi_j\Big|_F, \tag{375}$$

where '$|_F$' means 'the F-component of' (i.e. the coefficient of $\frac{1}{2}\theta\cdot\theta$ in the product). Here M_{ij} is taken to be symmetric in i and j. Then

$$\begin{aligned} W_{\text{quad}} &= \frac{1}{2}M_{ij}(\phi_i F_j + \phi_j F_i - \chi_i\cdot\chi_j) \\ &= M_{ij}\phi_i F_j - \frac{1}{2}M_{ij}\chi_i\cdot\chi_j. \end{aligned} \tag{376}$$

Referring back to the italicized comment at the end of the previous subsection, the fact that (376) is the F-component of a chiral superfield (which is the product of two other such superfields, in this case), guarantees that the terms in (376) provide a SUSY-invariant Action. And in fact they are precisely the terms involving M_{ij} in the W-Z model of section 8: see (262) with W_i given by the first term in (278), and W_{ij} given by the first term in (266). Note also that our W_{quad} has exactly the same form, as a function of Φ_i and Φ_j, as the M_{ij} part of W in (268) had, as a function of ϕ_i and ϕ_j.

Thus encouraged, let's go on to consider the product of three chiral superfields:

$$\Phi_i\Phi_j\Phi_k = [\phi_i\phi_j + \theta\cdot(\chi_i\phi_j + \chi_j\phi_i) + \frac{1}{2}\theta\cdot\theta(\phi_i F_j + \phi_j F_i - \chi_i\cdot\chi_j)][\phi_k + \theta\cdot\chi_k + \frac{1}{2}\theta\cdot\theta F_k]. \tag{377}$$

Because our interest is confined to obtaining candidates for SUSY-invariant Actions, we shall only be interested in the F component. Inspection of (377) yields the obvious terms

$$\phi_i\phi_j F_k + \phi_j\phi_k F_i + \phi_k\phi_i F_j - \chi_i \cdot \chi_j \phi_k. \tag{378}$$

In addition, the term $\theta \cdot (\chi_i\phi_j + \chi_j\phi_i)\theta \cdot \chi_k$ can be re-written as in (371) to give

$$-\frac{1}{2}\theta \cdot \theta(\chi_i\phi_j + \chi_j\phi_i) \cdot \chi_k. \tag{379}$$

So altogether

$$\Phi_i\Phi_j\Phi_k|_F = \phi_i\phi_j F_k + \phi_j\phi_k F_i + \phi_k\phi_i F_j - \chi_i \cdot \chi_j\phi_k - \chi_j \cdot \chi_k\phi_i - \chi_i\chi_k\phi_j. \tag{380}$$

Let's now consider the cubic analogue of (375), namely

$$W_{\text{cubic}} = \frac{1}{6} y_{ijk}\Phi_i\Phi_j\Phi_k\Big|_F, \tag{381}$$

where the coefficients y_{ijk} are totally symmetric in i, j and k. Then from (380) we immediately obtain

$$W_{\text{cubic}} = \frac{1}{2} y_{ijk}\phi_i\phi_j F_k - \frac{1}{2} y_{ijk}\chi_i \cdot \chi_j\phi_k. \tag{382}$$

Sure enough, the first term here is precisely the first term in (262) with W_i given by the second (y_{ijk}) term in (278), while the second term in (382) is the second term in (262) with W_{ij} given by the y_{ijk} term in (266). Note, again, that our W_{cubic} has exactly the same form, as a function of the Φ's, as the y_{ijk} part of the W in (268), as a function of the ϕ's.

Thus we have shown that all the interactions found in section 8 can be expressed as F-components of products of superfields, a result which guarantees the SUSY-invariance of the associated Action. Of course, we must also include the Hermitian conjugates of the terms considered here. Because all the interactions are generated from the superfield products in W_{quad} and W_{cubic}, such W's are called *superpotentials*. The full superpotential for the W-Z model is thus

$$W = \frac{1}{2} M_{ij}\Phi_i\Phi_j + \frac{1}{6} y_{ijk}\Phi_i\Phi_j\Phi_k, \tag{383}$$

it being understood that the F-component is to be taken in the Lagrangian.

The understanding is often made explicit by integrating over θ_1 and θ_2. Integrals over such anticommuting variables are defined by the following rules:

$$\int \mathrm{d}\theta_1 1 = 0; \int \mathrm{d}\theta_1\ \theta_1 = 1; \int \mathrm{d}\theta_1 \int \mathrm{d}\theta_2\ \theta_2\theta_1 = 1 \tag{384}$$

(see Appendix O of [12] for example. These rules imply that

$$\int \mathrm{d}\theta_1 \int \mathrm{d}\theta_2\ \frac{1}{2}\theta\cdot\theta = \int \mathrm{d}\theta_1 \int \mathrm{d}\theta_2\ \theta_2\theta_1 = 1. \tag{385}$$

On the other hand, we can write

$$\mathrm{d}\theta_1\,\mathrm{d}\theta_2 = -\mathrm{d}\theta_2\,\mathrm{d}\theta_1 = -\frac{1}{2}\mathrm{d}\theta\cdot\mathrm{d}\theta \equiv \mathrm{d}^2\theta. \tag{386}$$

It then follows that

$$\int \mathrm{d}^2\theta\ W = \text{coefficient of } \tfrac{1}{2}\theta\cdot\theta \text{ in } W \text{ (i.e. the } F \text{ component)}. \tag{387}$$

Such integrals are commonly used to project out the desired parts of superfield expressions.

As already noted, the functional form of (383) is the same as that of (268), which is why they are both called W. Note, however, that the W of (383) includes, of course, *all* the interactions of the W-Z model, not only those involving the ϕ fields alone. In the MSSM, superpotentials of the form (383) describe the non-gauge interactions of the fields - that is, in fact, interactions involving the Higgs supermultiplets; in this case the quadratic and cubic products of the Φ's must be constructed so as to be singlets (invariant) under the gauge groups.

It is time to consider other supermultiplets, in particular ones containing gauge fields, with a view to supersymmetrizing the gauge interactions of the SM.

10 Vector (or Gauge) Supermultiplets

Having developed a certain amount of superfield formalism, it might seem sensible to use it now to discuss supermultiplets containing vector (gauge) fields. But although this is of course perfectly possible (see for example [19] chapter 3), it is actually fairly complicated, and we prefer the 'try it and see' approach that we used in section 3, which (as before) establishes the appropriate SUSY transformations more intuitively. We begin with a simple example, a kind of vector analogue of the model of section 3.

10.1 The free Abelian gauge supermultiplet

Consider a simple massless U(1) gauge field $A^\mu(x)$, like that of the photon. The spin of such a field is 1, but on-shell it contains only two (rather than three) degrees of freedom, both transverse to the direction of propagation. As we saw in section 6, we expect that SUSY will partner this field with a spin-1/2 field, also with two on-shell degrees of freedom. Such a fermionic partner of a gauge field is called generically a 'gaugino'. This one is a photino, and we'll denote its field by λ, and take it to be L-type. Being in the same multiplet as the photon, it must have the same 'internal' quantum numbers as the photon, in particular it must be electrically neutral. So it doesn't have any coupling to the photon. The photino must also have the same mass as the photon, namely zero. The Lagrangian is therefore just a sum of the Maxwell term for the photon, and the appropriate free massless spinor term for the photino:

$$\mathcal{L}_{\gamma\lambda} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda, \tag{388}$$

where as usual $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. We now set about investigating what might be the SUSY transformations between A^μ and λ, such that the Lagrangian (388) (or the corresponding Action) is invariant.

We anticipate that, as with the chiral supermultiplet, we shall not be able consistently to ignore the off-shell degree of freedom of the gauge field - but we shall start by doing so. First, consider $\delta_\xi A^\mu$. This has to be a 4-vector, and also a real rather than complex quantity, linear in ξ and ξ^*. We try (recalling the 4-vector combination from section 2.2)

$$\delta_\xi A^\mu = \xi^\dagger\bar{\sigma}^\mu\lambda + \lambda^\dagger\bar{\sigma}^\mu\xi, \tag{389}$$

where ξ is also an L-type spinor, but has dimension $M^{-1/2}$ as in (131). The spinor field λ has dimension $M^{3/2}$, so (389) is consistent with A^μ having the desired dimension M^1.

What about $\delta_\xi\lambda$? This must presumably be proportional to A^μ - or better, since λ is gauge-invariant, to the gauge-invariant quantity $F^{\mu\nu}$, so we try

$$\delta_\xi\lambda \sim \xi F^{\mu\nu}. \tag{390}$$

Since the dimension of $F^{\mu\nu}$ is M^2, we see that the dimensions already balance on both sides of (390), so there is no need to introduce any derivatives. But we do need to absorb the two Lorentz indices μ and ν on the RHS, and leave ourselves with something transforming correctly as an L-type spinor. This can be neatly done by recalling (section 2.2) that the quantity $\bar{\sigma}^\nu \xi$ transforms as an R-type spinor ψ, while $\sigma^\mu \psi$ transforms as an L-type spinor. So we try

$$\delta_\xi \lambda = C \sigma^\mu \bar{\sigma}^\nu \xi F_{\mu\nu}, \tag{391}$$

whee C is a constant to be determined. Then we also have

$$\delta_\xi \lambda^\dagger = C^* \xi^\dagger \bar{\sigma}^\nu \sigma^\mu F_{\mu\nu}. \tag{392}$$

Consider the SUSY variation of the Maxwell term in (388). Using the antisymmetry of $F^{\mu\nu}$ we have

$$\begin{aligned} \delta_\xi \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) &= -\frac{1}{2} F_{\mu\nu} (\partial^\mu \delta_\xi A^\nu - \partial^\nu \delta_\xi A^\mu) \\ &= -F_{\mu\nu} \partial^\mu \delta_\xi A^\nu \\ &= -F_{\mu\nu} \partial^\mu (\xi^\dagger \bar{\sigma}^\nu \lambda + \lambda^\dagger \bar{\sigma}^\nu \xi). \end{aligned} \tag{393}$$

The variation of the spinor term is

$$\begin{aligned} &\mathrm{i}(\delta_\xi \lambda^\dagger) \bar{\sigma}^\mu \partial_\mu \lambda + \mathrm{i} \lambda^\dagger \bar{\sigma}^\mu \partial_\mu (\delta_\xi \lambda) \\ &= \mathrm{i}(C^* \xi^\dagger \bar{\sigma}^\nu \sigma^\mu F_{\mu\nu}) \bar{\sigma}^\rho \partial_\rho \lambda + \mathrm{i} C \lambda^\dagger \bar{\sigma}^\rho \partial_\rho (\sigma^\mu \bar{\sigma}^\nu \xi F_{\mu\nu}). \end{aligned} \tag{394}$$

The ξ part of (393) must cancel the ξ part of (394) (or else their sum must be expressible as a total derivative), and the same is true of the $\xi^\dagger$ parts. So consider the $\xi^\dagger$ part of (394). It is

$$\mathrm{i} C^* \xi^\dagger \bar{\sigma}^\nu \sigma^\mu \bar{\sigma}^\rho \partial_\rho \lambda F_{\mu\nu} = -\mathrm{i} C^* \xi^\dagger \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho \partial_\rho \lambda F_{\mu\nu}. \tag{395}$$

Now the σ's are just Pauli matrices, together with the identity matrix, and we know that products of two Pauli matrices will give either the identity matrix or a third Pauli matrix. Hence products of three σ's as in (395) must be expressible as a linear combination of σ's. The identity we need is

$$\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho = g^{\mu\nu} \bar{\sigma}^\rho - g^{\mu\rho} \bar{\sigma}^\nu + g^{\nu\rho} \bar{\sigma}^\mu - \mathrm{i} \epsilon^{\mu\nu\rho\delta} \bar{\sigma}_\delta. \tag{396}$$

When (396) is inserted into (395), some simplifications occur. First, the term involving $....g^{\mu\nu}....F_{\mu\nu}$ vanishes, because $g^{\mu\nu}$ is symmetric in its indices while $F_{\mu\nu}$ is antisymmetric. Next, we can do a partial integration to re-write $\partial_\rho \lambda F_{\mu\nu}$ as $-\lambda \partial_\rho F_{\mu\nu} = -\lambda(\partial_\rho \partial_\mu A_\nu - \partial_\rho \partial_\nu A_\mu)$. The first of these two terms is symmetric under interchange of ρ and μ, and the second is symmetric under interchange of ρ and ν. But they are both multiplied by $\epsilon^{\mu\nu\rho\delta}$, which is antisymmetric under the interchange of either of these pairs of indices. Hence this whole term vanishes, and (395) becomes

$$-\mathrm{i}C^*\xi^\dagger[-\bar{\sigma}^\nu\partial^\mu\lambda + \bar{\sigma}^\mu\partial^\nu\lambda]F_{\mu\nu}. \tag{397}$$

In the second term here, interchange the indices μ and ν throughout, and then use the antisymmetry of $F_{\nu\mu}$: you find that the second term equals the first, so that this '$\xi^\dagger$' part of the variation of the fermionic part of $\mathcal{L}_{\gamma\lambda}$ is

$$2\mathrm{i}C^*\xi^\dagger\bar{\sigma}^\nu\partial^\mu\lambda F_{\mu\nu}. \tag{398}$$

This will cancel the $\xi^\dagger$ part of (393) if $C = \mathrm{i}/2$, and so the required SUSY transformations are (389) and

$$\delta_\xi\lambda = \frac{1}{2}\mathrm{i}\sigma^\mu\bar{\sigma}^\nu\xi F_{\mu\nu}, \tag{399}$$

$$\delta_\xi\lambda^\dagger = -\frac{1}{2}\mathrm{i}\xi^\dagger\bar{\sigma}^\nu\sigma^\mu F_{\mu\nu}. \tag{400}$$

However, if we try to calculate (as in section 7) $\delta_\eta\delta_\xi - \delta_\xi\delta_\eta$ as applied to the fields A^μ and λ, we shall find that consistent results are not obtained unless the free-field equations of motion are assumed to hold, which is not satisfactory. Off-shell, A^μ has a third degree of freedom, and so we expect to have to introduce one more auxiliary field, call it $D(x)$, which is a real scalar field with one degree of freedom. We add to $\mathcal{L}_{\gamma\lambda}$ the extra (non-propagating) term

$$\mathcal{L}_D = \frac{1}{2}D^2. \tag{401}$$

We now have to consider SUSY transformations including D.

First note that the dimension of D is M^2, the same as for F. This suggests that D transforms in a similar way to F, as given by (251). However, D is a real field, so we modify (251) by adding the hermitian conjugate term, arriving at

$$\delta_\xi D = -\mathrm{i}(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda - (\partial_\mu\lambda)^\dagger\bar{\sigma}^\mu\xi). \tag{402}$$

As in the case of $\delta_\xi F$, this is also a total derivative. Analogously to (255) and (257), we expect to modify (399) and (400) so as to include additional terms

$$\delta_\xi\lambda = \xi D, \quad \delta_\xi\lambda^\dagger = \xi^\dagger D. \tag{403}$$

The variation of $\mathcal{L}_D$ is then

$$\delta_\xi\left(\frac{1}{2}D^2\right) = D\delta_\xi D = -\mathrm{i}D(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda - (\partial_\mu\lambda)^\dagger\bar{\sigma}^\mu\xi), \tag{404}$$

and the variation of the fermionic part of $\mathcal{L}_{\gamma\lambda}$ gets an additional contribution which is

$$\mathrm{i}\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda D + \mathrm{i}\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\xi D. \tag{405}$$

The first term of (405) cancels the first term of (404), and the second terms also cancel after either one has been integrated by parts.

10.2 Non-Abelian gauge supermultiplets

The preceding example is clearly unrealistic physically, but it will help us in guessing the SUSY transformations in the physically relevant non-Abelian case. For definiteness, we'll mostly consider an SU(2) gauge theory, such as occurs in the electroweak sector of the SM. We begin by recalling some necessary facts about non-Abelian gauge theories.

For an SU(2) gauge theory, the Maxwell field strength tensor $F_{\mu\nu}$ of U(1) is generalized to (see for example [12] chapter 13)

$$F^\alpha_{\mu\nu} = \partial_\mu W^\alpha_\nu - \partial_\nu W^\alpha_\mu - g\epsilon^{\alpha\beta\gamma}W^\beta_\mu W^\gamma_\nu, \tag{406}$$

where α, β and γ have the values 1,2 and 3, the gauge field $\boldsymbol{W}_\mu = (W^1_\mu, W^2_\mu, W^3_\mu)$ is an SU(2) triplet (or 'vector', thinking of it in SO(3) terms), and g is the gauge coupling constant. We are writing the SU(2) indices as superscripts rather than subscripts, but this has no mathematical significance; rather, it is to avoid confusion, later, with the spinor index of the gaugino field λ^α_a. Equation (406) can alternatively be written in 'vector' notation as

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu\boldsymbol{W}_\nu - \partial_\nu\boldsymbol{W}_\mu - g\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu. \tag{407}$$

If the gauge group was SU(3) there would be 8 gauge fields (gluons, in the QCD case), and in general for SU(N) there are N^2-1. Gauge fields always belong to a particular representation of the gauge group, namely the *regular* or *adjoint* one, which has as many components as there are generators of the group: see pages 400-401 of [12].

An infinitesimal gauge transformation on the gauge fields W^α_μ takes the form

$$W'^\alpha_\mu(x) = W^\alpha_\mu(x) - \partial_\mu\epsilon^\alpha(x) - g\epsilon^{\alpha\beta\gamma}\epsilon^\beta(x)W^\gamma_\mu(x), \tag{408}$$

where we have here indicated the x-dependence explicitly, to emphasize the fact that this is a local transformation, in which the three infinitesimal parameters $\epsilon^\alpha(x)$ depend on x. In U(1) we would have only one such $\epsilon(x)$, the second term in (408) would be absent, and the field strength tensor $F_{\mu\nu}$ would be gauge-invariant. In SU(2), the corresponding tensor (407) transforms by

$$F_{\mu\nu}^{\alpha\prime}(x) = F_{\mu\nu}^{\alpha}(x) - g\epsilon^{\alpha\beta\gamma}\epsilon^{\beta}(x)F_{\mu\nu}^{\gamma}(x), \tag{409}$$

which is nothing but the statement that $\boldsymbol{F}_{\mu\nu}$ transforms as an SU(2) triplet. Note that (409) involves no derivative of $\boldsymbol{\epsilon}(x)$, such as appears in (408), even though the transformations being considered are local ones. This fact shows that the simple generalization of the Maxwell Lagrangian in terms of $\boldsymbol{F}_{\mu\nu}$,

$$-\frac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}^{\alpha}F^{\mu\nu\alpha} \tag{410}$$

is invariant under local SU(2) transformations - i.e. is SU(2) gauge-invariant.

We now need to generalize the simple U(1) SUSY model of the previous subsection. Clearly the first step is to introduce an SU(2) triplet of gauginos, $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \lambda^3)$, to partner the triplet of gauge fields. Under an infinitesimal SU(2) gauge transformation, λ^α transforms as in (409):

$$\lambda^{\alpha\prime}(x) = \lambda^{\alpha}(x) - g\epsilon^{\alpha\beta\gamma}\epsilon^{\beta}(x)\lambda^{\gamma}(x). \tag{411}$$

The gauginos are of course not gauge fields and so their transformation does not include any derivative of $\boldsymbol{\epsilon}(x)$. So the straightforward generalization of (388) would be

$$\mathcal{L}_{W\lambda} = -\frac{1}{4}F_{\mu\nu}^{\alpha}F^{\mu\nu\alpha} + \mathrm{i}\lambda^{\alpha\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\lambda^{\alpha}. \tag{412}$$

But although the first term of (412) is SU(2) gauge-invariant, the second is not, because the gradient will act on the x-dependent parameters $\epsilon^\beta(x)$ in (411) to leave uncancelled $\partial_\mu\epsilon^\beta(x)$ terms after the gauge transformation. The way to make this term gauge-invariant is to replace the ordinary gradient in it by the appropriate *covariant derivative* - see [12] page 47, for instance. The general recipe is

$$\partial_\mu \to D_\mu \equiv \partial_\mu + \mathrm{i}g\mathbf{T}^{(t)}\cdot\boldsymbol{W}_\mu, \tag{413}$$

where the three matrices $T^{(t)\alpha}, \alpha = 1, 2, 3$, are of dimension $2t+1 \times 2t+1$ and represent the generators of SU(2) when acting on a $2t+1$-component field, which is in the representation of SU(2) characterized by the 'isospin' t (see [12] section M.5). In the present case, the λ^α's belong in the triplet ($t = 1$) representation, for which the three 3×3 matrices $T^{(1)\alpha}$ are given by (see [12] equation (M.70))

$$\left(T^{(1)\alpha}\right)_{\beta\gamma \text{ element}} = -\mathrm{i}\epsilon^{\alpha\beta\gamma}. \tag{414}$$

Thus, in (412), we need to make the replacement

$$\begin{aligned}
\partial_\mu \lambda^\alpha \to (D_\mu \lambda)^\alpha &= \partial_\mu \lambda^\alpha + \mathrm{i}g(\mathbf{T}^{(1)} \cdot \boldsymbol{W}_\mu)_{\alpha\beta \text{ element}} \lambda^\beta \\
&= \partial_\mu \lambda^\alpha + \mathrm{i}g(-\mathrm{i}\epsilon^{\gamma\alpha\beta} W_\mu^\gamma)\lambda^\beta \\
&= \partial_\mu \lambda^\alpha + g\epsilon^{\gamma\alpha\beta} W_\mu^\gamma \lambda^\beta \\
&= \partial_\mu \lambda^\alpha - g\epsilon^{\alpha\beta\gamma} W_\mu^\beta \lambda^\gamma.
\end{aligned} \tag{415}$$

With this replacement for $\partial_\mu \lambda^\alpha$ in (412), the resulting $\mathcal{L}_{W\lambda}$ is SU(2) gauge-invariant.

What about making it also invariant under SUSY transformations? From the experience of the U(1) case in the previous subsection, we expect that we'll need to introduce the analogue of the auxiliary field D. In this case, we need a triplet of D's, D^α, balancing the third off-shell degree of freedom for each W_μ^α. So our shot at a SUSY- and gauge-invariant Lagrangian for an SU(2) gauge supermultiplet is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + \mathrm{i}\lambda^{\alpha\dagger}\bar{\sigma}^\mu (D_\mu \lambda)^\alpha + \frac{1}{2} D^\alpha D^\alpha. \tag{416}$$

Confusion must be avoided as between the covariant derivative and the auxiliary field!

What are reasonable guesses for the relevant SUSY transformations? We try the obvious generalizations of the U(1) case:

$$\begin{aligned}
\delta_\xi W^{\mu\alpha} &= \xi^\dagger \bar{\sigma}^\mu \lambda^\alpha + \lambda^{\alpha\dagger}\bar{\sigma}^\mu \xi, \\
\delta_\xi \lambda^\alpha &= \frac{1}{2}\mathrm{i}\sigma^\mu \bar{\sigma}^\nu \xi F_{\mu\nu}^\alpha + \xi D^\alpha \\
\delta_\xi D^\alpha &= -\mathrm{i}(\xi^\dagger \bar{\sigma}^\mu (D_\mu \lambda)^\alpha - (D_\mu \lambda)^{\alpha\dagger} \bar{\sigma}^\mu \xi);
\end{aligned} \tag{417}$$

note that in the last equation we have replaced the '∂_μ' of (402) by 'D_μ', so as to maintain gauge-invariance. This in fact works, just as it is! Quite remarkably, the Action for (416) is invariant under the transformations (417), and $(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta)$ can be consistently applied to all the fields W_μ^α, λ and D^α in this gauge supermultiplet. This supersymmetric gauge theory therefore has two sorts of interactions: (i) the usual self-interactions among the W fields as generated by the term (410); and (ii) interactions between the W's and the λ's generated by the covariant derivative coupling in (416). We stress again that the supersymmetry requires the gaugino partners to belong to the same representation of the gauge group as the gauge bosons themselves - i.e. to the regular, or adjoint, representation.

We are getting closer to the MSSM at last. The next stage is to build Lagrangians containing both chiral and gauge supermultiplets, in such a way that they (or the Actions) are invariant under both SUSY and gauge transformations.

11 Combining Chiral and Gauge Supermultiplets

We do this in two steps. First we introduce - via appropriate covariant derivatives - the couplings of the gauge fields to the scalars and fermions ('matter fields') in the chiral supermultiplets. This will account for the interactions between the gauge fields of the vector supermultiplets and the matter fields of the chiral supermultiplets. But there are also gaugino and D fields in the vector supermultiplets, and we need to consider whether there are any possible renormalizable interactions between the matter fields and gaugino and D fields, which are both gauge- and SUSY-invariant. Including such interactions is the second step in the programme of combining the two kinds of supermultiplets.

The essential points in such a construction are contained in the simplest case, namely that of a single U(1) (Abelian) vector supermultiplet and a single free chiral supermultiplet, the combination of which we shall now consider.

11.1 Combining one U(1) vector supermultiplet and one free chiral supermultiplet

The first step is accomplished by taking the Lagrangian of (260), for only a single supermultiplet, replacing ∂_μ by D_μ where (compare (413))

$$D_\mu = \partial_\mu + \mathrm{i}qA_\mu, \tag{418}$$

where q is the U(1) coupling constant (or charge), and adding on the Lagrangian for the U(1) vector supermultiplet (i.e. (388) together with (401)). This produces the Lagrangian

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mathrm{i}\chi^\dagger\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2. \tag{419}$$

We now have to consider possible interactions between the matter fields ϕ and χ, and the other fields λ and D in the vector supermultiplet. Any such interaction terms must certainly be Lorentz-invariant, renormalizable (i.e. have mass dimension less than or equal to 4), and gauge-invariant. Given some terms with these characteristics, we shall then have to examine whether we can include them in a SUSY-preserving way.

Since the fields λ and D are neutral, any gauge-invariant couplings between them and the charged fields ϕ and χ must involve neutral bilinear combinations of the latter fields, namely $\phi^\dagger\phi$, $\phi^\dagger\chi$, $\chi^\dagger\phi$ and $\chi^\dagger\chi$. These have mass dimension 2, 5/2, 5/2 and 3 respectively. They have to be coupled to the fields λ and D which have dimension 3/2 and 2 respectively, so as to make quantities with dimension no greater than 4. This rules out the bilinear $\chi^\dagger\chi$, and allows just three possible Lorentz- and gauge-invariant renormalizable couplings: $(\phi^\dagger\chi)\cdot\lambda$, $\chi^\dagger\cdot(\chi^\dagger\phi)$, and $\phi^\dagger\phi D$. In the first of these the Lorentz invariant is formed as the '$\cdot$' product of the L-type quantity $\phi^\dagger\chi$ and the L-type spinor λ, while in the second it is formed as a '$\lambda^\dagger\cdot\chi^\dagger$'-type product. We take the sum of the first two couplings to obtain a Hermitian interaction, and arrive at the possible allowed interaction terms

$$Aq[(\phi^\dagger\chi)\cdot\lambda + \lambda^\dagger\cdot(\chi^\dagger\phi)] + Bq\phi^\dagger\phi D. \tag{420}$$

The coefficients A and B are now to be determined by requiring that the complete Lagrangian of (419) together with (420) is SUSY-invariant (note that for convenience we have extracted an explicit factor of q from A and B).

