A few aspects of bosonization in light front field theory

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Introduction

It is a remarkable property of two-dimensional field theory in the conventional formulation (which uses $x^\mu = (t, x)$ and quantization at a spacelike surface) that there is no sharp distiction between bosonic and fermionic field variables. Thus, there exists a kind of "duality" in the description of corresponding dynamics, as has been explicitly found by Coleman [1] and Mandelstam [2] for the case of sine-Gordon and Thirring models, and implicitly by Klaiber [3], Del'Antonio, Frishman and Zwanziger [4] and others. These results have later on been made more rigorous by Schroer and Truong [5] and Lehmann and Stehr [6]. For the case of the massive Schwinge model, bosonization was demonstrated by Coleman, Jackiw and Susskind [7, 8], Kogut and Susskind [9] and recently by Melnikov and Weinstein [10] (massless case).

The bosonization corresponding rules are summarized by

$$j^\mu(x) = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi(x),$$

$$j_5^{\mu}(x) = \frac{1}{\sqrt{\pi}} \partial_\nu \phi(x),$$

$$\overline{\psi}(x)\psi(x) = c : \cos 2\sqrt{\pi}\phi(x) :,$$

$$\overline{\psi}(x)\gamma^5\psi(x) = c : \sin 2\sqrt{\pi}\phi(x) :,$$

$$\psi_{L(R)}(x) = \sqrt{\frac{c\mu}{2\pi}} : \exp\big[-i\sqrt{\pi}\Big(\int_{-\infty}^{x} d\xi \dot{\phi}(\xi) \pm \phi(x) \Big) \big] : \qquad (1)$$

Some elements of bosonization have been studied in the LF formulation (Pauli and Elsner) but no systematic study was done.

Here:

An attempt to construct the fermion-boson correspondence in $D = 2$ within the Hamiltonian LF field theory of massive fields.

- Free fermions - continuum

- Free fermions - DLCQ
- Massive Schwinger model
- Massive Thirring model

Free fermions

In the following, we will sketch a simple light-front treatment of the bosonisation of a free massive LF fermi field in continuum formulation. In essence, the task is to find a suitable exponential operator of a scalar field which satisfies the correct anticommutator and has a fermionic form of the two-point correlation function. The construction is based on the presence of a Schwinger term in the commutator of the plus (dynamical) component of a free vector current. Let us study the continuum version of the free theory in a detail. The Lagrangian, Hamiltonian and field equations are

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^{\mu} \overleftrightarrow{\partial}_{\mu} \psi - m\overline{\psi}\psi \tag{2}$$

and its LF form is

$$\mathcal{L}_{lf} = i\psi_2^{\dagger} \overleftrightarrow{\partial}_{+} \psi_2 + i\psi_1^{\dagger} \overleftrightarrow{\partial}_{-} \psi_1 - m(\psi_2^{\dagger}\psi_1 + \psi_1^{\dagger}\psi_2). \tag{3}$$

We are using the chiral representation, where $\gamma^0 = \sigma^1, \gamma^1 = i\sigma^2, \gamma^5 = \gamma^0\gamma^1 = diag(-1,1)$ and $\psi_+^{\dagger} = (0, \psi_2^{\dagger}), \psi_-^{\dagger} = (\psi_1^{\dagger}, 0)$. The fermi field components are obtained by the projection $\psi_+ = \Lambda_+\psi, \psi_- = \Lambda_-\psi$, where $\Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm$. The LF Hamiltonian takes the form

$$P^- = m\int\limits_{-\infty}^{+\infty}\frac{dx^-}{2}\big[\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2\big]. \tag{4}$$

It vanishes for $m = 0$. The Dirac equation can be decomposed into two equations by means of the projectors $\Lambda_\pm$. ψ_2 satisfies the dynamical field equation while ψ_1 obeys a constraint:

$$2i\partial_+\psi_2 = m\psi_1, \quad 2i\partial_-\psi_1 = m\psi_2. \tag{5}$$

The solution of the latter is

$$\psi_1(x) = \frac{m}{4i}\int\limits_{-\infty}^{+\infty}\frac{dy^-}{2}\epsilon(x^- - y^-)\psi_2(x^+, y^-). \tag{6}$$

The field expansion of the independent component is

$$\psi_2(x^+,x^-) = \int_0^\infty \frac{dp^+}{4\pi\sqrt{p^+}} \left(b(p^+,x^+)e^{-\frac{i}{2}p^+x^-} + d^\dagger(p^+,x^+)e^{\frac{i}{2}p^+x^-} \right). \quad (7)$$

