Chiral Symmetry in Light-Cone Field Theory

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Introduction

Structure of the light-cone dispersion relation of a particle

$$p_+ = \frac{m^2 + p_\perp^2}{2p_-}$$

determines to a large extent the structure of light-cone field theory. Both, the kinematical nature of the vacuum and the non-locality of light-cone field theory have their origin in this dispersion relation. Consequences of symmetries in the context of non-local field theories are not properly understood. In particular a consistent light-cone formulation of phases with spontaneously broken symmetries remains to be developed. Important tools in the analysis of symmetries are the Ward-Takahashi (WT) identities and their consequences such as the Gell-Mann–Oakes–Renner relation

$$m_\pi^2 f_\pi^2 = -m_q \langle 0|\bar{\psi}\psi|0\rangle$$

WT identities relate properties of the Nambu-Goldstone (NG) bosons to properties of the constituents and the vacuum and in particular connect the NG creation operator with the generator of the symmetry.

Path-Integrals for Fermionic Light-Cone Field Theories

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\partial_\mu \gamma^\mu - m)\psi + \mathcal{L}_{\mathrm{int}}(\bar{\psi}, \psi)$$

$$\mathcal{L} = \mathrm{i}\varphi^\dagger \partial_+ \varphi + \mathrm{i}\chi^\dagger \partial_- \chi + \frac{\mathrm{i}}{\sqrt{2}} \left(\chi^\dagger \partial_m \varphi - \varphi^\dagger \partial_m^\dagger \chi\right) + \mathcal{L}_{\mathrm{int}}(\varphi, \chi)$$

$$\partial_m = \sigma_3 \partial_1 + \mathrm{i}\partial_2 - \mathrm{i}\sigma_1 m$$

Chiral Rotations

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi \qquad \varphi \to \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\varphi\,, \quad \chi \to \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\chi$$

Axial Current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

$$j_5^+ = \varphi^\dagger\sigma_3\varphi\,, \quad j_5^- = \chi^\dagger\sigma_3\chi\,,$$

$$j_5^1 = \frac{1}{\sqrt{2}}(\varphi^\dagger\chi + \chi^\dagger\varphi)\,, \quad j_5^2 = \frac{\mathrm{i}}{\sqrt{2}}(\chi^\dagger\sigma_3\varphi - \varphi^\dagger\sigma_3\chi)$$

Fermion bilinears relevant for realization of symmetry

$$\mathrm{i}\,\bar{\psi}\gamma_5\psi = -\frac{1}{\sqrt{2}}\left(\varphi^\dagger\sigma_2\chi + \chi^\dagger\sigma_2\varphi\right)$$

$$\bar{\psi}\psi = -\frac{1}{\sqrt{2}}\left(\varphi^\dagger\sigma_1\chi + \chi^\dagger\sigma_1\varphi\right)$$

Generating Functional

$$Z[\eta,\gamma] = \int D[\varphi,\chi]\,\exp\mathrm{i}\{S[\varphi,\chi] + s[\eta,\gamma,\varphi,\chi]\}$$

$$s[\eta,\gamma,\varphi,\chi] = \int \mathrm{d}^4x\,(\eta^\dagger\varphi + \varphi^\dagger\eta + \gamma^\dagger\chi + \chi^\dagger\gamma)$$

Integrating out the ”constrained” (bad) degrees of freedom defines the generating functional of light-cone field theory

$$Z[\eta] = \int D[\varphi]\ \exp\left\{\mathrm{i}S[\varphi] + \mathrm{i}(\varphi^\dagger\eta + \eta^\dagger\varphi)\right\}$$

with

$$\exp\left\{\mathrm{i}S[\varphi]\right\} = \int D[\chi]\ \exp\left\{\mathrm{i}S[\varphi,\chi]\right\}$$

Chiral symmetry

$$S[\mathrm{e}^{\mathrm{i}\alpha\sigma_3}\varphi, \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\chi] = S[\varphi,\chi]$$

preserved in integrating out χ

$$S[\mathrm{e}^{\mathrm{i}\alpha\sigma_3}\varphi] = S[\varphi]$$

NJL - Model on the Light cone

$$\mathcal{L}_{\text{int}} = \frac{g^2}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}\mathrm{i}\gamma_5\psi)^2\right]$$
$$= \frac{g^2}{4}\left[\left(\varphi^\dagger\sigma_1\chi + \chi^\dagger\sigma_1\varphi\right)^2 + \left(\varphi^\dagger\sigma_2\chi + \chi^\dagger\sigma_2\varphi\right)^2\right]$$

