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17.7 $\phi_2$, or $\alpha$

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In the Standard Model of particle physics (SM) the CKM matrix results in a set of nine unitary relationships, six of which are triangles in a complex plane (Chapter 16). The imaginary components of these triangles are manifestations of a single complex phase that dictates the amount of CP violation in the theory. The measurement of $\phi_0$ described in Chapter 17.6 establishes one of the angles of the unitarity triangle associated with $B_d$ decays, the so-called Unitarity triangle. To fully over-constrain the SM one needs to measure the other two angles and the sides of the Unitarity triangle. The second angle of the Unitarity triangle to be measured is $\phi_2$, which is the subject of this chapter. Together the measurements of $\phi_1$ and $\phi_2$ are sufficient to test the predictions of the SM. Constraints on the third angle, $\phi_3$, are discussed in Chapter 17.8 and how one typically interprets these results in the context of the SM is reviewed in Chapter 25.1.

A probe that can be used to determine $\phi_2$ is the measurement of the time-dependent CP asymmetry in $B^0 \to \pi^+\pi^-$ transitions. This CP violation is produced by the interference of the dominant box diagram for $B^0 \to \overline{B}^0$ mixing with the tree diagram $b \bar{d} \to w\pi d\bar{d}$. If these were the only contributing diagrams, the resulting CP asymmetry parameters would be $S = \sin 2\phi_2$ and $C = 0$. However the situation is not so simple: this final state is also produced by higher order weak transitions, and of particular relevance is the one-loop diagram usually called the ‘penguin’ diagram (Shifman, Vainshtein, and Zakharov, 1977). The presence of penguin contributions with weak phases that differ from the leading order tree results in theoretical uncertainties on $\phi_2$ that are sometimes referred to as ‘penguin pollution’ in the literature. The penguin contribution affects $B(B^0 \to \pi^+\pi^-)$ and its CP asymmetry (Bigi, Khoze, Uraltsev, and Sanda, 1989). Gronau and London (1990) emphasized isospin symmetry to correct for the effect of penguin contributions when extracting $\phi_2$. At first it was thought that penguin contributions are very small
in the SM for $B^0 \rightarrow \pi\pi$, since the amplitude $b \rightarrow dq\bar{q}$ is suppressed by a factor of $|\lambda| = |V_{ub}|$ relatively to $b \rightarrow s\bar{q}q$. However, data showed that the $B^0 \rightarrow \pi^+\pi^-$ rate is larger than had been initially expected, and this is explained by

the presence of a sizeable penguin contribution; therefore

penguin amplitudes can significantly affect the extraction of $\phi_2$. Thus the measurement of $\phi_2$ with $B \rightarrow \pi\pi$ requires a more complicated approach than the measurement of $\phi_1$ with $B \rightarrow J/\psi K_{S}$.

The theoretical issues associated with this approach are described in Section 17.7.1.1, and the corresponding experimental treatment is summarized in Section 17.7.3.1. It is worth noting that new physics (NP) could enhance penguin contributions significantly. Experimentally one could identify such contributions by observing a significant difference between values of $\phi_2$ obtained using different decay modes.

The $CP$ asymmetries in all $\bar{b}d \rightarrow \pi\pi\bar{b}\bar{d}$ decays depend on $\phi_2$, but the impact of penguin amplitudes in general can be different for different final states. When it was recognized that the measurement of $\phi_2$ via $B^0 \rightarrow \pi\pi$ would be less sensitive than anticipated due to a large penguin amplitude it became necessary to explore experimentally and theoretically more difficult scenarios in the hope that nature was kind enough to permit measurement of this angle in another way. As a result, the $B$ Factories approached the problem using a rather different technique than that of a time-dependent analysis of the Dalitz plot of $B^0 \rightarrow \pi^+\pi^-\pi^0$. The theoretical issues related to this measurement are introduced in Section 17.7.1.2, while the corresponding experimental discussion can be found in Section 17.7.4. Given current data samples the primary importance of including $B^0 \rightarrow \pi^+\pi^-\pi^0$ transitions in the extraction of $\phi_2$ is to help resolve discrete ambiguities in the weak phase arising from the interpretation of other measurements. The resulting constraints obtained from $B$ Factories data do not add a significant amount of information to improve the accuracy of the SM solution for $\phi_2$, however they suppress these discrete ambiguities.

The decay $B^0 \rightarrow \pi^+\pi^-\pi^0$ is expected to ultimately dominate the experimental determination of $\phi_2$ and to provide a sensitive probe for the impact of NP and its features as a non-leading source of $CP$ violation.

After several years of data taking it became apparent that extraction of $\phi_2$ from the $B$ Factories is a difficult enterprise. Thus it was realized that one has to think about other final states and $BABAR$ started to investigate other related options such as $B \rightarrow \rho\rho$ decays. They were previously dismissed by the community as experimentally and theoretically too challenging to be a viable alternative compared with the already ambitious attempts to study $B \rightarrow \pi\pi$ and $B \rightarrow \rho\pi$. When the experimental work commenced, the outcome of this endeavor was not entirely clear; however, there were hints that indicated these modes could be more promising than originally thought. The presence of two vector particles in the final state meant that one would have to perform a full angular analysis of the final state (see Chapter 12) in addition to constraining penguin contributions. However, it was possible to piece together sufficient information from various sources in order to motivate attempting the measurement of $\phi_2$ with $B \rightarrow \rho\rho$ decays. Ultimately a full angular analysis was not required to constrain $\phi_2$ as the fraction of longitudinally polarized events was found to almost completely dominate (Section 17.7.3.2). The result of this approach turned out to provide the most stringent constraint on $\phi_2$, where the efforts of $BABAR$ and Belle are summarized in Section 17.7.3.2. The time-dependent analysis of $B^0 \rightarrow \rho^0\rho^0$ promises to help resolve some of the discrete ambiguities inherent in the isospin analysis and is discussed in Section 17.7.3.3. As a further development one constrains $\phi_2$ using final states including vector and axial-vectors particles, in particular using $B \rightarrow a_1(1260)\pi$ decays, where one can determine the impact of penguin contributions with the aid of $SU(3)$ flavor symmetry. This theoretical approach is discussed in Section 17.7.1.3. Time-dependent measurements of $B \rightarrow a_1(1260)\pi$ and the complementary studies of $B \rightarrow K_1\pi$ decays are used to control penguin pollution as discussed in Section 17.7.5. A complementary cross-check using $SU(3)$ for $B \rightarrow pp$ and $K^+\rho$ decays is discussed in Section 17.7.6.

In contrast to the initial expectations of the B Factories where it was anticipated that $\pi\pi$ final states would provide a measurement of $\phi_2$ and $\rho\pi$ would be used to resolve ambiguities, ‘reality’ told a different story. Measurements of $B \rightarrow \rho\rho$ decays dominate the determination of the angle $\phi_2$ and $B \rightarrow a_1(1260)\pi$ decays provide additional precision on the overall measurement of this angle. The study of $\rho\pi$ final states provides additional discrimination: the power to resolve between some of the discrete ambiguities, as originally expected. The $B$ Factories have been able to make an accurate measurement of $\phi_2$, using $B$ decays to $\pi\pi$, $\rho\pi$, and $a_1$ final states, as discussed in Section 17.7.7.