To implement this programme we need to specify the SUSY transformations of the fields. At first sight, this seems straightforward enough: we use

(389), (399), (400) and (402) for the fields in the vector supermultiplet, and we 'covariantize' the transformations used for the chiral supermultiplet. For the latter, then, we provisionally assume

$$\delta_\xi \phi = \xi \cdot \chi, \quad \delta_\xi \chi = -\mathrm{i}\sigma^\mu(\mathrm{i}\sigma_2)\xi^{\dagger\mathrm{T}} D_\mu \phi + \xi F, \quad \delta_\xi F = -\mathrm{i}\xi^\dagger \bar{\sigma}^\mu D_\mu \chi, \tag{421}$$

together with the analogous transformations for the Hermitian conjugate fields. As we shall see, however, there is no choice we can make for A and B in (420) such that the complete Lagrangian is invariant under these transformations. One may not be too surprised by this: after all, the transformations for the chiral supermultiplet were found for the case $q = 0$, and it is quite possible, one might think, that one or more of the transformations in (421) have to be modified by pieces proportional to q. Indeed, we shall find that the transformation for F does need to be so modified. But there is a more important reason for the 'failure' to find a suitable A and B. The transformations of (389), (399), (400) and (402), on the one hand, and those of (421) on the other, certainly do ensure the SUSY-invariance of the gauge and chiral parts of (419) respectively, in the limit $q = 0$. But there is no *a priori* reason - at least in our 'brute-force' approach - why the 'ξ' parameter in one set of transformations should be exactly the same as that in the other. Either 'ξ' can be rescaled by a constant multiple, and the relevant sub-Lagrangian will remain invariant. However, when we *combine* the Lagrangians and include (420), for the case $q \neq 0$, we shall see that the requirement of overall SUSY-invariance fixes the relative scale of the two 'ξ's' (up to a sign), and without a rescaling in one or the other transformation we cannot get a SUSY-invariant theory. For definiteness we shall keep the 'ξ' in (421) unmodified, and introduce a real scale parameter α into the transformations for the vector supermultiplet, so that they now become

$$\delta_\xi A^\mu = \alpha(\xi^\dagger \bar{\sigma}^\mu \lambda + \lambda^\dagger \bar{\sigma}^\mu \xi) \tag{422}$$

$$\delta_\xi \lambda = \frac{\alpha \mathrm{i}}{2}(\sigma^\mu \bar{\sigma}^\nu \xi) F_{\mu\nu} + \alpha \xi D \tag{423}$$

$$\delta_\xi \lambda^\dagger = -\frac{\alpha \mathrm{i}}{2}(\xi^\dagger \bar{\sigma}^\nu \sigma^\mu) F_{\mu\nu} + \alpha \xi^\dagger D \tag{424}$$

$$\delta_\xi D = -\alpha \mathrm{i}(\xi^\dagger \bar{\sigma}^\mu \partial_\mu \lambda - (\partial_\mu \lambda^\dagger)\bar{\sigma}^\mu \xi). \tag{425}$$

Consider first the SUSY variation of the 'A' part of (420). This is

$$\begin{array}{l} Aq[(\delta_\xi \phi^\dagger)\chi \cdot \lambda + \phi^\dagger(\delta_\xi \chi) \cdot \lambda + \phi^\dagger \chi \cdot (\delta_\xi \lambda) \\ (\delta_\xi \lambda^\dagger) \cdot \chi^\dagger \phi + \lambda^\dagger \cdot (\delta_\xi \chi^\dagger)\phi + \lambda^\dagger \cdot \chi^\dagger(\delta_\xi \phi)]. \end{array} \tag{426}$$

Among these terms there are two which are linear in q and D, arising from $\phi^\dagger\chi\cdot(\delta_\xi\lambda)$ and its Hermitian conjugate, namely

$$Aq[\alpha\phi^\dagger\chi\cdot\xi D+\alpha\xi^\dagger\cdot\chi^\dagger D\phi]. \tag{427}$$

Similarly, the variation of the 'B' part is

$$\begin{aligned}Bq[(\delta_\xi\phi^\dagger)\phi D+\phi^\dagger(\delta_\xi\phi)D+\phi^\dagger\phi(\delta_\xi D)]=Bq[\chi^\dagger\cdot\xi^\dagger\phi D+\\ \phi^\dagger\xi\cdot\chi D+\phi^\dagger\phi(-\alpha\mathrm{i})(\xi^\dagger\bar\sigma^\mu\partial_\mu\lambda-(\partial_\mu\lambda^\dagger)\bar\sigma^\mu\xi)].\end{aligned} \tag{428}$$

The 'D' part of (428) will cancel the term (427) if (using $\chi^\dagger\cdot\xi^\dagger=\xi^\dagger\cdot\chi^\dagger$ and $\xi\cdot\chi=\chi\cdot\xi$)

$$A\alpha=-B. \tag{429}$$

Next, note that the first and last terms of (426) produce the changes

$$Aq[\chi^\dagger\cdot\xi^\dagger\,\chi\cdot\lambda+\lambda^\dagger\cdot\chi^\dagger\,\xi\cdot\chi]. \tag{430}$$

Meanwhile, there is a corresponding change coming from the variation of the term $-q\chi^\dagger\bar\sigma^\mu\chi A_\mu$, namely

$$-q\chi^\dagger\bar\sigma^\mu\chi(\delta_\xi A_\mu)=-q\alpha\chi^\dagger\bar\sigma^\mu\chi(\xi^\dagger\bar\sigma_\mu\lambda+\lambda^\dagger\bar\sigma_\mu\xi). \tag{431}$$

This can be simplified with the help of the exercise:
Exercise Show that

$$(\chi^\dagger\bar\sigma^\mu\chi)(\lambda^\dagger\bar\sigma_\mu\xi)=2(\chi^\dagger\cdot\lambda^\dagger)(\chi\cdot\xi). \tag{432}$$

So (431) becomes

$$-2q\alpha[\chi^\dagger\cdot\xi^\dagger\,\chi\cdot\lambda+\chi^\dagger\cdot\lambda^\dagger\,\chi\cdot\xi], \tag{433}$$

which will cancel (430) if (again using $\chi\cdot\xi=\xi\cdot\chi$ and $\chi^\dagger\cdot\lambda^\dagger=\lambda^\dagger\cdot\chi^\dagger$)

$$A=2\alpha. \tag{434}$$

So far, there is nothing to prevent us from choosing $\alpha=1$, say, in (429) and (434). However, a constraint on α arises when we consider the variation of the $A^\mu-\phi$ interaction term in (419), namely

$$-\mathrm{i}q\delta_\xi(A^\mu\phi^\dagger\partial_\mu\phi-(\partial_\mu\phi)^\dagger A^\mu\phi). \tag{435}$$

The terms in $\delta_\xi A^\mu$ yield a change

$$\mathrm{i}q\alpha[(\partial_\mu\phi^\dagger)(\xi^\dagger\bar\sigma^\mu\lambda+\lambda^\dagger\bar\sigma^\mu\xi)\phi-(\xi^\dagger\bar\sigma^\mu\lambda+\lambda^\dagger\bar\sigma^\mu\xi)\phi^\dagger\partial_\mu\phi]. \tag{436}$$

A similar change arises from the terms $Aq[\phi^\dagger(\delta_\xi\chi)\cdot\lambda+\lambda^\dagger\cdot(\delta_\xi\chi^\dagger)\phi]$ in (426), namely

$$Aq[\phi^\dagger(-\mathrm{i}\sigma^\mu\mathrm{i}\sigma_2\xi^{\dagger\mathrm{T}}\partial_\mu\phi)\cdot\lambda+\lambda^\dagger\cdot\partial_\mu\phi^\dagger\xi^{\mathrm{T}}(-\mathrm{i}\sigma_2\mathrm{i}\sigma^\mu\phi)]. \tag{437}$$

The first spinor dot product is

$$\xi^\dagger(-\mathrm{i}\sigma_2)(-\mathrm{i}\sigma^{\mu\mathrm{T}})(-\mathrm{i}\sigma_2)\lambda=\mathrm{i}\xi^\dagger\bar{\sigma}^\mu\lambda, \tag{438}$$

using (193). The second spinor product is the Hermitian conjugate of this, so that (437) yields a change

$$Aq\mathrm{i}[\phi^\dagger(\partial_\mu\phi)\xi^\dagger\bar{\sigma}^\mu\lambda-(\partial_\mu\phi^\dagger)\phi\,\lambda^\dagger\bar{\sigma}^\mu\xi]. \tag{439}$$

Along with (436) and (439) we must also group the last two terms in (428), which we write out again here for convenience

$$Bq[\phi^\dagger\phi(-\alpha\mathrm{i})(\xi^\dagger\bar{\sigma}^\mu\partial_\mu\lambda-(\partial_\mu\lambda^\dagger)\bar{\sigma}^\mu\xi)], \tag{440}$$

and integrate by parts to yield

$$\alpha\mathrm{i}Bq\{[(\partial_\mu\phi^\dagger)\phi+\phi^\dagger\partial_\mu\phi](\xi^\dagger\bar{\sigma}^\mu\lambda)-[(\partial_\mu\phi^\dagger)\phi+\phi^\dagger\partial_\mu\phi](\lambda^\dagger\bar{\sigma}^\mu\xi)\}. \tag{441}$$

Consider now the terms involving the quantity $\xi^\dagger\bar{\sigma}^\mu\lambda$ in (436), (439) and (441), which are

$$\mathrm{i}q\alpha[(\partial_\mu\phi^\dagger)\phi-\phi^\dagger\partial_\mu\phi]+Aq\mathrm{i}\phi^\dagger\partial_\mu\phi+\alpha\mathrm{i}Bq[(\partial_\mu\phi^\dagger)\phi+\phi^\dagger\partial_\mu\phi]. \tag{442}$$

These will all cancel if the condition (434) holds, and if in addition

$$B=-1. \tag{443}$$

From (434) and (429) it now follows that

$$\alpha^2=\frac{1}{2}. \tag{444}$$

We conclude that, as promised, the combined Lagrangian will not be SUSY-invariant unless we modify the scale of the transformations of the gauge supermultiplet, relative to those of the chiral supermultiplet, by a non-trivial factor, which we choose (in agreement with what seems to be the usual convention - see [22] equations (3.57)-(3.59)) to be

$$\alpha=-\frac{1}{\sqrt{2}}. \tag{445}$$

With this choice, the coefficient A is determined to be

$$A = -\sqrt{2}, \tag{446}$$

and our combined Lagrangian is fixed.

We have, of course, not given a complete analysis of all the terms in the SUSY variation of our Lagrangian, an exercise we leave to the dedicated reader - who will find that (with one more adjustment to the SUSY transformations) all the variations do indeed vanish (after partial integrations in some cases, as usual). The need for the adjustment appears when we consider the variation associated with the terms $Aq[\phi^\dagger(\delta_\xi\chi)\cdot\lambda + \lambda^\dagger\cdot(\delta_\xi\chi^\dagger)\phi]$ in (426), which includes the term

$$Aq[\phi^\dagger\xi\cdot\lambda F + \lambda^\dagger\cdot\xi^\dagger F^\dagger\phi]. \tag{447}$$

This cannot be cancelled by any other variation, and we therefore have to modify the transformation for F and $F^\dagger$ so as to generate a cancelling term from the variation of $F^\dagger F$ in the Lagrangian. This requires

$$\delta_\xi F = -\sqrt{2}q\lambda^\dagger\cdot\xi^\dagger\phi + \text{previous transformation} \tag{448}$$

and

$$\delta_\xi F^\dagger = -\sqrt{2}q\xi\cdot\lambda\phi^\dagger + \text{previous transformation}, \tag{449}$$

where we have now inserted the known value of A.

In summary then, our SUSY-invariant combined chiral and U(1) gauge supermultiplet Lagrangian is

$$\begin{aligned}\mathcal{L} = {} & (D_\mu\phi)^\dagger(D^\mu\phi) + \mathrm{i}\chi^\dagger\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2 \\ & -\sqrt{2}q[(\phi^\dagger\chi)\cdot\lambda + \lambda^\dagger\cdot(\chi^\dagger\phi)] - q\phi^\dagger\phi D. \end{aligned} \tag{450}$$

Note that the terms in the last line of (450) are interactions whose strengths are fixed by SUSY to be proportional to the gauge coupling constant q, even though they don't have the form of ordinary gauge interactions; the terms coupling the photino λ to the matter fields may be thought of as arising from supersymmetrizing the usual coupling of the gauge field to the matter fields.

The equation of motion for the field D is

$$D = q\phi^\dagger\phi. \tag{451}$$

Since no derivatives of D enter, we may (as in the W-Z case for F_i and $F_i^\dagger$, c.f. equations (279) and (280)) eliminate the auxliary field D from the Lagrangian by using (451). The effect of this is clearly to replace the two terms involving D in (450) by the single term

$$-\frac{1}{2}q^2(\phi^\dagger\phi)^2. \tag{452}$$

This is a '$(\phi^\dagger\phi)^2$' type of interaction, just as in the Higgs potential (4), but here appearing with a coupling constant which is not an unknown parameter, but is determined by the gauge coupling q. In the next section we shall see that the same feature persists in the more realistic non-Abelian case. Since the Higgs mass is (for a fixed vev of the Higgs field) determined by the $(\phi^\dagger\phi)^2$ coupling - see (3) for example - it follows that there is likely to be less arbitrariness in the mass of the Higgs in the MSSM than in the SM. We shall see in section 16, when we examine the Higgs sector of the MSSM, that this is indeed the case.

11.2 The non-Abelian case

Once again, we proceed in two steps. We start from the W-Z Lagrangian for a collection of chiral supermultiplets labelled by i, and including the superpotential terms:

$$\partial_\mu\phi_i^\dagger\partial^\mu\phi_i + \chi_i^\dagger \mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi_i + F_i^\dagger F_i + \left[\frac{\partial W}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\phi_j}\chi_i\cdot\chi_j + \mathrm{h.c.}\right] \tag{453}$$

into which we introduce the gauge couplings via the covariant derivatives

$$\begin{aligned} \partial_\mu\phi_i &\rightarrow D_\mu\phi_i = \partial_\mu\phi_i + \mathrm{i}gA_\mu^\alpha(T^\alpha\phi)_i \qquad (454)\\ \partial\chi_i &\rightarrow D_\mu\chi_i = \partial_\mu\chi_i + \mathrm{i}gA_\mu^\alpha(T^\alpha\chi)_i, \qquad (455) \end{aligned}$$

where g and A_μ^α are the gauge coupling constant and gauge fields (for example, g_s and gluon fields for QCD), and the T^α are the hermitian matrices representing the generators of the gauge group in the representation to which, for given i, ϕ_i and χ_i belong (for example, if ϕ_i and χ_i are SU(2) doublets, the T^α's would be the $\tau^\alpha/2$, with α running from 1 to 3). Recall that SUSY requires that ϕ_i, χ_i and F_i must all be in the same representation of the relevant gauge group. Of course, if - as is the case in the SM - some matter fields interact with more than one gauge field, then all the gauge couplings must be included in the covariant derivatives. There is no covariant derivative for the auxiliary fields F_i, because their ordinary derivatives don't appear in (453). To (455) we need to add the Lagrangian for the gauge supermultiplet(s), equation (416), and then (in the second step) additional 'mixed' interactions as in (420).

We therefore need to construct all possible Lorentz- and gauge-invariant renormalizable interactions between the matter fields and the gaugino (λ^α) and auxiliary (D^α) fields, as in the U(1) case. We have the specific particle content of the SM in mind, so we need only consider the cases in which the matter fields are either singlets under the gauge group (for example, the R parts of quark and lepton fields), or belong to the fundamental representation of the gauge group (that is, the triplet for SU(3) and the doublet for SU(2)). For matter fields in singlet representations, there is no possible gauge-invariant coupling between them and λ^α or D^α, which are in the regular representation. For matter fields in the fundamental representation, however, we can form bilinear combinations of them which transform according to the regular representation, and these bilinears can be 'dotted' into λ^α and D^α to give gauge singlets (i.e. gauge-invariant couplings). We must also arrange the couplings to be Lorentz invariant, of course.

The bilinear combinations of the ϕ_i and χ_i which transform as the regular representation are (see for example [12] sections 12.1.3 and 12.2)

$$\phi_i^\dagger T^\alpha \phi_i, \ \phi_i^\dagger T^\alpha \chi_i, \ \chi_i^\dagger T^\alpha \phi_i, \text{ and } \chi_i^\dagger T^\alpha \chi_i, \tag{456}$$

where for example $T^\alpha = \tau^\alpha/2$ in the case of SU(2), and where the τ^α, ($\alpha = 1, 2, 3$) are usual the Pauli matrices used in the isospin context. These bilinears are the obvious analogues of the ones considered in the U(1) case; in particular they have the same dimension. Following the same reasoning, then, the allowed additional interaction terms are

$$Ag[(\phi_i^\dagger T^\alpha \chi_i) \cdot \lambda^\alpha + \lambda^{\alpha\dagger} \cdot (\chi_i^\dagger T^\alpha \phi_i)] + Bg(\phi_i^\dagger T^\alpha \phi_i) D^\alpha, \tag{457}$$

where A and B are coefficients to be determined by the requirement of SUSY-invariance.

In fact, however, a consideration of the SUSY transformations in this case shows that they are essentially the same as in the U(1) case (apart from straightforward changes involving the matrices T^α). The upshot is that, just as in the U(1) case, we need to change the SUSY transformations of (417) by replacing ξ by $-\xi/\sqrt{2}$, and by modifying the transformation of $F_i^\dagger$ to

$$\delta_\xi F_i^\dagger = -\sqrt{2} g \phi_i^\dagger T^\alpha \xi \cdot \lambda^\alpha + \text{previous transformation}, \tag{458}$$

and similarly for $\delta_\xi F_i$. The coefficients A and B in (457) are then $-\sqrt{2}$ and -1 respectively, as in the U(1) case, and the combined SUSY-invariant Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{gauge + chiral}} &= \mathcal{L}_{\text{gauge}}(\text{equation (416)}) \\ &+\mathcal{L}_{\text{W-Z, covariantized}}(\text{equation (453), with } \partial_\mu \to D_\mu \text{ as in (454) and (455)}) \\ &-\sqrt{2} g[(\phi_i^\dagger T^\alpha \chi_i) \cdot \lambda^\alpha + \lambda^{\alpha\dagger} \cdot (\chi_i^\dagger T^\alpha \phi_i)] - g(\phi_i^\dagger T^\alpha \phi_i) D^\alpha. \end{aligned} \tag{459}$$

We draw attention to an important consequence of the terms $-\sqrt{2}g[\ldots]$ in (459), for the case in which the chiral multiplets (ϕ_i, χ_i) are the two Higgs supermultiplets $H_{\rm u}$ and $H_{\rm d}$, containing higgs and higgsino fields (see Table 1 below). When the scalar Higgs fields $H_{\rm u}^0$ and $H_{\rm d}^0$ acquire vevs, these terms will be bilinear in the higgsino and gaugino fields, implying that mixing will occur among these fields as a consequence of electroweak symmetry breaking. We shall discuss this in section 17.

The equation of motion for the field D^α is

$$D^\alpha = g \sum_i (\phi_i^\dagger T^\alpha \phi_i), \tag{460}$$

where the sum over i (labelling a given chiral supermultiplet) has been reinstated explicitly. As before, we may eliminate these auxiliary fields from the Lagrangian by using (460). The complete scalar potential (as in '$\mathcal{L} = \mathcal{T} - \mathcal{V}$') is then

$$\mathcal{V}(\phi_i, \phi_i^\dagger) = |W_i|^2 + \frac{1}{2} \sum_{\rm G} \sum_\alpha \sum_{i,j} g_{\rm G}^2 (\phi_i^\dagger T_{\rm G}^\alpha \phi_i)(\phi_j^\dagger T_{\rm G}^\alpha \phi_j), \tag{461}$$

where in the summation we have recalled that more than one gauge group G will enter, in general, given the SU(3)$\times$SU(2)$\times$U(1) structure of the SM, with different couplings $g_{\rm G}$ and generators $T_{\rm G}$. The first term in (461) is called the 'F-term', for obvious reasons; it is determined by the fermion mass terms M_{ij} and Yukawa couplings (see (278)). The second term is called the 'D-term', and is determined by the gauge interactions. There is no room for any other scalar potential, *independent* of these parameters appearing in other parts of the Lagrangian. It is worth emphasizing that $\mathcal{V}$ is a sum of squares, and is hence always greater than or equal to zero for every field configuration. We shall see in section 16 how the form of the D-term allows an important bound to be put on the mass of the lightest Higgs boson in the MSSM.

12 The MSSM

We have now introduced all the interactions appearing in the MSSM, apart from specifying the superpotential W. We already had a brief look at the particle multiplets in section 4 - let's begin by reviewing them again.

All the SM fermions - i.e. the quarks and the leptons - have the property that their L ('χ') parts are $\mathrm{SU(2)_L}$ doublets, while their R ('ψ') parts are $\mathrm{SU(2)_L}$ singlets. So these weak gauge group properties suggest that we should treat the L and R parts separately, rather than together as in a Dirac 4-component spinor. The basic 'building block' is therefore the chiral supermultiplet, suitably 'gauged'.

We have set up the chiral multiplet to involve an L-type spinor χ: this is clearly fine for $e_{\mathrm{L}}^-, \mu_{\mathrm{L}}^-, u_{\mathrm{L}}, d_{\mathrm{L}}$, etc., but what about $e_{\mathrm{R}}^-, \mu_{\mathrm{R}}^-$, etc. ? These *R-type particle fields* can be accommodated within the 'L-type' convention for chiral supermultiplets by regarding them as the charge conjugates of *L-type antiparticle fields*, which we use instead. Charge conjugation was mentioned in section 2.3; see also section 20.5 of [12] (but note that we are here using $C_0 = -\mathrm{i}\gamma_2$). If (as is often done) we denote the field by the particle name, then we have $e_{\mathrm{R}}^- \equiv \psi_{\mathrm{e}^-}$, while $e_{\mathrm{L}}^+ \equiv \chi_{\mathrm{e}^+}$. On the other hand, if we regard e_{R}^- as the charge conjugate of e_{L}^+, then (compare equation (111))

$$e_{\mathrm{R}}^- \equiv \psi_{\mathrm{e}^-} = (e_{\mathrm{L}}^+)^{\mathrm{c}} \equiv \mathrm{i}\sigma_2 \chi_{\mathrm{e}^+}^{\dagger\mathrm{T}}. \tag{462}$$

To remind ourselves of how this works (see also section 2.4), consider a Dirac mass term for the electron:

$$\begin{aligned} \bar{\Psi}_{\mathrm{e}^-}\Psi_{\mathrm{e}^-} = \psi_{\mathrm{e}^-}^\dagger \chi_{\mathrm{e}^-} + \chi_{\mathrm{e}^-}^\dagger \psi_{\mathrm{e}^-} &= (\mathrm{i}\sigma_2 \chi_{\mathrm{e}^+}^{\dagger\mathrm{T}})^\dagger \chi_{\mathrm{e}^-} + \chi_{\mathrm{e}^-}^\dagger \mathrm{i}\sigma_2 \chi_{\mathrm{e}^+}^{\dagger\mathrm{T}} \\ &= \chi_{\mathrm{e}^+}^{\mathrm{T}}(-\mathrm{i}\sigma_2)\chi_{\mathrm{e}^-} + \chi_{\mathrm{e}^-}^\dagger \mathrm{i}\sigma_2 \chi_{\mathrm{e}^+}^{\dagger\mathrm{T}} \\ &= \chi_{\mathrm{e}^+} \cdot \chi_{\mathrm{e}^-} + \chi_{\mathrm{e}^-}^\dagger \cdot \chi_{\mathrm{e}^+}^\dagger. \end{aligned} \tag{463}$$

So it's all expressed in terms of χ's. It is also useful to note that

$$\bar{\Psi}_{\mathrm{e}^-}\gamma_5 \Psi_{\mathrm{e}^-} = -\chi_{\mathrm{e}^+} \cdot \chi_{\mathrm{e}^-} + \chi_{\mathrm{e}^-}^\dagger \cdot \chi_{\mathrm{e}^+}^\dagger. \tag{464}$$

In Table 1 we list the chiral supermultiplets appearing in the MSSM (our y is twice that of [22], following the convention of [12] chapter 22). Note that the 'bar' on the fields in this Table is merely a label, signifying 'antiparticle', not (for example) Dirac conjugation. The subscript $_i$ can be added to the names to signify the family index: for example, $u_{1\mathrm{L}} = u_{\mathrm{L}}, u_{2\mathrm{L}} =$

Names		spin 0	spin 1/2	$SU(3)_c$, $SU(2)_L$, $U(1)_y$
squarks, quarks	Q	$(\tilde{u}_L, \tilde{d}_L)$	(u_L, d_L)	**3**, **2**, 1/3
(× 3 families)	$\bar{u}$	$\tilde{\bar{u}}_L \sim \tilde{u}_R^\dagger$	$\bar{u}_L \sim (u_R)^c$	$\bar{\mathbf{3}}$, **1**, -4/3
	$\bar{d}$	$\tilde{\bar{d}}_L \sim \tilde{d}_R^\dagger$	$\bar{d}_L \sim (d_R)^c$	$\bar{\mathbf{3}}$, **1**, 2/3
sleptons, leptons	L	$(\tilde{\nu}_{eL}, \tilde{e}_L)$	(ν_{eL}, e_L)	**1**, **2**, -1
(× 3 families)	$\bar{e}$	$\tilde{\bar{e}}_L \sim \tilde{e}_R^\dagger$	$\bar{e}_L \sim (e_R)^c$	**1**, **1**, 2
higgs, higgsinos	H_u	(H_u^+, H_u^0)	$(\tilde{H}_u^+, \tilde{H}_u^0)$	**1**, **2**, 1
	H_d	(H_d^0, H_d^-)	$(\tilde{H}_d^0, \tilde{H}_d^-)$	**1**, **2**, -1

Table 1: Chiral supermultiplet fields in the MSSM.

Names	spin 1/2	spin 1	$SU(3)_c$, $SU(2)_L$, $U(1)_y$
gluinos, gluons	$\tilde{g}$	g	**8**, **1**, 0
winos, W bosons	$\widetilde{W}^\pm, \widetilde{W}^0$	$W^\pm$, W^0	**1**, **3**, 0
bino, B boson	$\tilde{B}$	B	**1**, **1**, 0

Table 2: Gauge supermultiplet fields in the MSSM.

$c_L, u_{3L} = t_L$, and similarly for leptons. In Table 2, similarly, we list the gauge supermultiplets of the MSSM. After electroweak symmetry breaking, the W^0 and the B fields mix to produce the physical Z^0 and γ fields, while the corresponding 's'-fields mix to produce a zino ($\tilde{Z}^0$) degenerate with the Z^0, and a massless photino $\tilde{\gamma}$.

So, knowing the gauge groups, the particle content, and the gauge transformation properties, all we need to do to specify any proposed model is to give the superpotential W. *The MSSM is specified by the choice*

$$W = y_u^{ij} \bar{u}_i Q_j \cdot H_u - y_d^{ij} \bar{d}_i Q_j \cdot H_d - y_e^{ij} \bar{e}_i L_j \cdot H_d + \mu H_u \cdot H_d. \tag{465}$$

The fields appearing in (465) are the chiral superfields indicated under the 'Names' column of Table 1. We can alternatively think of W as being the same function of the scalar fields in each chiral supermultiplet, as explained in section 9.4. In either case, the y's are 3×3 matrices in family (or generation) space, *and are exactly the same Yukawa couplings as those which enter the SM* (see for example section 22.7 of [12]).[13]These couplings give masses to the quarks and leptons when the Higgs fields acquire vacuum expectation

[13]We stress once again - see section 4 and footnote 9 - that whereas in the SM we can use one Higgs doublet and its charge conjugate doublet (see section 22.6 of [12]), this is

values: there are no 'Lagrangian' masses for the fermions, since these would explicitly break the $SU(2)_L$ gauge symmetry. The '$\cdot$' notation here means the SU(2)-invariant coupling of two doublets.[14] Also, colour indices have been suppressed, so that '$\bar{u}_i Q_j$', for example, is really $\bar{u}_{\alpha i} Q^{\alpha}_j$, where the upstairs $\alpha = 1, 2, 3$ is a colour $\mathbf{3}$ (triplet) index, and the downstairs α is a colour $\bar{\mathbf{3}}$ (antitriplet) index.

