Perturbation theory in a p = 0 condensate

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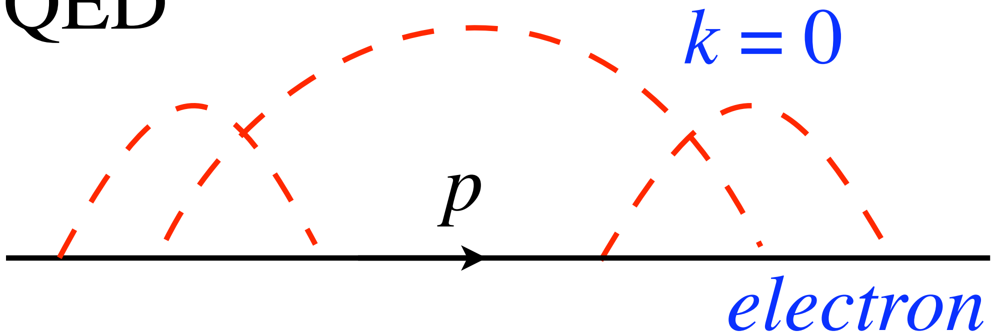
The calculation

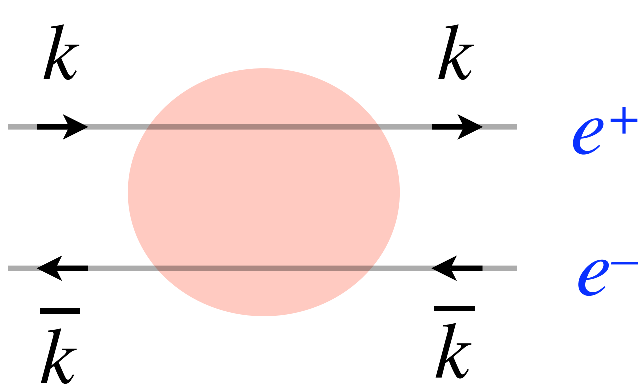
We add a term to the on-shell photon (QED) and gluon (QCD) propagators at zero momentum:

$$D^{\mu\nu}_{ab}(p) = -g^{\mu\nu}\delta_{ab}\left[\frac{i}{p^2+i\varepsilon} + \Lambda^2(2\pi)^4\delta^4(p)\right]$$

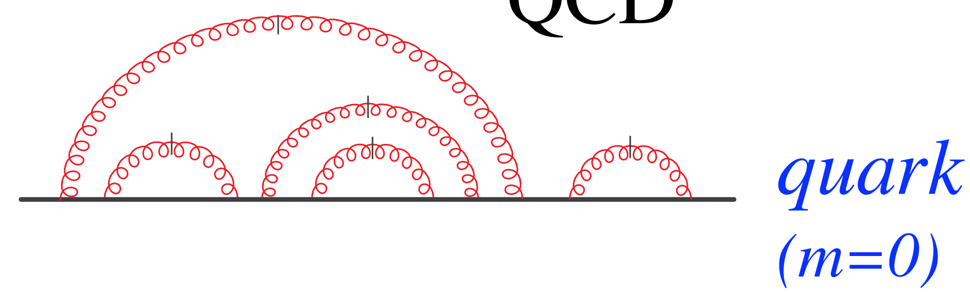
We find the exact Λ -dependence of the one- and two-fermion propagators at “dressed” Born level, from all loop corrections of the form

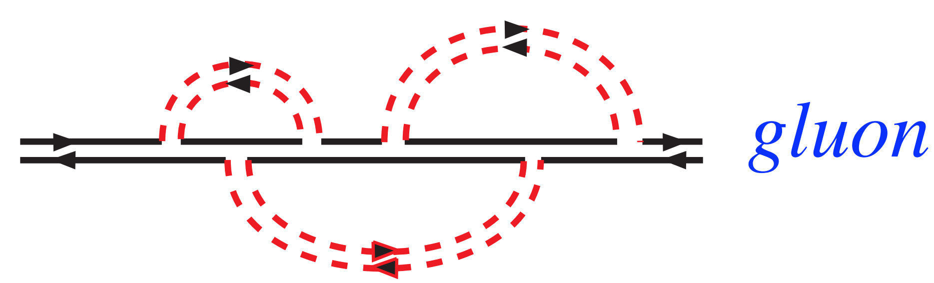
QED





QCD





Only planar diagrams, *i.e.*, leading order in 1/N

Motivation

The Feynman iε prescription at the pole of the free propagator defines the perturbative expansion around an empty state (the perturbative vacuum).

If there are particles in the in- and out-states the prescription changes. *E.g.*, for a scalar propagator

$$D_k(x-y) \equiv \frac{1}{\mathcal{N}}\langle 0|a_k\, T[\phi(x)\phi(y)]\, a_k^\dagger|0\rangle \;; \qquad \mathcal{N} = \langle 0|a_k\, a_k^\dagger|0\rangle$$

is

$$\begin{aligned} D_k(p) &= \mathcal{P}\frac{i}{p^2-m^2} + \pi\delta(p^2-m^2) \\ &+ \frac{(2\pi)^4}{\mathcal{N}}\left[\delta^3(\boldsymbol{p}-\boldsymbol{k})\delta(p^0-E_k) + \delta^3(\boldsymbol{p}+\boldsymbol{k})\delta(p^0+E_k)\right] \end{aligned}$$

Motivated by the gluon condesate of the true QCD vacuum we wish to study perturbative expansions around a nontrivial, non-empty state.

