Wavelet Bases in Few-Body Scattering Calculations

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Outline

- Background
- Scaling functions
- Multiscale analysis and wavelets
- Wavelet numerical analysis
- Applications

What are wavelets?

They are orthonormal basis functions that are used in data compression algorithms.

JPEG digital images are tables of expansion coefficients in a wavelet basis.

FBI fingerprint files are stored as expansion coefficients in a wavelet basis.

Our interest in wavelets?

Scattering problems in Poincaré invariant quantum mechanics can be formulated as integral equations with compact kernels.

Compact operators can be uniformly approximated by finite-dimensional matrices.

Large matrices can be uniformly approximated by sparse matrices in a wavelet basis.

Wavelet bases can be used to improve the efficiency of momentum-space scattering calculations.

General considerations

Wavelet bases are natural for problems (quantum field theory, phase transitions) where many scales are strongly coupled.

Basis functions are related to fixed points of a linear renormalization group equation.

Basis functions have a fractal structure, and are not amenable to standard numerical methods.

Elements of wavelet numerical analysis

Unitary operators, D and T on $L^2(\mathbb{R})$:

$$D(f)(x) := \sqrt{\frac{1}{2}} f(\frac{x}{2}) \qquad T(f)(x) := f(x-1)$$

Scaling equation:

$$\boxed{D(\phi)(x) = \sum_{l=0}^{2k-1} h_l T^l(\phi)(x).} \tag{1}$$

The scaling function, $\phi(x)$, is a fixed point of (1) with normalization

$$\int \phi(x) dx = 1.$$

h_l **are constant coefficients that determine the type of wavelet (we use Daubechies' wavelets).**

k **is a finite integer.**

A necessary condition for the existence of a fixed point of the scaling equation is:

$$\sum_{l=1}^{2k-1} h_l = \sqrt{2}.$$

Daubechies' scaling coefficients, $k = 1, 2, 3$

h_l	**k=1**	**k=2**	**k=3**
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Computing $\phi(x)$

Theorem: support$[\phi(x)] \in [0, 2k-1]$

Theorem: $k > 1 \quad \Rightarrow \quad \phi(x)$ **continuous.**

$$\phi(n) = \sqrt{2} \sum_{l=0}^{2k-1} \sqrt{2} h_l \phi(2n - l).$$

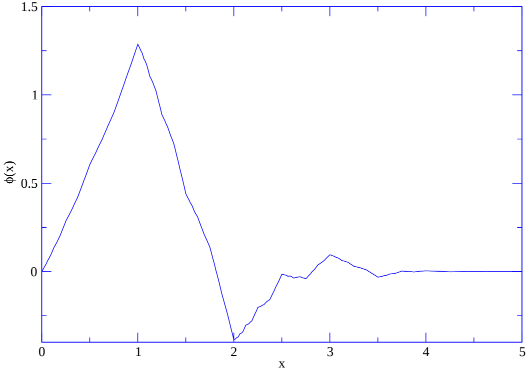
$$\sum_n \phi(n) = 1.$$

Solving above equations gives $\phi(n),\ n \in [0, 2k-1]$.

$$\phi(\frac{n}{2}) = \sum_{l=0}^{2k-1} \sqrt{2} h_l \phi(n - l).$$

Induction gives exact values at all dyadic rationals.

Daubechies' K=3 scaling function



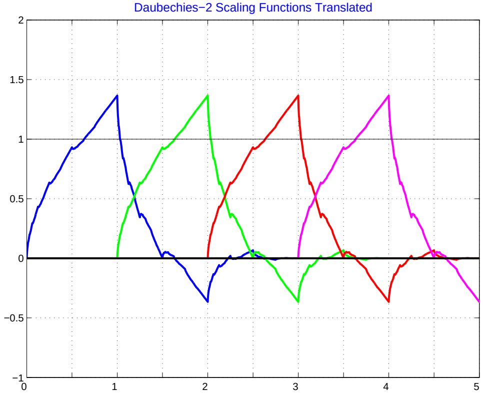
Approximation spaces

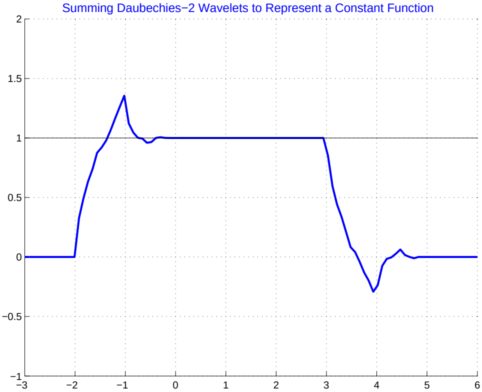
- $\phi_{mn}(x) := D^m T^n \phi(x)$.