17.7.1 Introduction

The angle $\phi_2$ can be inferred from time-dependent $CP$ asymmetries in charmless $b \rightarrow u$ transitions. Feynman diagrams describing these decays, such as $B^0 \rightarrow \pi\pi$ and $B^0 \rightarrow pp$, are shown in Fig. 17.7.1. Interference between the leading tree amplitude and the amplitude of $B^0 - B^0$ mixing (Fig. ??) provides access to the observable $\phi_2$. As explained above, if the tree amplitude was the only decay amplitude (as is the case for $B^0 \rightarrow J/\psi K_{S}$), the $S$ parameter in $B^0 \rightarrow \pi^+\pi^-$ would be equal to sin$2\phi_2$ and $C$ zero (see Eq. ??). However, the penguin contributions to charmless $B$ decays cannot be ignored. In general one measures sin$2\phi_2^{\text{eff}}$ instead sin$2\phi_2$, where $\phi_2^{\text{eff}}$ is related to $\phi_2$ up to a shift $\Delta\phi_2$ resulting from penguin amplitudes with a different weak phase to that of the leading order tree contribution, i.e. $\Delta\phi_2 = \phi_2^{\text{eff}} - \phi_2$. We have to understand how to control the penguin contributions and determine the difference between $\phi_2^{\text{eff}}$ and $\phi_2$.

In the following, we discuss three complementary techniques to extract the angle $\phi_2$ from time-dependent $CP$ asymmetry measurements in $B \rightarrow 2\pi$, $3\pi$ and $4\pi$ decays.

- Isospin analysis in $B \rightarrow \pi\pi$ and $B \rightarrow pp$;
Dalitz analysis in $B \to \rho \pi$;  
- SU(3) analysis of $B \to a_1(1260)\pi(K)$ and SU(3) constraints in charmless $B$ decays to two vector meson final states.

The combined accuracy on $\phi_2$ obtained by the $B$ Factories is discussed in Section 17.7.7. Some time-integrated measurements are required to constrain penguin contributions in various decays; those are discussed in Section 17.4.

### 17.7.1 Isospin analysis of $B \to \pi \pi$ and $B \to \rho \rho$

The $CP$ asymmetry in $B^0 \to \pi^+\pi^-$ depends on $\phi_2$. However (unlike $B^0 \to J/\psi K^0_s$) in the SM one expects direct $CP$ violation to be manifest in $B^0 \to \pi^+\pi^-$, hence:

$$\frac{\Gamma(B \to \pi^+\pi^-) - \Gamma(B \to \pi^-\pi^+)}{\Gamma(B \to \pi^+\pi^-) + \Gamma(B \to \pi^-\pi^+)} = C \cos \Delta m_d \Delta t - S \sin \Delta m_d \Delta t$$  \hfill (17.7.1)

with

$$C = \frac{1 - |\lambda|^2}{1 + |\lambda|^2}, \quad S = \frac{2 \Im \left( \frac{q}{p} \mathcal{R}(\pi^+\pi^-) \right)}{1 + |\lambda|^2}$$  \hfill (17.7.2)

where $\lambda = (q/p)\mathcal{A}/A$ as noted in Chapter 10, and from Chapter 16 we recall that

$$0 \leq C^2 + S^2 \leq 1.$$  \hfill (17.7.4)

Here $\mathcal{R}$ refers to the amplitude of the $B^0$ decay to the final state normalized by the $B^0$ decay to the same one (Eq. 17.7.7). $CP$ violation is manifest if $0 < C^2 + S^2$, i.e. if either of the asymmetry parameters are non-zero. Without penguin contributions one predicts in the SM

$$\left| \frac{q}{p} \mathcal{R}(\pi^+\pi^-) \right| \simeq 1$$  \hfill (17.7.5)

$$\Im \left( \frac{q}{p} \mathcal{R}(\pi^+\pi^-) \right) \simeq \sin 2\phi_2,$$  \hfill (17.7.6)

where for this simplified case we have

$$\mathcal{R}(\pi^+\pi^-) \equiv \frac{A(B^0 \to \pi^+\pi^-)}{A(B^0 \to \pi^-\pi^+)}.$$  \hfill (17.7.7)

However, penguin amplitudes do contribute. In this case one finds $\left| \frac{q}{p} \mathcal{R}(\pi^+\pi^-) \right| \neq 1$ and therefore $C^2 \neq 0$. Our control of the quantitative impact of penguin amplitudes and in general non-perturbative QCD is rather limited. It has been predicted by Bigi, Khoze, Uraltsev, and Sanda (1989) that:

$$-\Im \left( \frac{q}{p} \mathcal{R}(B^0 \to \pi^+\pi^-) \right) \simeq \left| \frac{q}{p} \mathcal{R}(B^0 \to \pi^+\pi^-) \right| \sin 2\phi_2$$  \hfill (17.7.8)

with $\left| \mathcal{R}(B^0 \to \pi^+\pi^-) \right| \sim 1$.

One technique for measuring $\phi_2$ is to study time dependent $CP$ asymmetries in $B^0 \to \pi^+\pi^-$ decays (Aubert (2002); Abe (2003b)). The data show (see Sec. 17.7.3.1) that:

$$S = -0.65 \pm 0.07,$$  \hfill (17.7.9)

$$C = -0.30 \pm 0.05,$$  \hfill (17.7.10)

which are consistent with expectation from the SM. One can conclude that penguin amplitudes are significant (however it is not possible to determine if those contributions are from the SM or from NP). Therefore one cannot directly obtain $\phi_2$ from the time-dependent analysis and use this with the value of $\phi_1$ from $B^0 \to J/\psi K^0_s$, discussed in Section 17.6, to construct the SM unitarity triangle. A complementary study in $B^0 \to \rho^+\rho^-$ was pioneered by BABAR (Aubert, 2004c) with the hope of being able to contribute to the measurement of $\phi_2$. However once again the extraction of this angle is complicated by the presence of both tree and penguin amplitudes, with different weak phases. An isospin analysis of the $\pi \pi$ or $\rho \rho$ system is necessary (Gronau and London, 1990) to disentangle the tree contribution, and hence determine $\phi_2$ as explained in the following.