We take the vacuum particles to be photons (or gluons) with $\boldsymbol{k} = 0$.

Why $k = 0$?

- To preserve Lorentz invariance at each order.

- To enable dressing Green functions to all orders,
  effectively shifting the point of expansion.

Apart from the soft dressing with $k = 0$ lines (replacing the $i\varepsilon$ prescription), the perturbative expansion is determined in the standard way by the QED or QCD lagrangian.

The dressed Green functions (are required to) have the standard perturbative behavior at short distances.

They exhibit new features at long distances, including (for QED) exponential behavior reminiscent of soft hadron physics.

The rewards of perturbation theory

Amplitudes calculated using the (standard) perturbative expansion

- Have the correct symmetries (Lorentz, gauge, etc.)
- Have proper analyticity
- Are unitary ($SS^\dagger = I$)

These requirements are very complex: For physical theories there are few alternative methods of building amplitudes satisfying these constraints.

However, for QCD the asymptotic states are hadrons. How can the above properties be ensured for hadronic amplitudes?

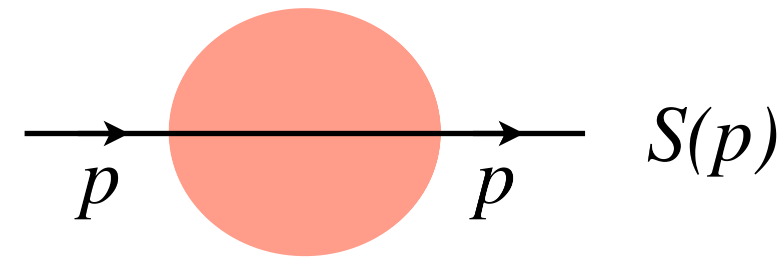
- Conversely, it is not clear what analyticity and unitarity imposes on quark and gluon Green functions in a confining theory.

- We hope to gain some insight into this through the dressed PT.

Preview: some properties of QED at dressed Born level

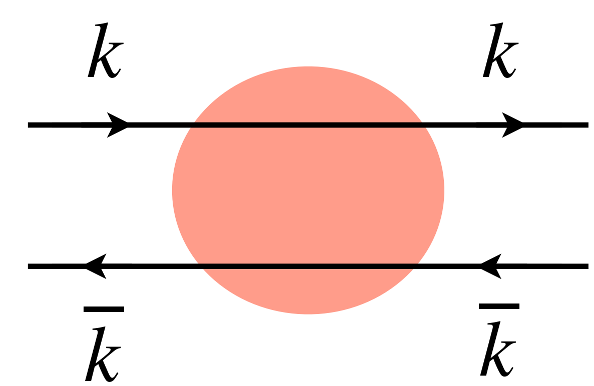
Electron propagator, of mass m and momentum p:

- Reduces to free propagator for $|p^2| \to \infty$ on one sheet
- Has no pole at $p^2 = m^2$
- Has (exponentially suppressed) cut for $p^2 < 0$
- Behaves as $\exp[m^2/(2e^2\Lambda^2)]\, p\cdot\gamma/p^4$ for $p^2 \to 0$



Electron – positron propagator, of momenta k, $\bar{k}$:

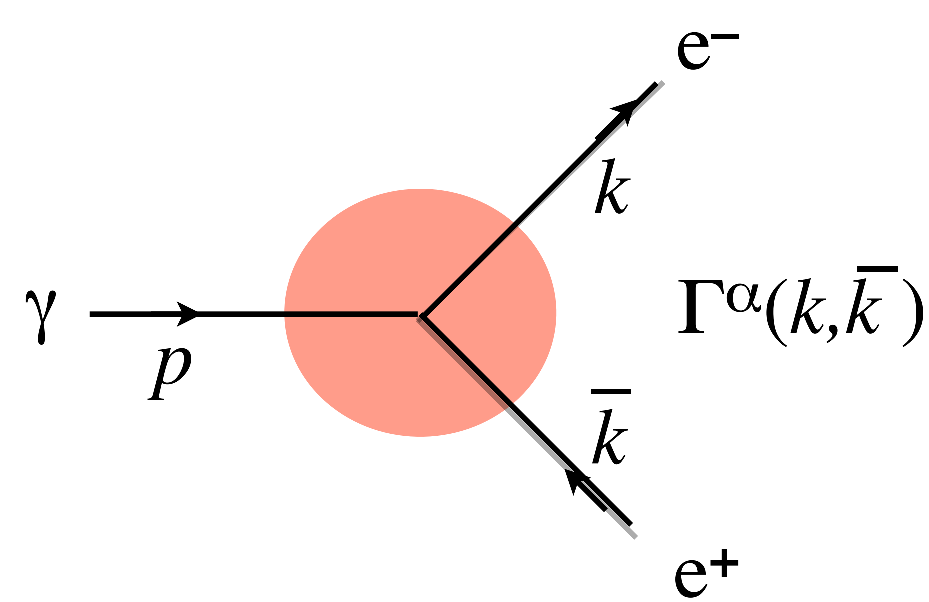
- Reduces to perturbative result for at short distance
- Is enhanced close to "threshold": $\sqrt{(k-\bar{k})^2} \simeq \sqrt{k^2} + \sqrt{\bar{k}^2}$