Introduce auxiliary scalar $\sigma(x)$ and pseudoscalar fields $\pi(x)$ corresponding to the two bilinears to eliminate the four fermion interaction in the path integral

$$Z[\eta,\gamma] = \int D[\varphi,\chi,\sigma,\pi]\,\exp\left\{\mathrm{i}S[\varphi,\chi,\sigma,\pi] + \mathrm{i}s\right\}\ ,$$

with action

$$S[\varphi,\chi,\sigma,\pi] = \int \mathrm{d}^4x\{\mathrm{i}\varphi^\dagger\partial_+\varphi + \mathrm{i}\chi^\dagger\partial_-\chi$$
$$+ \frac{\mathrm{i}}{\sqrt{2}}(\chi^\dagger\partial_\perp\varphi - \varphi^\dagger\partial_\perp^\dagger\chi) - \frac{1}{2}(\sigma^2+\pi^2)\}$$

and

$$\partial_\perp = \sigma_3\partial_1 + \mathrm{i}\partial_2 - \mathrm{i}g(\sigma\sigma_1 + \pi\sigma_2) - \mathrm{i}m\sigma_1 \,.$$

Breakdown of chiral symmetry by spontaneous mass generation - $\sigma(x)$ develops a finite vacuum expectation value.

Chiral rotations of the spinors accompanied by rotations of auxiliary fields

$$\begin{aligned} \sigma &\rightarrow \sigma' = \sigma\cos 2\alpha + \pi\sin 2\alpha \,, \\ \pi &\rightarrow \pi' = \pi\cos 2\alpha - \sigma\sin 2\alpha \,, \end{aligned}$$

Integrating out the field χ

$$Z[\eta,\gamma] = \int D[\varphi,\sigma,\pi]\ \exp\left\{\mathrm{i}S[\varphi,\sigma,\pi] + \mathrm{i}s[\eta]\right\}$$

$$S[\varphi,\sigma,\pi] = \int \mathrm{d}^4x \left(\mathrm{i}\varphi^\dagger\partial_+\varphi + \mathrm{i}\chi^\dagger\partial_-\chi - \frac{1}{2}(\sigma^2+\pi^2)\right)$$

Composite field

$$\chi = \chi[\varphi] = -\frac{1}{\sqrt{2}\partial_-}\partial_\perp\varphi$$

Light-cone energy of given field configuration

$$p_+ = \int \mathrm{d}^3x \left(\mathrm{i}\chi^\dagger\partial_-\chi + \frac{1}{2}(\sigma^2+\pi^2)\right)$$

Hidden non-local "chiral symmetry" of free, massive fermions on the light-cone

$$\mathcal{L} = \mathrm{i}\varphi^\dagger \partial_+ \varphi + \mathrm{i}\chi^\dagger \partial_- \chi \,, \quad \chi = -\frac{1}{\sqrt{2}\partial_-}\partial_m \varphi$$

$$\partial_m = \sigma_3 \partial_1 + \mathrm{i}\partial_2 - \mathrm{i}m\sigma_1 \,, \quad \partial_m^{-1} = \frac{\partial_m^\dagger}{-\Delta_\perp + m^2}$$

Non-local, unitary, global symmetry transformation

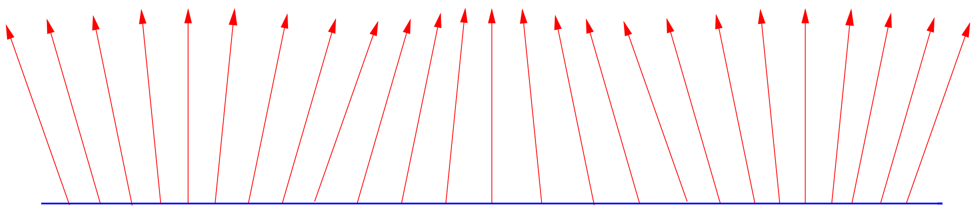
$$\varphi' = U\varphi = \partial_m^{-1}\, e^{i\alpha\sigma_3}\, \partial_m\, \varphi \,, \quad \chi' = e^{i\alpha\sigma_3}\chi \,, \quad UU^\dagger = 1 \,, \quad \mathcal{L}' = \mathcal{L}$$