The all-charged modes ($B^0 \to \pi^+\pi^-$ and $B^0 \to \rho^+\rho^-$) are dominated by the external tree ($T$) and gluonic penguin ($P$) amplitudes, while the all-neutral modes ($B^0 \to \pi^0\pi^0$ and $B^0 \to \rho^0\rho^0$) are very sensitive to the $P$ contribution, since the internal tree diagram $(C)$ is color-suppressed. The amplitudes $A^{00} \equiv A(B^0 \to h^0\bar{h}^0)$, $A^{+0} \equiv A(B^0 \to h^+h^-)$, and $A^{++} \equiv A(B^0 \to h^+h^0)$, where $h = \pi, \rho$, and their complex conjugates, obey the Gronau-London isospin relation (Gronau and London, 1990). Here
we note that $I = 1/2$ only for $u$ and $d$ quarks and as a result the isospin decomposition of $B \to \pi \pi$ decays follows the corresponding $K \to \pi \pi$ case. Bose statistics forbids $I = 1$ $\pi \pi$ final states, which simplifies the isospin construction of these decays. The tree topologies shown in Fig. 17.7.1 come from operators that describe $\Delta I = 1/2$ or $\Delta I = 3/2$ transitions, and so as a $B$ meson has $I = 1/2$, the corresponding final states can either be $I = 0$ or 2. In contrast the penguin contributions come from a $\Delta I = 1/2$ operator, hence these can only be $I = 0$. Both $h^+ h^0$ and $h^- h^0$ final states can be $I = 0$ or 2, and so in general these decays may proceed via both tree and penguin transitions. In contrast, the $h^- h^0$ final state is $I = 2$ and therefore can only have tree contributions. The resulting isospin relations obtained by Gronau and London are:

$$A^{+-}/\sqrt{2} + A^{00} = A^{+0},$$  \hspace{1cm} (17.7.1b)

$$A^{++}/\sqrt{2} + A^{00} = A^{++},$$  \hspace{1cm} (17.7.12)

each of which can be represented by a triangle in a complex plane (Fig. 17.7.2). The $CP$ conjugate relation is usually shown with a tilde replacing the bar to denote that the bases of the two isospin triangles have been aligned such that $A^{+0} = A^{+0}$, which explicitly neglects any effect coming from electroweak (EW) penguins.\footnote{Electroweak penguins have the same topology as the gluonic penguin shown in Fig 17.7.1, but are mediated by a photon or $Z^0$ boson. It is expected that EW penguins are small and can be neglected. This assumption can be tested by constraining the level of direct $CP$ violation found in $B^+ \to h^+ h^0$ decays, which is predicted to be zero in the absence of any EW penguin contribution.}

The isospin analysis of the vector-vector modes $B \to \rho \rho$ is more complicated than that for $B \to \pi \pi$. The $\rho \rho$ final states include three contributions of different $CP$ eigenvalues. As a result there are three isospin analyses that can be performed, one for each of the transversity amplitudes (i.e., one for the longitudinal and two for the transverse parts following the discussion in Chapter 12). Naive factorization expectations (Suzuki, 2002) indicated that one would expect the longitudinal polarization ($CP$ even) to dominate over the transverse one (a $CP$ admixture), which had the implication that analysis of these decays could be simplified from a full angular treatment to a simplified one where only the fraction of longitudinally polarized events needed to be extracted from data (see Chapter 12). However, the polarization measurements of charmless $B$ decays available at the time were not straightforward and did not all support the expectation of nearly a 100% longitudinal polarization contribution (see Section 17.4). It had also been noted in (Aleksan et al., 1995) that using naive factorization calculations one obtains a definite hierarchy of penguin contributions in $B \to \pi \pi$, $\rho \pi$ and $\rho \rho$ final states. The results of these calculations implied that the penguin contributions would be largest for $B \to \pi \pi$ decays and smallest for $B \to \rho \rho$. However at the time the $B$ Factories started taking data this message had not been widely appreciated by the community. Given that one expects the ratio of amplitudes of $\rho \rho$ to $\pi \pi$ decays to be $O(f_\rho^2/f_\pi^2) \sim 2.5$, one could piece together a credible theoretically motivated scenario that indicated the $\rho \rho$ final states might be an attractive alternative way to measure $\phi_2$, if one could overcome the experimental challenges. Fortunately, the longitudinal polarization in $B^0 \to \rho^+ \rho^-$ final state has been found to be consistent with unity (Aubert (2004b); Somov (2006)). Moreover, the neutral branching fraction $B^0 \to \rho^0 \rho^0$ was found to be relatively small (Aubert (2007b, 2008a); Chiang (2008)), which constrains the penguin uncertainty in the $B \to \rho \rho$ system significantly (see Section 17.4 for the details of the branching fraction measurements of the decays $B^0 \to \rho^0 \rho^0$ and $B^+ \to \rho^+ \rho^0$). Shortly after the observation of $B^0 \to \rho^0 \rho^0$ by BABAR performed a time-dependent $CP$ asymmetry measurement as a proof of principle that one could indeed constrain $\phi_2$ (Aubert, 2004c) using larger data samples. It has been noted by Falk et al. (2004) that there could be a small $I = 1$ component to $B \to \rho \rho$, which could be tested by measuring $S$ as a function of the dif-

![Fig. 17.7.2. Gronau-London isospin triangles for $B^0 \to hh$ (solid lines) and $\bar{B}^0 \to hh$ (dashed lines), drawn for illustration purposes (not to scale). The $B^0$ amplitudes are denoted with a tilde to highlight that the two triangles have been rotated relative to each other so that the $h^+ h^0$ and $h^- h^0$ amplitudes are aligned.](image)
ference between the mass of the two $\rho$'s. Any departure from uniformity would indicate that there is an $I = 1$ component, in which case the isospin construct required to correct for penguins would require some modification.

The analysis methodology outlined in the remainder of this chapter in terms of the study of four body final states relies on the quasi-two-body approximation, which is sufficient for work at the $B$ Factories. However, it should be borne in mind that in the future one will want to probe the impact of NP in $4\pi$ and $2\pi K\bar{K}$ final states to search for possible non-leading sources of CP violation. Amplitude analyses will be required for such searches and future Super Flavor Factories will have to adopt a more general approach for such analyses.

17.7.1.2 Dalitz analysis of $B \to \rho\pi$

The $B$ Factories have performed analyses of the quasi-two-body final states $B \to \rho\pi$ to validate our theoretical control over extracting $\phi_2$. A familiar picture can be seen at the quark level: in the SM the relevant transitions are $B^0 \to B^\pm$ amplitude and this interesting part of the transition is complicated by the presence of $b \to d$ penguin terms. Thus the presence of penguins complicates the interpretation of any measured CP asymmetries, and hence complicates the extraction of $\phi_2$.

A proposed analysis of quasi-two-body final states (Lipkin, Nir, Quinn, and Snyder, 1991; Snyder and Quinn, 1993) relies on the isospin symmetry of the rates of all $B \to \rho\pi$ modes. The decay channels $B^+ \to \rho^+\pi^+$ and $B^0 \to \rho^0\pi^+$ have been first observed by Belle (Gordon, 2002) and $\bar{B}\bar{A}$R (Aubert, 2004a). Evidence for the $B^0 \to \rho^0\pi^0$ mode, which was expected to be small, has been reported by Belle (Dragic, 2004) with a rate higher than an upper bound obtained by $\bar{B}\bar{A}$R (Aubert, 2004a). However, these two results are in agreement at the level of 1.5$\sigma$. The remaining mode $B^- \to \rho^-\pi^0$ has two neutral pions in the final state that makes it a challenging measurement. $\bar{B}\bar{A}$R has reported the observation of this mode (Aubert, 2004a). These analyses are described in detail in Section 17.4.