Photon vertex satisfies Ward identity

- Check of gauge invariance

$$p_\alpha \Gamma^\alpha(k,\bar{k}) = i\left[S(\bar{k}) - S(k)\right]$$



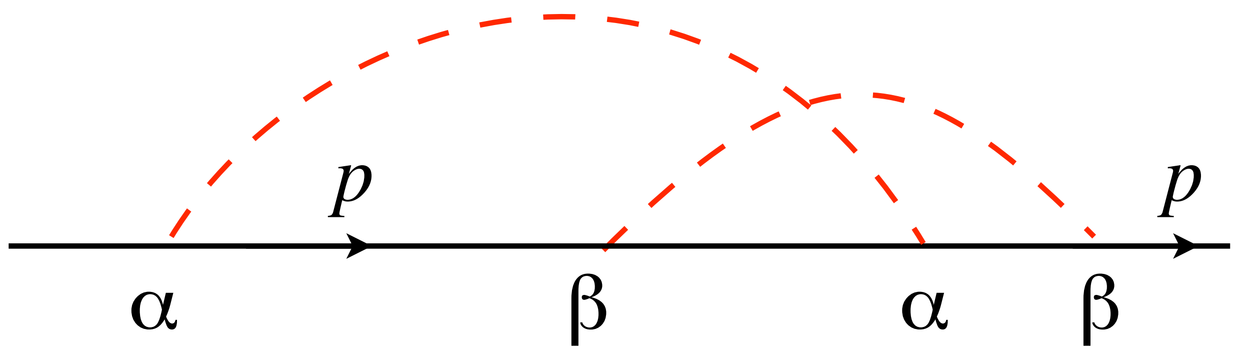
Derivation of the dressed electron propagator

There are N_n distinct ways of dressing an electron with n photon lines, where

$$N_n = \frac{(2n)!}{2^n\, n!} \simeq \sqrt{2}\,(2n)^n\, e^{-n}$$

$N_2 = 3$

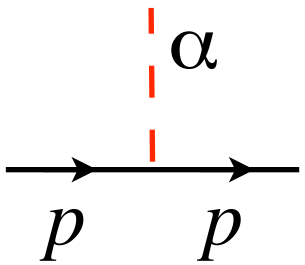
The contraction of the Lorentz indices depends on the ordering of the vertices on the propagator



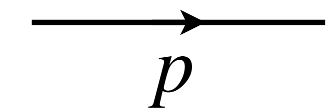
$\mathcal{O}\left(\mu^4\right)$ $\mu^2 = e^2\,\Lambda^2$

Summing all contributions from $n = 0$ to $n = \infty$ seems daunting.
However, there is a trick...

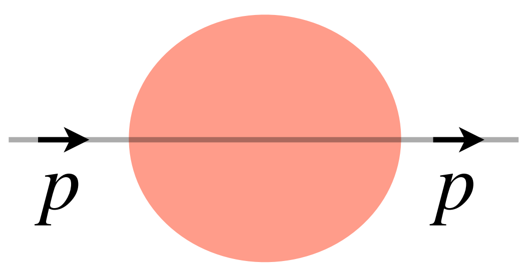
Coupling a zero momentum photon to a fermion is equivalent to differentiation:



$$= \frac{i}{\not p - m}(-ie\gamma^\alpha)\frac{i}{\not p - m} = -e\frac{\partial}{\partial p_\alpha}\frac{i}{\not p - m} = -e\frac{\partial}{\partial p_\alpha}$$

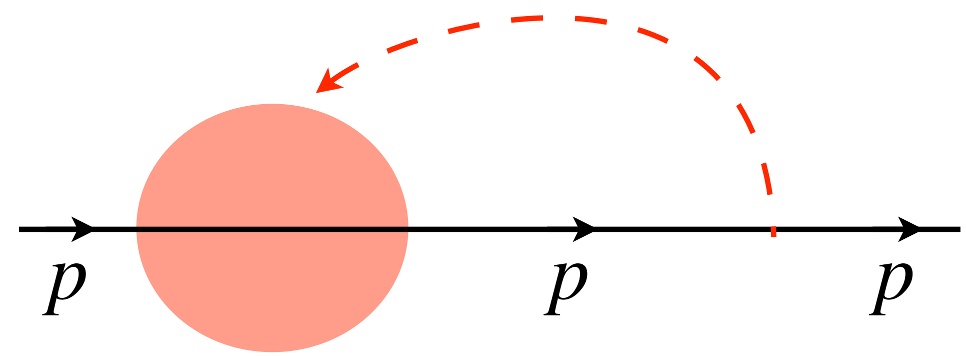


Differentiating the propagator at any order inserts the photon at all positions:



$\mathcal{O}\left(\mu^{2n}\right)$

=



$\mathcal{O}\left(\mu^{2n-2}\right)$

Sum over *n* gives a differential equation for the dressed electron propagator:

$$(\not p - m)\, S(p) = i + \mu^2\, \gamma^\alpha \frac{\partial}{\partial p^\alpha} S(p)$$

Pure gauge?

The above condition on propagator

$$(\not{p} - m - \mu^2 \not{\partial}_p) S(p) = i$$

may be compared with equation of motion in coordinate space

$$(i\not{\partial} - m - e\not{A}\,) S(x) = i\delta^4(x)$$

giving $\quad eA^\nu = i\mu^2 x^\nu = i\mu^2 \partial^\nu (\frac{1}{2}x^2)$

and suggesting that A^μ is a pure gauge (with complex gauge parameter).