- $(\phi_{mn}, \phi_{mk}) = \delta_{nk}$ **(fixed** m**)**.

- $\{\phi_{mn}\}_n$ **can locally pointwise represent polynomials of degree** $< k$ **for each** m.

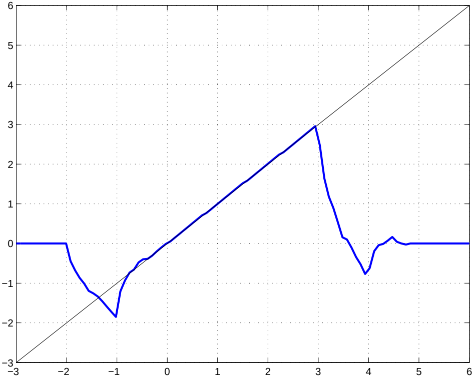
- $\mathcal{V}_m := \text{span}\{\phi_{mn}\}_n \cap L^2(\mathbb{R}) =$ **"scale** m **subspace" of** $L^2(\mathbb{R})$.


Daubechies−2 Scaling Functions Translated
2
1.5
1
0.5
0
−0.5
−1
0
1
2
3
4
5



Summing Daubechies−2 Wavelets to Represent a Constant Function
2
1.5
1
0.5
0
−0.5
−1
−3
−2
−1
0
1
2
3
4
5
6

Summing Daubechies−2 Wavelets to Represent a Linear Function



Multiscale analysis

$$m > n \quad \Rightarrow \quad \mathcal{V}_n \supset \mathcal{V}_m$$

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{V}_{n-1} \supset \mathcal{V}_n \supset \mathcal{V}_{n+1} \supset \cdots \supset \emptyset$$

Theorem: $\lim_{n\to-\infty} \mathcal{V}_n = L^2(\mathbb{R})$

$$\mathcal{V}_n = \mathcal{V}_{n+1} \oplus \mathcal{W}_{n+1}$$

$$\Downarrow$$

$$\mathcal{V}_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus \mathcal{V}_{n+m}$$

$$L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{W}_n = \mathcal{V}_m \oplus \left(\bigoplus_{n=-\infty}^{m} \mathcal{W}_n \right)$$

Wavelets

$\mathcal{W}_n$ **are wavelet spaces**

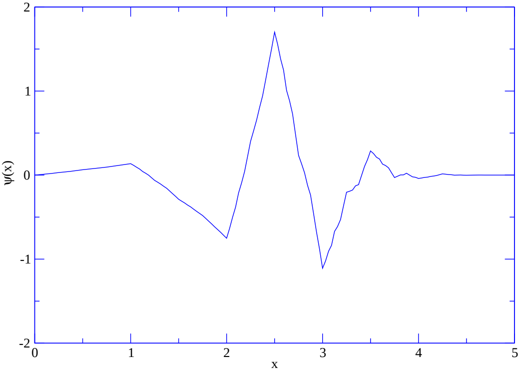
$$D\psi(x) = \sum_{l=0}^{2k-1} (-)^l h_{2k-l-1} T^l \phi(x)$$

$$\psi_{ml}(x) = D^m T^l \psi(x)$$

$\{\psi_{ml}\}_l$ **orthonormal basis for** $\mathcal{W}_m$

$\psi(x)$ **is called the "Mother" wavelet**

Daubechies' $k = 3$ mother wavelet



$$\textbf{support } [\psi(x)] = \textbf{support } [\phi(x)]$$

h_l **are determined up to space reflection by the requirements**

$$(\psi, x^n) = 0, \quad n = 0, \cdots, k-1 \qquad (\phi, T^m \phi) = \delta_{m0}$$

$$\Downarrow$$

$$\sum_m m^n (-)^m h_{l-m} = 0 \qquad \sum_{l=0}^{2k-1} h_{l-2m} h_l = \delta_{m0}$$

$$\sum_{l=1}^{2k-1} h_l = \sqrt{2}$$

Properties

$\phi_m(x)$ **can locally pointwise represent polynomials of degree** $< k$.