The solution of the constraint equation (6) for ψ_1 then becomes:

$$\psi_1(x^+,x^-) = m\int_0^\infty \frac{dp^+}{4\pi p^+\sqrt{p^+}} \left(b(p^+,x^+)e^{-\frac{i}{2}p^+x^-} - d^\dagger(p^+,x^+)e^{\frac{i}{2}p^+x^-} \right). \quad (8)$$

It is implicitly understood in both cases that small imaginary parts $\pm i\epsilon$ have been added to $x^\pm$ in the exponentials for the reasons of convergence (see later), The time dependence of the annihilation (creation) operators can easily by determined from a combination of both equations (5) as

$$b(p^+,x^+) = b(p^+,0)\exp(-\frac{i}{2}\hat{p}^-x^+), \quad \hat{p}^- = \frac{m^2}{p^+} \quad (9)$$

and similarly for $d(p^+, x^+)$. The non-zero anticommutation relations between the Fock operators are

$$\{b(p^+), b(p'^+\} = \{d(p^+), d(p'^+\} = 2\pi p^+ \delta(p^+ - p'^+). \tag{10}$$

This can be obtained by Dirac-Bergmann procedure or by a simple postulation.

We shall need the two-point correlation functions and anticommutators at unequal LF time for the both field components.

$$S_{22}(x) = \langle 0|\psi_2(0,0)\psi_2^\dagger(x^+, x^-)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi} e^{\frac{i}{2}p^+(x^- + i\epsilon) + \frac{i}{2}\frac{m^2}{p^+}(x^+ + i\delta)}. \tag{11}$$

One has to distinguish four combinations of signs of x^+, x^-, leading to space-like or time-like $x^2 = x^+x^-$. The integral is then found to be

$$S_{22}(x) = -\frac{m}{8}|\mathrm{sgn}(x^+) + \mathrm{sgn}(x^-)|\sqrt{\frac{x^+}{x^-}} i\Big[J_1(m\sqrt{x^2}) + i\mathrm{sgn}(x^+)N_1(m\sqrt{x^2})\Big] -$$

$$-\frac{im}{4\pi}\Big(\mathrm{sgn}(x^{+})-\mathrm{sgn}(x^{-})\Big)K(x^{+},x^{-}), \tag{12}$$

where K is equal to

$$K(x^{+},x^{-})=\sqrt{\frac{x^{+}+i\epsilon}{x^{-}-i\epsilon}}K_1(m\sqrt{(x^{+}+i\epsilon)(x^{-}-i\epsilon)}\ \text{ for }\ x^{+}>0,x^{-}<0 \tag{13}$$

or to a complex conjugate expression if $x^{+}<0,x^{+}>0$. $J_1(z)$($K_1(z)$) is the (modified) Bessel function of the order 1.

As the next step, let us consider the commutation relation between two dynamical current components

$$j^{+}(x)=2:\psi_2^{\dagger}(x)\psi_2(x):. \tag{14}$$

It is convenient first to Fourier transform the current,

$$j^{+}(x^{-})=\int\limits_0^\infty\frac{dp^{+}}{4\pi}[A(p^{+})e^{-\frac{i}{2}p^{+}x^{-}}+A^{\dagger}(p^{+})e^{\frac{i}{2}p^{+}x^{-}}], \tag{15}$$

where the composite operator $A(k^+)$ is given by

$$A(k^+) = \int\limits_{-\infty}^{+\infty} \frac{dx^-}{2} j^+(x^-) e^{\frac{i}{2}k^+x^-} = \int\limits_0^{k^+} \frac{dp^+}{4\pi p^+} d(p^+) b(k^+ - p^+) +$$

$$+ \int\limits_0^{\infty} \frac{dp^+}{4\pi p^+} \left[b^\dagger(p^+) b(p^+ + k^+) - d^\dagger(p^+) d(p^+ + k^+) \right]. \qquad (16)$$