However, demanding that *S(p)* approaches the perturbative propagator at short distances (large p^2) we find a non-trivial solution in momentum space.

In terms of dimensionless momentum and mass, where $\mu \equiv e\,\Lambda$:

$$\boxed{\hat{p} = \frac{p}{\mu} \qquad \hat{m} = \frac{m}{\mu}}$$

$$(\not{p} - m - \not{\partial}_p)S(p) = \frac{i}{\mu}$$

where I omit the ^ on the dimensionless variables *p* and *m*.
Lorentz symmetry implies

$$-iS(p) = a(p^2)\not{p} + b(p^2)$$

A general solution has exponential behaviors $exp(\pm p^2)$ at large p^2 .
The special solution

$$S(p) = (\not{p} + m - \not{\partial}_p)\frac{i}{2\mu}\int_0^\infty dt \exp\left[-\frac{tp^2}{2} + \frac{tm^2}{2(1+t)}\right]$$

approaches $\dfrac{1}{\mu}\dfrac{i}{\not{p} - m}$ for $p^2 \to \infty$ with $\mathrm{Re}(p^2) > 0$
(where the integral converges)

For $\mathrm{Im}(p^2) > 0$ (and $\mathrm{Im}(p^2) < 0$) the same propagator is given by

$$S(p) = (\not{p} + m - \not{\partial}_p)\frac{\pm 1}{2\mu}\int_0^\infty dt \exp\left[\pm\frac{i}{2}\left(tp^2 - \frac{tm^2}{1 \mp it}\right)\right]$$

From these expressions it can be seen that *S(p)* approaches $\dfrac{1}{\mu}\dfrac{i}{\not{p} - m}$ for $|p^2| \to \infty$ on the whole 'physical' sheet.

S(p) has a discontinuity along the negative real axis given by

$$\mathrm{Disc}S(p^2 < 0) = -\frac{\pi m^2}{\mu p^2}\left[\not{p}J_2\left(m\sqrt{-p^2}\right) - \sqrt{-p^2}J_1\left(m\sqrt{-p^2}\right)\right]\exp(p^2+m^2)$$

The discontinuity is exponentially damped for $p^2 \to -\infty$.

For $p^2 \to +\infty$ Disc *S(p)* grows exponentially, implying that *S(p)* increases exponentially on sheets other than the physical one.

S(p) is regular at $p^2 = m^2$ for any $\mu^2 > 0$ (recall that $m = \hat{m} = m/\mu$ above).

Near $p^2 = 0$ $S(p)$ behaves as

$$S(p^2 \to 0) = \frac{i}{\mu}\exp(\tfrac{1}{2}m^2)$$

$$\times\left[\frac{\not{p}}{p^2}\left(1 - \frac{m^2}{2} + \frac{2}{p^2}\right) + \frac{m}{p^2} - \frac{m^4}{16}\not{p}\log p^2 + \frac{m^3}{4}\log p^2 + \mathcal{O}\left((p^2)^0\right)\right]$$

Thus the leading singularity goes like $\not{p}/p^4$, and is exponentially enhanced in $\hat{m} \equiv m/\mu$. The limits $p^2 = p^2/\mu^2 \to 0$ and $\mu^2 \to 0$ do not commute.

For a massless fermion (m = 0) the propagator reduces (in D = 3+1) to

$$S(p)\Big|_{m=0} = (\not{p} - \not{\partial}_p)\frac{i}{\mu\,\not{p}} = \frac{i}{\mu\,\not{p}}\left(1 + \frac{2}{p^2}\right)$$

which has no discontinuity nor exponential behavior, but is similarly singular at $p^2 = 0$.

Summary of the dressed electron propagator

- The calculation was done for QED, but may be illustrative of the effects of expanding around a non-empty state also for QCD.

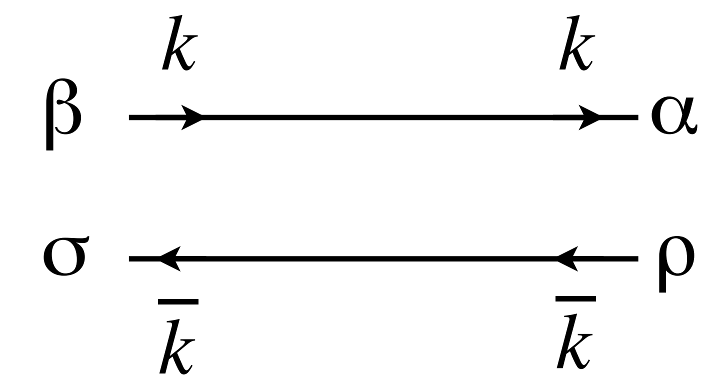
- The propagator reduces to the standard perturbative one at short distances, but has novel features at long distances.

- The absence of a simple pole in p^2 implies that the (on-shell) fermion does not propagate to asymptotic times.

- It is important to identify the true asymptotic states (if any!). We next consider dressing the e^+e^- (double) propagator.