$\psi_{mn}(x)$ **are orthogonal to polynomials of degree** $< k$.

$$\mathcal{V}_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+l} \oplus \mathcal{V}_{n+l}$$

Approximations

Fix scale n of approximate solution

$$\mathcal{V}_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus \mathcal{V}_{n+m}$$

$$W : \mathcal{V}_n \rightarrow \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus \mathcal{V}_{n+m}$$

Wavelet transform W ($O(N)$ orthogonal transformation)

$$f(x) = \sum_m a_m \phi_{nm}(x) = \sum_m a'_m \phi_{n+l,m}(x) + \sum_m \sum_{k=1}^{l} b_{n+k,m} \psi_{n+k,m}(x)$$

Sparse matrices

$$a_n \underbrace{\leftrightarrow}_{W} a'_n, b_{nm}$$

b_{nm} small if $f(x)$ can be well approximated by degree $k-1$ polynomial on the support of $\psi_{nm}(x)$.

$$\textbf{supp}(\psi_{nm}(x)) \subseteq [2^n m, 2^n(m+2k-1)]$$

$$\mathcal{W}M\mathcal{W}^T = M_1 + M_2 \qquad \|M_2\| < \epsilon,\ M_1 \textbf{ sparse}$$

Wavelet numerical analysis

Moments

$$\langle x^0 \rangle_\phi := (x^0, \phi) = 1$$

$$\langle x^m \rangle_\phi := (x^m, \phi) = (Dx^m, D\phi) = 2^{-m-1/2} \sum_l h_l (x^m, T^l \phi) \quad \Rightarrow$$

$$\langle x^m \rangle_\phi = \frac{1}{\sqrt{2}} \frac{1}{2^m - 1} \sum_{l=0}^{2k-1} \sum_{n=0}^{m-1} h_l l^{m-n} \frac{m!}{n!(m-n)!} \langle x^n \rangle_\phi$$

recursion

$$\Downarrow$$

$$\langle x^m \rangle_\phi, \quad \langle x^n \rangle_{\phi_{lm}}, \quad \langle x^n \rangle_{\psi_{lm}} \quad \textbf{exact}$$

Low-degree polynomials

$$x^m = \sum_n c_n^m \phi_n(x)$$

exact for $m < k$

$$c_n^m = (T^n \phi, x^m) = \sum_{l=0}^{m} \frac{m! n^{m-l}}{l!(m-l)!} \langle x^l \rangle_\phi$$

$$c_n^0 = 1; \quad c_n^1 = n + \langle x^l \rangle_\phi, \quad \cdots$$

Smoothness

$$\phi(x) = \sqrt{2} \sum_{l=0}^{2k-1} h_l \phi(2x - l)$$

$$\Downarrow$$

$$\phi(n) = \sqrt{2} \sum_m H_{nm} \phi(m) \qquad H_{mn} := h_{2n-m}$$

$$\frac{d^l \phi}{dx^l}(n) = 2^l \sqrt{2} \sum_m H_{nm} \frac{d^l \phi}{dx^l}(m)$$

ϕ has l derivatives if H_{mn} has an eigenvalue $\lambda = 2^l \sqrt{2}$

Derivatives

$$x = \sum_n c_n^1 \phi_n(x) \Rightarrow 1 = \sum_n c_n^1 \frac{d\phi_n}{dx}(x) \qquad c_n^1 = n + \langle x^1 \rangle_\phi$$

$$D(\frac{d\phi}{dx})(x) = 2 \sum_{l=0}^{2k-1} h_l T^l (\frac{d\phi}{dx})(x).$$

$$\Downarrow$$

$$\frac{d\phi}{dx}(x) \qquad (\frac{d\phi}{dx}, \phi_n),$$

$$\frac{d\phi}{dx}(x) \approx \sum_n (\frac{d\phi}{dx}, \phi_n)\phi_n(x)$$

Nonlinearities

$$\phi_{n_2}(x)\phi_{n_2}(x) \approx \sum_{n_3} \Gamma^{n_3}_{n_1,n_2}\phi_{n_3}(x)$$