By a direct calculation one finds

$$[A(p^+), A^\dagger(p'^+)] = 2p^+ \delta(p'^+ - p^+), \quad [A(p^+),\ A(p'^+)] = 0 \qquad (17)$$

and as a consequence also

$$[j^+(x^-), j^+(y^-)] = \frac{i}{\pi} \partial_-^x (x^- - y^-). \qquad (18)$$

The latter relation is the basis for deriving the bosonized version of the fermi field operator ψ_2. Indeed, consider the exponential operator

$$\Phi(x^-) = c_1 e^{-ic_0\phi(x^-)} = c_1 c_2 : e^{-ic_0\phi(x^-)} : \tag{19}$$

where the constant c_2 coming from the normal ordering will be found below. The scalar field ϕ given by

$$\phi(x^-) = \sqrt{\pi} \int\limits_{-\infty}^{+\infty} \frac{dy^-}{2} \frac{1}{2} \epsilon(x^- - y^-) j^+(y^-). \tag{20}$$

The constants c_0 and c_1 are left as free parameters for the moment. The operator ϕ satisfies the commutation relation for the canonical scalar LF field:

$$[\phi(0, x^-), \phi(0, y^-)] = -\frac{i}{8} \epsilon(x^- - y^-), \tag{21}$$

The crucial point for deriving the above commutator is the Schwinger term in the current-current commutator. Inserting the momentum representation of the antisymmetric sign function

$$\epsilon(x^- - y^-) = \frac{2i}{\pi}\int_0^\infty \frac{dp^+}{2}\frac{1}{p^+}\left[e^{-\frac{i}{2}p^+(x^- - i\epsilon)} - e^{\frac{i}{2}p^+(x^- + i\epsilon)}\right] \tag{22}$$

into Eq.(20) one finds

$$\phi(x^-) = \frac{i}{4\sqrt{\pi}}\int_0^\infty dq^+ \frac{1}{q^+}\left[A(q^+)e^{-\frac{i}{2}q^+x^-} - A^\dagger(q^+)e^{\frac{i}{2}q^+x^-}\right] \tag{23}$$

and

$$\Phi(x^-) = c_1 c_2 \exp[-\hat{A}^\dagger(x^-)]\exp[\hat{A}(x^-)] \tag{24}$$

with

$$\hat{A}(x^-) = \frac{c_0}{4\sqrt{\pi}}\int_0^\infty dq^+ \frac{A(q^+)}{q^+}e^{-\frac{i}{2}q^+x^-}. \tag{25}$$

Can the operator $\Phi(x^-)$ satisfy anticommutation relation ? It turns out that the constant c_0 may be chosen in such a way that it can. Indeed, use the Baker-

Campbell-Hausdorff formula $\exp(A)\exp(B) = \exp[A,B]\exp(B)\exp(A)$ in the product $\Phi(x^-)\Phi(y^-)$ to interchange the order of the two operators. One gets

$$\begin{aligned}
&\Phi(x^-)\Phi(y^-) = c_1^2 c_2^2 e^{-\hat{A}^\dagger(x^-)} e^{\hat{A}(x^-)} e^{-\hat{A}^\dagger(y^-)} e^{\hat{A}(y^-)} = \\
&= c_1^2 c_2^2 \exp\big\{-[\hat{A}(x^-), \hat{A}^\dagger(y^-)]\big\} \exp\big\{-[\hat{A}^\dagger(x^-), \hat{A}(y^-)]\big\} \times \\
&\times e^{-\hat{A}^\dagger(y^-)} e^{\hat{A}(y^-)} e^{-\hat{A}^\dagger(x^-)} e^{\hat{A}(x^-)} = \\
&\quad = \exp[-\frac{c_0^2}{4\pi}\int_0^\infty \frac{dq^+}{q^+} e^{-\frac{i}{2}q^+(x^- - y^-)}] \exp[\frac{c_0^2}{4\pi}\int_0^\infty \frac{dq^+}{q^+} e^{\frac{i}{2}q^+(x^- - y^-)}] \Phi(y^-)\Phi(x^-).
\end{aligned} \tag{26}$$

Taking into account the definition of the sign function (22) and choosing $c_0 = 2\sqrt{\pi}$, the two exponentials in the last line of the above equations combine to $\exp[i\pi\epsilon(x^- - y^-)] = -1$ yielding the desired anticommutation property.