Differential equation for the $e^+ e^-$ propagator

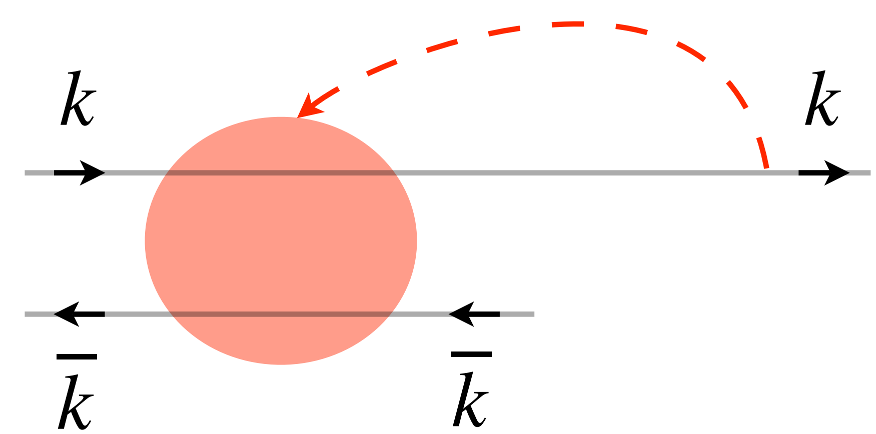
A differential equation for the free propagator



$$iG_{\alpha\beta,\rho\sigma}(\mu=0) = \left(i\frac{\not{k}+m}{k^2-m^2}\right)_{\alpha\beta}\left(i\frac{\not{\bar{k}}+m}{\bar{k}^2-m^2}\right)_{\rho\sigma}$$

when fully dressed with zero-momentum photons
may be derived as for the single fermion propagator:

$$(\not{k} - m - \not{\partial}_k)G(k,\bar{k}) = S(\bar{k})$$



where $\bar{k} = k - p$. However, now

$G(k^2, \bar{k}^2, p^2)$ depends on three invariants, and has an involved Dirac structure.
Solving the partial differential equation(s) seems daunting.

But there is another trick...

Alternative derivation of the dressed electron propagator

The equation of motion must be satisfied by $S_D(x) \equiv S_0(x)\exp(-x^2/2)$, *i.e.*, the undressed propagator multiplied by the gauge factor. In momentum space

$$S_D(p) = \frac{i}{(2\pi)^2\mu}\int_{-\infty}^{\infty} d^4q\,\frac{\not{q}-m}{q^2+m^2}\exp[-\tfrac{1}{2}(q-p)^2] \qquad (eucl.)$$

where the integral converges for euclidean metric, $g^{\mu\nu} = \delta^{\mu\nu}$. The equation of motion is easily seen to be satisfied:

$$\left(\not{p}+m+\mu^2\gamma^\alpha\frac{\partial}{\partial p^\alpha}\right)S_D(p) = -i \quad (eucl.)$$

However, $S_D(p) \propto \exp[|p^2|/2\mu^2]$ for $p^2 \to -\infty$ (or $\mu \to 0$ with $p^2 < 0$) and thus does not approach the undressed propagator asymptotically in all of the p^2 plane.

With Feynman parametrization we get (in Minkowski metric)

$$\frac{1}{q^2+m^2} = \frac{1}{2}\int_0^\infty du\,\exp[-\tfrac{1}{2}u(q^2+m^2)]$$

$$S_D(p) = -(\not{p}+m-\not{\partial}_p)\frac{i}{2\mu}\int_{-1}^{0} dt\,\exp\left[-\frac{tp^2}{2}+\frac{tm^2}{2(1+t)}\right]$$

which exhibits the exponential behavior for $p^2 \to \infty$ and satisfies the equation of motion due to the identity

$$(\not{p}-m-\not{\partial}_p)S_D(p) = \frac{i}{\mu}\int_{-1}^{0} dt\,\frac{d}{dt}\left\{(1+t)^2\exp\left[-\frac{tp^2}{2}+\frac{tm^2}{2(1+t)}\right]\right\} = \frac{i}{\mu}$$

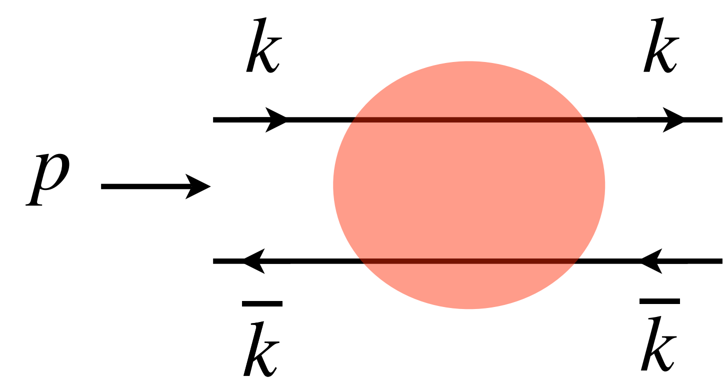
Our well-behaved propagator differed only in the choice of integration limits:

$$S(p) = (\not{p}+m-\not{\partial}_p)\frac{i}{2\mu}\int_0^\infty dt\,\exp\left[-\frac{tp^2}{2}+\frac{tm^2}{2(1+t)}\right]$$

The integration region can similarly be fixed "by hand" for higher point Green functions.