[In parenthesis, we note a possibly confusing aspect of the labelling adopted for the Higgs fields. In the conventional formulation of the SM, the Higgs field $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ generates mass for the 'down' quark, say, via a Yukawa interaction of the form (suppressing family labels)

$$b\,\bar{q}_L \phi d_R + \text{h.c.} \tag{466}$$

In this case, the SU(2) dot product is simply $q_L^\dagger \phi$, which is plainly invariant under $q_L \to U q_L$, $\phi \to U\phi$. Now $q_L^\dagger \phi = u^\dagger \phi^+ + d^\dagger \phi^0$; so when ϕ^0 develops a vev, (466) contributes

$$b\langle \phi^0 \rangle \bar{d}_L d_R + \text{h.c.} \tag{467}$$

which is a d-quark mass. Why, then, do we label our Higgs field $\begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$ with a subscript 'u' rather than 'd'? The point is that, in the SUSY version (465), the SU(2) dot product involving the superfield H_u is taken with the superfield Q which has the quantum numbers of the quark doublet q_L rather than the antiquark doublet $q_L^\dagger$. If we revert for the moment to the procedure of section 8, and write W just in terms of the corresponding scalar fields, the first term in (465) is

$$y_u^{ij} \tilde{\bar{u}}_{Li} (\tilde{u}_{Lj} H_u^0 - \tilde{d}_{Lj} H_u^+). \tag{468}$$

---

not allowed in SUSY, because W cannot depend on the complex conjugate of any field (which would appear in the charge conjugate). By convention, the MSSM does not include Dirac-type neutrino mass terms, neutrino masses being generally regarded as 'beyond the SM' physics.

[14] To take an elementary example: Consider the isospin part of the deuteron's wavefunction. It has $I = 0$ - i.e. it is the SU(2)-invariant coupling of the two doublets $N^{(1)} = \begin{pmatrix} p^{(1)} \\ n^{(1)} \end{pmatrix}$, $N^{(2)} = \begin{pmatrix} p^{(2)} \\ n^{(2)} \end{pmatrix}$. This $I = 0$ wavefunction is, as usual, $\frac{1}{\sqrt{2}}(p^{(1)} n^{(2)} - n^{(1)} p^{(2)})$, which (dropping the $1/\sqrt{2}$) we may write as $N^{(1)\mathrm{T}} \mathrm{i}\tau_2 N^{(2)} \equiv N^{(1)} \cdot N^{(2)}$. Clearly this isospin-invariant coupling is basically the same as the Lorentz-invariant spinor coupling '$\chi^{(1)} \cdot \chi^{(2)}$', which is why we use the same '$\cdot$' notation for both - we hope without causing confusion.

The first term here will, via (267) and (262), generate a term in the Lagrangian (c.f. (307))

$$
\begin{aligned}
&-\frac{1}{2}y_{\mathrm{u}}^{ij}(\chi_{\bar{\mathrm{u}}_{\mathrm{L}i}}\cdot\chi_{\mathrm{u}_{\mathrm{L}j}}+\chi_{\mathrm{u}_{\mathrm{L}i}}\cdot\chi_{\bar{\mathrm{u}}_{\mathrm{L}j}})H_{\mathrm{u}}^{0}+\mathrm{h.c.}\\
&=-y_{\mathrm{u}}^{ij}(\chi_{\bar{\mathrm{u}}_{\mathrm{L}i}}\cdot\chi_{\mathrm{u}_{\mathrm{L}j}})H_{\mathrm{u}}^{0}+\mathrm{h.c.}
\end{aligned}
\tag{469}
$$

When H_{u}^0 develops a (real) vacuum value v_{u} (see section 16), this will become a Dirac-type mass term for the u-quark (compare (463)):

$$
-(m_{\mathrm{u}ij}\chi_{\bar{\mathrm{u}}_{\mathrm{L}i}}\cdot\chi_{\mathrm{u}_{\mathrm{L}j}}+\mathrm{h.c.})
\tag{470}
$$

where

$$
m_{\mathrm{u}ij}=v_{\mathrm{u}}y_{\mathrm{u}}^{ij}.
\tag{471}
$$

Transforming to the basis which diagonalizes the mass matrices then leads to flavour mixing exactly as in the SM (see section 22.7 of [12], for example).]

In summary, then, at the cost of only one new parameter, μ, we have got an exactly supersymmetric extension of the SM.

The fermion masses are evidently proportional to the relevant y parameter, so since the top, bottom and tau are the heaviest fermions in the SM, it is sometimes useful to consider an approximation in which the only non-zero y's are

$$
y_{\mathrm{u}}^{33}=y_{\mathrm{t}};\quad y_{\mathrm{d}}^{33}=y_{\mathrm{b}};\quad y_{\mathrm{e}}^{33}=y_{\tau}.
\tag{472}
$$

In terms of the $\mathrm{SU(2)_L}$ weak isospin fields, this gives (for the scalar fields, and omitting the μ term)

$$
W\approx y_{\mathrm{t}}[\tilde{\bar{t}}_{\mathrm{L}}(\tilde{t}_{\mathrm{L}}H_{\mathrm{u}}^{0}-\tilde{b}_{\mathrm{L}}H_{\mathrm{u}}^{+})]-y_{\mathrm{b}}[\tilde{\bar{b}}_{\mathrm{L}}(\tilde{t}_{\mathrm{L}}H_{\mathrm{d}}^{-}-\tilde{b}_{\mathrm{L}}H_{\mathrm{d}}^{0})]-y_{\mathrm{t}}[\tilde{\bar{\tau}}_{\mathrm{L}}(\tilde{\nu}_{\tau\mathrm{L}}H_{\mathrm{d}}^{-}-\tilde{\tau}_{\mathrm{L}}H_{\mathrm{d}}^{0})].
\tag{473}
$$

The minus signs in W have been chosen so that the terms $y_{\mathrm{t}}\tilde{\bar{t}}_{\mathrm{L}}\tilde{t}_{\mathrm{L}}$, $y_{\mathrm{b}}\tilde{\bar{b}}_{\mathrm{L}}\tilde{b}_{\mathrm{L}}$ and $y_{\tau}\tilde{\bar{\tau}}_{\mathrm{L}}\tilde{\tau}_{\mathrm{L}}$ have the correct sign to generate mass terms for the top, bottom and tau when $\langle H_{\mathrm{u}}^0\rangle\neq 0$ and $\langle H_{\mathrm{d}}^0\rangle\neq 0$.

It is worth recalling that in such a SUSY theory, in addition to the Yukawa couplings of the SM, which couple the Higgs fields to quarks and to leptons, there must also be similar couplings between Higgsinos, squarks and quarks, and between Higgsinos, sleptons and leptons (i.e. we change two ordinary particles into their superpartners). There are also scalar quartic interactions with strength proportional to y_{t}^2, as noted in section 8, arising from the term '$|W_i|^2$' in the scalar potential (461). In addition, there are scalar quartic interactions proportional to the squares of the gauge couplings g and g' coming from the 'D-term' in (461). These include (Higgs)^4 coupings such as are postulated in the SM, but now appearing with coefficients which are determined in terms of the parameters g and g' already present in the model. The important phenomenological consequences of this will be discussed in section 16.

Although there are no conventional mass terms in (465), there is one term which is quadratic in the fields, the so-called 'μ term', which is the SU(2)-invariant coupling of the two different Higgs superfield doublets:

$$W(\mu \text{ term}) = \mu H_{\rm u} \cdot H_{\rm d} = \mu(H_{\rm u1}H_{\rm d2} - H_{\rm u2}H_{\rm d1}) \tag{474}$$

where the subscripts 1 and 2 denote the isospinor component. This is the only such bilinear coupling of the Higgs fields allowed in W, because the other possibilities $H_{\rm u}^{\dagger} \cdot H_{\rm u}$ and $H_{\rm d}^{\dagger} \cdot H_{\rm d}$ involve Hermitian conjugate fields, which would violate SUSY (see footnotes 9 and 11). As always, we need the F-component of (474), which is (see (374))

$$\mu[(H_{\rm u}^{+}F_{\rm d}^{-} - H_{\rm u}^{0}F_{\rm d}^{0} + H_{\rm d}^{0}F_{\rm u}^{0} - H_{\rm d}^{-}F_{\rm u}^{+}) - (\tilde{H}_{\rm u}^{+} \cdot \tilde{H}_{\rm d}^{-} - \tilde{H}_{\rm u}^{0} \cdot \tilde{H}_{\rm d}^{0})], \tag{475}$$

and we must include also the Hermitian conjugate of (475). The second term in (475) will contribute to (off-diagonal) Higgsino mass terms. The first term has the general form 'W_iF_i' of section 8, and hence (see (280)) it leads to the following term in the scalar potential, involving the Higgs fields:

$$|\mu|^2(|H_{\rm u}^{+}|^2 + |H_{\rm d}^{-}|^2 + |H_{\rm u}^{0}|^2 + |H_{\rm d}^{0}|^2). \tag{476}$$

But these terms all have the (positive) sign appropriate to a standard '$m^2\phi^{\dagger}\phi$' bosonic mass term, *not* the negative sign needed for electroweak symmetry breaking via the Higgs mechanism (recall the discussion following equation (4)). This means that our SUSY-invariant Lagrangian cannot accommodate electroweak symmetry breaking.

Of course, SUSY itself - in the MSSM application we are considering - cannot be an exact symmetry, since we've not yet observed the s-partners of the SM fields. We shall discuss SUSY breaking briefly in section 15, but it is clear from the above that some SUSY-breaking terms will be needed in the Higgs potential, in order to allow electroweak symmetry breaking.

This actually poses something of a puzzle [28]. The parameter μ should presumably lie roughly in the range 100 GeV - 1 TeV, or else we'd need delicate cancellations between the positive $|\mu|^2$ terms and the negative SUSY-breaking terms necessary for electroweak symmetry breaking (see a similar argument in section 1.1). We saw in section 1.1 that the general 'no fine-tuning' argument suggested that SUSY-breaking masses should not be much greater than 1 TeV. But the μ term doesn't break SUSY! We are faced with an apparent difficulty: where does this low scale for the SUSY-respecting parameter μ come from? References to some proposed solutions to this 'μ problem' are given in [22] section 5.1, where some further discussion is also given of the various interactions present in the MSSM; see also [23] section 4.2, and particularly the review of the μ problem in [29].

13 Gauge Coupling Unification in the MSSM

As mentioned in section 1.2(b), the idea [30] that the three scale-dependent ('running') SU(3)×SU(2)×U(1) gauge couplings of the SM should converge to a common value - or *unify* - at some very high energy scale does not, in fact, prove to be the case for the SM itself, but it does work very convincingly in the MSSM [31]. The evolution of the gauge couplings is determined by the numbers and types of the gauge and matter multiplets present in the theory, which we have just now given for the MSSM; we can therefore proceed to describe this celebrated result.

The couplings α_3 and α_2 are defined by

$$\alpha_3 = g_{\rm s}^2/4\pi, \qquad \alpha_2 = g^2/4\pi \tag{477}$$

where $g_{\rm s}$ is the $\mathrm{SU(3)_c}$ gauge coupling of QCD and g is that of the electroweak $\mathrm{SU(2)_L}$. The definition of the third coupling α_1 is a little more complicated. It obviously has to be related in some way to g'^2, where g' is the gauge coupling of the $\mathrm{U}(1)_y$ of the SM. The constants g and g' appear in the $\mathrm{SU(2)_L}$ covariant derivative (see equation (22.21) of [12] for example)

$$D_\mu = \partial_\mu + \mathrm{i}g(\boldsymbol{\tau}/2)\cdot\boldsymbol{W}_\mu + \mathrm{i}g'(y/2)B_\mu. \tag{478}$$

The problem is that, strictly within in the SM framework, the scale of 'g'' is arbitrary: we could multiply the weak hypercharge generator y by an arbitrary constant c, and divide g' by c, and nothing would change. In contrast to this, the normalization of whatever couplings multiply the three generators τ^1, τ^2 and τ^3 in (478) is fixed by the normalization of the τ's:

$$\mathrm{Tr}(\frac{\tau^\alpha}{2}\frac{\tau^\beta}{2}) = \frac{1}{2}\delta_{\alpha\beta}. \tag{479}$$

Since each generator is normalized to the same value, the same constant g must multiply each one - no relative rescalings are possible. Within a 'unified' framework, therefore, we hypothesize that some multiple of y, say $Y = c(y/2)$, is one of the generators of a larger group (SU(5) for instance), which also includes the generators of $\mathrm{SU(3)_c}$ and $\mathrm{SU(2)_L}$, all being subject to a common normalization condition; there is then only one (unified) gauge coupling. The quarks and leptons of one family will all belong to a single representation of the larger group, though this need not necessarily be the fundamental representation. All that matters is that the generators all have a common normalization. For example, we can demand the condition

$$\mathrm{Tr}(c^2(y/2)^2) = \mathrm{Tr}(t_3)^2 \tag{480}$$

say, where t_3 is the third $\mathrm{SU(2)_L}$ generator (any generator will give the same result), and the Trace is over all states in the representation - here, u, d, ν_e and e^-. The Traces are simply the sums of the squares of the eigenvalues. On the RHS of (480) we obtain

$$3(\frac{1}{4} + \frac{1}{4}) + \frac{1}{4} + \frac{1}{4} = 2 \tag{481}$$

where the '3' comes from colour, while on the LHS we find from Table 1

$$c^2(\frac{3}{36} + \frac{3}{36} + \frac{3.4}{9} + \frac{3.1}{9} + \frac{1}{4} + 1 + \frac{1}{4}) = c^2\frac{20}{6}. \tag{482}$$

It follows that

$$c = \sqrt{\frac{3}{5}}, \tag{483}$$

so that the correctly normalized generator is

$$Y = \sqrt{\frac{3}{5}}\, y/2. \tag{484}$$

The B_μ term in (478) is then

$$\mathrm{i}g'\sqrt{\frac{5}{3}}\, Y\, B_\mu, \tag{485}$$

indicating that the correctly normalized α_1 is

$$\alpha_1 = \frac{5}{3}\frac{g'^2}{4\pi} \equiv \frac{g_1^2}{4\pi}. \tag{486}$$

Equation (486) can also be interpreted as a prediction for the weak angle θ_W at the unification scale: since $g \tan\theta_W = g' = \sqrt{3/5}g_1$ and $g = g_1$ at unification, we have $\tan\theta_W = \sqrt{3/5}$, or

$$\sin^2\theta_W(\text{unification scale}) = \frac{3}{8}. \tag{487}$$

We are now ready to consider the running of the couplings α_i. To one loop order, the renormalization group equation (RGE) has the form (for an introduction, see chapter 15 of [12] for example)

$$\frac{d\alpha_i}{dt} = -\frac{b_i}{2\pi}\alpha_i^2 \tag{488}$$

where $t = \ln Q$ and Q is the 'running' energy scale, and the coefficients b_i are determined by the gauge group and the matter multiplets to which the gauge bosons couple. For SU(N) gauge theories with matter in the fundamental representation, we have (see [32] for example)

$$b_N = \frac{11}{3}N - \frac{1}{3}n_f - \frac{1}{6}n_s \tag{489}$$

where n_f is the number of left-handed fermions (counting, as usual, right-handed ones as left-handed antiparticles), and n_s is the number of complex scalars, which couple to the gauge bosons. For a $U(1)_Y$ gauge theory in which the fermionic matter particles have charges Y_f and the scalars have charges Y_s, the corresponding formula is

$$b_1 = -\frac{2}{3}\sum_f Y_f^2 - \frac{1}{3}\sum_s Y_s^2. \tag{490}$$

To examine unification, it is convenient to rewrite (488) as

$$\frac{d}{dt}(\alpha_i^{-1}) = \frac{b_i}{2\pi} \tag{491}$$

which can be immediately integrated to give

$$\alpha_i^{-1}(Q) = \alpha_i^{-1}(Q_0) + \frac{b_i}{2\pi}\ln(Q/Q_0), \tag{492}$$

where Q_0 is the scale at which running commences. We see that the inverse couplings run linearly with $\ln Q$. Q_0 is taken to be m_Z, where the couplings are well measured. 'Unification' is then the hypothesis that, at some higher scale $Q_{\rm U} = m_{\rm U}$, the couplings are equal:

$$\alpha_1(m_{\rm U}) = \alpha_2(m_{\rm U}) = \alpha_3(m_{\rm U}) \equiv \alpha_{\rm U}. \tag{493}$$

This implies that the three equations (492), for $i = 1, 2, 3$, become

$$\alpha_{\rm U}^{-1} = \alpha_3^{-1}(m_{\rm Z}) + \frac{b_3}{2\pi}\ln(m_{\rm U}/m_{\rm Z}) \tag{494}$$

$$\alpha_{\rm U}^{-1} = \alpha_2^{-1}(m_{\rm Z}) + \frac{b_2}{2\pi}\ln(m_{\rm U}/m_{\rm Z}) \tag{495}$$

$$\alpha_{\rm U}^{-1} = \alpha_1^{-1}(m_{\rm Z}) + \frac{b_1}{2\pi}\ln(m_{\rm U}/m_{\rm Z}). \tag{496}$$

Eliminating $\alpha_{\rm U}$ and $\ln(m_{\rm U}/m_{\rm Z})$ from these equations gives one condition relating the measured constants $\alpha_i^{-1}(m_{\rm Z})$ and the calculated numbers b_i, which is

$$\frac{\alpha_3^{-1}(m_{\rm Z}) - \alpha_2^{-1}(m_{\rm Z})}{\alpha_2^{-1}(m_{\rm Z}) - \alpha_1^{-1}(m_{\rm Z})} = \frac{b_2 - b_3}{b_1 - b_2}. \tag{497}$$

Checking the truth of (497) is one simple way of testing unification quantitatively (at least, at this one-loop level).

Let's call the LHS of (497) $B_{\rm exp}$, and the RHS $B_{\rm th}$. For $B_{\rm exp}$, we use the data

$$\sin^2\theta_{\rm W}(m_{\rm Z}) = 0.231 \tag{498}$$

$$\alpha_3(m_{\rm Z}) = 0.119, \qquad \text{or} \qquad \alpha_3^{-1}(m_{\rm Z}) = 8.40 \tag{499}$$

$$\alpha_{\rm em}^{-1}(m_{\rm Z}) = 128. \tag{500}$$

We are not going to bother with errors here, but the uncertainty in $\alpha_3(m_{\rm Z})$ is about 2%, and that in $\sin^2\theta_{\rm W}(m_{\rm Z})$ and $\alpha_{\rm em}(m_{\rm Z})$ is much less. Here $\alpha_{\rm em}$ is defined by $\alpha_{\rm em} = e^2/4\pi$ where $e = g\sin\theta_{\rm W}$. Hence

$$\alpha_2^{-1}(m_{\rm Z}) = \alpha_{\rm em}^{-1}(m_{\rm Z})\sin^2\theta_{\rm W}(m_{\rm Z}) = 29.6. \tag{501}$$

Finally,

$$g'^2 = g^2\tan^2\theta_{\rm W} \tag{502}$$

and hence

$$\alpha_1^{-1}(m_{\rm Z}) = \frac{3}{5}\alpha'^{-1}(m_{\rm Z}) = \frac{3}{5}\alpha_2^{-1}(m_{\rm Z})\cot^2\theta_{\rm W}(m_{\rm Z}) = 59.12. \tag{503}$$

From these values we obtain

$$B_{\mathrm{exp}} = 0.72. \tag{504}$$

Now let's look at B_{th}. First, consider the SM. For $\mathrm{SU}(3)_{\mathrm{c}}$ we have

$$b_3^{\mathrm{SM}} = 11 - \frac{1}{3}12 = 7. \tag{505}$$

For $\mathrm{SU}(2)_{\mathrm{L}}$ we have

$$b_2^{\mathrm{SM}} = \frac{22}{3} - 4 - \frac{1}{6} = \frac{19}{6}, \tag{506}$$

and for $\mathrm{U}(1)_Y$ we have

$$b_1^{\mathrm{SM}} = -\frac{2}{3}\frac{3}{5}\sum_{\mathrm{f}}(y_{\mathrm{f}}/2)^2 - \frac{1}{3}\frac{3}{5}\sum_{\mathrm{s}}(y_{\mathrm{s}}/2)^2 \tag{507}$$

$$= -\frac{2}{5}3\frac{20}{6} - \frac{1}{5}\frac{1}{2} = -\frac{41}{10}. \tag{508}$$

Hence, in the SM, the RHS of (497) gives

$$B_{\mathrm{th}} = \frac{115}{218} = 0.528, \tag{509}$$

which is in very poor accord with (504).

What about the MSSM? Formula (489) must be modified to take account of the fact that, in each SU(N), the gauge bosons are accompanied by gauginos in the regular representation of the group. Their contribution to b_N is $-2N/3$. In addition, we have to include the scalar partners of the quarks and of the leptons, in the fundamental representations of SU(3) and SU(2); and we must not forget that we have two Higgs doublets, both accompanied by Higgsinos, all in the fundamental representation of SU(2). These changes give

$$b_3^{\mathrm{MSSM}} = 7 - 2 - \frac{1}{6}12 = 3, \tag{510}$$

and

$$b_2^{\mathrm{MSSM}} = \frac{19}{6} - \frac{4}{3} - \frac{1}{6}12 - \frac{1}{3}2 - \frac{1}{6} = -1. \tag{511}$$

It is interesting that the sign of b_2 has been reversed. For b_1^{MSSM}, there is no contribution from the gauge bosons or their fermionic partners. The left-handed fermions contribute as in (507), and are each accompanied by corresponding scalars, so that

$$b_1^{\rm MSSM}(\text{fermions and sfermions}) = -\frac{3}{5}10 = -6. \tag{512}$$

The Higgs and Higginos contribute

$$b_1^{\rm MSSM}(\text{Higgs and Higgsinos}) = -\frac{3}{5}4\,\frac{1}{4} = -\frac{3}{5}. \tag{513}$$

In total, therefore,

$$b_1^{\rm MSSM} = -\frac{33}{5}. \tag{514}$$

From (510), (511) and (514) we obtain

$$B_{\rm th}^{\rm MSSM} = \frac{5}{7} = 0.74 \tag{515}$$

which is in excellent agreement with (504).

This has been by no means a 'professional' calculation. One should consider two-loop corrections. Also, SUSY must be broken, presumably at a scale of order 1 TeV or less, and the resulting mass differences between the particles and their s-partners will lead to 'threshold' corrections. Similarly, the details of the theory at the high scale (in particular, its breaking) may be expected to lead to (high energy) threshold corrections. A recent analysis by Pokorski [33] concludes that the present data are in good agreement with the predictions of supersymmetric unification, for reasonable estimates of such corrections.

Returning to (495) and (496), and inserting the values of $\alpha_2^{-1}(m_{\rm Z})$ and $\alpha_1^{-1}(m_{\rm Z})$, we can obtain an estimate of the unification scale $m_{\rm U}$. We find

$$\ln(m_{\rm U}/m_{\rm Z}) = \frac{10\pi}{28}[\alpha_1^{-1}(m_{\rm Z}) - \alpha_2^{-1}(m_{\rm Z})] \simeq 33.1, \tag{516}$$

which implies

$$m_{\rm U} \simeq 2.2 \times 10^{16}\text{GeV}, \tag{517}$$

a value relatively close to the Planck scale $m_{\rm P} \simeq 1.2 \times 10^{19}$ GeV.

Of course, one can make up any number of models yielding the experimental value $B_{\rm exp}$. But there is no denying that the correct prediction (515) is an unforced consequence simply of the matter content of the MSSM, and agreement was clearly not inevitable. It does seem to provide support both for the inclusion of supersymmetric particles in the RGE, and for gauge unification. Grand unified theories are reviewed by Raby in [34].

14 R-parity

As stated in the section 12, the 'minimal' supersymmetric extension of the SM is specified by the choice (465) for the superpotential. There are, however, other gauge-invariant and renormalizable terms which could also be included in the superpotential, namely ([22] section 5.2)

$$W_{\Delta L=1} = \lambda_e^{ijk} L_i \cdot L_j \bar{e}_k + \lambda_L^{ijk} L_i \cdot Q_j \bar{d}_k + \mu_L^i L_i \cdot H_{\rm u} \tag{518}$$

and

$$W_{\Delta B=1} = \lambda_B^{ijk} \bar{u}_i \bar{d}_j \bar{d}_k. \tag{519}$$

The superfields Q carry baryon number $B = 1/3$ and $\bar{u}$, $\bar{d}$ carry $B = -1/3$, while L carries lepton number $L = 1$ and $\bar{e}$ carries $L = -1$. Hence the terms in (518) violate lepton number conservation by one unit of L, and those in (519) violate baryon number conservation by one unit of B. Now, B- and L-violating processes have never been seen experimentally. If both the couplings λ_L and λ_B were present, the proton could decay via channels such as $e^+\pi^0$, $\mu^+\pi^0, \ldots$ etc. The non-observance of such decays places strong limits on the strengths of such couplings, which would have to be extraordinarily small (being renormalizable, the couplings are dimensionless, and there is no natural suppression by a high scale such as would occur in a non-renormalizable term). It is noteworthy that in the SM, there are no possible renormalizable terms in the Lagrangian which violate B or L - this is indeed a nice bonus provided by the SM. We could of course just impose B and L conservation as a principle, thus forbidding (518) and (519). But in fact both are known to be violated by non-perturbative electroweak effects, which are negligible at ordinary energies but which might be relevant in the early universe. Neither B nor L can therefore be regarded as a fundamental symmetry. Instead, people have come up with an alternative symmetry, which forbids (518) and (519), while allowing all the interactions of the MSSM.

This symmetry is called R-parity, which is multiplicatively conserved, and is defined by

$$R = (-)^{3B+L+2s} \tag{520}$$

where s is the spin of the particle. One quickly finds that R is +1 for all conventional matter particles, and (because of the $(-)^{2s}$ factor) -1 for all their s-partners. Since the product of $(-)^{2s}$ is +1 for the particles involved in any interaction vertex, by angular momentum conservation, it's clear that both (518) and (519) do not conserve R-parity, while the terms in (465) do.

In fact, every interaction vertex in (465) contains an even number of $R = -1$ sparticles, which has important phenomenological consequences:

- The lightest sparticle ('LSP') is absolutely stable, and if electrically uncharged it could be an attractive candidate for non-baryonic dark matter.
- The decay products of all other sparticles must contain an odd number of LSP's.
- In accelerator experiments, sparticles can only be produced in pairs.

In the context of the MSSM, the LSP must lack electromagnetic and strong interactions; otherwise, LSP's surviving from the Big Bang era would have bound to nuclei forming objects with highly unusual charge to mass ratios, but searches for such exotics have excluded all models with stable charged or strongly interacting particles unless their mass exceeds several TeV, which is unacceptably high for the LSP. An important implication is that in collider experiments LSP's will carry away energy and momentum while escaping detection. Since all sparticles will decay into at least one LSP (plus SM particles), and since in the MSSM sparticles are pair-produced, it follows that at least $2m_{\tilde{\chi}_1^0}$ missing energy will be associated with each SUSY event, where $m_{\tilde{\chi}_1^0}$ is the mass of the LSP (often taken to be a neutralino - see section 17). In $\mathrm{e}^-\mathrm{e}^+$ machines, the total visible energy and momentum can be well measured, and the beams have very small spread, so that the missing energy and momentum can be well correlated with the energy and momentum of the LSP's. In hadron colliders, the distribution of energy and longitudinal momentum of the partons (i.e. quarks and gluons) is very broad, so in practice only the missing transverse momentum (or missing transverse energy $\not{E}_{\mathrm{T}}$) is useful.

Further aspects of R-parity are discussed in [22].