The situation is slightly more complicated for the anticommutor between $\Phi(x)$ and $\Phi^\dagger(y)$ which we should find to be proportional to $\delta(x^- - y^-)$. We remind here the form of the two-point function of the massive LF scalar field $\tilde{\phi}(x)$ of a mass μ that will be needed. The field expansion is

$$\tilde{\phi}(x^-) = \int_0^\infty \frac{dq^+}{4\pi q^+}\left[a(q^+)e^{-\frac{i}{2}q^+x^- - \frac{i}{2}\frac{\mu^2}{q^+}x^+} + a^\dagger(q^+)e^{\frac{i}{2}q^+x^- + \frac{i}{2}\frac{\mu^2}{q^+}x^+}\right], \tag{27}$$

where small imaginary parts for $x^\pm$ are understood. Using the Fock commutation relation $[a(p^+), a^\dagger(p'^+)] = 4\pi p^+\delta(p^+ - p'^+)$, one straightforwardly finds for the two-point function

$$D(x) = \langle 0|\tilde{\phi}(0,0)\tilde{\phi}(x^+,x^-)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi p^+} e^{\frac{i}{2}p^+(x^- + i\epsilon) + \frac{i}{2}\frac{\mu^2}{p^+}(x^+ + i\epsilon)}. \tag{28}$$

In analogy with the fermionic calculation, the integral is

$$D(x) = -\frac{i}{8}|\mathrm{sgn}(x^+) + \mathrm{sgn}(x^-)|\Big(N_0(\mu\sqrt{x^2}) - i\mathrm{sgn}(x^+)J_0(\mu\sqrt{x^2})\Big) + \\ + \frac{1}{4\pi}|\mathrm{sgn}(x^+) - \mathrm{sgn}(x^-)|\tilde{K}(x^+,x^-), \tag{29}$$

where $\tilde{K}$ is equal to

$$\tilde{K}(x^+, x^-) = K_0(\mu\sqrt{(x^+ - i\epsilon)(x^- + i\epsilon)}, \quad \text{if} \quad x^+ > 0, x^- < 0 \tag{30}$$

or to a complex conjugate expression if $x^+ < 0, x^- > 0$. $J_0(z)$ $(K_0(z)$ is the modified Bessel function of the order 0. This expression is useful because the 2-point correlation function of $\phi(x)$ is

$$\langle 0|\phi(0,0)\phi(x^+, x^-)|0\rangle = \frac{1}{2}D(x). \tag{31}$$

The calculation of the anticommutator in the bosonized form requires a small separation in the LF time between the two operators Φ. As in the previous calculation, we get, using the BCH formulae,

$$\begin{aligned}\Phi(x^+, x^-)\Phi^\dagger(0,0) &= \exp\left[\hat{A}(x^+, x^-), \hat{A}^\dagger(0,0)\right] : \Phi(x^+, x^-)\Phi^\dagger(0,0) :, \\ \Phi^\dagger(0,0)\Phi(x^+, x^-) &= \exp\left[\hat{A}(0,0), \hat{A}^\dagger(x^+, x^-)\right] : \Phi^\dagger(0,0)\Phi(x^+, x^-) : \end{aligned} \tag{32}$$

It follows that

$$\Phi(x^+,x^-)\Phi^\dagger(0,0)+\Phi^\dagger(0,0)\Phi(x^+,x^-)=:\Phi(x^+,x^-)\Phi^\dagger(0,0):\times$$

$$\left[\exp\left(\int\limits_0^\infty \frac{dk^+}{k^+}e^{-\frac{i}{2}k^+x^- -\frac{i}{2}\frac{\mu^2}{k^+}x^+}\right)+\exp\left(\int\limits_0^\infty \frac{dk^+}{k^+}e^{\frac{i}{2}k^+x^- +\frac{i}{2}\frac{\mu^2}{k^+}x^+}\right)\right]. \quad (33)$$

From Eqs.(28) and (30) we see that the integrals $\hat{E}_1$, $\hat{E}_1^*$ in the exponentials of the latter expression are related to the function $D(x)$. For $x^2<0$ we have

$$\hat{E}_1 = 2K_0\left(\mu\sqrt{(x^+-i\epsilon)(x^-+i\epsilon)}\right). \quad (34)$$