$$\Gamma^{n_3}_{n_1,n_2} = \int \phi_{n_1}(x)\phi_{n_2}(x)\phi_{n_3}(x)dx =$$

$$\sum_{l_1,l_2,l_3} \sqrt{2}h_{l_1}h_{l_2}h_{l_3}\Gamma^{2n_3+l_3}_{2n_1+l_1,2n_2+l_2}$$

$$\sum_{n_3} \Gamma^{n_3}_{n_1,n_2} = \delta_{n_1,n_2} \qquad \Gamma^{n_3}_{n_1,n_2} = \Gamma^{0}_{n_1-n_3,n_2-n_3}$$

$$\Downarrow$$

$$\Gamma^{n_3}_{n_1,n_2}, \cdots$$

Boundary integrals

$$B_m = \int_m^\infty \phi(x)dx = \frac{1}{2}\sum_{l=1}^{2k-1} h_l B_{2m-l}$$

$$B_m = 0 \qquad m \geq 2k-1 \qquad B_m = 1 \qquad m \leq 0$$

$$B^m = \int_{-\infty}^m \phi(x)dx = 1 - B_m$$

Poles

$$J_n := (\phi_n, \frac{1}{x - i0^+}) = (D\phi_n(x), D\frac{1}{x - i0^+}) =$$

$$\sqrt{2}\sum_{l=0}^{2k-1} h_l(\phi_{2n+l}, \frac{1}{x}) = \sqrt{2}\sum_{l=0}^{2k-1} h_l J_{2n+l}$$

large n

$$J_n = \frac{1}{n}\sum_{m=0}^{\infty} \frac{\langle x^m \rangle_\phi}{n^m}$$

$$i\pi = \int_{-m}^{m} \frac{dx}{x - i0^+} = \sum_n J_n + \textbf{boundary terms}$$

Logarithmic singularities

$$L_n := (\phi_n, \ln) = (D\phi_n, D\ln) =$$

$$\frac{1}{\sqrt{2}} \sum_{l=0}^{2k-1} h_l L_{2n-l} - \ln(2)$$

large n

$$L_n = \ln(n) - \sum_{m=1}^{\infty} (-1)^m \frac{\langle x^m \rangle_\phi}{m n^m}$$

One-point quadrature

$$k > 1 \Rightarrow \langle x^n \rangle_\phi = \langle x \rangle_\phi^n, \qquad n = 0, 1, 2$$

$$\Downarrow$$

$$\int \phi(x) P(x) dx \approx P(\langle x^n \rangle_\phi)$$

exact for $P(x)$ a polynomial of degree 2!

Moving singularities

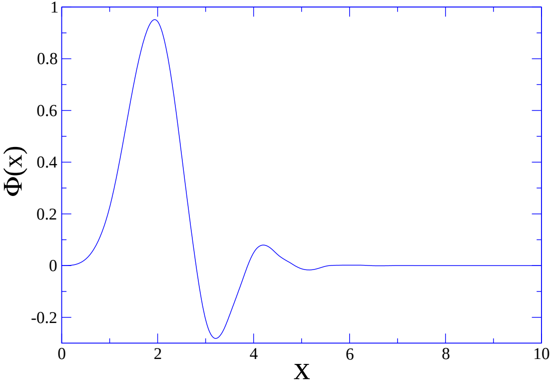
$$J_{mn} := \int \frac{\phi_m(x)\phi_n(y)}{k-x-y}dxdy =$$

$$\int \frac{\Phi(x)}{k-x-m-n}dx$$

$$\Phi(x) := \int \phi(y)\phi(x-y)dy$$

$$D\Phi(x) = \sum_l r_l T^l \Phi(x) \qquad \int \Phi(x)dx = 1$$

Autocorrelation function



$$f(x) = g(x) + \int \frac{K(x,y)}{y - i0^+} f(y) dy$$

$$f \approx \sum f_n \phi_{mn}(x) \qquad x_n = \langle x^1 \rangle_{\phi_{mn}} = 2^m (\langle x^1 \rangle_\phi + n) \qquad w = 2^{m/2}$$

$$f_l = g(x_l) + \sum_n \left(\frac{K(x_l, y_n) - K(x_l, 0)}{y_n} + K(x_l, 0) J_n \right) w f_n$$

$$\mathcal{W} \Rightarrow \text{solve} \Rightarrow \mathcal{W}^{-1} \Rightarrow$$