Thus from the ansatz for the $e^+ e^-$ propagator in euclidean space,

$$G_D(k,\bar{k}) = \frac{i}{(2\pi)^2\mu^2}\int_{-\infty}^{\infty} d^4q\,\frac{1}{\not{k}+\not{q}+m}\,\frac{1}{\bar{\not{k}}+\not{q}+m}\,\exp(-\tfrac{1}{2}q^2)$$



where $\bar{k} = k - p$, we find in minkowski space
(in the case of $m = 0$)

$$G(k,\bar{k})\Big|_{m=0} = \frac{i}{\mu^2}[\not{k} - \not{\partial}_k]\,[\bar{\not{k}} - \not{\partial}_k]\int_0^{\infty} dt\,\frac{\exp(-t/4)}{(t+2k\cdot\bar{k})^2 - \lambda(p^2,k^2,\bar{k}^2)}$$

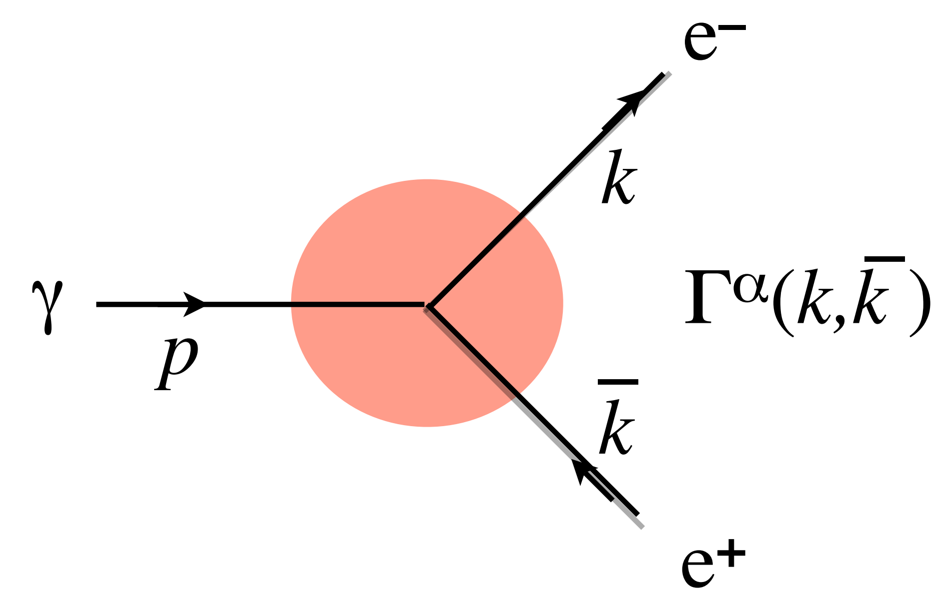
where $\lambda(p^2,k^2,\bar{k}^2) = (p^2-k^2-\bar{k}^2)^2 - 4k^2\bar{k}^2$

G satisfies $(\not{k} - \not{\partial}_k)G(k,\bar{k}) = S(\bar{k})$ with the physical propagator S.

For k^2, $\bar{k}^2 \to \infty$ at fixed p^2, $iG(\bar{k},k) \to S(k)\,\bar{S}(k)$ (correct short distance limit)

Ward identity

Coupling the dressed e+ e– propagator to the photon we get the dressed γ e+ e– vertex. It is then straightforward to check the Ward identity



$$p_\alpha \Gamma^\alpha(k,\bar{k}) = i\left[S(\bar{k}) - S(k)\right]$$

Threshold enhancement

From the expression of the dressed e+ e− propagator:

$$G(k,\bar{k})\Big|_{m=0} = \frac{i}{\mu^2}[\not{k}-\not{\partial}_k]\,[\bar{\not{k}}-\not{\partial}_k]\int_0^\infty dt\,\frac{\exp(-t/4)}{(t+2k\cdot\bar{k})^2-\lambda(p^2,k^2,\bar{k}^2)}$$

where

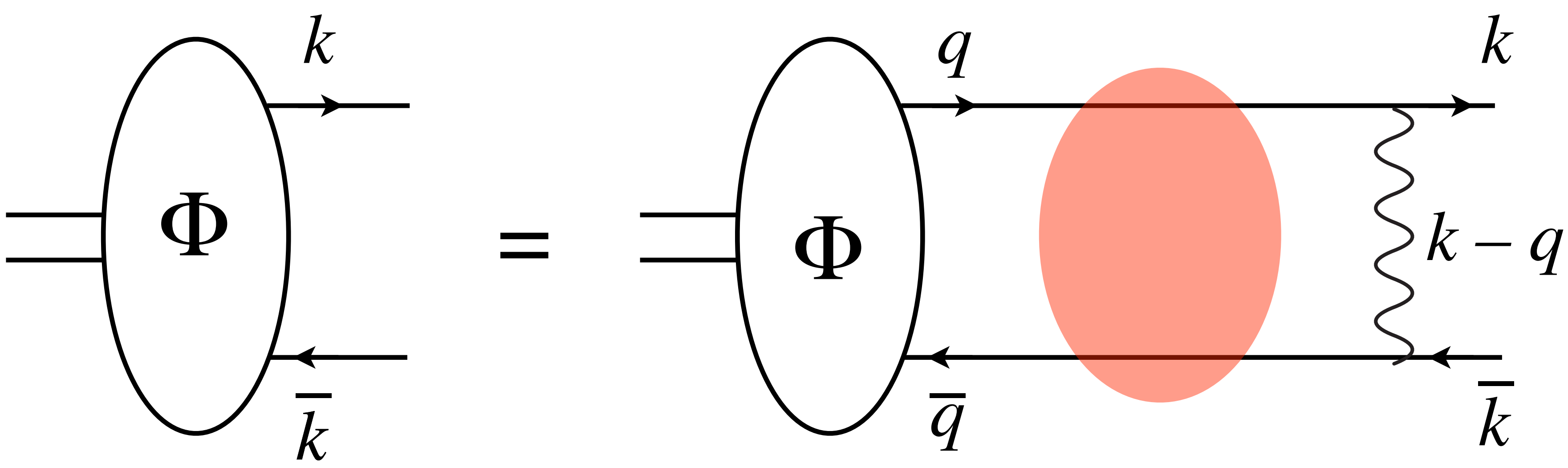
$$\lambda(p^2,k^2,\bar{k}^2) = \left[(p^2-\left(\sqrt{k^2}+\sqrt{\bar{k}^2}\right)^2\right]\left[(p^2-\left(\sqrt{k^2}-\sqrt{\bar{k}^2}\right)^2\right]$$