15 SUSY breaking

Since SUSY is manifestly not an exact symmetry of the known particle spectrum, the issue of SUSY-breaking must be addressed before the MSSM can be applied phenomenologically. We know only two ways in which a symmetry can be broken: either (a) by explicit symmetry-breaking terms in the Lagrangian, or (b) by spontaneous symmetry breaking, such as occurs in the case of the chiral symmetry of QCD, and is hypothesized to occur for the electroweak symmetry of the SM via the Higgs mechanism. In the electroweak case, the introduction of explicit mass terms for the fermions and massive gauge bosons would spoil renormalizability, which is why in this case spontaneous symmetry breaking (which preserves renormalizability) is preferred theoretically - and indeed is strongly indicated by experiment, via the precision measurement of finite radiative corrections. We shall give a brief introduction to spontaneous SUSY-breaking, since it presents some novel features as compared, say, to the more 'standard' QCD and electroweak cases. But in fact there is no consensus on how 'best' to break SUSY spontaneously, and in practice one is reduced to introducing explicit SUSY-breaking terms as in approach (a) after all, which parametrize the low-energy effects of the unknown breaking mechanism presumed (usually) to operate at some high mass scale. We shall see in section 15.2 that these SUSY-breaking terms are quite constrained by the requirement that they do not re-introduce divergences which would spoil the SUSY solution to the hierarchy problem; nevertheless, over 100 parameters are needed to characterize them.

15.1 Breaking SUSY Spontaneously

The fundamental requirement for a symmetry in field theory to be spontaneously broken (see for example [12] Part 7) is that a field, which is not invariant under the symmetry, should have a non-vanishing vacuum expectation value. That is, if the field in question is denoted by ϕ', then we require $\langle 0|\phi'(x)|0\rangle \neq 0$. Since ϕ' is not invariant, it must belong to a symmetry multiplet of some kind, along with other fields, and it must be possible to express ϕ' as

$$\phi'(x) = \mathrm{i}[Q, \phi(x)] \tag{521}$$

where Q is a generator of the symmetry group, and ϕ is a suitable field in the multiplet to which ϕ' belongs. So then we have

$$\langle 0|\phi'|0\rangle = \langle 0|\mathrm{i}[Q, \phi]|0\rangle = \langle 0|\mathrm{i}Q\phi - \mathrm{i}\phi Q|0\rangle \neq 0. \tag{522}$$

Now the vacuum state $|0\rangle$ is usually assumed to be such that $Q|0\rangle = 0$, since this implies that $|0\rangle$ is invariant under the transformation generated by Q. But if we take $Q|0\rangle = 0$, we violate (522). Hence for spontaneous symmetry breaking we have to assume $Q|0\rangle \neq 0$ - that is, the vacuum is not invariant under the symmetry.

In the case of SUSY, this means that we require

$$Q_a|0\rangle \neq 0, \quad Q_b^\dagger|0\rangle \neq 0 \tag{523}$$

for the SUSY generators of section 5. The condition (523) has a remarkable consequence in SUSY, which is strikingly different from all other examples of spontaneous symmetry breaking. The SUSY algebra (210) is

$$\{Q_a, Q_b^\dagger\} = (\sigma^\mu)_{ab} P_\mu. \tag{524}$$

So we have

$$\begin{aligned} Q_1 Q_1^\dagger + Q_1^\dagger Q_1 = (\sigma^\mu)_{11} P_\mu &= P_0 + P_3 \\ Q_2 Q_2^\dagger + Q_2^\dagger Q_2 = (\sigma^\mu)_{22} P_\mu &= P_0 - P_3. \end{aligned} \tag{525}$$

It follows that

$$P_0 = \frac{1}{2}(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2) = H, \tag{526}$$

where H is the Hamiltonian of the theory considered. Hence we find

$$\begin{aligned} \langle 0|H|0\rangle &= \frac{1}{2}(\langle 0|Q_1 Q_1^\dagger|0\rangle + \langle 0|Q_1^\dagger Q_1|0\rangle + \ldots) \\ &= \frac{1}{2}(\ |(Q_1^\dagger|0\rangle)|^2 + |(Q_1|0\rangle)|^2 + \ldots) \\ &> 0, \end{aligned} \tag{527}$$

the strict inequality in the last step following from the basic symmetry-breaking assumption (523). We deduce the remarkable result: *when SUSY is spontaneously broken, the vacuum energy is necessarily positive.*[15] On the other hand, when SUSY is exact, so that $Q_a|0\rangle = Q_b^\dagger|0\rangle = 0$, we obtain $\langle 0|H|0\rangle = 0$ - the vacuum energy of a (globally) SUSY-invariant theory is zero.

For SUSY to be spontaneously broken, therefore, the scalar potential $\mathcal{V}$ must have no SUSY-respecting minimum (assuming the kinetic bits of the

[15]This is true for global SUSY - i.e. the case in which the parameters $\xi, \xi^\dagger$ in SUSY transformations don't depend on the space-time coordinate x. In the local case, which is essentially supergravity, it turns out that the vacuum has exactly zero energy in the spontaneously broken case.

Hamiltonian don't contribute in the vacuum). For, if it did, such a SUSY-respecting configuration would necessarily have zero energy, and since by hypothesis this is the minimum value of $\mathcal{V}$, SUSY-breaking (which requires $\mathcal{V} > 0$) will simply not happen, on energy grounds.

What kinds of field ϕ' could have a non-zero value in the SUSY case? Returning to (521), with Q now a SUSY generator, we consider all such possible commutation relations, beginning with those for the chiral supermultiplet. The commutation relations of the Q's with the fields are determined by the SUSY transformations which are

$$\begin{aligned}\delta_\xi \phi &= \mathrm{i}[\xi \cdot Q, \phi] = \xi \cdot \chi \\ \delta_\xi \chi &= \mathrm{i}[\xi \cdot Q, \chi] = -\mathrm{i}\sigma^\mu \mathrm{i}\sigma_2 \xi^* \partial_\mu \phi + \xi F \\ \delta_\xi F &= \mathrm{i}[\xi \cdot Q, F] = -\mathrm{i}\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi.\end{aligned} \tag{528}$$

Considering the terms on the RHS of each of the three relations in (528), we cannot have $\langle 0|\chi|0\rangle \neq 0$ since χ isn't a scalar field, and such a vev for a spinor would break Lorentz invariance; we can't have $\langle 0|\partial_\mu \phi|0\rangle \neq 0$ either, because ϕ is assumed constant in the vacuum; so this leaves

$$\langle 0|F|0\rangle \neq 0 \tag{529}$$

as the only possibility! This is called 'F-type SUSY breaking', since it is the auxiliary field F which acquires a vev.

Recall now that in W-Z models, with superpotentials of the form (268) such as are used in the MSSM, we had

$$F_i = -\left(\frac{\partial W}{\partial \phi_i}\right)^\dagger = -(M_{ij}\phi_j + \frac{1}{2} y_{ijk}\phi_j\phi_k)^\dagger, \tag{530}$$

and $\mathcal{V}(\phi) = |F_i|^2$, which has an obvious minimum when all the ϕ's are zero. Hence with this form of W, SUSY can't be spontaneously broken. To get spontaneous SUSY breaking, we must add a constant to F_i, that is a linear term in W (see footnote 10). Even this is tricky, and it needs ingenuity to produce a simple working model . One (due to O'Raifeartaigh [35]) employs three chiral supermultiplets, and takes W to be

$$W = m\phi_1\phi_3 + g\phi_2(\phi_3^2 - M^2). \tag{531}$$

This produces

$$-F_1^\dagger = m\phi_3, \; -F_2^\dagger = g(\phi_3^2 - M^2), \; -F_3^\dagger = m\phi_1 + 2\phi_2\phi_3. \tag{532}$$

Hence

$$\begin{aligned} \mathcal{V} &= |F_1|^2 + |F_2|^2 + |F_3|^2 \\ &= m^2|\phi_3|^2 + g^2|\phi_3^2 - M^2|^2 + |m\phi_1 + 2\phi_2\phi_3|^2. \end{aligned} \tag{533}$$

The first two terms in (533) cannot both vanish at once, and so there is no possible field configuration giving $\mathcal{V} = 0$, which is the SUSY-preserving solution. Instead, there is a SUSY-breaking minimum at

$$\phi_1 = \phi_3 = 0, \tag{534}$$

which are interpreted as the corresponding vev's, with ϕ_2 undetermined (a so-called 'flat' direction in field space). This solution gives

$$\langle 0|F_1^\dagger|0\rangle = \langle 0|F_3^\dagger|0\rangle = 0, \tag{535}$$

but

$$\langle 0|F_2^\dagger|0\rangle = gM^2. \tag{536}$$

The minimum value of $\mathcal{V}$ is g^2M^4, which is strictly positive, as expected. Note that the parameter M does indeed have the dimensions of a mass: it can be understood as signifying the scale of spontaneous SUSY breaking, via $\langle 0|F_2^\dagger|0\rangle \neq 0$, much as the Higgs vev sets the scale of electroweak symmetry breaking.

In the SM, or MSSM, all the terms in W must be gauge-invariant - but there is no field in the SM which is itself gauge-invariant (i.e. all its gauge quantum numbers are zero). Hence in the SM or MSSM we cannot have a linear term in W, and must look beyond these models if we want to pursue this form of SUSY breaking.

In this F-type SUSY breaking, then, we have

$$0 \neq \langle 0|[Q, \chi(x)]|0\rangle = \sum_n \langle 0|Q|n\rangle\langle n|\chi(x)|0\rangle - \langle 0|\chi(x)|n\rangle\langle n|Q|0\rangle, \tag{537}$$

where $|n\rangle$ is a complete set of states. It can be shown that (537) implies that there must exist among the states $|n\rangle$ a *massless* state $|\tilde{g}\rangle$ which couples to the vacuum via the generator Q: $\langle 0|Q|\tilde{g}\rangle \neq 0$. This is the SUSY version of Goldstone's theorem - see for example section 17.4 of [12]. The theorem states that when a symmetry is spontaneously broken, one or more massless particles must be present, which couple to the vacuum via a symmetry generator. In the non-SUSY case, they are (Goldstone) bosons; in the SUSY case, since the generators are fermionic, they are fermions - 'Goldstinos'. You can check that the fermion spectrum in the above model contains a massless field χ_2 - it is in fact in a supermultiplet along with F_2, the auxiliary field which gained a vev, and the scalar field ϕ_2, where ϕ_2 is the field direction along which the potential is 'flat' - a situation analogous to that for the standard Goldstone 'wine-bottle' potential, where the massless mode is associated with excitations round the flat rim of the bottle.

Exercise Show that the mass spectrum of the O'Raifeartaigh model consists of (a) 6 real scalar fields with tree-level masses 0,0 (the real and imaginary parts of the complex field ϕ_2) m^2, m^2 (ditto for the complex field ϕ_1) and $m^2 - 2g^2M^2, m^2 + 2g^2M^2$ (the no longer degenerate real and imaginary parts of the complex field ϕ_3); (b) 3 L-type fermions with masses 0 (the Goldstino χ_2), $|\mu|, |\mu|$ (linear combinations of the fields χ_1 and χ_3). (Hint: for the scalar masses, take $\langle\phi_2\rangle = 0$ for convenience, expand the potential about the point $\phi_1 = \phi_2 = \phi_3 = 0$, and examine the quadratic terms. For the fermions, the mass matrix of (280) is $W_{13} = W_{31} = m$, all other components vanishing; diagonalize the mass term by introducing the linear combinations $\chi_- = (\chi_1 - \chi_3)/\sqrt{2}, \chi_+ = (\chi_1 + \chi_3)/\sqrt{2}$. See also [19] section 2.10).)

In the absence of SUSY breaking, a single massive chiral supermultiplet consists (as in the W-Z model of section 8) of a complex scalar field (or equivalently two real scalar fields) degenerate in mass with an L-type spin-1/2 field. It is interesting that in the O'Raifeartaigh model the masses of the '3' supermultiplet, after SUSY breaking, obey the relation

$$(m^2 - 2g^2M^2) + (m^2 + 2g^2M^2) = 2m^2 = 2m^2_{\chi_3}, \tag{538}$$

which is evidently a generalization of the relation that would hold in the SUSY-preserving case $g = 0$. Indeed, there is a general sum rule for the tree-level $(\text{mass})^2$ values of scalars and chiral fermions in theories with spontaneous SUSY breaking [36]:

$$\sum m^2_{\text{real scalars}} = 2\sum m^2_{\text{chiral fermions}}, \tag{539}$$

where it is understood that the sums are over sectors with the same electric charge, colour charge, baryon number and lepton number. Unfortunately, (539) implies that this kind of SUSY breaking cannot be phenomenologically viable, since it requires the existence of (for example) scalar partners of right-handed d-type quarks, with masses of at most a few GeV - and this is excluded experimentally.

We also need to consider possible SUSY breaking via terms in a gauge supermultiplet. This time the transformations are

$$\begin{aligned}
\delta_\xi W^{\mu\alpha} &= \mathrm{i}[\xi\cdot Q, W^{\mu\alpha}] = -\frac{1}{\sqrt{2}}(\xi^\dagger\bar{\sigma}^\mu\lambda^\alpha + \lambda^{\alpha\dagger}\bar{\sigma}^\mu\xi) \\
\delta_\xi \lambda^\alpha &= \mathrm{i}[\xi\cdot Q, \lambda^\alpha] = -\frac{\mathrm{i}}{2\sqrt{2}}\sigma^\mu\bar{\sigma}^\nu\xi F^\alpha_{\mu\nu} + \frac{1}{\sqrt{2}}\xi D^\alpha \\
\delta_\xi D^\alpha &= \mathrm{i}[\xi\cdot Q, D^\alpha] = \frac{\mathrm{i}}{\sqrt{2}}(\xi^\dagger\bar{\sigma}^\mu(D_\mu\lambda)^\alpha - (D_\mu\lambda)^{\alpha\dagger}\bar{\sigma}^\mu\xi). \qquad (540)
\end{aligned}$$

Inspection of (540) shows that, as for the chiral supermultiplet, only the auxiliary fields can have a non-zero vev:

$$\langle 0|D^\alpha|0\rangle, \qquad (541)$$

which is called D-type symmetry breaking.

At first sight, however, such a mechanism can't operate in the MSSM, for which the scalar potential is as given in (461). 'F-type' SUSY breaking comes from the first term $|W_i|^2$, D-type from the second, and the latter clearly has a SUSY-preserving minimum at $\mathcal{V}=0$ when all the fields vanish. But there is another possibility, rather like the 'linear term in W' trick used for F-type breaking, which was discovered by Fayet and Iliopoulos [37] for the U(1) gauge case. The auxiliary field D of a U(1) gauge supermultiplet is gauge-invariant, and a term in the Lagrangian proportional to D is SUSY-invariant too, since (see (425)) it transforms by a total derivative. Suppose, then, that we add a term M^2D, the *Fayet-Iliopoulos term*, to the Lagrangian (459). The part involving D is now

$$\mathcal{L}_D = M^2 D + \frac{1}{2}D^2 - g_1 D\sum_i e_i\phi_i^\dagger\phi_i, \qquad (542)$$

where e_i are the U(1) charges of the scalar fields ϕ_i in units of g_1, the U(1) coupling constant. Then the equation of motion for D is

$$D = -M^2 + g_1\sum_i e_i\phi_i^\dagger\phi_i. \qquad (543)$$

The corresponding potential is now

$$\mathcal{V}_D = \frac{1}{2}(-M^2 + g_1\sum_i e_i\phi_i^\dagger\phi_i)^2. \qquad (544)$$

Consider for simplicity the case of just one scalar field ϕ, with charge eg_1. If $eg_1 > 0$ there will be a SUSY-preserving solution, i.e. with $\mathcal{V}_D = 0$, at $|\langle 0|\phi|0\rangle| = (M^2/eg_1)^{1/2}$. This is actually a Higgs-type breaking of the U(1) symmetry, and it will also generate a mass for the U(1) gauge field. On the other hand, if $eg_1 < 0$, we find $\mathcal{V}_D = \frac{1}{2}M^4$ when $\langle 0|\phi|0\rangle = 0$, which is a U(1)-preserving and SUSY-breaking solution. In fact, we then have

$$\mathcal{L}_D = -\frac{1}{2}M^4 - |eg_1|M^2\phi^\dagger\phi + \dots \tag{545}$$

showing that the ϕ field has a mass $M(|eg_1|)^{1/2}$. The gaugino field λ and the gauge field A^μ remain massless, and λ can be interpreted as a Goldstino.

This mechanism can't be used in the non-Abelian case, because no term of the form $M^2 D^\alpha$ can be gauge-invariant. Could we have D-term breaking in the $\mathrm{U}(1)_y$ sector of the MSSM? Unfortunately not. What we want is a situation in which the scalar fields in (544) do not acquire vev's (for example, because they have large mass terms in the superpotential), so that the minimum of (544) forces D to have a non-zero (vacuum) value, thus breaking SUSY. In the MSSM, however, the squark and slepton fields have no superpotential mass terms, and so wouldn't be prevented from acquiring vev's *en route* to minimizing (544). But this would imply the breaking of any symmetry associated with quantum numbers carried by these fields, for example colour, which is not acceptable.

One common viewpoint seems to be that SUSY breaking could occur in a sector that is weakly coupled to the chiral supermultiplets of the MSSM. For example, it could be (a) via gravitational interactions (presumably at the Planck scale, so that SUSY-breaking mass terms would enter as (the vev of an F- or D-type parameter having dimension $M^2)/M_{\mathrm{P}}$, which gives $\sqrt{(\mathrm{vev})} \sim 10^{10}$ GeV, say), or (b) via electroweak gauge interactions. These possibilities are discussed in [22] section 6. A more recent review, with additional SUSY-breaking mechanisms, is contained in [23] section 3.

15.2 Soft SUSY-breaking Terms

In any case, however the necessary breaking of SUSY is effected, we can always look for a parametrization of the SUSY-breaking terms which should be present at 'low' energies, and do phenomenology with them. It is a vital point that such phenomenological SUSY-breaking terms in the (now effective) Lagrangian should be 'soft', as the jargon has it - that is, they should have positive mass dimension, for example '$M^2\phi^2$', '$M\phi^3$', '$M\chi\cdot\chi$', etc. The reason is that such terms (which are super-renormalizable) will not introduce new divergences into, for example, the relations between the dimensionless coupling constants which follow from SUSY, and which guarantee the stability of the mass hierarchy, which was one of the prime motivations for SUSY in the first place. As we saw section 1.1, a typical one-loop radiative correction to a scalar mass^2 is

$$\delta m^2 \sim (\lambda_{\rm scalar} - g^2_{\rm fermion})\Lambda^2 \tag{546}$$

where Λ is the u-v cutoff. In SUSY we essentially have $\lambda_{\rm scalar} = g^2_{\rm fermion}$, and the dependence on Λ becomes safely logarithmic. Suppose, on the other hand, that the dimensionless couplings $\lambda_{\rm scalar}$ or $g_{\rm fermion}$ themselves received divergent one-loop corrections, arising from renormalizable (rather than super-renormalizable) SUSY-breaking interactions.[16] Then $\lambda_{\rm scalar}$ and $g_{\rm fermion}$ would differ by terms of order $\ln\Lambda$, with the result that the mass shift (546) becomes very large indeed, once more. In general, soft SUSY-breaking terms maintain the cancellations of quadratically divergent radiative corrections to scalar mass^2, to all orders in perturbation theory [38]. This means that corrections δm^2 go like $m^2_{\rm soft}\ln(\Lambda/m_{\rm soft})$, where $m_{\rm soft}$ is the typical mass scale of the soft SUSY-breaking terms. This is a stable shift in the sense of the hierarchy problem, provided of course that (as remarked in section 1.1) the new mass scale $m_{\rm soft}$ is not much greater than 1 TeV, say. The origin of this mass scale remains unexplained.

The forms of possible soft SUSY breaking terms are quite limited. They are as follows.

(a) Gaugino masses for each gauge group:

$$-\frac{1}{2}(M_3\tilde{g}^\alpha\cdot\tilde{g}^\alpha + M_2\tilde{W}^\alpha\cdot\tilde{W}^\alpha + M_1\tilde{B}\cdot\tilde{B} + {\rm h.c.}) \tag{547}$$

where in the first (gluino) term α runs from 1 to 8 and in the second (wino) term it runs from 1 to 3, the dot here signifying the Lorentz invariant spinor product.

[16] One example of such a renormalizable SUSY-breaking interaction would be the Standard Model Yukawa interaction that generates mass for 'up' fermions and which involves the charge-conjugate of the Higgs doublet that generates mass for the 'down' fermions (see footnote 11). The argument being given here implies that we do *not* want to generate 'up' masses this way, but rather via a second, independent, Higgs field.

(b) Squark $(\text{mass})^2$ terms:

$$-m^2_{\tilde{\mathrm{Q}}ij}\tilde{Q}^\dagger_i \cdot \tilde{Q}_j - m^2_{\tilde{\bar{\mathrm{u}}}ij}\tilde{\bar{u}}^\dagger_i\tilde{\bar{u}}_j - m^2_{\tilde{\bar{\mathrm{d}}}ij}\tilde{\bar{d}}^\dagger_i\tilde{\bar{d}}_j, \tag{548}$$

where i and j are family labels, and the first term is an $\mathrm{SU}(2)_\mathrm{L}$-invariant dot product of scalar doublets partnering L-type quark doublets (see footnote 13 and the 'In parenthesis' paragraph following that footnote).

(c) Slepton $(\text{mass})^2$ terms:

$$-m^2_{\tilde{\mathrm{L}}ij}\tilde{L}^\dagger_i \cdot \tilde{L}_j - m^2_{\tilde{\bar{\mathrm{e}}}ij}\tilde{\bar{e}}^\dagger_i\tilde{\bar{e}}_j. \tag{549}$$

(d) Higgs $(\text{mass})^2$ terms:

$$-m^2_{\mathrm{H_u}}H^\dagger_\mathrm{u} \cdot H_\mathrm{u} - m^2_{\mathrm{H_d}}H^\dagger_\mathrm{d} \cdot H_\mathrm{d} - (bH_\mathrm{u} \cdot H_\mathrm{d} + \mathrm{h.c.}) \tag{550}$$

where the $\mathrm{SU}(2)_\mathrm{L}$ invariant dot products are

$$H^\dagger_\mathrm{u} \cdot H_\mathrm{u} = |H^+_\mathrm{u}|^2 + |H^0_\mathrm{u}|^2 \tag{551}$$

and similarly for $H^\dagger_\mathrm{d} \cdot H_\mathrm{d}$, while

$$H_\mathrm{u} \cdot H_\mathrm{d} = H^+_\mathrm{u}H^-_\mathrm{d} - H^0_\mathrm{u}H^0_\mathrm{d}. \tag{552}$$

(e) Triple scalar couplings

$$-a^{ij}_\mathrm{u}\tilde{\bar{u}}_i\tilde{Q}_j \cdot H_\mathrm{u} + a^{ij}_\mathrm{d}\tilde{\bar{d}}_i\tilde{Q}_j \cdot H_\mathrm{d} + a^{ij}_\mathrm{e}\tilde{\bar{e}}_i\tilde{L}_j \cdot H_\mathrm{d} + \mathrm{h.c.} \tag{553}$$

The five $(\text{mass})^2$ matrices are in general complex, but must be Hermitian so that the Lagrangian is real. All the terms (547)-(553) manifestly break SUSY: the mass terms only involve part of the relevant supermultiplets, and the triple scalar couplings involve three 'ϕ' or '$\phi^\dagger$' fields, rather than (c.f. (305)) the combinations '$\phi^2\phi^\dagger$' or '$\phi^{2\dagger}\phi$'.

On the other hand, it is important to emphasize that the terms (547) - (553) do respect the SM gauge symmetries. The b term in (550) and the triple scalar couplings in (553) have the same form as the 'μ' and 'Yukawa' couplings in the (gauge-invariant) superpotential (465), but here involving just the scalar fields, of course. It is particularly noteworthy that gauge-invariant mass terms are possible for all these superpartners, in marked contrast to the situation for the known SM particles. Consider (547) for instance. The gluinos are in the regular representation of a gauge group, like their gauge boson superpartners: for example, in SU(2) the winos are in the $t = 1$ ('vector') representation. In this representation, the transformation matrices can be chosen to be real (the generators are pure imaginary, $(T_i^{(1)})_{jk} = -\mathrm{i}\epsilon_{ijk}$), which means that they are orthogonal rather than unitary, just like rotation matrices in ordinary 3-D space. Thus quantities of the form '$\widetilde{\boldsymbol{W}} \cdot \widetilde{\boldsymbol{W}}$' are invariant under SU(2) transformations, including local ones since no derivatives are involved; similarly for the gluinos and the bino. Coming to (548) and (549), squark and slepton mass terms of this form are allowed if i and j are family indices, and the m_{ij}^2's are Hermitian matrices in family space, since under a gauge transformation $\phi_i \to U\phi_i,\ \phi_j \to U\phi_j$, where $U^\dagger U = 1$, and the ϕ's stand for a squark or slepton flavour multiplet. Higgs mass terms like $-m_{\mathrm{H_u}}^2 H_{\mathrm{u}}^\dagger H_{\mathrm{u}}$ are of course present in the SM already, and (as we saw in section 12 - see the remarks following equation (476)) from the perspective of the MSSM we need to include such SUSY-violating terms in order to have any chance of breaking electroweak symmetry spontaneously (the parameter '$m_{\mathrm{H_u}}^2$' can of course have either sign). The b term in (550) is like the SUSY-invariant μ term of (474), but it only involves the Higgs, not the Higgsinos, and is hence SUSY-breaking.

The upshot of these considerations is that mass terms which break SUSY, but preserve electroweak symmetry, can be written down for all the so-far unobserved particles of the MSSM. By contrast, of course, similar mass terms for the known particles of the SM would all break electroweak symmetry explicitly, which is unacceptable (non-renormalizability/unitarity violations): the masses of the known SM particles must all arise via spontaneous breaking of electroweak symmetry. Thus it could be argued that, from the viewpoint of the MSSM, it is natural that the known particles have been found, since they are 'light', with a scale associated with electroweak symmetry breaking. The masses of the undiscovered particles, on the other hand, are associated with SUSY breaking, which can be significantly higher.[17] As against this, it must be repeated that electroweak symmetry breaking is not possible while preserving SUSY: the Yukawa-like terms in (465) do respect SUSY, but will not generate fermion masses unless some Higgs fields have a non-zero vev, and this won't happen with a potential of the form (461) (see also (476)); similarly, the gauge-invariant couplings (454) are part of a SUSY-invariant

[17]The Higgs is an interesting special case (taking it to be unobserved as yet). In the SM its mass is arbitrary (though see footnote 1), but in the MSSM the lightest Higgs particle is predicted to be no heavier than about 140 GeV (see the following section).

theory, but the electroweak gauge boson masses require a Higgs vev in (454). So some, at least, of the SUSY-breaking parameters must have values not too far from the scale of electroweak symmetry breaking, if we don't want fine tuning. From this point of view, then, there seems no very clear distinction between the scales of electroweak and of SUSY breaking.

Unfortunately, although the terms (547) - (553) are restricted in form, there are nevertheless quite a lot of possible terms in total, when all the fields in the MSSM are considered, and this implies very many new parameters. In fact, Dimopoulos and Sutter [39] counted a total of 105 parameters describing masses, mixing angles and phases, after allowing for all allowed redefinitions of bases. It is worth emphasizing that this massive increase in parameters is entirely to do with SUSY breaking, the SUSY-invariant (but unphysical) MSSM Lagrangian having only one new parameter (μ) with respect to the SM.