A better approach uses a Dalitz plot analysis of $B \to 3\pi$ final states, which relaxes the quasi-two-body approximation and use information from the interference between resonances in the corners of the Dalitz plot. Snyder and Quinn (1993) pointed out that a time-dependent Dalitz plot analysis (TDPA) of $B^0 \to \rho\pi \to \pi^+\pi^-\pi^0$ offers a unique way to determine the angle $\phi_2$ without discrete ambiguities. The TDPA uses isospin and takes into account contamination from $b \to d$ penguin transitions. In addition, one can use measurements of $B^+ \to \rho^+\pi^0$ and $\rho^-\pi^+$ states to provide additional information to constrain $\phi_2$ (Gronau, 1991; Lipkin, Nir, Quinn, and Snyder, 1991).

Technicalities required to perform a TDPA can be found in Chapter 13.

A preliminary TDPA was reported by $\bar{B}\bar{A}$R at ICHEP in 2004 using a data sample of 213 million $B\bar{B}$ pairs. Subsequent analyses have been published by both Belle (Kusaka, 2007) and $\bar{B}\bar{A}$R (Aubert, 2007d) using larger data samples.

Future flavor factories should study the Dalitz plots for $B \to 3\pi$ states beyond intermediate $\rho\pi$ contributions and also explore $B \to K\bar{K}\pi$ to search for possible signs of NP as a non-leading source of CP violation.

17.7.1.3 $B \to a_1(1260)\pi$, $B^0 \to a_1(1260)K$ constraints

The last set of decay modes considered at the $B$ Factory experiments for the extraction of $\phi_2$ is $B^0 \to a_1^0(1260)\pi^\pm$, with $a_1^0(1260) \to \pi^+\pi^0\pi^\pm$. As with the previous examples these decays proceed mainly via $b \to ud$ tree amplitudes which can be used to measure time-dependent CP asymmetries and allow one to extract the angle $\phi_2$. As with the other modes discussed in this section the existence of non-trivial penguin amplitudes complicates the extraction of $\phi_2$. Gronau and Zupan (2004) proposed an SU(3)-based procedure for extracting $\phi_2$ in the presence of penguin contributions that the $B$ Factories have followed. This procedure requires measurements of $B$ meson decays into the axial-vector plus pseudoscalar final states $a_1\pi$, $a_1K$, and $K\pi$.

$\bar{B}\bar{A}$R (Aubert, 2006a) and Belle (Dalseno, 2012) measure the branching fraction of the $B^0$ meson decay to $a_1^0(1260)$ $\pi^\pm$ to be relatively large ($\sim 3 \times 10^{-5}$; see Section 17.4). Following on from the observation of this decay mode $\bar{B}\bar{A}$R performed a set of measurements of $a_1\pi$ and $a_1K$ to extract the angle $\phi_2$. This includes the time-dependent asymmetry measurement of $B$ decays to $a_1^0(1260)\pi^\pm$ (Aubert, 2007c), and observation of both $B^+ \to a_1^+ (1260) K^0$ and $B^0 \to a_1^- (1260) K^+$ decays (Aubert, 2008b). The final piece of information required to constrain $\phi_2$ using this approach is the branching fraction of $B$ decays to $K_{1A}\pi$, which was also measured by $\bar{B}\bar{A}$R (Aubert, 2010), where $K_{1A}$ denotes the axial vector excited $K$ meson states.

As with $B \to \rho\pi$ decays, $B$ meson decays to $a_1^0(1260)\pi^\pm$ final states are not CP eigenstate decays, so to extract $\phi_2$ from these channels one needs to simultaneously consider $B^0(\bar{B}^0) \to a_1^0(1260)\pi^\pm$ and $B^0(\bar{B}^0) \to a_1^0(1260)\pi^\mp$ transitions (Aleksejev, Dunietz, Kayser, and Le Diberder, 1991). One might cope with the difficulty due to the contribution of penguin amplitudes by using isospin symmetry (Gardner, 1999; Gronau, 1991; Gronau and London, 1990; Gronau and Zupan, 2004; Lipkin, Nir, Quinn, and Snyder, 1991) or a TDPA (Quinn and Silva, 2000; Snyder and Quinn, 1993) or approximate SU(3) flavor symmetry (Charles, 1999; Gronau, London, Sinha, and Sinha, 2001; Grossman and Quinn, 1998).

A full isospin analysis (Gardner, 1999; Gronau, 1991; Gronau and London, 1990; Gronau and Zupan, 2004; Lipkin, Nir, Quinn, and Snyder, 1991) requires the precise measurement of the branching fractions and asymmetries in the five modes (and their CP conjugates) $B^0 \to a_1^0(1260)\pi^\pm$, $a_1^0(1260)\pi^\mp$, $a_1^0(1260)\pi^0$, $B^+ \to a_1^+(1260)\pi^\mp$, $a_1^+(1260)\pi^\mp$. Currently the poor precision of most of these measurements (Aubert, 2007c,e) does not permit the application of this method.
As pointed out in the references (Quinn and Silva, 2000; Snyder and Quinn, 1993) the angle φ2 may be excited without ambiguities from a TDPA. This method has been successfully applied to the decay B0 → π+π−π0 by both experiments. This process could be applied to the decay B0 → π+π−π0π0 with contributions from a1(1260), a′1(1260)π0, a1(1260)π0π0, and ρ0ρ0 amplitudes or to the decay B0 → π+π−π0π+ with contributions from a1(1260)π+ a′1(1260)π+π0, and ρ0ρπ amplitudes. Such analyses would be difficult because of the four particles in the final state, the small overlapping region of the phase space of the pions from the a1(1260) and a′1(1260) mesons, uncertainties in the a1(1260) meson parameters and line shape, the small number of signal events and the expected large background.

The method chosen by BABAR for the study of B0 → a1(1260)π± decays follows the quasi-two-body approximation. The decays B0(B0) → a1(1260)π± have been reconstructed with a1(1260) → π+π−π±. The other subdecay modes with a1(1260) → π±π0π0 could be used to enhance statistics, however these are ignored as they have low reconstruction efficiency and large background. From a time-dependent CP analysis one extracts an effective angle φ2eff which, in analogy with the approaches described above, is an approximate measure of the angle φ2. Details on this approach for the decays B0 → a1(1260)π± are discussed by Gronau and Zupan (2004). Applying flavor SU(3) symmetry one can determine an upper bound on ∆φ2 = |φ2 − φ2eff| by relating the B0 → a1(1260)π± decays to decay rates with those of the ∆S = 1 transitions involving the same SU(3) multiplet of a1(1260), B0 → a1(1260)K and B0 → K1Aπ. The K1A meson is a nearly equal admixture of the K1(1270) and K1(1400) resonances (Amsler et al., 2008). The rates of B0 → K1Aπ decays can be derived from the decay rates of B0 → K1(1270)π and B0 → K1(1400)ππ.