Taking into account the form of the Bessel function $K_0(z)$ for small z, $K_0(z)\approx -\gamma_E - ln(\frac{z}{2})+O(z^2)$, where γ_E is the Euler's constant, we find

$$\left\{\Phi(x^+,x^-),\Phi^\dagger(0,0)\right\}=e^{-2\gamma_E}\frac{4}{\mu^2}:\Phi(x^+,x^-)\Phi^\dagger(0,0):\times$$

$$\times \left[\frac{1}{(x^+ - i\epsilon)(x^- + i\epsilon)} + + \frac{1}{(x^+ + i\epsilon)(x^- - i\epsilon)} \right] \tag{35}$$

which for $x^+ = 0$ reduces to the equal-time anticommutator

$$\left\{ \Phi(0, x^-), \Phi^\dagger(0,0) \right\} = c_2^2 e^{-2\gamma_E} \frac{8\pi}{\epsilon\mu^2} \delta(x^-), \tag{36}$$

because : $\Phi(0,0)\Phi^\dagger(0,0)$:= 1 and we have used the relation $(x^- + i\epsilon)^{-1} - (x^- - i\epsilon)^{-1} = -2i\pi\delta(x^-)$. The normalization constant c_1 can now be determined from the requirement that the anticommutator (36) is equal to $1/2\delta(x^-)$. One finds $c_1 = \mu\sqrt{\epsilon}\frac{e^{\gamma_E}}{4\sqrt{\pi}}$. Since the constant c_2 was

$$\exp\Big(-\frac{1}{2}[\hat{A}(x), \hat{A}^\dagger(x)]\Big) = \exp\Big(-\frac{1}{2}K_0(\mu\epsilon)\Big) = \sqrt{\frac{\mu\epsilon}{2}} \tag{37}$$

the final expression for the bosonized dynamical Fermi field component is

$$\Phi(x) = C : e^{-i2\sqrt{\pi}\phi(x)} :$$

$$C = c_1 c_2 = e^{\gamma_E}\sqrt{\frac{\mu}{8\pi}}. \tag{38}$$

It is interesting to note that a small imaginary part of the time argument of the bosonized fermion field was used also by Mandelstam:

$$\psi_L(x) = (c\mu/2\pi)^{1/2} e^{\mu/8\epsilon} : \exp\big[-2i\pi \int_{-\infty}^{x} d\xi \dot\phi(\xi) - \frac{i}{2}\beta\phi(x)\big] : \tag{39}$$

has been found in the original Mandelstam's paper [2] starting from the commutation relation

$$\big[\phi^+(x, t+dt), \phi^-(y,t)\big] = \Delta_+\big((x-y)^2 - (dt+i\epsilon)^2\big), \tag{40}$$

where

$$\Delta_+ = -\frac{1}{4\pi}\ln\big\{c^2\mu^2\big[x^2 - (dt+i\epsilon)^2\big]\big\} + O(x^2). \tag{41}$$

μ is a mass parameter and β defines the interacting term $\cos\big(\beta\phi(x)\big)$ of the sine-Gordon model.

The last two properties we have to understand is if one really has $j^- = -\partial_+\phi$ and if one can write the mass term as something proportional to $\cos(\phi)$.

From Eq.(20) we easily get, using partial integration

$$j^+(x) = \frac{2}{\sqrt{\pi}}\partial_-\phi(x). \tag{42}$$

In a similar way, assuming the vector-current conservation $\partial_+ j^+(x) + \partial_- j^-(x) = 0$, we find

$$\partial_+\phi(x) = -\sqrt{\pi}\int_{-\infty}^{+\infty}\frac{dy^-}{2}\frac{1}{2}\epsilon(x^- - y^-)\partial_- j^-(x^+, y^-) = -\frac{\sqrt{\pi}}{2}j^-(x^-). \tag{43}$$

The Eqs.(42,43) are summarized in the familiar statement

$$j^\mu(x) = \frac{\epsilon^{\mu\nu}}{\sqrt{\pi}}\partial_\nu\phi(x). \tag{44}$$

To derive the bosonized form of the mass term, one has to take into account the fact that the second component of the Fermi field satisfies an equation of constraint in the LF formulation. This implies

$$\chi(x) = \frac{m}{4i} \int\limits_{-\infty}^{+\infty} \frac{dy^-}{2} \epsilon(x^- - y^-) \Phi(x^+, y^-) \tag{45}$$

The free fermionic Hamiltonian becomes

$$P^- = \frac{m^2}{4i} \int\limits_{-\infty}^{+\infty} \frac{dx^-}{2} \int\limits_{-\infty}^{+\infty} \frac{dy^-}{2} \epsilon(x^- - y^-) \Big[\Phi^\dagger(x) \Phi(x^+, y^-) - \Phi^\dagger(x^+, y^-) \Phi(x) \Big] \tag{46}$$

or

$$P^- = \frac{m^2}{4i} C^2 \int\limits_{-\infty}^{+\infty} \frac{dx^-}{2} \int\limits_{-\infty}^{+\infty} \frac{dy^-}{2} \epsilon(x^- - y^-) \Big[: e^{i\sqrt{\pi}\phi(x^-)} : \ : e^{-i\sqrt{\pi}\phi(y^-)} : - \\ : e^{i\sqrt{\pi}\phi(y^-)} : \ : e^{-i\sqrt{\pi}\phi(x^-)} : \Big] \tag{47}$$