One may well be dismayed by such an apparently huge arbitrariness in the theory, but this impression is in a sense misleading since extensive regions of parameter space are in fact excluded phenemenologically. This is because generic values of most of the new parameters allow flavour changing neutral current (FCNC) processes, or new sources of CP violation, at levels which are excluded by experiment. For example, if the matrix $\mathbf{m}^2_{\tilde{\mathrm{e}}}$ in (549) has a non-suppressed off-diagonal term such as[18]

$$(m^2_{\tilde{\mathrm{e}}})_{\mathrm{e}\mu}\tilde{\tilde{e}}^\dagger_{\mathrm{L}}\tilde{\tilde{\mu}}_{\mathrm{L}} \tag{554}$$

(in the basis in which the lepton masses are diagonal), then unacceptably large lepton flavour changing ($\mu \to \mathrm{e}$) will be generated. We can, for instance, envisage a loop diagram contributing to $\mu \to \mathrm{e}+\gamma$, in which the μ first decays virtually to $\tilde{\tilde{\mu}}_{\mathrm{L}}$ + bino through one of the couplings in (459), the $\tilde{\tilde{\mu}}_{\mathrm{L}}$ then changing to $\tilde{\tilde{\mathrm{e}}}_{\mathrm{L}}$ via (554), followed by $\tilde{\tilde{\mathrm{e}}}_{\mathrm{L}}$ re-combining with the bino to make an electron, after emitting a photon. The upper limit on the branching ratio for $\mu \to \mathrm{e} + \gamma$ is 1.2×10^{-11}, and our loop amplitude will be many orders of magnitude larger than this, even for sleptons as heavy as 1 TeV. Similarly, the squark $(\mathrm{mass})^2$ matrices are tightly constrained both as to flavour mixing and as to CP-violating complex phases by data on $\mathrm{K}^0 - \bar{\mathrm{K}}^0$ mixing, $\mathrm{D}^0 - \bar{\mathrm{D}}^0$ and $\mathrm{B}^0 - \bar{\mathrm{B}}^0$ mixing, and the decay $\mathrm{b} \to \mathrm{s}\gamma$. For a recent survey, with further references, see [23] section 5.

[18]Perhaps more physically, $\tilde{\tilde{\mathrm{e}}}_{\mathrm{L}}$ is the slepton partner of the e_{R}, and $\tilde{\tilde{\mu}}_{\mathrm{L}}$ that of μ_{R}.

The existence of these strong constraints on the SUSY-breaking parameters at the SM scale suggests that whatever the actual SUSY-breaking mechanism might be, it should be such as to lead naturally to the suppression of such dangerous off-diagonal terms. One framework which guarantees this is the 'minimal supergravity (mSUGRA)' theory [40] [41] [42], in which the parameters (547) - (553) take a particularly simple form at the Planck scale:

$$M_3 = M_2 = M_1 = m_{1/2}; \tag{555}$$

$$\mathbf{m}^2_{\tilde{\mathrm{Q}}} = \mathbf{m}^2_{\tilde{\bar{\mathrm{u}}}} = \mathbf{m}^2_{\tilde{\bar{\mathrm{d}}}} = \mathbf{m}^2_{\tilde{\mathrm{L}}} = \mathbf{m}^2_{\tilde{\bar{\mathrm{e}}}} = m_0^2\,\mathbf{1}, \tag{556}$$

where '**1**' stands for the unit matrix in family space;

$$m^2_{\mathrm{H_u}} = m^2_{\mathrm{H_d}} = m_0^2; \tag{557}$$

and

$$\mathbf{a}_{\mathrm{u}} = A_0\mathbf{y}_{\mathrm{u}},\ \mathbf{a}_{\mathrm{d}} = A_0\mathbf{y}_{\mathrm{d}},\ \mathbf{a}_{\mathrm{e}} = A_0\mathbf{y}_{\mathrm{e}} \tag{558}$$

where the **y** matrices are those appearing in (465). Relations (556) imply that at m_{P} all squark and sleptons are degenerate in mass (independent of both flavour and family, in fact) and so, in particular, squarks and sleptons with the same electroweak quantum numbers can be freely transformed into each other by unitary transformations. All mixings can then be eliminated, apart from that originating via the triple scalar terms. But conditions (558) ensure that only the squarks and sleptons of the (more massive) third family can have large triple scalar couplings. If $m_{1/2}, A_0$ and b of (550) all have the same complex phase, the only CP-violating phase in the theory will be the usual CKM one (leaving CP-violation in the neutrino sector aside). Somewhat weaker conditions than (555) - (558) would also suffice to accommodate the phenomenological constraints. (For completeness, we mention other SUSY-breaking mechanisms that have been proposed: gauge-mediated [43], gaugino-mediated [44] and anomaly-mediated [45] symmetry breaking.)

We must now remember, of course, that if we use this kind of effective Lagrangian to calculate quantities at the electroweak scale, in perturbation theory, the results will involve logarithms of the form[19]

$$\ln[(\text{high scale, for example } m_{\mathrm{P}})/\text{low scale } m_{\mathrm{Z}}], \tag{559}$$

[19] Expression (559) may be thought of in the context either of running the quantities 'down' in scale - i.e. from a supposedly 'fundamental' high scale $Q_0 \sim m_{\mathrm{P}}$ to a low scale $\sim m_{\mathrm{Z}}$; or - as in (494) - (496) - of running 'up' from a low scale $Q_0 \sim m_{\mathrm{Z}}$ to a high scale $\sim m_{\mathrm{P}}$ (in order, perhaps, to try and infer high-scale physics from weak-scale input). Either way, a crucial hypothesis is, of course, that no new physics intervenes between $\sim m_{\mathrm{Z}}$ and $\sim m_{\mathrm{P}}$.

coming from loop diagrams, which can be large enough to invalidate perturbation theory. As usual, such 'large logarithms' must be re-summed by the renormalization group technique (see chapter 15 of [12] for example). This amounts to treating all couplings and masses as running parameters, which evolve as the energy scale changes according to RG equations, whose coefficients can be calculated perturbatively. Conditions such as (555) - (558) are then interpreted as boundary conditions on the parameters at the high scale.

This implies that after evolution to the SM scale the relations (555) - (558) will no longer hold, in general. However, RG corrections due to gauge interactions will not introduce flavour-mixing or CP-violating phases, while RG corrections due to Yukawa interactions are quite small except for the third family. It seems to be generally the case that if FCNC and CP-violating terms are suppressed at a high Q_0, then supersymmetric contributions to FCNC and CP-violating observables are not in conflict with present bounds, though this may change as the bounds are improved.

15.3 RGE Evolution of the Parameters in the (Softly Broken) MSSM

It is fair to say that the apparently successful gauge unification in the MSSM (section 13) encourages us to apply a similar RG analysis to the other MSSM couplings and to the soft parameters (555) - (558). One-loop RGEs for the MSSM are given in [23] Appendix C.6; see also [22] section 7.1.

A simple example is provided by the gaugino mass parameters M_i ($i = 1, 2, 3$) whose evolution (at 1-loop order) is determined by an equation very similar to (488) for the running of the α_i, namely

$$\frac{\mathrm{d}M_i}{\mathrm{d}t} = -\frac{b_i}{2\pi}\alpha_i M_i. \tag{560}$$

From (488) and (560) we obtain

$$\frac{1}{\alpha_i}\frac{\mathrm{d}M_i}{\mathrm{d}t} - M_i\frac{1}{\alpha_i^2}\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = 0, \tag{561}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_i/\alpha_i) = 0. \tag{562}$$

It follows that the three ratios (M_i/α_i) are RG-scale independent at 1-loop order. In mSUGRA-type models, then, we can write

$$\frac{M_i(Q)}{\alpha_i(Q)} = \frac{m_{1/2}}{\alpha_i(m_{\rm P})}, \tag{563}$$

and since all the α_i's are already unified below $m_{\rm P}$ we obtain

$$\frac{M_1(Q)}{\alpha_1(Q)} = \frac{M_2(Q)}{\alpha_2(Q)} = \frac{M_3(Q)}{\alpha_3(Q)} \tag{564}$$

at any scale Q, up to small 2-loop corrections and possible threshold effects at high scales.

Applying (564) at $Q = m_{\rm Z}$ we find

$$M_1(m_{\rm Z}) = \frac{\alpha_1(m_{\rm Z})}{\alpha_2(m_{\rm Z})} M_2(m_{\rm Z}) = \frac{5}{3}\tan^2\theta_{\rm W}(m_{\rm Z}) \simeq 0.5 M_2(m_{\rm Z}) \tag{565}$$

and

$$M_3(m_{\rm Z}) = \frac{\alpha_3(m_{\rm Z})}{\alpha_2(m_{\rm z})} M_2(m_{\rm Z}) = \frac{\sin^2\theta_{\rm W}(m_{\rm Z})}{\alpha_{\rm em}(m_{\rm Z})}\alpha_3(m_{\rm Z}) M_2(m_{\rm Z}) \simeq 3.5 M_2(m_{\rm Z}) \tag{566}$$

where we have used (498) - (500). Equations (565) and (566) may be summarized as

$$M_3(m_{\rm Z}) : M_2(m_{\rm Z}) : M_1(m_{\rm Z}) \simeq 7 : 2 : 1. \tag{567}$$

This simple prediction is common to most supersymmetric phenomenology. It implies that the gluino is expected to be heavier than the states associated with the electroweak sector. (The latter are 'neutralinos', which are mixtures of the neutral Higgsinos $(\tilde{H}_{\rm u}^0, \tilde{H}_{\rm d}^0)$ and neutral gauginos $(\tilde{B}, \tilde{W}^0)$, and 'charginos', which are mixtures of the charged Higgsinos $(\tilde{H}_{\rm u}^+, \tilde{H}_{\rm d}^-)$ and winos $(\tilde{W}^+, \tilde{W}^-)$.)

A second significant example concerns the running of the scalar masses. Here the gauginos contribute to the RHS of '${\rm d}m^2/{\rm d}t$' with a negative coefficient, which tends to increase the mass as the scale Q is lowered. On the other hand, the contributions from fermion loops have the opposite sign, tending to decrease the mass at low scales. The dominant such contribution is provided by top quark loops since $y_{\rm t}$ is so much larger than the other

Yukawa couplings. If we retain only the top quark Yukawa coupling, the 1-loop evolution equations for $m^2_{\rm H_u}$, $m^2_{\tilde{\rm Q}_3}$ and $m^2_{\tilde{\bar{\rm u}}_3}$ are

$$\frac{{\rm d}m^2_{\rm H_u}}{{\rm d}t} = [\frac{3X_{\rm t}}{4\pi} - 6\alpha_2 M_2^2 - \frac{6}{5}\alpha_1 M_1^2]/4\pi \tag{568}$$

$$\frac{{\rm d}m^2_{\tilde{\rm Q}_3}}{{\rm d}t} = [\frac{X_{\rm t}}{4\pi} - \frac{32}{3}\alpha_3 M_3^2 - 6\alpha_2 M_2^2 - \frac{2}{15}\alpha_1 M_1^2]/4\pi \tag{569}$$

$$\frac{{\rm d}m^2_{\tilde{\bar{\rm u}}_3}}{{\rm d}t} = [\frac{2X_{\rm t}}{4\pi} - \frac{32}{3}\alpha_3 M_3^2 - \frac{32}{15}\alpha_1 M_1^2]/4\pi, \tag{570}$$

where

$$X_{\rm t} = 2|y_{\rm t}|^2(m^2_{\rm H_u} + m^2_{\tilde{\rm Q}_3} + m^2_{\tilde{\bar{\rm u}}_3} + A_0^2] \tag{571}$$

and we have used (558). In contrast, the corresponding equation for $m^2_{\rm H_d}$, to which the top quark loop does not contribute, is

$$\frac{{\rm d}m^2_{\rm H_d}}{{\rm d}t} = [-6\alpha_2 M_2^2 - \frac{6}{5}\alpha_1 M_1^2]/4\pi. \tag{572}$$

Since the quantity $X_{\rm t}$ is positive, its effect is always to decrease the appropriate (mass)^2 parameter at low scales. From (568) - (570) we can see that, of the three masses, $m^2_{\rm H_u}$ is (a) decreased the most because of the factor 3, and (b) increased the least because the gluino contribution (which is larger than those of the other gauginos) is absent. On the other hand, $m^2_{\rm H_d}$ will always tend to increase at low scales. The possibility then arises that $m^2_{\rm H_u}$ could run from a positive value at $Q_0 \sim 10^{16}$ GeV to a negative value at the electroweak scale, while all the other (mass)^2 parameters of the scalar particles remain positive[20]. This can indeed happen, thanks to the large value of the top quark mass (or equivalently the large value of $y_{\rm t}$): see [46] [47] [48] [49] [50] [51]. Such a negative (mass)^2 value would tend to destabilize the point $H^0_{\rm u} = 0$, providing a strong (though not conclusive - see the next section) indication that this is the trigger for electroweak symmetry breaking.

The parameter $y_{\rm t}$ in (568)-(570), and the other Yukawa couplings in (465), all run too; consideration of the RGEs for these couplings provides some further interesting results. If (for simplicity) we make the approximations

[20]Negative values for the squark (mass)^2 parameters would have the undesirable consequence of spontaneously breaking the colour SU(3).

that only third-family couplings are significant, and ignore contributions from α_1 and α_2, the 1-loop RGEs for the parameters $y_{\rm t}$, $y_{\rm b}$ and y_τ are

$$\begin{aligned}
\frac{{\rm d}y_{\rm t}}{{\rm d}t} &= \frac{y_{\rm t}}{4\pi}[(6y_{\rm t}^2 + y_{\rm b}^2)/4\pi - \frac{16}{3}\alpha_{\rm s}] && (573)\\
\frac{{\rm d}y_{\rm b}}{{\rm d}t} &= \frac{y_{\rm b}}{4\pi}[(6y_{\rm b}^2 + y_{\rm t}^2 + y_\tau^2)/4\pi - \frac{16}{3}\alpha_{\rm s}] && (574)\\
\frac{{\rm d}y_\tau}{{\rm d}t} &= \frac{y_\tau}{16\pi^2}[4y_\tau^2 + 3y_{\rm b}^2]. && (575)
\end{aligned}$$

As in equations (568) - (570) the Yukawa couplings and the gauge coupling $\alpha_{\rm s}$ enter the RHS of (573) - (575) with opposite signs; the former tend to increase the y's at high scales, while $\alpha_{\rm s}$ tends to reduce $y_{\rm t}$ and $y_{\rm b}$. It is then conceivable that, starting at low scales with $y_{\rm t} > y_{\rm b} > y_\tau$, the three y's might unify at or around $m_{\rm U}$. Indeed, there is some evidence that the condition $y_{\rm b}(m_{\rm U}) = y_\tau(m_{\rm U})$, which arises naturally in many GUT models, leads to good low-energy phenomenology [52] [53] [54] [55].

Further unification with $y_{\rm t}(m_{\rm U})$ must be such as to be consistent with the known top quark mass at low scales. To get a rough idea of how this works, we return to the relation (471), and similar ones for $m_{{\rm d}ij}$ and $m_{{\rm e}ij}$, which in the mass-diagonal basis give

$$y_{\rm t} = \frac{m_{\rm t}}{v_{\rm u}}, \quad y_{\rm b} = \frac{m_{\rm b}}{v_{\rm d}}, \quad y_\tau = \frac{m_\tau}{v_{\rm d}}, \qquad (576)$$

where $v_{\rm d}$ is the vev of the field $H_{\rm d}^0$. It is clear that the viability of $y_{\rm t} \approx y_{\rm b}$ will depend on the value of the additional parameter $v_{\rm u}/v_{\rm d}$ (denoted by $\tan\beta$ - see the following section). It seems that 'Yukawa unification' at $m_{\rm U}$ may work in the parameter regime where $\tan\beta \approx m_{\rm t}/m_{\rm b}$ [56] [57] [58] [59] [60] [61] [62].

In the following section we shall discuss the Higgs sector of the MSSM where - even without assumptions such as (555) - (558) - only a few parameters enter, and one important prediction can be made: namely, an upper bound on the mass of the lightest Higgs boson, which is well within reach of the LHC.

16 The Higgs Sector and Electroweak Symmetry Breaking in the MSSM

16.1 The scalar potential and the conditions for electroweak symmetry breaking

We largely follow the treatment in Martin [22] section 7.2. The first task is to find the potential for the scalar Higgs fields in the MSSM. As frequently emphasized, there are two complex Higgs $\mathrm{SU(2)_L}$ doublets which we are denoting by $H_{\mathrm{u}} = (H_{\mathrm{u}}^{+}, H_{\mathrm{u}}^{0})$ which has weak hypercharge $y = 1$, and $H_{\mathrm{d}} = (H_{\mathrm{d}}^{0}, H_{\mathrm{d}}^{-})$ which has $y = -1$. The classical (tree-level) potential for these scalar fields is made up of several terms. First, quadratic terms arise from the SUSY-invariant ('F-term') contribution (476) which involves the μ parameter from (465), and also from SUSY-breaking terms of the type (550). The latter two contributions are

$$m_{\mathrm{H_u}}^2(|H_{\mathrm{u}}^{+}|^2 + |H_{\mathrm{u}}^{0}|^2) + m_{\mathrm{H_d}}^2(|H_{\mathrm{d}}^{0}|^2 + |H_{\mathrm{d}}^{-}|^2), \tag{577}$$

where despite appearances it must be remembered that the arbitrary parameters '$m_{\mathrm{H_u}}^2$' and '$m_{\mathrm{H_d}}^2$' may have either sign, and

$$b(H_{\mathrm{u}}^{+}H_{\mathrm{d}}^{-} - H_{\mathrm{u}}^{0}H_{\mathrm{d}}^{0}) + \mathrm{h.c.} \tag{578}$$

To these must be added the quartic SUSY-invariant 'D-terms' of (461), of the form $(\mathrm{Higgs})^2\ (\mathrm{Higgs})^2$, which we need to evaluate for the electroweak sector of the MSSM.

There are two groups G, $\mathrm{SU(2)_L}$ with coupling g and $\mathrm{U(1)}_y$ with coupling $g'/2$ (in the convention of [12] - see equation (22.21) of that reference). For the first, the matrices T^{α} are just $\tau^{\alpha}/2$, and we must evaluate

$$\begin{aligned}&\sum_{\alpha}(H_{\mathrm{u}}^{\dagger}(\tau^{\alpha}/2)H_{\mathrm{u}} + H_{\mathrm{d}}^{\dagger}(\tau^{\alpha}/2)H_{\mathrm{d}})(H_{\mathrm{u}}^{\dagger}(\tau^{\alpha}/2)H_{\mathrm{u}} + H_{\mathrm{d}}^{\dagger}(\tau^{\alpha}/2)H_{\mathrm{d}})\\ &= (H_{\mathrm{u}}^{\dagger}(\boldsymbol{\tau}/2)H_{\mathrm{u}})\cdot(H_{\mathrm{u}}^{\dagger}(\boldsymbol{\tau}/2)H_{\mathrm{u}}) + (H_{\mathrm{d}}^{\dagger}(\boldsymbol{\tau}/2)H_{\mathrm{d}})\cdot(H_{\mathrm{d}}^{\dagger}(\boldsymbol{\tau}/2)H_{\mathrm{d}})\\ &+2(H_{\mathrm{u}}^{\dagger}(\boldsymbol{\tau}/2)H_{\mathrm{u}})\cdot(H_{\mathrm{d}}^{\dagger}(\boldsymbol{\tau}/2)H_{\mathrm{d}}).\end{aligned} \tag{579}$$

If we write

$$H_{\mathrm{u}} = \begin{pmatrix} a \\ b \end{pmatrix}, H_{\mathrm{d}} = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{580}$$

then brute force evaluation of the matrix and dot products in (579) yields the result

$$\frac{1}{4}\{[(|a|^2+|b|^2)-(|c|^2+|d|^2)]^2+4(ac^*+bd^*)(a^*c+b^*d)\}, \tag{581}$$

so that the SU(2) contribution is (581) multiplied by $g^2/2$. The U(1) contribution is

$$\frac{1}{2}(g'/2)^2[H_{\rm u}^\dagger H_{\rm u}-H_{\rm d}^\dagger H_{\rm d}]^2=\frac{g'^2}{8}[(|a|^2+|b|^2)-(|c|^2+|d|^2)]^2. \tag{582}$$

Re-writing (581) and (582) in the notation of the fields, and including the quadratic pieces, the complete potential for the scalar fields in the MSSM is

$$\begin{aligned}&\mathcal{V}=(|\mu|^2+m_{\rm H_u}^2)(|H_{\rm u}^+|^2+|H_{\rm u}^0|^2)+(|\mu|^2+m_{\rm H_d}^2)(|H_{\rm d}^0|^2+|H_{\rm d}^-|^2)+\\&[b(H_{\rm u}^+H_{\rm d}^- - H_{\rm u}^0H_{\rm d}^0)+\text{h.c.}]+\frac{(g^2+g'^2)}{8}(|H_{\rm u}^+|^2+|H_{\rm u}^0|^2-|H_{\rm d}^0|^2-|H_{\rm d}^-|^2)^2\\&+\frac{g^2}{2}|H_{\rm u}^+H_{\rm d}^{0\dagger}+H_{\rm u}^0H_{\rm d}^{-\dagger}|^2.\end{aligned} \tag{583}$$

We prefer not to re-write $(|\mu|^2+m_{\rm H_u}^2)$ and $(|\mu|^2+m_{\rm H_d}^2)$ as m_1^2 and m_2^2, say, so as to retain a memory of the fact that $|\mu|^2$ arises from a SUSY-invariant term, and is necessarily positive, while $m_{\rm H_u}^2$ and $m_{\rm H_d}^2$ are SUSY-breaking and of either sign *a priori*.

We must now investigate whether - and if so under what conditions - this potential can have a minimum which (like that of the simple Higgs potential (4) of the SM) breaks the $\text{SU(2)}_{\rm L}\times\text{U(1)}_y$ electroweak symmetry down to $\text{U(1)}_{\rm em}$.

We can use the gauge symmetry to simplify the algebra somewhat. As in the SM (see for example sections 17.6 and 19.6 of [12]) we can reduce a possible vev of one component of either $H_{\rm u}$ or $H_{\rm d}$ to zero by an $\text{SU(2)}_{\rm L}$ transformation. We choose $H_{\rm u}^+=0$ at the minimum of $\mathcal{V}$. The conditions $H_{\rm u}^+=0$ and $\partial\mathcal{V}/\partial H_{\rm u}^+=0$ then imply that, at the minimum of the potential, either

$$H_{\rm d}^-=0 \tag{584}$$

or

$$b+\frac{g^2}{2}H_{\rm d}^{0\dagger}H_{\rm u}^{0\dagger}=0. \tag{585}$$

The second condition (585) implies that the b term in (583) becomes

$$g^2|H_{\rm u}^0|^2|H_{\rm d}^0|^2 \tag{586}$$

which is definitely positive, and unfavourable to symmetry-breaking. As we shall see, condition (584) leads to a negative b-contribution. Accepting alternative (584) then, it follows that neither $H_{\rm u}^+$ nor $H_{\rm d}^-$ acquire a vev, which means (satisfactorily) that electromagnetism is not spontaneously broken. We can now ignore the charged components, and concentrate on the potential for the neutral fields which is

$$\begin{aligned}\mathcal{V}_{\rm n} &= (|\mu|^2 + m_{\rm H_u}^2)|H_{\rm u}^0|^2 + (|\mu|^2 + m_{\rm H_d}^2)|H_{\rm d}^0|^2 \\ &-(bH_{\rm u}^0H_{\rm d}^0 + {\rm h.c.}) + \frac{(g^2+g'^2)}{8}(|H_{\rm u}^0|^2 - |H_{\rm d}^0|^2)^2.\end{aligned} \tag{587}$$

This is perhaps an appropriate point to note that the coefficient of the quartic term is *not* a free parameter, but is determined by the known electroweak couplings ($(g^2+g'^2)/8 \approx 0.065$). This is of course in marked contrast to the case of the SM, where the coefficient $\lambda/4$ in (4) is a free parameter. Recalling from (3) that, in the SM, the mass of the Higgs boson is proportional to $\sqrt{\lambda}$, for given Higgs vev, this suggests that in the MSSM there should be a relatively light Higgs particle. As we shall see, this is indeed the case, though the larger field content of the Higgs sector in the MSSM makes the analysis more involved.

Consider now the b-term in (587), which is the only one that depends on the phases of the fields. Without loss of generality, b may be taken to be real and positive, any possible phase of b being absorbed into the relative phase of $H_{\rm u}^0$ and $H_{\rm d}^0$. For a minimum of $\mathcal{V}_{\rm n}$, the product $H_{\rm u}^0H_{\rm d}^0$ must be real and positive too, which means that (at the minimum) the vev's of $H_{\rm u}^0$ and $H_{\rm d}^0$ must have equal and opposite phases. Since these fields have equal and opposite hypercharges, we can make a $\mathrm{U}(1)_y$ gauge transformation to reduce both their phases to zero. All vev's and couplings can therefore be chosen to be real, which means that **CP** is not spontaneously broken by the 2-Higgs potential of the MSSM, any more than it is in the 1-Higgs potential of the SM.[21]

[21] While this is true at tree level, **CP** symmetry could be broken significantly by radiative corrections, specifically via loops involving third generation squarks [63]; this would imply that the three neutral Higgs eigenstates would not have well defined **CP** quantum numbers (for the usual, **CP** conserving, case, see comments following equation (660) below).

The scalar potential now takes the more manageable form

$$\mathcal{V}_{\rm n} = (|\mu|^2 + m^2_{\rm H_u})x^2 + (|\mu|^2 + m^2_{\rm H_d})y^2 - 2bxy + \frac{(g^2 + g'^2)}{8}(x^2 - y^2)^2, \quad (588)$$

where $x = |H^0_{\rm u}|, y = |H^0_{\rm d}|$; it depends on three parameters, $|\mu|^2 + m^2_{\rm H_u}$, $|\mu|^2 + m^2_{\rm H_d}$ and b. We want to identify the conditions required for the stable minimum of $\mathcal{V}_{\rm n}$ to occur at non-zero values of x and y. First note that, along the special ('flat') direction $x = y$, the potential will be unbounded from below (no minimum) unless

$$2|\mu|^2 + m^2_{\rm H_u} + m^2_{\rm H_d} > 2b > 0. \quad (589)$$

Hence $(|\mu|^2 + m^2_{\rm H_u})$ and $(|\mu|^2 + m^2_{\rm H_d})$ cannot both be negative. This implies, referring to (588), that the point $x = y = 0$ cannot be a maximum of $\mathcal{V}_{\rm n}$. If $(|\mu|^2 + m^2_{\rm H_u})$ and $(|\mu|^2 + m^2_{\rm H_d})$ are both positive, then the origin is a minimum (which would be an unwanted symmetry-preserving solution) unless

$$(|\mu|^2 + m^2_{\rm H_u})(|\mu|^2 + m^2_{\rm H_d}) < b^2, \quad (590)$$

which is the condition for the origin to be a saddle point. (590) is automatically satisfied if either $(|\mu|^2 + m^2_{\rm H_u})$ or $(|\mu|^2 + m^2_{\rm H_d})$ is negative.

The b-term favours electroweak symmetry breaking, but it is not required to be non-zero. What can be said about $m^2_{\rm H_u}$ and $m^2_{\rm H_d}$? A glance at conditions (589) and (590) shows that they cannot both be satisfied if $m^2_{\rm H_u} = m^2_{\rm H_d}$, a condition that is typically taken to hold at a high scale $\sim 10^{16}$ GeV. However, the parameter $m^2_{\rm H_u}$ is, in fact, the one whose renormalization group evolution can drive it to negative values at the electroweak scale, as discussed at the end of the previous section. It is clear that a negative value of $m^2_{\rm H_u}$ will tend to help condition (590) to be satisfied, but it is neither necessary nor sufficient ($|\mu|$ may be too large or b too small). A 'large' negative value for $m^2_{\rm H_u}$ is a significant factor, but it falls short of a demonstration that electroweak symmetry breaking *will* occur via this mechanism.