Spontaneously Broken Chiral Symmetry

Vacua of different chiral orientation

Generation of low energy excitations by soft (long wavelength) modulations of chiral orientation

$$U_5[\alpha] = \exp\left(\mathrm{i}\int \mathrm{d}^3x\,\psi^\dagger(x)\gamma_5\alpha(x)\psi(x)\right)$$



WT-Identities in Covariant or Equal Time Quantization

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\partial_\mu \gamma^\mu - m)\psi + \mathcal{L}_{\mathrm{int}}(\bar{\psi}, \psi)$$

Local change of integration variables ψ

$$\psi(x) \;\rightarrow\; e^{i\alpha(x)\gamma_5}\psi(x)$$

$$\delta\mathcal{L}(x) \;\rightarrow\; -j^5_\mu(x)\partial^\mu\alpha(x) - m\bar{\psi}\left(e^{2i\alpha(x)\gamma_5} - 1\right)\psi(x)$$

induces changes of the chirally symmetric, local Lagrangian which vanish for infinite wavelength of α. The value of the functional integral does not change

$$0 = \int D[\psi][e^{i\int d^4x\left(\mathcal{L}+\delta\mathcal{L}+\bar{\eta}e^{i\alpha\gamma_5}\psi+\bar{\psi}e^{i\alpha\gamma_5}\eta\right)} - e^{i\int d^4x\left(\mathcal{L}+\bar{\eta}\psi+\bar{\psi}\eta\right)}]$$

Expansion in α yields fundamental functional identity

$$0 = \int D[\psi]e^{i\int d^4x\left(\mathcal{L}+\bar{\eta}\psi+\bar{\psi}\eta\right)} \quad [i\partial^\mu j^5_\mu(x) + 2m\bar{\psi}(x)\gamma_5\psi(x) \\ -\bar{\eta}(x)\gamma_5\psi(x) - \bar{\psi}(x)\gamma_5\eta(x)]$$

Chiral Ward identities result from functional differentiation with respect to the sources. Application of the operator

$$\gamma^5_{\beta\alpha}\frac{\delta}{\delta\bar{\eta}_\alpha(y)}\frac{\delta}{\delta\eta_\beta(y)}|_{\eta=\bar{\eta}=0}$$

$$\begin{aligned}0 = \int D[\psi]e^{i\int d^4x\mathcal{L}} \quad & [\bar{\psi}(y)\gamma_5\psi(y)\left(i\partial^\mu j^5_\mu(x) + 2m\bar{\psi}(x)\gamma_5\psi(x)\right) \\ & -2i\delta(x-y)\bar{\psi}(y)\psi(y)]\end{aligned}$$

leads to the Gell-Mann Oakes Renner relation. For small momenta the axial current is pion dominated

$$j^\mu_5(x) \sim f_\pi \partial^\mu \phi(x)$$

WT-Identities in Light-Cone Quantization

Attempt to extend global chiral transformation of φ (and of the auxiliary fields) to soft, local transformations

$$\varphi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)\sigma_3}\varphi(x)$$

and to derive thereby appropriate WT-identities fails. No pion dominance. Besides Nambu-Goldstone- also axial vector particles contribute. No simple relation between Nambu-Goldstone field and axial current. Non-locality induced by the light-cone dispersion relation prevents soft modulations to generate low energy excitations

$$\begin{aligned} p'_+ &= \int \mathrm{d}^3x\{\mathrm{i}\chi'^\dagger\partial_-\chi' + \frac{1}{2}(\sigma^2+\pi^2)\} \\ &= \int \mathrm{d}^3x\{\frac{\mathrm{i}}{2}\varphi^\dagger \mathrm{e}^{-\mathrm{i}\alpha(x)\sigma_3}\,\partial'_\perp\,\frac{1}{\partial_-}\,\partial'_\perp\,\mathrm{e}^{\mathrm{i}\alpha(x)\sigma_3}\varphi + \frac{1}{2}(\sigma^2+\pi^2)\} \end{aligned}$$