It is instructive to compare this result with the derivation of the bosonized form of the mass in the conventional field theory. In one (a little heuristic) approach (Ilieva and Pervushin) one uses the commutators between pseudoscalar density J_5 and the axial current density j_5^0:

$$\begin{aligned}
&J_5(x) = \psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L = F_+\big(\phi(x)\big) - F_-\big(\phi(x)\big), \\
&j_0^5 = \psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R, \\
&\Big[j_5^0(x), F_+\big(\phi(y)\big)\Big] = 2\delta(x-y) F_+\big(\phi(y)\big), \\
&\Big[j_5^0(x), F_-\big(\phi(y)\big)\Big] = -2\delta(x-y) F_-\big(\phi(y)\big).
\end{aligned} \tag{48}$$

The Fermi field $\psi(x)$ has the upper and lower components ψ_R and ψ_L and satisfies the canonical anticommutation relation. Since from the current bosonization one has $j_5^0(x) = \frac{1}{\sqrt{\pi}}\partial_0\phi(x) = \frac{1}{\sqrt{\pi}}\Pi_\phi(x)$, the two commutators can be rewritten as

$$\frac{1}{\sqrt{\pi}}\Big[\Pi_\phi(x), F_\pm\Big(\phi(y)\Big)\Big] = \pm 2\delta(x-y) F_\pm\big(\phi(y)\big). \tag{49}$$

The corresponding solution is

$$F_{\pm}\big(\phi(x)\big) = c\exp\big\{\pm 2i\sqrt{\pi}\phi(x)\big\} \tag{50}$$

so that $J(x) = c\cos 2\sqrt{\pi}\phi(x)$ (c is a constant). Note that this result is based on the canonical commutator of the scalar field

$$\Big[\phi(x), \Pi_{\phi}(y)\Big] = i\delta(x - y) \tag{51}$$

and cannot be used in the LF theory where $\Pi \neq \partial_{+}\phi$. Thus one can expect a modification of the bosonized mass term in the LF formulation.

Discrete formulation

A similar contruction can be derived in the discretized (infrared-regularized) formulation. The corresponding starting expressions read

$$\psi_2(0,x^-) = \frac{1}{\sqrt{2L}} \sum_{n=\frac{1}{2}}^{\infty} \left[b_n e^{-\frac{i}{2}p_n^+ x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} + d_n^\dagger e^{\frac{i}{2}p_n^+ x^- + \frac{i}{2}\frac{m^2}{p^+}x^+} \right], \tag{52}$$

$$j^+(x) = \frac{1}{L}\left[A_0 + \sum_{m=1}^{\infty} \left(A_m e^{-\frac{i}{2}p_m^+ x^-} - A_m^\dagger e^{\frac{i}{2}p_m^+ x^-} \right) \right], \tag{53}$$

where $A_0 = Q$. By a direct calculation one finds

$$\left[A_m, A_n^\dagger \right] = m\delta_{m,n} \tag{54}$$

and from (53) also the current-current commutator with the Schwinger term

$$\left[j^+(x^-), j^+(y^-)\right] = \frac{1}{\pi}\partial_-^x \delta_n(x^- - y^-). \tag{55}$$

From the definition of the scalar field $\phi(x)$ we find

$$\phi(x) = \frac{x^-}{2L}Q + \sum_{m=1}^{\infty} \frac{1}{p_m^+}\left[A_m\left(e^{-\frac{i}{2}p_m^+ x^-} - (-1)^m\right) - A_m^\dagger\left(e^{\frac{i}{2}p_m^+ x^-} - (-1)^m\right)\right],$$

$$A_m = \sum_{k=\frac{1}{2}}^{\infty}\left(b_k^\dagger b_{k+m} - d_k^\dagger d_{k+m}\right) + \sum_{k=\frac{1}{2}}^{m-\frac{1}{2}} d_{m-k} b_k. \tag{56}$$