Having established the conditions (589) and (590) required for $|H^0_{\rm u}|$ and $|H^0_{\rm d}|$ to have non-zero vevs, say $v_{\rm u}$ and $v_{\rm d}$ respectively, we can now proceed to write down the equations determining these vevs which follow from imposing the stationary conditions

$$\frac{\partial \mathcal{V}_{\rm n}}{\partial x} = \frac{\partial \mathcal{V}_{\rm n}}{\partial y} = 0. \quad (591)$$

Performing the differentiations and setting $x = v_\mathrm{u}$ and $y = v_\mathrm{d}$ we obtain

$$(|\mu|^2 + m^2_{\mathrm{H_u}})v_\mathrm{u} = bv_\mathrm{d} + \frac{1}{4}(g^2 + g'^2)(v^2_\mathrm{d} - v^2_\mathrm{u})v_\mathrm{u} \quad (592)$$

$$(|\mu|^2 + m^2_{\mathrm{H_d}})v_\mathrm{d} = bv_\mathrm{u} - \frac{1}{4}(g^2 + g'^2)(v^2_\mathrm{d} - v^2_\mathrm{u})v_\mathrm{d}. \quad (593)$$

One combination of v_u and v_d is fixed by experiment, since it determines the mass of the W and Z bosons, just as in the SM. The relevant terms in the electroweak sector are

$$(D_\mu H_\mathrm{u})^\dagger(D^\mu H_\mathrm{u}) + (D_\mu H_\mathrm{d})^\dagger(D^\mu H_\mathrm{d}) \quad (594)$$

where (see equation (22.21) of [12])

$$D_\mu = \partial_\mu + \mathrm{i}g(\boldsymbol{\tau}/2)\cdot\boldsymbol{W}_\mu + \mathrm{i}(g'/2)yB_\mu. \quad (595)$$

The mass terms for the vector particles come (in unitary gauge) from inserting the vevs for H_u and H_d, and defining

$$Z^\mu = (-g'B^\mu + gW^\mu_3)/(g^2 + g'^2)^{1/2}. \quad (596)$$

One finds

$$m^2_\mathrm{Z} = \frac{1}{2}(g^2 + g'^2)(v^2_\mathrm{u} + v^2_\mathrm{d}) \quad (597)$$

$$m^2_\mathrm{W} = \frac{1}{2}g^2(v^2_\mathrm{u} + v^2_\mathrm{d}). \quad (598)$$

Hence (see equations (22.29)-(22.32) of [12])

$$(v^2_\mathrm{u} + v^2_\mathrm{d})^{1/2} = \left(\frac{2m^2_\mathrm{W}}{g^2}\right)^{1/2} = 174\mathrm{GeV}. \quad (599)$$

Equations (592) and (593) may now be written as

$$(|\mu|^2 + m^2_{\mathrm{H_u}}) = b\cot\beta + (m^2_\mathrm{Z}/2)\cos 2\beta \quad (600)$$

$$(|\mu|^2 + m^2_{\mathrm{H_d}}) = b\tan\beta - (m^2_\mathrm{Z}/2)\cos 2\beta, \quad (601)$$

where

$$\tan\beta \equiv v_\mathrm{u}/v_\mathrm{d}. \quad (602)$$

It is easy to check that (600) and (601) satisfy the necessary conditions (589) and (590). We may use (600) and (601) to eliminate the parameters $|\mu|$ and b in favour of $\tan\beta$, but the phase of μ is not determined. Because both v_u and v_d are real and positive, the angle β lies between 0 and $\pi/2$.

We are now ready to calculate the mass spectrum.

16.2 The tree-level masses of the scalar Higgs states in the MSSM

In the SM, there are four real scalar degrees of freedom in the Higgs doublet (5); after electroweak symmetry breaking (i.e. given a non-zero Higgs vev), three of them become the longitudinal modes of the massive vector bosons $W^\pm$ and Z^0, while the fourth is the neutral Higgs boson of the SM, the mass of which is found by considering quadratic deviations away from the symmetry-breaking minimum (see chapter 19 of [12] for example). In the MSSM, there are 8 real scalar degrees of freedom. Three of them are massless, and just as in the SM they get 'swallowed' by the $W^\pm$ and Z^0. The masses of the other five are again calculated by expanding the potential about the minimum, up to second order in the fields. Though straightforward, the work is complicated by the fact that the quadratic deviations are not diagonal in the fields, so that some diagonalization has to be done before the physical masses can be extracted.

To illustrate the procedure, consider the Lagrangian

$$\mathcal{L}_{12} = \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2), \tag{603}$$

where $V(\phi_1, \phi_2)$ has a minimum at $\phi_1 = v_1, \phi_2 = v_2$. We expand V about the minimum, retaining only quadratic terms, and discarding an irrelevant constant; this yields

$$\begin{aligned}\mathcal{L}_{12,\text{quad}} = \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2}\frac{\partial^2 V}{\partial \phi_1^2}(\phi_1 - v_1)^2 \\ -\frac{1}{2}\frac{\partial^2 V}{\partial \phi_2^2}(\phi_2 - v_2)^2 - \frac{\partial^2 V}{\partial \phi_1 \partial \phi_2}(\phi_1 - v_1)(\phi_2 - v_2)\end{aligned} \tag{604}$$

where the derivatives are evaluated at the minimum (v_1, v_2). Defining

$$\tilde{\phi}_1 = \sqrt{2}(\phi_1 - v_1), \quad \tilde{\phi}_2 = \sqrt{2}(\phi_2 - v_2), \tag{605}$$

(604) can be written as

$$\mathcal{L}_{12,\text{quad}} = \frac{1}{2}\partial_\mu \tilde{\phi}_1 \partial^\mu \tilde{\phi}_1 + \frac{1}{2}\partial_\mu \tilde{\phi}_2 \partial^\mu \tilde{\phi}_2 - \frac{1}{2}(\tilde{\phi}_1 \, \tilde{\phi}_2)\mathbf{M}^{\text{sq}} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix}, \tag{606}$$

where the $(\text{mass})^2$ matrix $\mathbf{M}^{\text{sq}}$ is given by

$$\mathbf{M}^{\text{sq}} = \frac{1}{2}\begin{pmatrix} V''_{11} & V''_{12} \\ V''_{12} & V''_{22} \end{pmatrix} \tag{607}$$

where

$$V''_{ij} = \frac{\partial^2 V}{\partial\phi_i \partial\phi_j}(v_1, v_2). \tag{608}$$

The matrix $\mathbf{M}^{\rm sq}$ is real and symmetric, and can be diagonalized via an orthogonal transformation of the form

$$\begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix}. \tag{609}$$

If the eigenvalues of $\mathbf{M}^{\rm sq}$ are m_+^2 and m_-^2, we see that in the new basis (606) becomes

$$\mathcal{L}_{12,\rm quad} = \frac{1}{2}\partial_\mu\phi_+\partial^\mu\phi_+ + \frac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_- - \frac{1}{2}(\phi_+)^2 m_+^2 - \frac{1}{2}(\phi_-)^2 m_-^2, \tag{610}$$

from which it follows (via the equations of motion for ϕ_+ and ϕ_-) that m_+^2 and m_-^2 are the squared masses of the modes described by ϕ_+ and ϕ_-.

We apply this formalism first to the pair of fields $({\rm Im}H_{\rm u}^0, {\rm Im}H_{\rm d}^0)$. The part of our scalar potential involving this pair is

$$\begin{aligned}\mathcal{V}_{\rm A} &= (|\mu|^2 + m_{\rm H_u}^2)({\rm Im}H_{\rm u}^0)^2 + (|\mu|^2 + m_{\rm H_d}^2)({\rm Im}H_{\rm d}^0)^2 + 2b({\rm Im}H_{\rm u}^0)({\rm Im}H_{\rm d}^0) \\ &+ \frac{(g^2+g'^2)}{8}[({\rm Re}H_{\rm u}^0)^2 + ({\rm Im}H_{\rm u}^0)^2 - ({\rm Re}H_{\rm d}^0)^2 - ({\rm Im}H_{\rm d}^0)^2]^2. \end{aligned} \tag{611}$$

Evaluating the second derivatives at the minimum point, we find the elements of the (mass)^2 matrix:

$$M_{11}^{\rm sq} = |\mu|^2 + m_{\rm H_u}^2 + \frac{(g^2+g'^2)}{4}(v_{\rm u}^2 - v_{\rm d}^2) = b\cot\beta, \tag{612}$$

where we have used (592), and similarly

$$M_{12}^{\rm sq} = b, \quad M_{22}^{\rm sq} = b\tan\beta. \tag{613}$$

The eigenvalues of $\mathbf{M}^{\rm sq}$ are easily found to be

$$m_+^2 = 0, \quad m_-^2 = 2b/\sin 2\beta. \tag{614}$$

The eigenmode corresponding to the massless state is

$$\sqrt{2}[\sin\beta({\rm Im}H_{\rm u}^0) - \cos\beta({\rm Im}H_{\rm d}^0)], \tag{615}$$

and this will become the longitudinal state of the Z^0. The orthogonal combination

$$\sqrt{2}[\cos\beta(\mathrm{Im}H_\mathrm{u}^0) + \sin\beta(\mathrm{Im}H_\mathrm{d}^0)] \tag{616}$$

is the field of a scalar particle 'A^0', with mass

$$m_{\mathrm{A}^0} = (2b/\sin 2\beta)^{1/2}. \tag{617}$$

In discussing the parameter space of the Higgs sector of the MSSM, the pair of parameters $(b, \tan\beta)$ is usually replaced by the pair $(m_{\mathrm{A}^0}, \tan\beta)$.

Next, consider the charged pair $(H_\mathrm{u}^+, H_\mathrm{d}^{-\dagger})$. In this case the relevant part of the Lagrangian is

$$\mathcal{L}_{\mathrm{ch,\,quad}} = (\partial_\mu H_\mathrm{u}^+)^\dagger(\partial^\mu H_\mathrm{u}^+) + (\partial_\mu H_\mathrm{d}^-)^\dagger(\partial^\mu H_\mathrm{d}^-) - \frac{\partial^2\mathcal{V}}{\partial H_\mathrm{u}^{+\dagger}\partial H_\mathrm{u}^+} H_\mathrm{u}^{+\dagger}H_\mathrm{u}^+$$

$$- \frac{\partial^2\mathcal{V}}{\partial H_\mathrm{d}^{-\dagger}\partial H_\mathrm{d}^-} H_\mathrm{d}^{-\dagger}H_\mathrm{d}^- - \frac{\partial^2\mathcal{V}}{\partial H_\mathrm{u}^{+}\partial H_\mathrm{d}^-} H_\mathrm{u}^{+}H_\mathrm{d}^- - \frac{\partial^2\mathcal{V}}{\partial H_\mathrm{u}^{+\dagger}\partial H_\mathrm{d}^{-\dagger}} H_\mathrm{u}^{+\dagger}H_\mathrm{d}^{-\dagger}, \tag{618}$$

where we use (583) for $\mathcal{V}$, and the derivatives are evaluated at $H_\mathrm{u}^0 = v_\mathrm{u}$, $H_\mathrm{d}^0 = v_\mathrm{d}$, $H_\mathrm{u}^+ = H_\mathrm{d}^- = 0$. We write the potential terms as

$$(H_\mathrm{u}^{+\dagger}\ H_\mathrm{d}^-)\mathbf{M}_\mathrm{ch}^\mathrm{sq}\begin{pmatrix} H_\mathrm{u}^+ \\ H_\mathrm{d}^{-\dagger} \end{pmatrix} \tag{619}$$

where

$$\mathbf{M}_\mathrm{ch}^\mathrm{sq} = \begin{pmatrix} M_{++}^\mathrm{sq} & M_{+-}^\mathrm{sq} \\ M_{-+}^\mathrm{sq} & M_{--}^\mathrm{sq} \end{pmatrix} \tag{620}$$

with $M_{++}^\mathrm{sq} = \partial^2\mathcal{V}/\partial H_\mathrm{u}^{+\dagger}\partial H_\mathrm{u}^+$ etc. Performing the differentiations and evaluating the results at the minimum, we obtain

$$\mathbf{M}_\mathrm{ch}^\mathrm{sq} = \begin{pmatrix} b\cot\beta + \frac{g^2}{2}v_\mathrm{d}^2 & b + \frac{g^2}{2}v_\mathrm{u}v_\mathrm{d} \\ b + \frac{g^2}{2}v_\mathrm{u}v_\mathrm{d} & b\tan\beta + \frac{g^2}{2}v_\mathrm{u}^2 \end{pmatrix}. \tag{621}$$

This matrix has eigenvalues 0 and $m_\mathrm{W}^2 + m_{\mathrm{A}^0}^2$. The massless state corresponds to the superposition

$$G^+ = \sin\beta H_\mathrm{u}^+ - \cos\beta H_\mathrm{d}^{-\dagger}, \tag{622}$$

and it provides the longitudinal mode of the W^+ boson. There is a similar state $G^- \equiv (G^+)^\dagger$, which goes into the W^-. The massive (orthogonal) state is

$$H^+ = \cos\beta H_\mathrm{u}^+ + \sin\beta H_\mathrm{d}^{-\dagger}, \tag{623}$$

which has mass $m_{\mathrm{H}^+} = (m_{\mathrm{W}}^2 + m_{\mathrm{A}^0}^2)^{1/2}$, and there is a similar state $H^- \equiv (H^+)^\dagger$. Note that after diagonalization (618) becomes

$$(\partial_\mu G^+)^\dagger(\partial^\mu G^+) + (\partial H^+)^\dagger(\partial^\mu H^+) - m_{\mathrm{H}^+}^2 H^{+\dagger} H^+ \tag{624}$$

and the equation of motion for H^+ shows that $m_{\mathrm{H}^+}^2$ is correctly identified with the physical squared mass, without the various factors of 2 that appeared in our example (604)-(610) of two neutral fields.

Finally, we consider the coupled pair $(\mathrm{Re}H_{\mathrm{u}}^0 - v_{\mathrm{u}}, \mathrm{Re}H_{\mathrm{d}}^0 - v_{\mathrm{d}})$, which is of the same type as our example, and as the pair $(\mathrm{Im}H_{\mathrm{u}}^0, \mathrm{Im}H_{\mathrm{d}}^0)$. The (mass)^2 matrix is

$$\mathbf{M}_{\mathrm{h,H}}^{\mathrm{sq}} = \begin{pmatrix} b\cot\beta + m_{\mathrm{Z}}^2\sin^2\beta & -b - \frac{1}{2}m_{\mathrm{Z}}^2\sin 2\beta \\ -b - \frac{1}{2}m_{\mathrm{Z}}^2\sin 2\beta & b\tan\beta + m_{\mathrm{Z}}^2\cos^2\beta \end{pmatrix} \tag{625}$$

which has eigenvalues

$$m_{\mathrm{h}^0}^2 = \frac{1}{2}\{m_{\mathrm{A}^0}^2 + m_{\mathrm{Z}}^2 - [(m_{\mathrm{A}^0}^2 + m_{\mathrm{Z}}^2)^2 - 4m_{\mathrm{A}^0}^2 m_{\mathrm{Z}}^2\cos^2 2\beta]^{1/2}\} \tag{626}$$

and

$$m_{\mathrm{H}^0}^2 = \frac{1}{2}\{m_{\mathrm{A}^0}^2 + m_{\mathrm{Z}}^2 + [(m_{\mathrm{A}^0}^2 + m_{\mathrm{Z}}^2)^2 - 4m_{\mathrm{A}^0}^2 m_{\mathrm{Z}}^2\cos^2 2\beta]^{1/2}\}. \tag{627}$$

Equation (626) and (627) display the dependence of m_{h^0} and m_{H^0} on the parameters m_{A^0} and β. The corresponding eigenmodes will be given in the following subsection.

The crucial point now is that, whereas the masses m_{A^0}, m_{H^0} and $m_{\mathrm{H}^\pm}$ are unconstrained (since they all grow as $b/\sin\beta$ which can in principle be arbitrarily large), the mass m_{h^0} is bounded from above. Let us write $x = m_{\mathrm{A}^0}^2$, $a = m_{\mathrm{Z}}^2$; then

$$m_{\mathrm{h}^0}^2 = \frac{1}{2}\{x + a - [(x+a)^2 - 4ax\cos^2 2\beta]^{1/2}\}. \tag{628}$$

It is easy to verify that this function has no stationary point for finite values of x. Further, for small x we find

$$m_{\mathrm{h}^0}^2 \approx x\cos^2 2\beta, \tag{629}$$

while for large x

$$m_{\mathrm{h}^0}^2 \to a\cos^2 2\beta - (a^2/4x)\sin^2 4\beta. \tag{630}$$

Hence the maximum value of $m^2_{\mathrm{h}^0}$, reached as $m^2_{\mathrm{A}^0} \to \infty$, is $a \cos^2 2\beta$, that is

$$m_{\mathrm{h}^0} \leq m_{\mathrm{Z}} |\cos 2\beta|. \tag{631}$$

Note that the RHS actually vanishes for $\beta = \pi/4$ i.e. for $\tan\beta = 1$.

This is the promised upper bound on the mass of one of the neutral Higgs bosons in the MSSM, and it is surely a remarkable result [64], [65]. The bound (631) has, of course[22], already been exceeded by the current experimental lower bound [67]

$$m_{\mathrm{H}} \geq 114.4 \text{ GeV } (95\% \text{ c.l.}). \tag{632}$$

Fortunately for the MSSM, the tree-level mass formulae derived above receive significant 1-loop corrections, particularly in the case of the h^0, whose mass is shifted upwards by a substantial amount [68] [69] [70] [71]. However, m_{h^0} is still minimized for $\tan\beta \approx 1$. The quantitative mass shift depends on additional MSSM parameters entering in the loops, but if these are tuned so as to maximize m_{h^0} for each value of m_{A^0} and $\tan\beta$ [72], the experimental lower bound (632) on m_{H} (assuming it to be so) can in principle be used to obtain exclusion limits on $\tan\beta$. This depends rather sensitively on the top quark mass. A recent summary [73] which includes leading two-loop effects and takes the average top squark squared mass to be $(2\mathrm{Tev})^2$, concludes that in the '$m^{\max}_{\mathrm{h}^0}$' scenario [72], with $m_{\mathrm{t}} = 179.4$ GeV, there is no constraint on $\tan\beta$, and $m_{\mathrm{h}^0} \leq 140$ GeV (with an accuracy of a few GeV). This is still an extremely interesting result. In the words of Drees [74]: "If experiments....fail to find at least one Higgs boson [in this energy region], the MSSM can be completely excluded, independent of the value of its 100 or so free parameters."

16.3 Tree-level Couplings of the h^0, H^0 and A^0 Bosons.

The phenomenolgy of the Higgs-sector particles depends, of course, not only on their masses but also on their couplings, which enter into production and decay processes. In this section we shall derive some of the more important couplings, for illustrative purposes.

[22] Well, maybe not! Drees [66] has recently suggested that the 2.3 σ excess of events around 98 GeV and the 1.7 σ excess around 115 GeV reported by the four LEP experiments [67] might actually be the h^0 and H^0 respectively.

First, note that after transforming to the mass-diagonal basis, the relation (471) and similar ones for $m_{\mathrm{d}ij}$ and $m_{\mathrm{e}ij}$ become

$$m_{\mathrm{u,c,t}} = v_{\mathrm{u}} y_{\mathrm{u,c,t}} \tag{633}$$

$$m_{\mathrm{d,s,b}} = v_{\mathrm{d}} y_{\mathrm{d,s,b}} \tag{634}$$

$$m_{\mathrm{e},\mu,\tau} = v_{\mathrm{d}} y_{\mathrm{e},\mu,\tau}. \tag{635}$$

In this basis, the Yukawa couplings in the superpotential are therefore (making use of (598))

$$y_{\mathrm{u,c,t}} = \frac{m_{\mathrm{u,c,t}}}{v_{\mathrm{u}}} = \frac{g m_{\mathrm{u,c,t}}}{\sqrt{2} m_{\mathrm{W}} \sin\beta} \tag{636}$$

$$y_{\mathrm{d,s,b}} = \frac{m_{\mathrm{d,s,b}}}{v_{\mathrm{d}}} = \frac{g m_{\mathrm{d,s,b}}}{\sqrt{2} m_{\mathrm{W}} \cos\beta} \tag{637}$$

$$y_{\mathrm{e},\mu,\tau} = \frac{m_{\mathrm{e},\mu,\tau}}{v_{\mathrm{d}}} = \frac{g m_{\mathrm{e},\mu,\tau}}{\sqrt{2} m_{\mathrm{W}} \cos\beta}. \tag{638}$$

Relations (636) and (637) suggest that very rough upper and lower bounds may be placed on $\tan\beta$ by requiring that neither y_{t} nor y_{b} is non-perturbatively large. For example, if $\tan\beta \geq 1$ then $y_{\mathrm{t}} \leq 1.4$, and if $\tan\beta \leq 50$ then $y_{\mathrm{b}} \leq 1.25$. Some GUT models can unify the running values of $y_{\mathrm{t}}, y_{\mathrm{b}}$ and y_τ at the unification scale; this requires $\tan\beta \approx m_{\mathrm{t}}/m_{\mathrm{b}} \simeq 40$.

To find the couplings of the MSSM Higgs bosons to fermions, we return to the Yukawa couplings (469) (together with the analogous ones for y_{d}^{ij} and y_{e}^{ij}), and expand H_{u}^0 and H_{d}^0 about their vacuum values. In order to get the result in terms of the physical fields h^0, H^0, however, we need to know how the latter are related to $\mathrm{Re}H_{\mathrm{u}}^0$ and $\mathrm{Re}H_{\mathrm{d}}^0$ - that is, we require expressions for the eigenmodes of the $(\mathrm{mass})^2$ matrix (625) corresponding to the eigenvalues $m_{\mathrm{h}^0}^2$ and $m_{\mathrm{H}^0}^2$ of (626) and (627). We can write (625) as

$$\mathbf{M}_{\mathrm{h,H}}^{\mathrm{sq}} = \frac{1}{2} \begin{pmatrix} A + Bc & -As \\ -As & A - Bc \end{pmatrix} \tag{639}$$

where $A = (m_{\mathrm{A}^0}^2 + m_{\mathrm{Z}}^2)$, $B = (m_{\mathrm{A}^0}^2 - m_{\mathrm{Z}}^2)$, $c = \cos 2\beta$, $s = \sin 2\beta$, and we have used (617). Expression (639) is calculated in the basis $(\sqrt{2}(\mathrm{Re}H_{\mathrm{u}}^0 - v_{\mathrm{u}}), \sqrt{2}(\mathrm{Re}H_{\mathrm{d}}^0 - v_{\mathrm{d}}))$. Let us denote the normalized eigenvectors of (639) by u_{h} and u_{H} where

$$u_{\mathrm{h}} = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}, \quad u_{\mathrm{H}} = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}, \tag{640}$$

with eigenvalues $m_{\mathrm{h}^0}^2$ and $m_{\mathrm{H}^0}^2$ respectively where

$$\begin{aligned} m_{\mathrm{h}^0}^2 &= \frac{1}{2}(A - C) && (641) \\ m_{\mathrm{H}^0}^2 &= \frac{1}{2}(A + C), && (642) \end{aligned}$$

with $C = [A^2 - (A^2 - B^2)c^2]^{1/2}$. The equation determining u_{h} is then

$$\begin{pmatrix} A + Bc & -As \\ -As & A - Bc \end{pmatrix} \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} = (A - C) \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}, \qquad (643)$$

which leads to

$$\begin{aligned} (C + Bc)\cos\alpha &= -As\sin\alpha && (644) \\ (-C + Bc)\sin\alpha &= As\cos\alpha. && (645) \end{aligned}$$

It is conventional to rewrite (644) and (645) more conveniently, as follows. Multiplying (644) by $\sin\alpha$ and (645) by $\cos\alpha$ and then subtracting the results, we obtain

$$\sin 2\alpha = -\frac{As}{C} = -\frac{(m_{\mathrm{A}^0}^2 + m_{\mathrm{Z}}^2)}{(m_{\mathrm{H}^0}^2 - m_{\mathrm{h}^0}^2)}\sin 2\beta. \qquad (646)$$

Again, multiplying (644) by $\cos\alpha$ and (645) by $\sin\alpha$ and adding the results gives

$$\cos 2\alpha = -\frac{Bc}{C} = -\frac{(m_{\mathrm{A}^0}^2 - m_{\mathrm{Z}}^2)}{(m_{\mathrm{H}^0}^2 - m_{\mathrm{h}^0}^2)}\cos 2\beta. \qquad (647)$$

Equations (646) and (647) serve to define the correct quadrant for the mixing angle α, namely $-\pi/2 \leq \alpha \leq 0$. Note that in the limit $m_{\mathrm{A}^0}^2 \gg m_{\mathrm{Z}}^2$ we have $\sin 2\alpha \approx -\sin 2\beta$ and $\cos 2\alpha \approx -\cos 2\beta$, and so

$$\alpha \approx \beta - \pi/2 \quad \text{for } m_{\mathrm{A}^0}^2 \gg m_{\mathrm{Z}}^2. \qquad (648)$$

The physical states are defined by

$$\begin{pmatrix} h^0 \\ H^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{Re}H_{\mathrm{u}}^0 - v_{\mathrm{u}} \\ \mathrm{Re}H_{\mathrm{d}}^0 - v_{\mathrm{d}} \end{pmatrix}, \qquad (649)$$

which we can write as

$$\mathrm{Re}H_{\mathrm{u}}^0 = [v_{\mathrm{u}} + \frac{1}{\sqrt{2}}(\cos\alpha\, h^0 + \sin\alpha\, H^0)] \tag{650}$$

$$\mathrm{Re}h_{\mathrm{d}}^0 = [v_{\mathrm{d}} + \frac{1}{\sqrt{2}}(-\sin\alpha\, h^0 + \cos\alpha\, H^0)]. \tag{651}$$

We also have, from (615) and (616),

$$\mathrm{Im}H_{\mathrm{u}}^0 = \frac{1}{\sqrt{2}}(\sin\beta\, H_{\mathrm{Z}} + \cos\beta\, A^0) \tag{652}$$

$$\mathrm{Im}H_{\mathrm{d}}^0 = \frac{1}{\sqrt{2}}(-\cos\beta\, H_{\mathrm{Z}} + \sin\beta\, A^0) \tag{653}$$

where H_{Z} is the massless field 'swallowed' by the Z^0.