Motivated by the B0 → a1π study BABAR performed a search for the related decay B0 → a1π using a data sample of 100 fb−1, but were unable to establish the presence of a significant signal (Aubert, 2006b). Future experiments may have sufficient data to isolate a clean sample of a1π and augment the list of channels used in the determination of φ2.

17.7.1.4 SU(3) constraints on φ2 using B0 → ρ+ρ− and B0 → K∗0ρ0π+ decays

A way to constrain penguin contributions to B0 → ρ+ρ− decays using SU(3) flavor symmetry was proposed by Beneke et al. (2006), and is referred to here as the BGRS method. The amplitude of this decay has SM contributions from both tree and penguin topologies, so may be written as

$$A(B^0 \to \rho^+\rho^-) = T e^{i\phi_3} + P e^{i\delta_{PT}},$$

(17.7.13)

where T and P are the magnitudes of the tree and penguin contributions to the decay, φ3 is the unitarity triangle angle introduced in Chapter 16, and δPT is the phase difference between the tree and penguin contributions. Interference between the amplitudes in Eq. (17.7.13) and those responsible for B0 → Bπ mixing results in the time-dependent asymmetry of this decay being sensitive to φ2 as discussed above. The SU(3) related decay B0 → K∗0ρ+ only proceeds via a penguin transition, so one can use knowledge of the branching fraction and longitudinal polarization of this decay to constrain the corresponding penguin contribution in ρ+ρ− up to SU(3) breaking corrections2.

In practice in order to constrain φ2 using this approach one needs to have seven experimental inputs in total: the branching fractions and fraction of longitudinally polarized events for the two decays, as well as S and C measured for ρ+ρ−, and finally the value of φ1 obtained from b → cπ transitions (see Section 17.6). These experimental inputs can be used to constrain the three unknowns: φ2, δPT, and rPT = |P/T| using

$$C = \frac{2r_{\text{PT}} \sin \delta_{\text{PT}} \sin(\phi_1 + \phi_2)}{1 - 2r_{\text{PT}} \cos \delta_{\text{PT}} \cos(\phi_1 + \phi_2) + r_{\text{PT}}^2},$$

(17.7.14)

$$S = \frac{\sin 2\phi_2 + 2r_{\text{PT}} \cos \delta_{\text{PT}} \sin(\phi_1 - \phi_2) - r_{\text{PT}}^2 \sin 2\phi_1}{1 - 2r_{\text{PT}} \cos \delta_{\text{PT}} \cos(\phi_1 + \phi_2) + r_{\text{PT}}^2},$$

(17.7.15)

and

$$\left(\frac{|V_{td}|f_{\rho-}}{|V_{cd}|f_{K+}}\right)^2 \frac{\Gamma_{1}(B^+ \to K^{*0}\rho^{+})}{\Gamma_{2}(B^0 \to \rho^{+}\rho^{-})} = \frac{1}{1 - 2r_{\text{PT}} \cos \delta_{\text{PT}} \cos(\phi_1 + \phi_2) + r_{\text{PT}}^2},$$

(17.7.16)

where F is an SU(3) breaking term, and f0 (fK+) is the ρ (K∗) decay constant. The factor F is estimated to be 0.9 ± 0.6 (Beneke et al., 2006). In fact it turns out that SU(3) breaking has little effect on the overall constraint obtained for φ2, and one can obtain a precision comparable to the isospin analysis approach even with 100% SU(3) breaking uncertainty. The decay widths Γi in Eq. (17.7.16) can be replaced by the corresponding branching fractions multiplied by the ratio of B0 to B± lifetimes. This approach provides a stringent constraint on φ2 that can be used as a cross-check of the traditional SU(2) isospin analysis. Results of using this approach can be found in Section 17.7.6, but given that the same inputs are used for this approach and the isospin analysis, one should take care not to combine the results obtained from the two methods when computing a global average for φ2.

17.7.2 Event reconstruction

The reconstruction of the charmless B decays and event selection follows a similar sequence in both BABAR and...
Belle. First, a sample of charged tracks and photons is selected. Typically, charged tracks are required to originate from the interaction region and to be identified as pions (Chapter 5). An electron veto is often applied. After an initial π⁰ selection based on two-photon candidates, the π⁰ candidates are kinematically constrained to the nominal π⁰ mass. Track and π⁰ candidates are combined to produce composite candidates [ρ, ω(1260)], and finally, signal B candidates are formed. The beam energy substituted mass m_{ES} and the energy difference ΔE are calculated for these candidates (see Chapter 7).

For time-dependent CP analyses, the flavor of the B candidates is determined using a flavor-tagging algorithm (Chapter 8), and the proper time difference Δt between the signal B and B_{tag} decay vertices is measured (Chapter 6). Finally, the signal yields and other decay properties (polarization, CP asymmetries) are determined in a multivariate maximum likelihood fit.

The continuum process e^+e^- → q̅q (q = u, d, s, c) is the main source of background for the charmless B decays. In order to suppress this background, charmless analyses employ multi-variate discriminants based on event topology, which tends to be isotropic for BB events and jet-like for q̅q events. A detailed description of these methods is given in Chapter 9.

Additional discrimination against the continuum background is provided by the output of the B-flavor tagging algorithms (Chapter 8). In Belle, the tag parameter r ranges from 0 to 1 and is a measure of the likelihood that the b flavor of the accompanying B meson is correctly assigned by the Belle flavor-tagging algorithm. Events with high values of r are well-tagged and are less likely to originate from continuum production. It is found that there is no strong correlation between r and any of the topological variables used above to separate signal from continuum. In BABAR, the background discrimination power arises from the difference between the tag efficiencies for signal and continuum background in seven tag categories (r_{tag} = 1…7) which is manifest in terms of a different signal purity in each tag category.

After the signal candidates are identified, the proper time difference Δt between the signal B and B_{tag} can be determined from the spatial separation between their decay vertices. The B_{tag} vertex is reconstructed from the remaining charged tracks in the event and its uncertainty dominates the Δt resolution σ_{Δt}. The typical proper time resolution is (σ_{Δt}) ≈ 0.7 ps. The distribution of the proper times provides further discrimination against the continuum backgrounds, which are characterized by small values of |Δt|.

### 17.7.3 Isospin analyses of B → ππ and B → ρρ

#### 17.7.3.1 B^0 → π^+π^−

The B Factories started performing time-dependent analyses of B^0 → π^+π^− early in their lifetime, and these results were updated on a number of occasions. The following describes only the most recent publications by BABAR (Lees, 2012) and Belle (Adachi, 2013), which use data samples of 467 × 10^6 and 772 × 10^6 BB pairs, respectively.