To show that the dynamical Fermi field component $\psi_2(x)$ has a representation in terms of a bosonic field, one can proceed as follows. First, calculate the commutator

$$\left[A_m, \psi_2(x)\right] = -e^{\frac{i}{2}p_m^+ x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\psi_2(x), \tag{57}$$

where a few nontrivial cancellations between different terms occured. Since $A_m|0\rangle = 0$, we immediately get that the state $\psi_2(x)|0\rangle$ is an eigenstate of the annihilation operator A_m:

$$A_m\psi_2(x)|0\rangle = -e^{\frac{i}{2}p_m^+x^- + \frac{i}{2}\frac{m^2}{p^+}x^+}\psi_2(x)|0\rangle \tag{58}$$

and hence it is a boson coherent state. Its form has to be

$$\psi_2(x)|0\rangle = \mathcal{N}\exp\Big\{-\sum_{m=1}^{\infty}\frac{A_m^\dagger}{m}e^{\frac{i}{2}p_m^+x^-}\Big\}|0\rangle. \tag{59}$$

or

$$\psi_2(x)|0\rangle = c'\exp\Big\{-\sum_{n=1}^{\infty}\frac{1}{n}\Big(A_n^\dagger e^{\frac{i}{2}p_n^+x^- + \frac{i}{2}\frac{m^2}{p_n^+}x^+} - A_n e^{-\frac{i}{2}p_n^+x^- - \frac{i}{2}\frac{m^2}{p_n^+}x^+}\Big)\Big\}|0\rangle. \tag{60}$$

In the next step, one can show that the same relation holds for an arbitrary Fock state generated by $A_n^\dagger$. So the bosonization correspondence holds as an operator relationship.

Interacting models

Let us try to generalize the above considerations to the case of an interacting model. First, consider the light-front massive Schwinger model in the continuum formulation and in the LC gauge $A^+ = 0$. The corresponding Lagrangian is

$$\mathcal{L}_{lf} = i\psi_2^\dagger \overleftrightarrow{\partial}_+ \psi_2 + i\psi_1^\dagger \overleftrightarrow{\partial}_- \psi_1 - m(\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2) + \frac{1}{2}(\partial_- A^-)^2 - \frac{e}{2}j^+ A^-. \quad (61)$$

Using the constraints

$$2i\partial_-\psi_1(x) = m\psi_2(x), \quad \partial_-^2 A^-(x) = -\frac{e}{2}j^+(x) \quad (62)$$

one gets the Hamiltonian in terms of dynamical variable $\psi_2(x)$ only ($x^+ = y^+$):

$$P^- = \int\limits_{-\infty}^{+\infty}\frac{dx^-}{2}\Big[-\frac{e^2}{4}j^+\frac{1}{\partial_-^2}j^+ + \frac{m^2}{2i}\int\limits_{-\infty}^{+\infty}\frac{dy^-}{2}\epsilon(x^- - y^-)\Big(\psi_2^\dagger(x)\psi_2(y) - \psi_2^\dagger(y)\psi_2(x)\Big)\Big]. \quad (63)$$

Expressing now the field ψ_2 and the current j^+ by their bosonic forms, we find

$$P^- = \int\limits_{-\infty}^{+\infty} \frac{dx^-}{2} \Big[\frac{e^2}{\pi}\phi^2 + \frac{m^2}{2i} C^2 \int\limits_{-\infty}^{+\infty} \frac{dy^-}{2} \epsilon(x^- - y^-) \Big(: e^{2i\sqrt{\pi}\phi(x^-)} :: e^{-2i\sqrt{\pi}\phi(y^-)} : -$$

$$- : e^{2i\sqrt{\pi}\phi(y^-)} :: e^{-2i\sqrt{\pi}\phi(x^-)} : \Big) \Big]. \tag{64}$$

Can the Hamiltonian be re-normal ordered with respect to the new mas $\frac{e^2}{\pi}$?