we see that G is enhanced close to the kinematic threshold for $p \to k+\bar{k}$
i.e., for

$$\sqrt{p^2} \simeq \sqrt{k^2}+\sqrt{\bar{k}^2}$$

We did not anticipate this property of the dressing, but it suggests a way to understand the non-relativistic quark model.

In the Bethe-Salpeter equation with single photon exchange



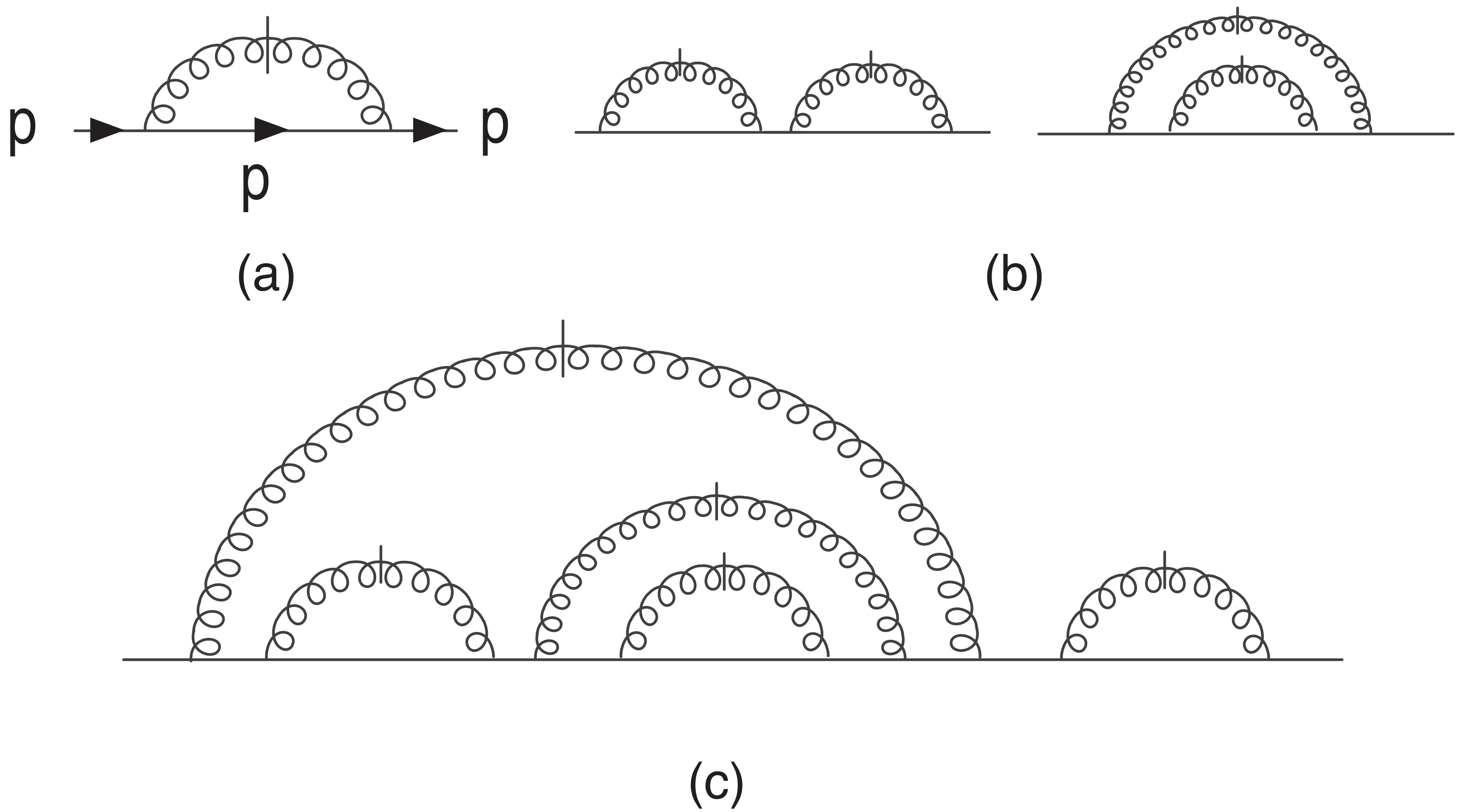
The small potential term $\propto \alpha$ on the rhs needs to be compensated by an enhanced propagator.

For positronium, this implies that both the electron and the positron are nearly on-shell.