The candidates for the CP-eigenstate decay B^0 → π^+π^− are constructed from two oppositely charged tracks originating from a common vertex. A good quality of the vertex fit is required (Chapter 6). The m_{ES} and ΔE requirements are loose relative to the signal resolution (m_{ES} > 5.24 GeV/c² and 0.2 GeV < ΔE < 0.15 GeV at Belle, and m_{ES} > 5.2 GeV/c² and |ΔE| < 0.15 GeV at BABAR), leaving sidebands for accurate determination of the background level.

The dominant background for this decay mode is continuum. To discriminate between signal and continuum background, multivariate discriminants composed of event-shape variables are used. The definitions of these variables appear in Chapter 9, Belle applies a very loose requirement on the Fisher discriminant KSFW, rejecting 90% of the continuum background while keeping 90% of the signal events. BABAR requires |cosθ_3| < 0.91 and R_2 < 0.7, and then constructs a Fisher discriminant from the L_0 and L_2 moments.

The selected samples contain not only B^0 → π^+π^− signal events but also B → K±π±, q̅q, and background from higher multiplicity B decays. The signal and background yields and the CP parameters are determined from unbinned extended maximum likelihood fits. The Belle fit is performed using the variables ΔE, m_{ES}, L(π±), KSFW, the B_{tag} q and Δt, where L(π±) are the identification likelihoods for each of the pions from the ππ candidate. BABAR uses the fit variables ΔE, m_{ES}, the L_0 and L_2 Fisher, the DIRC Cherenkov angles and dE/dx values for each track, the B_{tag} flavor, and Δt. The Δt amplitude of the signal events is described by

\[ A(Δt) = \frac{1}{4} e^{-|Δt|/τ_{B^0}} \times (1 ± C \cos Δm_d Δt ± S \sin Δm_d Δt), \]  

(17.7.17)

up to vertex position resolution and flavor mistag effects, which are included in the fit p.d.f.s. The Δt p.d.f.s for all other event types in the sample are CP conserving.

The two collaborations use different methods for presenting the event distributions and fit functions. Belle applies cuts on discriminating variables and plots the distributions of signal and background of the remaining events, while BABAR uses χ²-plots (Section ??). As an example, Fig. 17.7.3 shows the ΔE distributions of the BABAR data overlaid with components of the fit functions. The Δt distributions and time-dependent CP asymmetries are shown in Fig. 17.7.4 for the Belle results.

The fit to the Belle data yields 2964 ± 88 B^0 → π^+π^− events, 9205 ± 124 B^0 → K±π± events, and 23 ± 35 B^0 → K^+K^- events. In both analyses, most of the selected candidates (almost 98%) are from continuum background. The BABAR fit finds 1394 ± 54 π^+π^- events, 5410 ± 90

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Note the similarity with Eq. (??), the sign differences resulting from the fact that the CP eigenvalue of the decay is opposite that of B^0 → J/ψK_S^0.
$K^\pm \pi^\mp$ events, and $7 \pm 17 K^+ K^-$ events. The CP violation parameters obtained by Belle are

$$S = -0.64 \pm 0.08 \pm 0.03,$$

$$C = -0.33 \pm 0.06 \pm 0.03,$$  \hspace{1cm} (17.7.18)

and those obtained by B\(\bar{A}Bar\) are

$$S = -0.68 \pm 0.10 \pm 0.03,$$

$$C = -0.25 \pm 0.08 \pm 0.02,$$  \hspace{1cm} (17.7.19)

where the first error is statistical and the second is systematic.

The systematic uncertainties account for a variety of systematic effects. These include biases in $\Delta t$ due to misalignment, detector size scale, and beam profile; parameters that are fixed in the fit; the parameterization of the detector resolution function, particle-identification performance, and flavor tagging performance and CP violation in the $B_{\kappa g}$ decay.

Historically there was indication of some level of disagreement between the $B$ Factory measurements of $S$ and $C$ in this decay mode. While this difference was never large enough to claim a significant discrepancy between the results, the evolution of the parameters from one conference season to the next remained of wide interest to the community. Over time these measurements slowly regressed toward a common mean, and as one can see from the results presented here — results from the two experiments agree with each other within uncertainties. The $B$ Factory average of these results is

$$S = -0.66 \pm 0.07,$$

$$C = -0.30 \pm 0.05,$$  \hspace{1cm} (17.7.20)

where the resulting correlation between $S$ and $C$ for the average is $-8.1\%$.

### 17.7.3.2 $B^0 \rightarrow \rho^+ \rho^-$

The $B^0 \rightarrow \rho^+ \rho^-$ candidates are reconstructed by combining pairs of oppositely charged $\rho$ mesons, which in turn are selected using $\pi^\pm$ candidates and $\pi^0$ candidates. As the $\rho$ meson is a wide resonance, when reconstructing the signal final state some events contain multiple reconstructed $B$ candidates. Most of these candidates arise from combinations of $\pi^0$ mesons from the flavor tag $B$ meson of the event being incorrectly assigned to the signal candidate. In such events the $B$ candidate with the smallest sum $\sum_{\pi^\pm}(m_{\pi^\pm} - m_{\pi^0})^2$ is selected. Other incorrectly reconstructed $B$ candidates appear from events with mis-reconstructed $\pi^\pm$ tracks. These events contain mis-reconstructed vertices and may bias time-dependent $CP$ measurements if not accounted for appropriately. The fraction of signal decays in data samples selected for the time-dependent measurements of $B\bar{A}Bar$ and Belle that have at least one $\pi^0$ track incorrectly identified but pass all selection criteria is $13.8\%$ and $6.5\%$, respectively. Signal decays that have at least one $\pi$ meson incorrectly identified are referred to as mis-reconstructed signal.
Similar to other charmless $B$ decays the dominant background for the $B^0 \rightarrow \rho^+\rho^-$ channel originates from $e^+e^- \rightarrow q\bar{q}$ $(q = u,d,s,c)$ continuum events. The separation of continuum background from the signal events is done using event topology variables described in Chapter 9. The procedures adopted are described briefly in the following paragraphs.

The Belle analysis uses a Fisher discriminant formed from modified Fox-Wolfram moments and $\theta_B$, the polar angle in the CM frame between the $B$ direction and the beam axis. These two variables are combined into a signal to background likelihood ratio, $R$. The $p.d.f.s$ for signal and $q\bar{q}$ components are obtained from MC simulation and the data sideband, respectively, and used to fit the selected $B \rightarrow \rho^+\rho^-$ candidates.

In the BABAR analysis $q\bar{q}$ background is reduced by requiring $|\cos(\theta_T)| < 0.8$, where $\theta_T$ is the angle between the thrust axis of the candidate tracks and that of the remaining tracks in the event. Further signal to background separation is performed by using a multi-layer perceptron (neural network, NN, see Chapt. 4), which is trained and validated using off-peak data (background) and MC simulated events (signal). Eight topological variables, see Aubert (2007a), are included into this neural network, and the output is transformed by a 1 : 1 mapping tuned to facilitate $p.d.f.$ parameterization so that the neural network output can be used in a maximum likelihood fit.