$$f(x) = g(x) + \sum_n \left(\frac{K(x, y_n) - K(x, 0)}{y_n} + K(x, 0) J_n \right) w f_n$$

$$J_n := 2^{m/2} \int dx \frac{\phi_{mn}(x)}{x - i0^+}$$

$$f(x) = g(x) + \int \frac{K(x,y)}{y - i0^+} f(y) dy$$

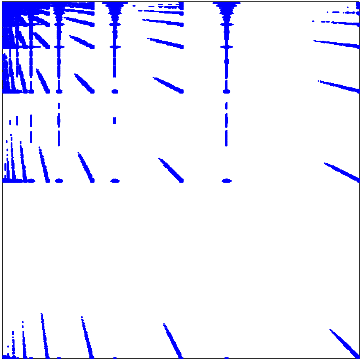
$$f \approx \sum f_n \phi_{mn}(x)$$

$$f_l = g(x_l) + \sum_{nk} K(x_l, y_n) \Gamma^l_{kn} J_k w f_n$$

$$\mathcal{W} \Rightarrow \text{solve} \Rightarrow \mathcal{W}^{-1} \Rightarrow$$

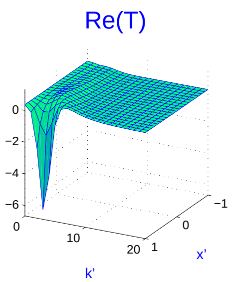
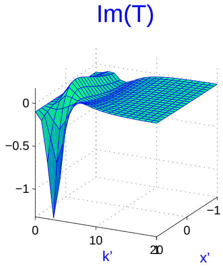
$$f(x) = g(x) + \sum_{nk} K(x, y_n) \Gamma^l_{kn} J_k w f_n$$

$$J_n := 2^{m/2} \int dx \frac{\phi_{mn}(x)}{x - i0^+}$$



K=3, E=10 MeV, J=-7

ϵ	percent	on-shell value	on-shell error	mean-square error
0	100	-125.00480	0	0
10^{-9}	17.78	-125.00480	1.05×10^{-8}	2.56×10^{-8}
10^{-8}	11.38	-125.00480	5.14×10^{-8}	2.44×10^{-7}
10^{-7}	6.6	-125.00475	4.49×10^{-7}	1.88×10^{-6}
10^{-6}	3.76	-125.00269	1.69×10^{-5}	2.08×10^{-5}
10^{-5}	2.14	-124.99030	.000116	.000228
10^{-4}	1.24	-124.85112	.00123	.00217
10^{-3}	.72	-123.82508	.00944	.0117
10^{-2}	.38	-125.25766	.00202	.128


Re(T)
0
−2
−4
−6
0
10
20
k'
−1
0
1
x'

Im(T)
0
−0.5
−1
0
10
20
k'
−1
0
1
x'

Quantum fields

$$\Phi(m,n) := \int \phi_{m,n}(x)\Phi(x)dx \qquad \Psi(m,n) := \int \psi_{m,n}(x)\Phi(x)dx$$

Operator products

$$\Phi(x) \approx \Phi(m,\vec{n})\phi_{m,\vec{n}}(x)$$

$$\Phi^2(x) \approx \sum_{n_3} \Phi(m,\vec{n}_1)\Phi(m,\vec{n}_2)\Gamma(m)^{\vec{n}_3}_{\vec{n}_1\vec{n}_2}\phi_{m,\vec{n}_3}(x)$$

Change of scale

$$\Phi(m,n) = W^{m1}_{nl}\Phi(m-1,l) + W^{m2}_{nl}\Psi(m-1,l)$$

- **Generalization of block-spin approximation. Has more smoothness.**
- **Scale invariance means that the dependence on the Ψ fields vanishes.**
- **Local operator products and derivatives of fields are described in terms of overlap integrals.**
- **Ψ fields describe physics lost on changing scales.**

Conclusion

- Wavelet bases have all of the advantages of spline bases with the additional properties:
  a. orthonormal basis.
  b. sparse matrix (even in momentum space).
  c. wavelet transform automatically finds structure.
  d. application to scattering with Malfliet-Tjon potential more accurate than what we can obtain with spline basis.
- Tested on two-body scattering using partial wave and direct three-dimensional integration.
- Interesting possibilities for discrete field theories.