We can now derive the couplings to fermions. For example, the Yukawa coupling (469) in the mass eigenstate basis, and for the third generation, is

$$-y_{\mathrm{t}}[\chi_{\bar{\mathrm{t}}\mathrm{L}} \cdot \chi_{\mathrm{tL}}(\mathrm{Re}H_{\mathrm{u}}^0 + \mathrm{i}\,\mathrm{Im}\,H_{\mathrm{u}}^0) + \chi_{\mathrm{tL}}^\dagger \cdot \chi_{\bar{\mathrm{t}}\mathrm{L}}^\dagger(\mathrm{Re}H_{\mathrm{u}}^0 - \mathrm{i}\,\mathrm{Im}\,H_{\mathrm{u}}^0)]. \tag{654}$$

Substituting (650) for $\mathrm{Re}H_{\mathrm{u}}^0$, the '$v_{\mathrm{u}}$' part simply produces the Dirac mass m_{u} via (470), while the remaining part gives

$$\begin{aligned} &-\frac{m_{\mathrm{t}}}{\sqrt{2}v_{\mathrm{u}}}(\chi_{\bar{\mathrm{t}}\mathrm{L}} \cdot \chi_{\mathrm{tL}} + \chi_{\mathrm{tL}}^\dagger \cdot \chi_{\bar{\mathrm{t}}\mathrm{L}}^\dagger)(\cos\alpha\, h^0 + \sin\alpha\, H^0) \\ &= -\left(\frac{gm_{\mathrm{t}}}{2m_{\mathrm{W}}}\right)\bar{t}t\left(\frac{\cos\alpha}{\sin\beta}\,h^0 + \frac{\sin\alpha}{\sin\beta}\,H^0\right), \end{aligned} \tag{655}$$

where '$\bar{t}t$' is the four-component Dirac bilinear. The corresponding expression in the SM would be just

$$-\left(\frac{gm_{\mathrm{t}}}{2m_{\mathrm{W}}}\right)\bar{t}t\,H_{\mathrm{SM}}, \tag{656}$$

where H_{SM} is the SM Higgs boson. Equation (655) shows how the SM coupling is modified in the MSSM. Simliarly, the coupling to the b quark is

$$-\left(\frac{gm_{\mathrm{b}}}{2m_{\mathrm{W}}}\right)\bar{b}b\left(-\frac{\sin\alpha}{\cos\beta}\,h^0 + \frac{\cos\alpha}{\cos\beta}\,H^0\right), \tag{657}$$

which is to be compared with the SM coupling

$$-\left(\frac{gm_{\mathrm{b}}}{2m_{\mathrm{W}}}\right)\bar{b}b\,H_{\mathrm{SM}}. \tag{658}$$

Finally the t-A^0 coupling is found by substituting (652) into (654), with the result

$$
\begin{aligned}
& -\mathrm{i}\frac{m_\mathrm{t}}{v_\mathrm{u}}(\chi_{\bar{\mathrm{t}}\mathrm{L}}\cdot\chi_{\mathrm{tL}} - \chi_{\mathrm{tL}}^\dagger\cdot\chi_{\bar{\mathrm{t}}\mathrm{L}}^\dagger)\frac{1}{\sqrt{2}}\cos\beta\, A^0 \\
& = \mathrm{i}\left(\frac{g m_\mathrm{t}}{2 m_\mathrm{W}}\right)\cot\beta\,\bar{t}\gamma_5 t\, A^0 \qquad (659)
\end{aligned}
$$

where we have used (464); and similarly the b-A^0 coupling is

$$
\mathrm{i}\left(\frac{g m_\mathrm{b}}{2 m_\mathrm{W}}\right)\tan\beta\,\bar{b}\gamma_5 b\, A^0. \qquad (660)
$$

Incidentally, the γ_5 coupling in (659) and (660) shows that the A^0 is a pseudoscalar boson (**CP** $= -1$), while the couplings (655) and (657) show that h^0 and H^0 are scalars (**CP**=+1). The limit of large m_{A^0} is interesting: in this case, α and β are related by (648), which implies

$$
\begin{aligned}
\sin\alpha & \approx -\cos\beta \qquad (661) \\
\cos\alpha & \approx \sin\beta. \qquad (662)
\end{aligned}
$$

It then follows from (655) and (657) that in this limit the couplings of h^0 become those of the SM Higgs, while the couplings of H^0 are the same as those of the A^0. For small m_{A^0} and large $\tan\beta$ on the other hand, the h^0 couplings differ substantially from the SM couplings, b-states being relatively enhanced and t-states being relatively suppressed, while the H^0 couplings are independent of β.

The couplings of the Higgs bosons to the gauge bosons are determined by the $SU(2)_L \times U(1)_y$ gauge invariance, that is by the terms (594) with D_μ given by (595). The terms involving $W_\mu^1, W_\mu^2, \mathrm{Re}H_\mathrm{u}^0$ and $\mathrm{Re}H_\mathrm{d}^0$ are easily found to be

$$
\frac{g^2}{4}(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu})[(\mathrm{Re}H_\mathrm{u}^0)^2 + (\mathrm{Re}H_\mathrm{d}^0)^2]. \qquad (663)
$$

Substituting (650) and (651), the v_u^2 and v_d^2 parts generate the W-boson $(\mathrm{mass})^2$ term via (598), while the W-W-(h^0,H^0) couplings are

$$
\begin{aligned}
& \frac{g^2}{4}(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu})\sqrt{2}[v_\mathrm{u}(\cos\alpha\, h^0 + \sin\alpha\, H^0) + v_\mathrm{d}(-\sin\alpha\, h^0 + \cos\alpha\, H^0)] \\
& = \frac{g m_\mathrm{W}}{2}(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu})[\sin(\beta-\alpha)\, h^0 + \cos(\beta-\alpha)\, H^0]. \qquad (664)
\end{aligned}
$$

Similarly, the Z-Z-(h^0,H^0) couplings are

$$\frac{g m_{\rm Z}}{2\cos\theta_{\rm W}} Z_\mu Z^\mu [\sin(\beta-\alpha)\, h^0 + \cos(\beta-\alpha)\, H^0]. \tag{665}$$

Again, these are the same as the couplings of the SM Higgs to W and Z, but modified by a factor $\sin(\beta-\alpha)$ for the h^0, and a factor $\cos(\beta-\alpha)$ for the H^0.[23] Once again, there is a simple large $m^2_{A^0}$ limit:

$$\sin(\beta-\alpha) \approx 1, \qquad \cos(\beta-\alpha) \approx 0, \tag{666}$$

showing that in this limit h^0 has SM couplings to gauge bosons, while the H^0 decouples from them entirely. At tree level, the A^0 has no coupling to pairs of gauge bosons.

The total widths of the MSSM Higgs bosons depend sensitively on $\tan\beta$. The h^0 decays mainly to fermion-antifermion pairs, with a width generally comparable to that of the SM Higgs, while the H^0 and A^0 are generally narrower than an SM Higgs of the same mass. The production rate at the LHC also depends on $\tan\beta$. The dominant production mechanism, as in the SM, is expected to be gluon fusion, proceeding via quark (or squark) loops. In the SM case, the top quark loop dominates; in the MSSM, if $\tan\beta$ is large and m_{A^0} not too large, the $b\bar{b}h$ coupling is relatively enhanced, as noted after equation (662), and the bottom quark loop becomes important. Searches for MSSM Higgs bosons are reviewed by Igo-Kemenes in [34].

17 SUSY Particles in the MSSM

In this final section, we shall give a very brief introduction to the physics of the various SUSY particle states in the MSSM. As in the scalar Higgs sector, the discussion is complicated by mixing phenomena. In particular, after $SU(2)_L \times U(1)_y$ breaking, mixing will in general occur between any two (or more) fields which have the same colour, charge and spin.

---

[23]This is essential for the viability of Drees's suggestion [66]: the excess of events near 98 GeV amounts to about 10% of the signal for a SM Higgs with that mass, and hence interpreting it as the h^0 requires that $\sin^2(\beta-\alpha) \approx 0.1$. It then follows that ZH^0 production at LEP would occur with nearly SM strength, if allowed kinematically. Hence the identification of the excess at around 115 GeV with the H^0.

17.1 Neutralinos

We consider first the sector consisting of the neutral higgsinos $\tilde{H}_{\rm u}^0$ and $\tilde{H}_{\rm d}^0$, and the neutral gauginos $\tilde{B}$ (bino) and $\tilde{W}^0$ (wino) (see Tables 1 and 2). These are all L-type spinor fields in our presentation (but they can equivalently be represented as Majorana fields, as discussed in section 2.3). In the absence of electroweak symmetry breaking, the $\tilde{B}$ and $\tilde{W}^0$ fields would have masses given by just the soft SUSY-breaking mass terms of (547):

$$-\frac{1}{2}M_1\tilde{B}\cdot\tilde{B}-\frac{1}{2}M_2\tilde{W}^0\cdot\tilde{W}^0+\text{h.c.} \tag{667}$$

However, bilinear combinations of one of $(\tilde{B},\tilde{W}^0)$ with one of $(\tilde{H}_{\rm u}^0,\tilde{H}_{\rm d}^0)$ are generated by the term '$-\sqrt{2}g[......]$' in (459), when the neutral scalar Higgs fields acquire a vev. Such bilinear terms will, as in the Higgs sector, appear as non-zero off-diagonal entries in the 4×4 mass matrix for the four fields $\tilde{B},\tilde{W}^0,\tilde{H}_{\rm u}^0$, and $\tilde{H}_{\rm d}^0$ - that is, they will cause mixing. After the mass matrix is diagonalized, the resulting four neutral mass eigenstates are called neutralinos, usually denoted by $\tilde{\chi}_i^0$ $(i=1,2,3,4)$, with the convention that the masses are ordered as $m_{\tilde{\chi}_1^0}<m_{\tilde{\chi}_2^0}<m_{\tilde{\chi}_3^0}<m_{\tilde{\chi}_4^0}$.

Consider for example the SU(2) contribution in (459) from the $H_{\rm u}$ supermultiplet, with $\alpha=3, T^3\equiv\tau^3/2,\lambda^3\equiv\tilde{W}^0$, which is

$$-\sqrt{2}g(H_{\rm u}^{+\dagger}\;H_{\rm u}^{0\dagger})\frac{\tau^3}{2}\begin{pmatrix}\tilde{H}_{\rm u}^+\\ \tilde{H}_{\rm u}^0\end{pmatrix}\cdot\tilde{W}^0+\text{h.c.} \tag{668}$$

When the field $H_{\rm u}^{0\dagger}$ acquires a vev $v_{\rm u}$ (which we have already chosen to be real), expression (668) contains the piece

$$+\frac{g}{\sqrt{2}}v_{\rm u}\tilde{H}_{\rm u}^0\cdot\tilde{W}^0+\text{h.c.}, \tag{669}$$

which we shall re-write as

$$-\frac{1}{2}[-\sin\beta\cos\theta_{\rm W}m_{\rm Z}](\tilde{H}_{\rm u}^0\cdot\tilde{W}^0+\tilde{W}^0\cdot\tilde{H}_{\rm u}^0)+\text{h.c.}, \tag{670}$$

using (602) and (597), and the result of the first Exercise in Notational Aside (1). In a gauge-eigenstate basis

$$\tilde{G}^0=\begin{pmatrix}\tilde{B}\\ \tilde{W}^0\\ \tilde{H}_{\rm d}^0\\ \tilde{H}_{\rm u}^0\end{pmatrix}, \tag{671}$$

this will contribute a mixing between the (2,4) and (4,2) components. Similarly, the U(1) contribution from the $H_{\rm u}$ supermultiplet, after electroweak symmetry breaking, leads to the mixing term

$$-\frac{g'}{\sqrt{2}} v_{\rm u} \tilde{H}^0_{\rm u} \cdot \tilde{B} + {\rm h.c.} \tag{672}$$

$$= -\frac{1}{2}[\sin\beta \sin\theta_{\rm W} m_{\rm Z}](\tilde{H}^0_{\rm u} \cdot \tilde{B} + \tilde{B} \cdot \tilde{H}^0_{\rm u}) + {\rm h.c.}, \tag{673}$$

which involves the (1,4) and (4,1) components. The SU(2) and U(1) contributions of the $H_{\rm d}$ supermultiplet to such bilinear terms can be evaluated similarly.

In addition to this mixing caused by electroweak symmetry breaking, mixing between $\tilde{H}^0_{\rm u}$ and $\tilde{H}^0_{\rm d}$ is induced by the SUSY-invariant 'μ term' in (475), namely

$$-\frac{1}{2}(-\mu)(\tilde{H}^0_{\rm u} \cdot \tilde{H}^0_{\rm d} + \tilde{H}^0_{\rm d} \cdot \tilde{H}^0_{\rm u}) + {\rm h.c.} \tag{674}$$

Putting all this together, mass terms involving the fields in G^0 can be written as

$$-\frac{1}{2}\tilde{G}^{0{\rm T}} \mathbf{M}_{\tilde{G}^0} \tilde{G}^0 + {\rm h.c.} \tag{675}$$

where

$$\mathbf{M}_{\tilde{G}^0} = \begin{pmatrix} M_1 & 0 & -c_\beta s_{\rm W} m_{\rm Z} & s_\beta s_{\rm W} m_{\rm Z} \\ 0 & M_2 & c_\beta c_{\rm W} m_{\rm Z} & -s_\beta c_{\rm W} m_{\rm Z} \\ -c_\beta s_{\rm W} m_{\rm Z} & c_\beta c_{\rm W} m_{\rm Z} & 0 & -\mu \\ s_\beta s_{\rm W} m_{\rm Z} & -s_\beta c_{\rm W} m_{\rm Z} & -\mu & 0 \end{pmatrix}, \tag{676}$$

with $c_\beta \equiv \cos\beta, s_\beta \equiv \sin\beta, c_{\rm W} \equiv \cos\theta_{\rm W}$, and $s_{\rm W} \equiv \sin\theta_{\rm W}$.

In general, the parameters M_1, M_2 and μ can have arbitrary phases. Most analyses, however, assume the 'gaugino unification' condition (563) which implies (565) at the electroweak scale, so that one of M_1 and M_2 is fixed in terms of the other. A redefinition of the phases of $\tilde{B}$ and $\tilde{W}^0$ then allows us to make both M_1 and M_2 real and positive. The entries proportional to $m_{\rm Z}$ are real by virtue of the phase choices made for the Higgs fields in section 16.1, which made $v_{\rm u}$ and $v_{\rm d}$ both real. It is usual to take μ to be real (so as to avoid unacceptably large **CP**-violating effects), but the sign of μ is unknown - and not fixed by Higgs-sector physics (see the sentence following equation (602)). The neutralino sector is then determined by three real parameters, M_1, $\tan\beta$ and μ (as well as by $m_{\rm Z}$ and $\theta_{\rm W}$, of course).

Clearly there is not a lot to be gained by pursuing the algebra of this 4×4 mixing problem, in general. A simple special case is that in which the m_Z-dependent terms in (676) are a relatively small perturbation on the other entries, which would imply that the neutralinos $\tilde{\chi}_1^0$ and $\tilde{\chi}_2^0$ are close to the weak eigenstates bino and wino respectively, with masses approximately equal to M_1 and M_2, while the higgsinos are mixed by the μ entries to form (approximately) the combinations

$$\tilde{H}_{\rm S}^0 = \frac{1}{\sqrt{2}}(\tilde{H}_{\rm d}^0 + \tilde{H}_{\rm u}^0), \quad \text{and} \quad \tilde{H}_{\rm A}^0 = \frac{1}{\sqrt{2}}(\tilde{H}_{\rm d}^0 - \tilde{H}_{\rm u}^0), \tag{677}$$

each having mass $\sim |\mu|$.

Assuming it is the LSP, the lightest neutralino, $\tilde{\chi}_1^0$, is an attractive candidate for non-baryonic dark matter [75].[24] Taking account of the restricted range of $\Omega_{\rm CDM}h^2$ consistent with the WMAP data, calculations show [76] [77] [78] that $\tilde{\chi}_1^0$'s provide the desired thermal relic density in certain quite well-defined regions in the space of the mSUGRA parameters ($m_{1/2}, m_0, \tan\beta$ and the sign of μ; A_0 was set to zero). Dark matter is reviewed by Drees and Gerbier in [34].

17.2 Charginos

The charged analogues of neutralinos are called 'charginos': there are two positively charged ones associated (before mixing) with $(\tilde{W}^+, \tilde{H}_{\rm u}^+)$, and two negatively charged ones associated with $(\tilde{W}^-, \tilde{H}_{\rm d}^-)$. Mixing between $\tilde{H}_{\rm u}^+$ and $\tilde{H}_{\rm d}^-$ occurs via the μ term in (475). Also, as in the neutralino case, mixing between the charged gauginos and higgsinos will occur via the '$-\sqrt{2}g[....]$' term in (459) after electroweak symmetry breaking. Consider for example the $H_{\rm u}$ supermultiplet terms in (459) involving $\tilde{W}^1$ and $\tilde{W}^2$, after the scalar Higgs $H_{\rm u}^0$ has acquired a vev $v_{\rm u}$. These terms are

$$-\frac{g}{\sqrt{2}}\{(0\, v_{\rm u})[\tau^1 \begin{pmatrix} \tilde{H}_{\rm u}^+ \\ \tilde{H}_{\rm u}^0 \end{pmatrix} \cdot \tilde{W}^1 + \tau^2 \begin{pmatrix} \tilde{H}_{\rm u}^+ \\ \tilde{H}_{\rm u}^0 \end{pmatrix} \cdot \tilde{W}^2]\} + \text{h.c.} \tag{678}$$

$$= -\frac{g}{\sqrt{2}} v_{\rm u} \tilde{H}_{\rm u}^+ \cdot (\tilde{W}^1 + {\rm i}\tilde{W}^2) + \text{h.c.} \tag{679}$$

[24]Other possibilities exist. For example, in gauge-mediated SUSY breaking, the gravitino is naturally the LSP. For this and other dark matter candidates within a softly-broken SUSY framework, see [23] section 6.

$$\equiv -gv_{\mathrm{u}}\tilde{H}_{\mathrm{u}}^{+} \cdot \tilde{W}^{-} + \mathrm{h.c.} \tag{680}$$

$$= -\frac{1}{2}\sqrt{2}s_{\beta}m_{\mathrm{W}}(\tilde{H}_{\mathrm{u}}^{+} \cdot \tilde{W}^{-} + \tilde{W}^{-} \cdot \tilde{H}_{\mathrm{u}}^{+}) + \mathrm{h.c.} \tag{681}$$

The corresponding terms from the H_{d} supermultiplet are

$$-gv_{\mathrm{d}}\tilde{H}_{\mathrm{d}}^{-} \cdot \tilde{W}^{+} + \mathrm{h.c.} \tag{682}$$

$$= -\frac{1}{2}\sqrt{2}c_{\beta}m_{\mathrm{W}}(\tilde{H}_{\mathrm{d}}^{-} \cdot \tilde{W}^{+} + \tilde{W}^{+} \cdot \tilde{H}_{\mathrm{d}}^{-}) + \mathrm{h.c.} \tag{683}$$

If we define a gauge-eigenstate basis

$$\tilde{g}^{+} = \begin{pmatrix} \tilde{W}^{+} \\ \tilde{H}_{\mathrm{u}}^{+} \end{pmatrix} \tag{684}$$

for the positively charged states, and similarly

$$\tilde{g}^{-} = \begin{pmatrix} \tilde{W}^{-} \\ \tilde{H}_{\mathrm{d}}^{-} \end{pmatrix} \tag{685}$$

for the negatively charged states, then the chargino mass terms can be written as

$$-\frac{1}{2}[\tilde{g}^{+\mathrm{T}}\mathbf{X}^{\mathrm{T}} \cdot \tilde{g}^{-} + \tilde{g}^{-\mathrm{T}}\mathbf{X} \cdot \tilde{g}^{+}] + \mathrm{h.c.}, \tag{686}$$

where

$$\mathbf{X} = \begin{pmatrix} M_2 & \sqrt{2}s_{\beta}m_{\mathrm{W}} \\ \sqrt{2}c_{\beta}m_{\mathrm{W}} & \mu \end{pmatrix}. \tag{687}$$

Since $\mathbf{X}^{\mathrm{T}} \neq \mathbf{X}$ (unless $\tan\beta = 1$), two distinct 2×2 matrices are needed for the diagonalization. Let us define the mass-eigenstate bases by

$$\tilde{\chi}^{+} = \mathbf{V}\tilde{g}^{+}, \quad \tilde{\chi}^{+} = \begin{pmatrix} \tilde{\chi}_1^{+} \\ \tilde{\chi}_2^{+} \end{pmatrix} \tag{688}$$

$$\tilde{\chi}^{-} = \mathbf{U}\tilde{g}^{-}, \quad \tilde{\chi}^{-} = \begin{pmatrix} \tilde{\chi}_1^{-} \\ \tilde{\chi}_2^{-} \end{pmatrix}, \tag{689}$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary. Then the second term in (686) becomes

$$-\frac{1}{2}\tilde{\chi}^{-\mathrm{T}}\mathbf{U}^{*}\mathbf{X}\mathbf{V}^{-1} \cdot \tilde{\chi}^{+}, \tag{690}$$

and we require

$$\mathbf{U}^*\mathbf{X}\mathbf{V}^{-1} = \begin{pmatrix} m_{\tilde{\chi}_1^\pm} & 0 \\ 0 & m_{\tilde{\chi}_2^\pm} \end{pmatrix}. \tag{691}$$

What about the first term in (686)? It becomes

$$-\frac{1}{2}\tilde{\chi}^{+\mathrm{T}}\mathbf{V}^*\mathbf{X}^\mathrm{T}\mathbf{U}^\dagger \cdot \tilde{\chi}^-. \tag{692}$$

But since $\mathbf{V}^*\mathbf{X}^\mathrm{T}\mathbf{U}^\dagger = (\mathbf{U}^*\mathbf{X}\mathbf{V}^{-1})^\mathrm{T}$ it follows that the expression (692) is also diagonal, with the same eigenvalues $m_{\tilde{\chi}_1^\pm}$ and $m_{\tilde{\chi}_2^\pm}$.

Now note that the Hermitian conjugate of (691) gives

$$\mathbf{V}\mathbf{X}^\dagger\mathbf{U}^\mathrm{T} = \begin{pmatrix} m^*_{\tilde{\chi}_1^\pm} & 0 \\ 0 & m^*_{\tilde{\chi}_2^\pm} \end{pmatrix}. \tag{693}$$

Hence

$$\mathbf{V}\mathbf{X}^\dagger\mathbf{X}\mathbf{V}^{-1} = \mathbf{V}\mathbf{X}^\dagger\mathbf{U}^\mathrm{T}\mathbf{U}^*\mathbf{X}\mathbf{V}^{-1} = \begin{pmatrix} |m_{\tilde{\chi}_1^\pm}|^2 & 0 \\ 0 & |m_{\tilde{\chi}_2^\pm}|^2 \end{pmatrix}, \tag{694}$$

and we see that the positively charged states $\tilde{\chi}^+$ diagonalize $\mathbf{X}^\dagger\mathbf{X}$. Similarly,

$$\mathbf{U}^*\mathbf{X}\mathbf{X}^\dagger\mathbf{U}^\mathrm{T} = \mathbf{U}^*\mathbf{X}\mathbf{V}^{-1}\mathbf{V}\mathbf{X}^\dagger\mathbf{U}^\mathrm{T} = \begin{pmatrix} |m_{\tilde{\chi}_1^\pm}|^2 & 0 \\ 0 & |m_{\tilde{\chi}_2^\pm}|^2 \end{pmatrix}, \tag{695}$$

and the negatively charged states $\tilde{\chi}^-$ diagonalize $\mathbf{X}\mathbf{X}^\dagger$. The eigenvalues of $\mathbf{X}^\dagger\mathbf{X}$ (or $\mathbf{X}\mathbf{X}^\dagger$) are easily found to be

$$\begin{pmatrix} |m_{\tilde{\chi}_1^\pm}|^2 \\ |m_{\tilde{\chi}_2^\pm}|^2 \end{pmatrix} = \frac{1}{2}[(M_2^2+|\mu|^2+2m_\mathrm{W}^2)\mp\{(M_2^2+|\mu|^2+2m_\mathrm{W}^2)^2-4|\mu M_2-m_\mathrm{W}^2\sin 2\beta|^2\}^{1/2}]. \tag{696}$$

It may be worth noting that, because $\mathbf{X}$ is diagonalized by the operation $\mathbf{U}^*\mathbf{X}\mathbf{V}^{-1}$, rather than by $\mathbf{V}\mathbf{X}\mathbf{V}^{-1}$ or $\mathbf{U}^*\mathbf{X}\mathbf{U}^\mathrm{T}$, these eigenvalues are not the squares of the eigenvalues of $\mathbf{X}$.

The expression (696) is not particularly enlightening, but as in the neutralino case it simplifies greatly if m_W can be regarded as a perturbation. Taking M_2 and μ to be real, the eigenvalues are then given approximately by $m_{\tilde{\chi}_1^\pm} \approx M_2$, and $m_{\tilde{\chi}_2^\pm} \approx |\mu|$ (the labelling assumes $M_2 < |\mu|$). In this limit, we have the approximate degeneracies $m_{\tilde{\chi}_1^\pm} \approx m_{\tilde{\chi}_2^0}$, and $m_{\tilde{\chi}_2^\pm} \approx m_{\tilde{H}_S^0} \approx m_{\tilde{H}_A^0}$. In general, the physics is sensitive to the ratio $M_2/|\mu|$.

As an illustration of possible signatures for neutralino and chargino production (at hadron colliders, for example), we mention the *trilepton signal* [79] [80] [81] [82] [83] [84], which arises from the production

$$\mathrm{p\bar{p}} \quad (\text{or} \quad \mathrm{pp}) \rightarrow \tilde{\chi}_1^\pm \tilde{\chi}_2^0 + X \tag{697}$$

followed by the decays

$$\tilde{\chi}_1^\pm \rightarrow l'^\pm \nu \tilde{\chi}_1^0 \tag{698}$$

$$\tilde{\chi}_2^0 \rightarrow l\bar{l}\tilde{\chi}_1^0. \tag{699}$$

Here the two LSPs in the final state carry away $2m_{\tilde{\chi}_1^0}$ of missing energy, which is observed as missing transverse energy, $\not{E}_\mathrm{T}$ (see section 14). In addition, there are three energetic, isolated leptons, and little jet activity. The expected SM background is small. Using the data sample collected from the 1992-3 run of the Fermilab Tevatron, D0 [85] and CDF [86] reported no candidate trilepton events after applying all selection criteria; the expected background was roughly 2 ± 1 events. Upper limits on the product of the cross section times the branching ratio (single tripleton mode) were set, for various regions in the space of MSSM parameters. Later searches using the data sample from the 1994-5 run [87] [88] were similarly negative.

17.3 Gluinos

Since the gluino $\tilde{g}$ is a colour octet fermion, it cannot mix with any other MSSM particle, even if R-parity is violated. So we get a (unique) break from mixing phenomena. We have already seen (section 15.3) that most models assume that the gluino mass is significantly greater than that of the neutralinos and charginos. A useful signature for gluino pair ($\tilde{g}\tilde{g}$) production is the *like-sign dilepton signal* [89] [90] [91]. This arises if the gluino decays with a significant branching ratio to hadrons plus a chargino, which then decays to lepton + ν + $\tilde{\chi}_1^0$. Since the gluino is indifferent to electric charge, the single lepton from each $\tilde{g}$ decay will carry either charge with equal probability. Hence many events should contain two like-sign leptons (plus jets plus $\not{E}_\mathrm{T}$). This has a low SM background, because in the SM isolated lepton pairs come from $\mathrm{W^+W^-}$, Drell-Yan or $\mathrm{t\bar{t}}$ production, all of which give opposite sign dileptons. Like-sign dilepton events can also arise from $\tilde{g}\tilde{q}$ and $\tilde{q}\bar{\tilde{q}}$ production.

CDF [92] reported no candidate events for like-sign dilepton pairs. Other searches based simply on dileptons (not required to be like-sign) plus two jets plus $\not{E}_{\rm T}$ [93] [94] reported no sign of any excess events. Results were expressed in terms of exclusion contours for mSUGRA parameters.

17.4 Squarks and Sleptons

The scalar partners of the SM fermions form the largest collection of new particles in the MSSM. Since separate partners are required for each chirality state of the massive fermions, there are altogether 21 new fields (the neutrinos are treated as massless here): four squark flavours and chiralities $\tilde{u}_{\rm L}, \tilde{u}_{\rm R}, \tilde{d}_{\rm L}, \tilde{d}_{\rm R}$ and three slepton flavours and chiralities $\tilde{\nu}_{\rm eL}, \tilde{e}_{\rm L}, \tilde{e}_{\rm R}$ in the first family, all repeated for the other two families.[25] These are all (complex) scalar fields, and so the 'L' and 'R' labels do not, of course, here signify chirality, but are just labels showing which SM fermion they are partnered with (and hence in particular what their SU(2)×U(1) quantum numbers are - see Table 1).

In principle, any scalars with the same electric charge, R-parity and colour quantum numbers can mix with each other, across families, via the soft SUSY-breaking parameters in (548), (549) and (553). This would lead to a 6×6 mixing problem for the u-type squark fields $(\tilde{u}_{\rm L}, \tilde{u}_{\rm R}, \tilde{c}_{\rm L}, \tilde{c}_{\rm R}, \tilde{t}_{\rm L}, \tilde{t}_{\rm R})$, and for the d-type squarks and the charged sleptons, and a 3×3 one for the sneutrinos. However, as we saw in section 15.2, phenomenological constraints imply that interfamily mixing among the SUSY states must be very small. As before, therefore, we shall adopt the 'mSUGRA' form of the soft parameters as given in equations (556) and (558), which guarantees the suppression of unwanted interfamily mixing terms (though one must remember that other, and more general, parametrizations are not excluded). As in the cases considered previously in this section, we shall also have to include various effects due to electroweak symmetry breaking.