The following components are distinguished in both Belle and BABAR analyses: signal and $\rho\pi\pi$ non-resonant decays, mis-reconstructed signal events, continuum background ($q\bar{q}$), charm $B$ background ($b \rightarrow c$), and charmless ($b \rightarrow u$) background. The fitted yield for the non-resonant $4\pi$ component was found to be consistent with zero by both experiments. The ($b \rightarrow u$) background is dominated by $B \rightarrow (\rho\pi, a_1\pi, a_1\rho, \rho^+\rho^-)$ decays. Unbinned extended ML fits are used for the data analyses.

The latest BABAR analysis is based on a data sample of 383.6 million $B\bar{B}$ pairs (Aubert, 2007a) and supersedes two previous BABAR analyses (Aubert, 2004c, 2005). The signal yield, longitudinal polarization fraction $f_L$, and $CP$ asymmetry parameters $C$ and $S$ are obtained simultaneously from an unbinned extended ML fit to 37,424 events.

The background discriminating variables are $m_{ES}$, $\Delta E$, $\Delta t$, $m_{\pi^{+}\pi^{-}e^{+}}$, $\cos(\theta_{k})$, and $NN$. A total of 165 different possible background contributions are considered for the $b \rightarrow u$ background. However, only components where more than one event is expected to contribute to the selected data sample are modeled individually. Those modes where less than one event is expected are collected together and modeled using an inclusive component (split into neutral and charged contributions). As a result the BABAR analysis incorporates 22 components for different background types explicitly including non-resonant $\pi^{+}\pi^{-}\pi^{0}\pi^{0}$ and $\rho\pi\pi$ final states as possible secondary signals. The fit results are $N_{w} = 729 \pm 60$ events, $f_L = 0.992 \pm 0.024$, $C = 0.01 \pm 0.15$, and $S = -0.17 \pm 0.20$. Distributions of $m_{ES}$, $\Delta E$, $\cos(\theta_{\text{wp}})$, and $m_{\pi^{+}\pi^{-}e^{+}}$ for the highest purity tagged events are shown in Fig. 17.7.5. For the $\Delta E$, $\cos(\theta_{k})$, and $m_{\pi^{+}\pi^{-}e^{+}}$ distributions in the plot, $m_{ES}$ is required to be larger than $5.27 \text{ GeV}/c^2$. The $\Delta t$ distribution for $B^0$ and $\bar{B}^0$ tagged events and the time-dependent decay-rate asymmetry are presented in Fig. 17.7.6.

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In the BABAR analysis $q\bar{q}$ background is reduced by requiring $|\cos(\theta_T)| < 0.8$, where $\theta_T$ is the angle between the thrust axis of the candidate tracks and that of the remaining tracks in the event. Further signal to background separation is performed by using a multi-layer perceptron (neural network, NN, see Chapt. 4), which is trained and validated using off-peak data (background) and MC simulated events (signal). Eight topological variables, see Aubert (2007a), are included into this neural network, and the output is transformed by a 1 : 1 mapping tuned to facilitate $p.d.f.$ parameterization so that the neural network output can be used in a maximum likelihood fit.

The following components are distinguished in both Belle and BABAR analyses: signal and $\rho\pi\pi$ non-resonant decays, mis-reconstructed signal events, continuum background ($q\bar{q}$), charm $B$ background ($b \rightarrow c$), and charmless ($b \rightarrow u$) background. The fitted yield for the non-resonant $4\pi$ component was found to be consistent with zero by both experiments. The ($b \rightarrow u$) background is dominated by $B \rightarrow (\rho\pi, a_1\pi, a_1\rho, \rho^+\rho^-)$ decays. Unbinned extended ML fits are used for the data analyses.

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The background discriminating variables are $m_{ES}$, $\Delta E$, $\Delta t$, $m_{\pi^{+}\pi^{-}e^{+}}$, $\cos(\theta_{k})$, and $NN$. A total of 165 different possible background contributions are considered for the $b \rightarrow u$ background. However, only components where more than one event is expected to contribute to the selected data sample are modeled individually. Those modes where less than one event is expected are collected together and modeled using an inclusive component (split into neutral and charged contributions). As a result the BABAR analysis incorporates 22 components for different background types explicitly including non-resonant $\pi^{+}\pi^{-}\pi^{0}\pi^{0}$ and $\rho\pi\pi$ final states as possible secondary signals. The fit results are $N_{w} = 729 \pm 60$ events, $f_L = 0.992 \pm 0.024$, $C = 0.01 \pm 0.15$, and $S = -0.17 \pm 0.20$. Distributions of $m_{ES}$, $\Delta E$, $\cos(\theta_{\text{wp}})$, and $m_{\pi^{+}\pi^{-}e^{+}}$ for the highest purity tagged events are shown in Fig. 17.7.5. For the $\Delta E$, $\cos(\theta_{k})$, and $m_{\pi^{+}\pi^{-}e^{+}}$ distributions in the plot, $m_{ES}$ is required to be larger than $5.27 \text{ GeV}/c^2$. The $\Delta t$ distribution for $B^0$ and $\bar{B}^0$ tagged events and the time-dependent decay-rate asymmetry are presented in Fig. 17.7.6.

The latest Belle measurements of the $CP$ asymmetry parameters are based on a data sample of 535 million $B\bar{B}$ pairs (Somov, 2007). Measurements of the polarization fraction and the fraction of $\rho\pi\pi$ non-resonant events were performed in (Somov, 2006). The analysis is organized into two steps. During the first step the yields of signal and background components are obtained using an unbinned extended ML fit to the three-dimensional ($m_{ES}$, $\Delta E$, $R$) distribution. A total of 176843 events are
selected for the analysis. The fit yields $N_{\rho^0+p+p+\pi^-} = 576 \pm 53$ events. During the second step the $CP$ asymmetry parameters $S$ and $C$ are determined from a fit to the $\Delta t$ distributions. The signal region used for the $\Delta t$ fit is $5.27 \text{ GeV}/c^2 < m_{ES} < 5.29 \text{ GeV}/c^2$, $-0.12 \text{ GeV} < \Delta E < 0.08 \text{ GeV}$, and $R > 0.15$, a loose requirement to suppress background-like events. The fractions of mis-reconstructed signal events are fixed to the expectation from MC simulation, the fraction of non-resonant $B \rightarrow \rho^0 \pi^0$ decays is fixed from the result obtained in Somov (2006), and the fractions of correctly reconstructed signal, $b \rightarrow c$ background, $q\bar{q}$ background and $b \rightarrow u$ background are normalized to the event fractions obtained from the $(m_{ES}, \Delta E, R)$ fit and are fixed in the $\Delta t$ fit. The event fractions and $\Delta t$ $p.d.f.s$ depend on the flavor tag category assigned to an event. A fit to the 18016 events in the $(m_{ES}, \Delta E, R)$ signal region gives $C = 0.16 \pm 0.21$ and $S = 0.19 \pm 0.30$ where the quoted uncertainties are statistical.