More successful strategy based on the hidden chiral symmetry - $m = 0$, $x_\perp$- independent $\alpha = \alpha(x^+, x^-)$

$$\chi = -\frac{1}{\sqrt{2}\partial_-}\partial_\perp \varphi \,, \quad \partial_\perp[\sigma', \pi'] = \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\partial_\perp[\sigma, \pi]\mathrm{e}^{-\mathrm{i}\alpha\sigma_3}$$

Change of coordinates

$$\varphi' = W\varphi \,, \quad W = \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\frac{1}{\partial_\perp}\mathrm{e}^{-\mathrm{i}\alpha\sigma_3}\partial_\perp$$

$$\chi' = \chi \,, \quad \sigma' = \sigma \,, \quad \pi' = \pi$$

In the symmetry broken phase - generated mass

$g\sigma = M$, $\pi = 0$

$$W \approx U = \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\partial_M^{-1}\mathrm{e}^{-\mathrm{i}\alpha\sigma_3}\partial_M \varphi(x) \,, \quad U U^\dagger = 1$$

Invariance of χ and of auxiliary fields implies that excitations generated by local modulations of global symmetry transformations have vanishing light-cone energy for $p_\perp = 0$ and arbitrary p_- and thus

$$m_\pi = 0$$

Infinitesimal transformations

$$\delta\varphi(x) = i\alpha(x^+, x^-)\Sigma_3\varphi(x)\,, \quad \delta\chi = \mathrm{i}\alpha(x^+, x^-)\frac{\sqrt{2}\,m}{\partial_-}\sigma_2\,\varphi$$

$$\Sigma_3 = \partial_M^{-1}\,[\sigma_3, \partial_M] = \frac{2M^2\sigma_3 - 2\mathrm{i}M\vec{\sigma}_\perp\vec{\partial}_\perp}{-\Delta_\perp + M^2}$$

in agreement with result from pion operator calculated in ladder approximation to BS-equation

Fundamental functional identity

$$0 \;=\; \int D[\varphi]e^{\mathrm{i}S[\varphi,\sigma,\pi]+\mathrm{i}s[\eta,\gamma]}\int \mathrm{d}^2x_\perp\,\Big[\left(\partial_+\varphi^\dagger(x) + \mathrm{i}\eta^\dagger(x)\right)(\Sigma_3\varphi)(x)$$

$$+\sqrt{2}m(\chi^\dagger\sigma_2\varphi)(x) + \mathrm{c.c.}\Big]$$

The Structure of light-cone WT identities

On the light-cone, formation of NG bosons is connected to the appearance of a new symmetry property. The (Goldstone) one-particle states with vanishing transverse momenta are degenerate with the ground state. This determines the general structure of the fundamental identity. Canonically, an operator $\Phi(x^+, x^-)$, the field operator of the NG particles with vanishing transverse momentum, must exist with the property

$$[\Phi(x^+, x^-), H]|0\rangle = 0\,, \quad \textit{i.e.}\,, \quad \frac{\partial}{\partial x^+}\Phi(x^+, x^-)|0\rangle = 0$$

formally, a WT identity

$$\frac{\partial}{\partial x^+}\langle 0|T\left(\Phi(x^+, x^-)\mathcal{O}(y)\right)|0\rangle \;=\; \delta(x^+ - y^+) \langle 0|\left[\Phi(x^+, x^-), \mathcal{O}(y)\right]|0\rangle$$

$\mathcal{O}(y)$ local in light-cone time, otherwise arbitrary

Functional formulation

$$0 = \frac{\delta}{\delta\omega(y)} \int D[\varphi] \mathrm{e}^{\mathrm{i}S+\mathrm{i}s} \Big[\frac{\partial}{\partial x^+} \Phi(x^+, x^-) - \delta(x^+ - y^+) \left(\Phi(x^+ + \epsilon^+, x^-) - \Phi(x^+ - \epsilon^+, x^-) \right) \Big] \Big|_{\epsilon=0}$$

where

$$\frac{\delta}{\delta\omega(y)} \mathrm{e}^{\mathrm{i}s} = \mathcal{O}(y) \mathrm{e}^{\mathrm{i}s}$$

Hidden Symmetry of the NJL model

$$\int \mathrm{d}^2 x_\perp \partial_\mu j_5^\mu = \int \mathrm{d}^2 x_\perp \left[\partial_+ \left(\varphi^\dagger \sigma_3 \varphi\right) + \partial_- \left(\chi^\dagger \sigma_3 \chi\right)\right] = 0\,.$$