Consider first the soft SUSY-breaking (mass)^2 parameters of the sfermions (squarks and sleptons) of the first family. In the model of (556) they are all degenerate at the high (Planck?) scale. The RGE evolution down to the electroweak scale is governed by equations of the same type as (569) and (570),

[25] In the more general family-index notation of section 15.2 (see equations (548), (549) and (553)), '$\tilde{Q}_1$' is the doublet $(\tilde{u}_{\rm L}, \tilde{d}_{\rm L})$, '$\tilde{Q}_2$' is $(\tilde{c}_{\rm L}, \tilde{s}_{\rm L})$, '$\tilde{Q}_3$' is $(\tilde{t}_{\rm L}, \tilde{b}_{\rm L})$, '$\tilde{\bar{u}}_1$' is $\tilde{u}_{\rm R}$, '$\tilde{\bar{d}}_1$' is $\tilde{d}_{\rm R}$ (and similarly for '$\tilde{\bar{u}}_{2,3}$' and '$\tilde{\bar{d}}_{2,3}$'), while '$\tilde{L}_1$' is $(\tilde{\nu}_{\rm eL}, \tilde{e}_{\rm L})$, '$\tilde{\bar{e}}_1$' is $e_{\rm R}$, etc.

but without the $X_{\rm t}$ terms: the corresponding terms for the first two families may be neglected because of their much smaller Yukawa couplings. Thus the soft masses of the first and second families evolve by purely gauge interactions, which (see the comment following equation (572)) tend to increase the masses at low scales. Their evolution can be parametrized (following [22] equations (7.65) - (7.69)) by

$$m^2_{\tilde{\rm u}_{\rm L},\tilde{\rm d}_{\rm L}} = m^2_{\tilde{\rm c}_{\rm L},\tilde{\rm s}_{\rm L}} = m_0^2 + K_3 + K_2 + \frac{1}{9}K_1 \quad (700)$$

$$m^2_{\tilde{\rm u}_{\rm R}} = m^2_{\tilde{\rm c}_{\rm R}} = m_0^2 + K_3 + \frac{16}{9}K_1 \quad (701)$$

$$m^2_{\tilde{\rm d}_{\rm R}} = m^2_{\tilde{\rm s}_{\rm R}} = m_0^2 + K_3 + \frac{4}{9}K_1 \quad (702)$$

$$m^2_{\tilde{\nu}_{\rm eL},\tilde{\rm e}_{\rm L}} = m^2_{\tilde{\nu}_{\mu{\rm L}},\tilde{\mu}_{\rm L}} = m_0^2 + K_2 + K_1 \quad (703)$$

$$m^2_{\tilde{\rm e}_{\rm R}} = m^2_{\tilde{\mu}_{\rm R}} = m_0^2 + 4K_1. \quad (704)$$

Here K_3, K_2 and K_1 are the RGE contributions from SU(3), SU(2) and U(1) gauginos respectively: all the chiral supermultiplets couple to the gauginos with the same ('universal') gauge couplings. The different numerical coefficients in front of the K_1 terms are the squares of the y-values of each field (see Table 1), which enter into the relevant loops. All the K's are positive, and are roughly of the same order of magnitude as the gaugino (mass)^2 parameter $m^2_{1/2}$, but with K_3 significantly greater than K_2, which in turn is greater than K_1 (this is because of the relative sizes of the different gauge couplings at the weak scale: $g_3^2 \sim 1.5, g_2^2 \sim 0.4, g_1^2 \sim 0.2$, see section 13). The large 'K_3' contribution is likely to be quite model-independent, and it is therefore reasonable to expect that squark (mass)^2 values will be greater than slepton ones.

Equations (700) - (704) give the soft (mass)^2 parameters for the fourteen states involved, in the first two families (we defer consideration of the third family for the moment). In addition to these contributions, however, there are further terms to be included which arise as a result of electroweak symmetry breaking. For the first two families, the most important such contributions are those coming from SUSY-invariant D-terms (see (461)) of the form (squark)^2(higgs)^2 and (slepton)^2(higgs)^2, after the scalar Higgs fields $H_{\rm u}^0$ and $H_{\rm d}^0$ have acquired vevs. Returning to equation (460), the SU(2) contribution to D^α is

$$D^\alpha = g\{(\tilde{u}_{\rm L}^\dagger \tilde{d}_{\rm L}^\dagger)\frac{\tau^\alpha}{2}\begin{pmatrix}\tilde{u}_{\rm L}\\ \tilde{d}_{\rm L}\end{pmatrix} + (\tilde{\nu}_{\rm eL}^\dagger \tilde{e}_{\rm L}^\dagger)\frac{\tau^\alpha}{2}\begin{pmatrix}\tilde{\nu}_{\rm eL}\\ \tilde{e}_{\rm L}\end{pmatrix}$$

$$+(H_{\rm u}^{+\dagger}H_{\rm u}^{0\dagger})\frac{\tau^\alpha}{2}\left(\begin{array}{c} H_{\rm u}^+ \\ H_{\rm u}^0 \end{array}\right) + (H_{\rm d}^{0\dagger}H_{\rm d}^{-\dagger})\frac{\tau^\alpha}{2}\left(\begin{array}{c} H_{\rm d}^0 \\ H_{\rm d}^- \end{array}\right)\} \tag{705}$$

$$\rightarrow g\{(\tilde{u}_{\rm L}^\dagger \tilde{d}_{\rm L}^\dagger)\frac{\tau^\alpha}{2}\left(\begin{array}{c} \tilde{u}_{\rm L} \\ \tilde{d}_{\rm L} \end{array}\right) + (\tilde{\nu}_{\rm eL}^\dagger \tilde{e}_{\rm L}^\dagger)\frac{\tau^\alpha}{2}\left(\begin{array}{c} \tilde{\nu}_{\rm eL} \\ \tilde{e}_{\rm L} \end{array}\right) - \frac{1}{2}v_{\rm u}^2\delta_{\alpha 3} + \frac{1}{2}v_{\rm d}^2\delta_{\alpha 3}\}, \tag{706}$$

after symmetry breaking. When this is inserted into the Lagrangian term $-\frac{1}{2}D^\alpha D^\alpha$, pieces which are quadratic in the scalar fields - and are therefore (mass)^2 terms - will come from cross terms between the '$\tau^\alpha/2$' and '$\delta_{\alpha 3}$' terms. These cross terms are proportional to $\tau^3/2$, and therefore split apart the $T^3 = +1/2$ weak isospin components from the $T^3 = -1/2$ components, but they are diagonal in the weak eigenstate basis. Their contribution to the sfermion (mass)^2 matrix is therefore

$$+\frac{1}{2}g^2\, 2\, \frac{1}{2}(v_{\rm d}^2 - v_{\rm u}^2)T^3 \tag{707}$$

where $T^3 = \tau^3/2$. Similarly, the U(1) contribution to 'D' is

$$D_y = g'\{\sum_{\tilde{f}} \frac{1}{2}\tilde{f}^\dagger y_{\tilde{f}}\tilde{f} - \frac{1}{2}(v_{\rm d}^2 - v_{\rm u}^2)\} \tag{708}$$

after symmetry breaking, where the sum is over all sfermions (squarks and sleptons). Expression (708) leads to the sfermion (mass)^2 term

$$+\frac{1}{2}g'^2 2\,(-\frac{1}{2}y)\frac{1}{2}(v_{\rm d}^2 - v_{\rm u}^2). \tag{709}$$

Since $y/2 = Q - T^3$, where Q is the electromagnetic charge, we can combine (707) and (709) to give a total (mass)^2 contribution for each sfermion:

$$\begin{aligned} \Delta_{\tilde{f}} &= \frac{1}{2}(v_{\rm d}^2 - v_{\rm u}^2)[(g^2 + g'^2)T^3 - g'^2 Q] \\ &= m_{\rm Z}^2 \cos 2\beta[T^3 - \sin^2\theta_{\rm W} Q], \end{aligned} \tag{710}$$

using (597). As remarked earlier, $\Delta_{\tilde{f}}$ is diagonal in the weak eigenstate basis, and the appropriate contributions simply have to be added to the RHS of equations (700) - (704). It is interesting to note that the splitting between the doublet states is predicted to be

$$-m_{\tilde{\rm u}_{\rm L}}^2 + m_{\tilde{\rm d}_{\rm L}}^2 = -m_{\tilde{\nu}_{\rm eL}}^2 + m_{\tilde{\rm e}_{\rm L}}^2 = -\cos 2\beta m_{\rm W}^2, \tag{711}$$

and similarly for the second family. On the assumption that $\tan\beta$ is most probably greater than 1 (see the comments following equation (638)), the 'down' states are heavier.

Sfermion (mass)^2 terms are also generated by SUSY-invariant F-terms, after symmetry breaking - that is, terms in the Lagrangian of the form

$$-\left|\frac{\partial W}{\partial \phi_i}\right|^2 \tag{712}$$

for every scalar field ϕ_i (see equations (278) and (280)); for these purposes we regard W of (465) as being written in terms of the scalar fields, as in section 8. Remembering that the Yukawa couplings are proportional to the associated fermion masses (see (471) and (633) - (635)), we see that on the scale expected for the masses of the sfermions, only terms involving the Yukawas of the third family can contribute significantly. Thus to a very good approximation we can write

$$\begin{aligned} W &\approx y_{\rm t}\tilde{t}^\dagger_{\rm R}(\tilde{t}_{\rm L}H^0_{\rm u} - \tilde{b}_{\rm L}H^+_{\rm u}) - y_{\rm b}\tilde{b}^\dagger_{\rm R}(\tilde{t}_{\rm L}H^-_{\rm d} - \tilde{b}_{\rm L}H^0_{\rm d}) - y_\tau\tilde{\tau}^\dagger_{\rm R}(\tilde{\nu}_{\tau{\rm L}}H^-_{\rm d} - \tilde{\tau}_{\rm L}H^0_{\rm d}) \\ &+ \mu(H^+_{\rm u}H^-{\rm d} - H^0_{\rm u}H^0_{\rm d}) \end{aligned} \tag{713}$$

as in (473). Then we have, for example,

$$-\left|\frac{\partial W}{\partial \tilde{t}^\dagger_{\rm R}}\right|^2 = -y_{\rm t}^2\tilde{t}^\dagger_{\rm L}\tilde{t}_{\rm L}|H^0_{\rm u}|^2 \to -y^2_{\rm t}v^2_{\rm u}\tilde{t}^\dagger_{\rm L}\tilde{t}_{\rm L} = -m^2_{\rm t}\tilde{t}^\dagger_{\rm L}\tilde{t}_{\rm L}, \tag{714}$$

after $H^0_{\rm u}$ acquires the vev $v_{\rm u}$. The L-type top squark ('stop') therefore gets a (mass)^2 term equal to the top quark (mass)^2. There will be an identical term for the R-type stop squark, coming from $-|\partial W/\partial\tilde{t}_{\rm L}|^2$. Similarly, there will be (mass)^2 terms $m^2_{\rm b}$ for $\tilde{b}_{\rm L}$ and $\tilde{b}_{\rm R}$, and m^2_τ for $\tilde{\tau}_{\rm L}$ and $\tilde{\tau}_{\rm R}$, though these are probably negligible in this context.

We also need to consider derivatives of W with respect to the Higgs fields. For example, we have

$$-\left|\frac{\partial W}{\partial H^0_{\rm u}}\right|^2 = -|y_{\rm t}\tilde{t}^\dagger_{\rm R}\tilde{t}_{\rm L} - \mu H^0_{\rm d}|^2 \to -|y_{\rm t}\tilde{t}^\dagger_{\rm R}\tilde{t}_{\rm L} - \mu v_{\rm d}|^2 \tag{715}$$

after symmetry breaking. The expression (715) contains the off-diagonal bilinear term

$$\mu v_{\rm d}y_{\rm t}(\tilde{t}^\dagger_{\rm R}\tilde{t}_{\rm L} + \tilde{t}^\dagger_{\rm L}\tilde{t}_{\rm R}) = \mu m_{\rm t}\cot\beta(\tilde{t}^\dagger_{\rm R}\tilde{t}_{\rm L} + \tilde{t}^\dagger_{\rm L}\tilde{t}_{\rm R}) \tag{716}$$

which mixes the R and L fields. Similarly, $-|\partial W/\partial H_{\rm d}^0|^2$ contains the mixing terms

$$\mu m_{\rm b} \tan\beta(\tilde{b}_{\rm R}^\dagger \tilde{b}_{\rm L} + \tilde{b}_{\rm L}^\dagger \tilde{b}_{\rm R}) \tag{717}$$

and

$$\mu m_\tau \tan\beta(\tilde{\tau}_{\rm R}^\dagger \tilde{\tau}_{\rm L} + \tilde{\tau}_{\rm L}^\dagger \tilde{\tau}_{\rm R}). \tag{718}$$

Finally, bilinear terms can also arise directly from the soft triple scalar couplings (553), after the scalar Higgs fields acquire vevs. Assuming the conditions (558), and retaining only the third family contribution as before, the relevant terms from (553) are

$$-A_0 y_{\rm t} v_{\rm u}(\tilde{t}_{\rm R}^\dagger \tilde{t}_{\rm L} + \tilde{t}_{\rm L}^\dagger \tilde{t}_{\rm R}) = -A_0 m_{\rm t}(\tilde{t}_{\rm R}^\dagger \tilde{t}_{\rm L} + \tilde{t}_{\rm L}^\dagger \tilde{t}_{\rm R}), \tag{719}$$

together with similar $\tilde{b}_{\rm L} - \tilde{b}_{\rm L}$ and $\tilde{\tau}_{\rm R} - \tilde{\tau}_{\rm L}$ mixing terms.

Putting all this together, then, the $(\text{mass})^2$ values for the squarks and sleptons of the first two families are given by the expressions (700) - (704), together with the relevant contribution $\Delta_{\tilde{f}}$ of (710). For the third family, we discuss the $\tilde{\rm t}$, $\tilde{\rm b}$ and $\tilde{\tau}$ sectors separately. The $(\text{mass})^2$ term for the top squarks is

$$-(\tilde{t}_{\rm L}^\dagger \tilde{t}_{\rm R}^\dagger)\, \mathbf{M}_{\tilde{\rm t}}^2 \begin{pmatrix} \tilde{t}_{\rm L} \\ \tilde{t}_{\rm R} \end{pmatrix}, \tag{720}$$

where

$$\mathbf{M}_{\tilde{\rm t}}^2 = \begin{pmatrix} m_{\tilde{\rm t}_{\rm L},\tilde{\rm b}_{\rm L}}^2 + m_{\rm t}^2 + \Delta_{\tilde{\rm u}_{\rm L}} & m_{\rm t}(A_0 - \mu\cot\beta) \\ m_{\rm t}(A_0 - \mu\cot\beta) & m_{\tilde{\rm t}_{\rm R}}^2 + m_{\rm t}^2 + \Delta_{\tilde{\rm u}_{\rm R}} \end{pmatrix}, \tag{721}$$

with

$$\Delta_{\tilde{\rm u}_{\rm L}} = (\frac{1}{2} - \frac{2}{3}\sin^2\theta_{\rm W}) m_{\rm Z}^2 \cos 2\beta \tag{722}$$

and

$$\Delta_{\tilde{\rm u}_{\rm R}} = -\frac{2}{3}\sin^2\theta_{\rm W} m_{\rm Z}^2 \cos 2\beta. \tag{723}$$

Here $m_{\tilde{\rm t}_{\rm L},\tilde{\rm b}_{\rm L}}^2$ and $m_{\tilde{\rm t}_{\rm R}}^2$ are given approximately by (569) and (570) respectively. In contrast to the corresponding equations for the first two families, the $X_{\rm t}$ term is now present, and will tend to reduce the running masses of $\tilde{\rm t}_{\rm L}$ and $\tilde{\rm t}_{\rm R}$ at low scales (the second more than the first), relative to those of the corresponding states in the first two families; on the other hand, the $m_{\rm t}^2$ term tends to work in the other direction.

The real symmetric matrix $\mathbf{M}^2_{\tilde{t}}$ can be diagonalized by the orthogonal transformation

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{\tilde{t}} & \sin\theta_{\tilde{t}} \\ -\sin\theta_{\tilde{t}} & \cos\theta_{\tilde{t}} \end{pmatrix} \begin{pmatrix} \tilde{t}_{\rm L} \\ \tilde{t}_{\rm R} \end{pmatrix}; \tag{724}$$

the eigenvalues are denoted by $m^2_{\tilde{t}_1}$ and $m^2_{\tilde{t}_2}$, with $m^2_{\tilde{t}_1} < m^2_{\tilde{t}_2}$. Because of the large value of $m_{\rm t}$ in the off-diagonal positions in (721), mixing effects in the stop sector are likely to be substantial, and will probably result in the mass of the lighter stop, $m_{\tilde{t}_1}$, being significantly smaller than the mass of any other squark. Of course, the mixing effect must not become too large, or else $m^2_{\tilde{t}_1}$ is driven to negative values, which would imply (as in the electroweak Higgs case) a spontaneous breaking of colour symmetry. This requirement places a bound on the magnitude of the unknown parameter A_0, which cannot be much greater than $m_{\tilde{t}_{\rm L},\tilde{b}_{\rm L}}$.

At e^+e^- colliders the $\tilde{t}_1$ production cross section depends on the mixing angle $\theta_{\tilde{t}}$; for example, the contribution from Z exchange actually vanishes when $\cos^2\theta_{\tilde{t}} = \frac{4}{3}\sin^2\theta_{\rm W}$ [95]. In contrast, $\tilde{t}_1$'s are pair-produced in hadron colliders with no mixing-angle dependence. Which decay modes of the $\tilde{t}_1$ dominate depends on the masses of charginos and sleptons. For example, if $m_{\tilde{t}_1}$ lies below all chargino and slepton masses, then the dominant decay is

$$\tilde{t}_1 \to c + \tilde{\chi}^0_1, \tag{725}$$

which proceeds through loops (a FCNC transition). If $m_{\tilde{t}_1} > m_{\tilde{\chi}^\pm}$,

$$\tilde{t}_1 \to b + \tilde{\chi}^\pm \tag{726}$$

is the main mode, with $\tilde{\chi}^\pm$ then decaying to $l\nu\tilde{\chi}^0_1$. D0 reported on a search for such light stops [96]; their signal was two acollinear jets plus $\not{E}_{\rm T}$ (they did not attempt to identify flavour). Improved bounds on the mass of the lighter stop were obtained by CDF [97] using a vertex detector to tag c- and b-quark jets. More recent searches are reported in [98] and [99]. The bounds depend sensitively on the (assumed) mass of the neutralino $\tilde{\chi}^0_1$; data is presented in the form of excluded regions in a $m_{\tilde{\chi}^0_1} - m_{\tilde{t}_1}$ plot.

Turning now to the $\tilde{b}$ sector, the (mass)^2 matrix is

$$\mathbf{M}^2_{\tilde{b}} = \begin{pmatrix} m^2_{\tilde{t}_{\rm L},\tilde{b}_{\rm L}} + m^2_{\rm b} + \Delta_{\tilde{d}_{\rm L}} & m_{\rm b}(A_0 - \mu\tan\beta) \\ m_{\rm b}(A_0 - \mu\tan\beta) & m^2_{\tilde{b}_{\rm R}} + m^2_{\rm b} + \Delta_{\tilde{d}_{\rm R}} \end{pmatrix}, \tag{727}$$

with

$$\Delta_{\tilde{\rm d}_{\rm L}} = (-\frac{1}{2} + \frac{1}{3}\sin^2\theta_{\rm W})m_{\rm Z}^2\cos 2\beta \tag{728}$$

and

$$\Delta_{\tilde{\rm d}_{\rm R}} = \frac{1}{3}\sin^2\theta_{\rm W} m_{\rm Z}^2\cos 2\beta. \tag{729}$$

Here, since $X_{\rm t}$ enters into the evolution of the mass of $\tilde{\rm b}_{\rm L}$ but not of $\tilde{\rm b}_{\rm R}$, we expect that the running mass of $\tilde{\rm b}_{\rm R}$ will be much the same as those of $\tilde{\rm d}_{\rm R}$ and $\tilde{\rm s}_{\rm R}$, but that $m_{\tilde{\rm b}_{\rm L}}$ may be less than $m_{\tilde{\rm d}_{\rm L}}$ and $m_{\tilde{\rm s}_{\rm L}}$. Similarly, the (mass)^2 matrix in the $\tilde{\tau}$ sector is

$$\mathbf{M}^2_{\tilde{\tau}} = \begin{pmatrix} m^2_{\tilde{\nu}_{\tau\rm L},\tilde{\tau}_{\rm L}} + m_\tau^2 + \Delta_{\tilde{\rm e}_{\rm L}} & m_\tau(A_0 - \mu\tan\beta) \\ m_\tau(A_0 - \mu\tan\beta) & m^2_{\tilde{\tau}_{\rm R}} + m_\tau^2 + \Delta_{\tilde{\rm e}_{\rm R}} \end{pmatrix}, \tag{730}$$

with

$$\Delta_{\tilde{\rm e}_{\rm L}} = (-\frac{1}{2} + \sin^2\theta_{\rm W})m_{\rm Z}^2\cos 2\beta \tag{731}$$

and

$$\Delta_{\tilde{\rm e}_{\rm R}} = \frac{1}{3}\sin^2\theta_{\rm W} m_{\rm Z}^2\cos 2\beta. \tag{732}$$

Mixing effects in the $\tilde{\rm b}$ and $\tilde{\tau}$ sectors depend on how large $\tan\beta$ is (see the off-diagonal terms in (727) and (730)). It seems that for $\tan\beta$ less than about 5(?), mixing effects will not be large, so that the masses of $\tilde{\rm b}_{\rm R}, \tilde{\tau}_{\rm R}$ and $\tilde{\tau}_{\rm L}$ will all be approximately degenerate with the corresponding states in the first two families, while $\tilde{\rm b}_{\rm L}$ will be lighter than $\tilde{\rm d}_{\rm L}$ and $\tilde{\rm s}_{\rm L}$. For larger values of $\tan\beta$, strong mixing may take place, as in the stop sector. In this case, $\tilde{\rm b}_1$ and $\tilde{\tau}_1$ may be significantly lighter than their analogues in the first two families (also, $\tilde{\nu}_{\tau\rm L}$ may be lighter than $\tilde{\nu}_{\rm eL}$ and $\tilde{\nu}_{\mu\rm L}$). Neutralinos and charginos will then decay predominantly to taus and staus, which is more challenging experimentally than (for example) the dilepton signal from (699).

The search for a light $\tilde{\rm b}_1$ decaying to ${\rm b}+\tilde{\chi}_1^0$ is similar to that for $\tilde{\rm t}_1 \to {\rm c}+\tilde{\chi}_1^0$. D0 [100] tagged b-jets through semi-leptonic decays to muons. They observed 5 candidate events consistent with the final state ${\rm b}\bar{\rm b} + \not{E}_{\rm T}$, as compared to an estimated background of 6.0$\pm$ 1.3 events from ${\rm t}\bar{\rm t}$ and W and Z production; results were presented in the form of an excluded region in the $(m_{\tilde{\chi}_1^0}, m_{\tilde{\rm b}_1})$ plane. Improved bounds were obtained in the CDF experiment [97].

Searches for SUSY particles are reviewed by Schmitt in [34], including in particular searches at LEP, which we have not discussed. In rough terms, the present status is that there is 'little room for SUSY particles lighter than m_Z.' With all LEP data analysed, and if there is still no signal from the Tevatron collaborations, it will be left to the LHC to provide definitive tests.

We have given here only a first orientation to the SUSY particle spectrum. Feynman rules for the interactions of these particles with each other and with the particles of the SM are given in [21], [23] and [101]. Representative calculations of cross sections for sparticle production at hadron colliders may be found in [102]. Experimental methods for measuring superparticle masses and cross sections at the LHC are summarized in [104].

17.5 Benchmarks for SUSY Searches

Assuming degeneracy between the first two families of sfermions, there are 25 distinct masses for the undiscovered states of the MSSM: 7 squarks and sleptons in the first two families, 7 in the third family, 4 Higgs states, 4 neutralinos, 2 charginos and 1 gluino. Many details of the phenomenology to be expected (production cross sections, decay branching ratios) will obviously depend on the precise ordering of these masses. These in turn depend, in the general MSSM, on a very large number (over 100) of parameters characterizing the soft SUSY-breaking terms, as noted in section 15.2. Any kind of representative sampling of such a vast parameter space is clearly out of the question. On the other hand, in order (for example) to use simulations to assess the prospects for detecting and measuring these new particles at different accelerators, some consistent model must be adopted [103]. This is because, very often, a promising SUSY signal in one channel, which has a small SM background, actually turns out to have a large background from other SUSY production and decay processes. Faced with this situation, it seems necessary to reduce drastically the size of the parameter space, by adopting one of the more restricted models for SUSY breaking, such as the mSUGRA one. Such models typically have only three or four parameters; for instance, in mSUGRA they are, as we have seen, $m_0, m_{1/2}, A_0, \tan\beta$, and the sign of μ.

But even a sampling of a 3- or 4-dimensional parameter space, in order (say) to simulate experimental signatures within a detector, is beyond present capabilities. This is why such studies are performed only for certain specific points in parameter space, or in some cases along certain lines. Such parameter sets are called 'benchmark sets'.

Various choices of benchmark have been proposed. To a certain extent, which one is likely to be useful depends on what is being investigated. For example, the '$m_{\rm h}^{\rm max}$-scenario' [72] referred to in section 16.2 is suitable for setting conservative bounds on $\tan\beta$ and $m_{\rm A^0}$, on the basis of the non-observation of the lightest Higgs state. Another approach is to require that the benchmark points used for studying collider phenomenology should be compatible with various experimental constraints - for example [105] the LEP searches for SUSY particles and for the Higgs boson, the precisely measured value of the anomalous magnetic moment of the muon, the decay $\rm b \to s\gamma$, and (on the assumption that $\tilde{\chi}_1^0$ is the LSP) the relic density $\Omega_{\tilde{\chi}_1^0} h^2$. The authors of [105] worked within the mSUGRA model, taking $A_0 = 0$ and considering 13 benchmark points (subject to these constraints) in the space of parameters $(m_0, m_{1/2}, \tan\beta, \text{sign } \mu)$. A more recent study [106] updates the analysis in the light of the more precise dark matter bounds provided by the WMAP data.

One possible drawback with this approach is that minor modifications to the SUSY-breaking model might significantly alter the cosmological bounds, or the rate for $\rm b \to s\gamma$, while having little effect on the collider phenomenology; thus important regions of parameter space might be excluded prematurely. In any case, it is clearly desirable to formulate benchmarks for other possibilities for SUSY-breaking, in particular. The 'Snowmass Points and Slopes' (SPS) [107] are a set of benchmark points and lines in parameter space, which include seven mSUGRA-type scenarios, two gauge-mediated symmetry-breaking scenarios (it should be noted that here the LSP is the gravitino), and one anomaly-mediated symmetry-breaking scenario. Another study [108] concentrates on models which imply that at least some superpartners are light enough to be detectable at the Tevatron (for 2 $\rm fb^{-1}$ integrated luminosity); such models are apparently common among effective field theories derived from the weakly coupled heterotic string.

The last two references conveniently provide diagrams or Tables showing the SUSY particle spectrum (i.e. the 25 masses) for each of the benchmark points. They are, in fact, significantly different, and may themselves be regarded as the benchmarks, rather than the values of the high-scale parameters which led to them. If and when sparticles are discovered, their masses and other properties may provide a window into the physics of SUSY-breaking. However, as emphasized in section 9 of [23], there are in principle not enough observables at hadron colliders to determine all the 105 parameters of the soft SUSY breaking Lagrangian; for this, data from future $\rm e^+e^-$ colliders will be required. Then again, the MSSM may not be nature's choice.

Endnote
This is not a review. No serious attempt has been made to compile a representative list of references. The 100 or so which follow have simply come to hand. This is an order of magnitude less than the number of references included in the review [23], and two orders less than the number of papers on supersymmetry/SUSY/MSSM indicated by SPIRES.

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