The corresponding threshold enhancement of the dressed $e^+ e^-$ propagator may give non-relativistic bound states with dynamically generated "constituent" masses $\sqrt{k^2}$, $\sqrt{\bar{k}^2}$.

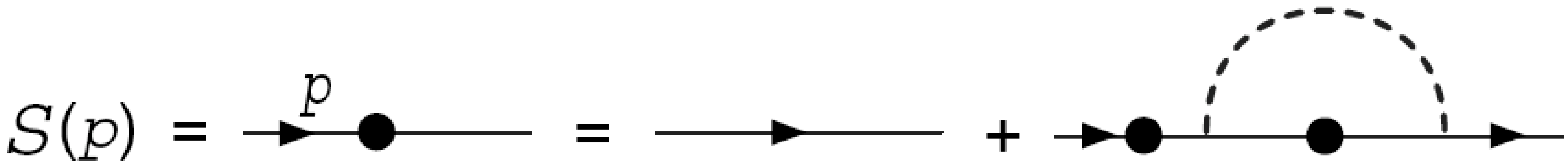
Planar QCD $(N \to \infty)$

For the massless quark propagator, we sum all planar diagrams of the form



There are much fewer planar diagrams than non-planar ones, resulting in a qualitatively different dressed propagator.

The diagrams can be directly summed, or one may note that it satisfies a Dyson-Schwinger type equation for quark propagator



$$iS(p) = \frac{i}{\not p} + C_F(-ig)^2(-\Lambda^2)\frac{i}{\not p}\gamma^\mu iS(p)\gamma_\mu iS(p)$$

This equation is algebraic and can be solved exactly, given the general form

$$-iS(p) = a(p^2)\not p + b(p^2)$$

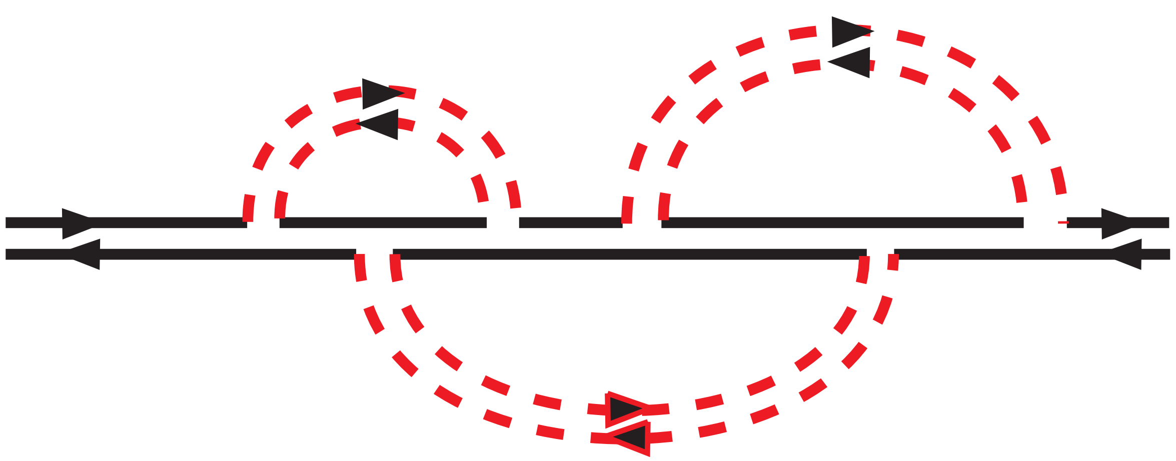
The massless (planar) dressed quark propagator is found to be

$$S(p) = \frac{2\not{p}}{p^2 + \sqrt{p^2(p^2 - 4\mu^2)}}$$

where $\mu^2 = g^2 N \Lambda^2$. This propagator was previously studied by

Munczek and Nemirovsky, PR D28 (1983) 181

The planar gluon propagator is dressed on both sides:



and does not satisfy a closed Dyson-Schwinger equation.

A sum of all planar diagrams gives the dressed gluon propagator as

$$iD^{\mu\nu}_{ab}(p) = \frac{-i}{p^2}\left[P^{\mu\nu}_T(p)\, d\left(-\frac{2\mu^2}{p^2}\right) + \xi\frac{p^\mu p^\nu}{p^2}\right]\delta_{ab}$$

where ζ is a covariant gauge parameter, $P^{\mu\nu}_T(p) = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$ and

$$d(x) = \frac{1}{2x}[1 - h(16x)] \qquad \text{with} \qquad h(x) = {}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}, 2, x)$$

The gluon propagator has a cut for $-32\mu^2 \le p^2 \le 0$, and since

$$d(x \to 0) = 1 + 2x + 10x^2 + \mathcal{O}\left(x^3\right)$$

it approaches the perturbative one in the limit $p^2 \to \infty$.

Concluding remarks

- We looked into the possibility of defining a perturbative expansion around a non-empty state.

- Dressing the free propagator with $p = 0$ gauge fields is in principle as justified as the Feynman $i\varepsilon$ prescription (”almost” a pure gauge).

- We found Lorentz and gauge invariant Green functions with the standard perturbative behavior at short distances, but new features at long distances.

- We may analogously dress higher point and higher order Green functions, and study, e.g., the high energy behavior of scattering amplitudes.

- The dressed double propagator is singular at threshold, $\sqrt{p^2} \simeq \sqrt{k^2} + \sqrt{\bar{k}^2}$, which may imply a bound state dynamics like the non-relativistic quark model.