Via Euler-Lagrange equations

$$\sqrt{2}\partial_+ \varphi = -\partial_\perp^\dagger \chi\,, \qquad \sqrt{2}\partial_- \chi = -\partial_\perp \varphi\,,$$

rewrite

$$\partial_- \left(\chi^\dagger \sigma_3 \chi\right) = \varphi^\dagger \left(\partial_\perp^\dagger \sigma_3 \frac{1}{\partial_\perp^\dagger}\right) \partial_+ \varphi + (\partial_+ \varphi)^\dagger \left(\frac{1}{\partial_\perp} \sigma_3 \partial_\perp\right) \varphi\,.$$

With $\partial_\perp \approx \partial_M$

$$\partial_\perp^\dagger \sigma_3 \frac{1}{\partial_\perp^\dagger} = \frac{1}{\partial_\perp} \sigma_3 \partial_\perp \to \frac{\partial_M^\dagger \sigma_3 \partial_M}{\partial_M^\dagger \partial_M}$$

$$\partial_+ \int \mathrm{d}^2 x_\perp \varphi^\dagger \Sigma_3 \varphi = 0\,,$$ infinity of conserved charges

Results

Functional differentiation of fundamental functional identity yields

Gell-Mann Oakes Renner relation on the light cone

$$p_+ G^+_\Sigma(p) - \mathrm{i}\sqrt{2} m G_5(p) = -\mathrm{i}\sqrt{2}\Gamma, \quad p = (p_+, p_-, p_\perp = 0)$$

with th 2-point function corresponding to the $\langle j_5(x)\ \bar\psi\gamma^5\psi(y)\rangle$ correlation function

$$G^+_\Sigma(x) = \langle 0|T(\varphi^\dagger(x)\Sigma^3\varphi(x)\ \big(\varphi^\dagger(0)\sigma_2\chi(0) + \mathrm{c.c.}\big))|0\rangle\,,$$

and the 2-point function corresponding to the $\langle\bar\psi\psi(x)\ \bar\psi\psi(y)\rangle$ correlation function

$$G_5(x) = \langle 0|T((\chi^\dagger(x)\sigma_2\varphi(x) + \mathrm{c.c.})\ \big(\varphi^\dagger(0)\sigma_2\chi(0) + \mathrm{c.c.}\big))|0\rangle$$

and the condensate

$$\Gamma = \frac{\mathrm{i}}{2\sqrt{2}}\langle 0|\varphi^\dagger(y)\Sigma_3\sigma_2\chi(y) - \chi^\dagger(y)\sigma_2\Sigma_3\varphi(y)|0\rangle$$

Quark Propagator and pion structure function

$$p_+F(p,q) = \Sigma_3(q)\theta(q_-)G(q) - \Sigma_3(q-p)\theta(q_- - p_-)G(q-p)$$

with three-point function $\pi \to q\,\bar{q}$

$$F_{\alpha\beta}(x,y) = \langle 0|T(\varphi^\dagger(0)\Sigma_3\varphi(0))\varphi_\alpha(x)\varphi^\dagger_\beta(y)|0\rangle\,,$$

and quark propagator

$$G(x) = \langle 0|T(\varphi(x)\varphi^\dagger(0))|0\rangle$$

Further applications for Gross-Neveu and 't Hooft models and calculation of the mass of the Schwinger particle from the anomaly in the Schwinger model

Remark concerning condensates

Condensate operators have to be defined as equal time limit of Heisenberg operators

$$\langle\bar\psi\psi\rangle = \lim_{\epsilon\to 0}\left[\langle\bar\psi\psi\rangle_\epsilon - \langle\bar\psi\psi\rangle_\epsilon^0\right] \ ,$$

with

$$\langle\bar\psi\psi\rangle_\epsilon = \langle 0|\bar\psi(\epsilon) P \mathrm{e}^{\mathrm{i}g\int_0^\epsilon \mathrm{d}x^\mu A_\mu}\psi(0)|0\rangle \ , \quad \epsilon^+ > 0$$

Light cone infrared singularity $(p_- = 0)$ determines short time behavior of condensate operators. Explicit calculation of condensate for NJl, Gross-Neveu and 't Hooft model.

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