Another possibility is to consider the massive Schwinger model in the Weyl gauge $A^- = 0$. The corresponding LF Hamiltonian is

$$P^- = \int\limits_{-\infty}^{+\infty} \frac{dx^-}{2} \left[\Pi^2_{A^+} + m \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) \right]. \tag{65}$$

The dependent component ψ_1 in (65) satisfies the constraint

$$2i\partial_- \psi_1(x) = m\psi_2(x) + e\psi_1 A^+(x), \tag{66}$$

which has a similar stucture than the fermionic constraint in the Thirring model (see below). It can be inverted by means of the Green's function $G(z^-; A^+)$:

$$\psi_1(x) = m \int_{-\infty}^{+\infty} \frac{dy^-}{2} G(x^- - y^-; A^+)\psi_2(x^+, y^-),$$
$$G(x^- - y^-; A^+) = \frac{1}{2i}\epsilon_a(x^- - y^-)e^{-ie\vartheta(x^-)+ie\vartheta(y^-)}$$
$$\vartheta(x^-) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{2}\epsilon(x^- - z^-)A^+(x^+, z^-). \tag{67}$$

A necessary condition in this formulation is to satisfy the Gauss' law as a condition on states:

$$\langle phys|G(x^-)|phys\rangle, \;\; G(x^-) = 2\partial_-\Pi_{A^+}(x^-) - ej^+(x^-). \tag{68}$$

Another model studied in connection with bosonization in conventional field theory is the massive Thirring model which was found to be equivalent to the sine-Gordon model by Coleman, Mandelstam, Schroer and others. It is defined by the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\overline{\psi}\psi - \frac{1}{2}g j_\mu j^\mu \tag{69}$$

where $j^\mu = \overline{\psi}\gamma^\mu\psi$. The corresponding LF expressions are

$$\begin{aligned} \mathcal{L}_{lf} &= i\psi_2^\dagger \overleftrightarrow{\partial}_+ \psi_2 + i\psi_1^\dagger \overleftrightarrow{\partial}_- \psi_1 - m\big(\psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2\big) - \frac{1}{2}g j^+ j^-, \\ j^+ &= 2 : \psi_2^\dagger\psi_2 :, \;\; j^- = 2 : \psi_1^\dagger\psi_1 : . \end{aligned} \tag{70}$$

The Euler-Lagrange equations read

$$\begin{aligned} 2i\partial_+\psi_2 &= m\psi_1 + g j^-\psi_2, \\ 2i\partial_-\psi_1 &= m\psi_2 + g j^+\psi_1. \end{aligned} \tag{71}$$

The latter equation is a constraint. It can be used to bring the LF Hamiltonian to the form

$$P^- = m \int\limits_{-\infty}^{+\infty} \frac{dx^-}{2} [\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2] \tag{72}$$

Note that the interaction term $j_\mu j^\mu$ disappeared from P^- and the interaction is contained solely in the $\psi_1(x)$ which is the solution of the constraint:

$$\psi_1(x) = \frac{m}{2i} \int\limits_{-\infty}^{+\infty} \frac{dy^-}{2} \epsilon(x^- - y^-) \exp\left\{\frac{ig}{2} \int\limits_{x^-}^{y^-} dz^- j^+(z^-)\right\} \psi_2(x^+, y^-). \tag{73}$$

Defining the Green's function G

$$G(x^- - y^-) = \frac{1}{2i} \epsilon(x^- - y^-) \exp\left\{ - ig\phi(x^-) + ig\phi(y^-)\right\},$$

$$\phi(x^-) = \sqrt{\pi} \int\limits_{-\infty}^{+\infty} \frac{dz^-}{2} \frac{1}{2} \epsilon(x^- - z^-) j^+(z^-), \tag{74}$$

the dependent Fermi field component can be written as

$$\psi_1(x) = m \int_{-\infty}^{+\infty} \frac{dy^-}{2} G(x^- - y^-)\psi_2(x^+, y^-). \tag{75}$$

Inserting the bosonic representation of $\psi_2(x)$ into P^-, we get

$$P^- = \frac{m^2}{2i} C^2 \int_{-\infty}^{+\infty} \frac{dx^-}{2} \int_{-\infty}^{+\infty} \frac{dy^-}{2} \epsilon(x^- - y^-) \Big(: e^{2i\sqrt{\pi}\phi(x^-)} : e^{-\frac{ig}{2}\phi(x^-) + \frac{ig}{2}\phi(y^-)} \times$$

$$\times : e^{-2i\sqrt{\pi}\phi(y^-)} : -h.c.\Big) \tag{76}$$

This LF Hamiltonian can be further treated by using operator identities.

Conclusions

- A strategy to bosonization in the LF Hamiltonian framework formulated
- Careful mathematical treatment - crucial role of $\pm i\epsilon$ (cf. its role for ETCR from PJ function and vanishing of surface terms in LF Poincare algebra)
- LF constraints (non-locality): bosonization formulae not completely the same as in the conventional field theory
- Further work (full treatment in a finite volume (DLCQ), fermion condensate ?)

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