The $B$ Factory combined values for $f_L$, $C$ and $S$ are

$$S = -0.05 \pm 0.17, \quad (17.7.21)$$

$$C = -0.06 \pm 0.13, \quad (17.7.22)$$

$$f_L = 0.978 \pm 0.023, \quad (17.7.23)$$

where the correlation between $S$ and $C$ is small. The measured $B^0 \rightarrow \rho^+ \rho^-$ branching fraction is given in Section 17.4.

17.7.3.3 $B^0 \rightarrow \rho^0 \rho^0$

The analyses reported here are based on the full data sample of both experiments, containing respectively 465 $\times$ $10^6$ $B\bar{B}$ pairs recorded with the $BABAR$ detector (Aubert, 2008a) and 772 $\times$ $10^6$ $B\bar{B}$ pairs collected with the Belle detector (Adachi, 2012). Two other analyses on a partial data sample were also published by $BABAR$ (Aubert, 2007b) and Belle (Chiang, 2008).

$B^0$ meson candidates are reconstructed from two $\rho^0$ candidates, each reconstructed from two oppositely charged pions. The analyses use six kinematic variables to reconstruct the signal: $m_{ES}$, $\Delta E$, the invariant $\pi^+\pi^-$ mass $m(\pi^+\pi^-)_{1,2}$, and the helicity angles $\theta_{1,2}$, defined as the angles between the $\pi^+$ and the $B$ flight direction in each $\rho^0$ rest frame. The $BABAR$ analysis applies the following kinematic selection: $5.245 < m_{ES} < 5.290 \text{ GeV}/c^2$, $|\Delta E| < 85 \text{ MeV}$, $0.55 < m(\pi^+\pi^-)_{1,2} < 1.05 \text{ GeV}/c^2$, and $|\cos\theta_{1,2}| < 0.98$. The Belle analysis requires that the invariant $\pi^+\pi^-$ mass lies within the signal window $m(\pi^+\pi^-)_{1,2}$ $< 0.705 \text{ GeV}/c^2$, $|\cos\theta_{1,2}| < 0.98$. In both cases the $\pi^+\pi^-$ mass window is chosen to accept $\rho^0$ into $\pi^+\pi^-$, $f_0(980)$ into $\pi^+\pi^-\gamma$ and non-resonant modes, and to exclude $K_S^0 \rightarrow \pi^+\pi^-$ and charm meson decays such as $D^0 \rightarrow \pi^+\pi^-$. Furthermore, to remove the peaking backgrounds from the $D_s^+$, $D^0$, $J/\psi$, and $K_S^0$ decays, corresponding mass vetoes are applied.

To distinguish $B\bar{B}$ events from the dominant jet-like continuum background, as described in Chapter 9, $BABAR$ uses a neural network-based discriminant, $N$, which combines eight topological variables, while Belle uses a Fisher discriminant, $F$, constructed from seven variables (Chapt. 4). These discriminants are used as inputs to the ML fits described below. In addition Belle places a loose requirement on $F$, which removes about 60% of the continuum background and 10% of the signal.

The $B$ meson can decay to $\rho^0\rho^0$ via two polarizations, longitudinal or transverse. These polarizations have different angular distributions and therefore, as described in Section ??, significantly different kinematics and average multiplicities of reconstructed candidates per event. Belle ($BABAR$) finds 1.17 and 1.03 (1.15 and 1.03) $B$ candidates per event for the longitudinal and transverse polarization, respectively. In case of multiple $B$ candidates, the one whose $m_{ES}$ is closest to the nominal $B$ mass is chosen for Belle, and the one that has the smallest $\chi^2$ for the four-pion vertex is selected for $BABAR$. The reconstruction efficiency for the signal is calculated from MC to be 21.1% (26.5%) for Belle and 22.3% (26.1%) for $BABAR$ for the longitudinal (transverse) polarization.

The branching ratio, $B(B^0 \rightarrow \rho^0\rho^0)$, as well as the fraction of longitudinal polarization, $f_L$, are extracted from an unbinned extended ML fit, where $\theta_{1,2}$ allows one to measure the polarization according to Eq. (??). Belle uses six discriminating variables in the fit $(\Delta E, m(\pi^+\pi^-)_{1,2}, m(\pi^+\pi^-)_{2,2}, \cos\theta_{1,2}, \cos\theta_{2}, F)$. In $BABAR$ the extended ML fit is used to extract not only $B(B^0 \rightarrow \rho^0\rho^0)$ and $f_L$, but also the coefficients of the time-dependent $CP$ asymmetry for the longitudinal signal, $C_L^{\rho^0\rho^0}$ and $S_L^{\rho^0\rho^0}$. Hence it uses ten variables: $m_{ES}$, $\Delta E$, $m(\pi^+\pi^-)_{1,2}$, $m(\pi^+\pi^-)_{2,2}$, $\cos\theta_{1,2}$, $\cos\theta_{2}$, $NN$, $c_{tag}$, $\Delta t$, and $\sigma_{\Delta t}$, where the tagging category $c_{tag}$ of the $B$-flavor tagging algorithm, introduced in Chapter 8, provides additional background discrimination power, and $\Delta t$ and its error $\sigma_{\Delta t}$ are added in order to include the time-dependent information.

Four categories of signal are distinguished in the fits: the longitudinally and transversely polarized signals and their respective mis-reconstructed components. The fractions of the mis-reconstructed signal are fixed according to MC expectations. Several kinds of background, including a variety of four pion final states, have to be dealt with. While $\Delta E$ and $m_{ES}$ are powerful in discriminating $B$ decays into four charged pions from other decays, $m(\pi^+\pi^-)$ helps to distinguish among the former event types. Belle considers 17 different event types in its fit. In addition to the four signal types, these are continuum, neutral or charged $B$’s decaying into charm or charmless final states and eight peaking background modes ($a_0^0(1260)\pi^\mp, a_0^0(1260)\pi^\pm, b_1^0(1235)\pi^\mp, b_1^0(1235)\pi^\pm, \rho^0\pi^0\pi^-, \rho^0\pi^0\pi^+, \rho^0\pi^0\pi^0, f_0(500), f_0(980), f_0(1010)$). The fraction of charged to neutral $B$ mesons decaying into final states with (or without) charm mesons is determined from MC. The branching fraction of $B^{0} \rightarrow a_0^0(1260)\pi^\mp$ is fixed to the published value $(33.2 \pm 3.0 \pm 3.8) \times 10^{-6}$ (Aubert, 2006a) and the ones of $B^0 \rightarrow a_0^0(1260)\pi^0$.

\footnote{At least one track from the signal decay is replaced by one from the accompanying tag $B$ meson in the event for a signal event to be mis-reconstructed.}
765 and $B^0 \rightarrow B^+_c \pi^-$ to values based on measured upper limits and theoretical expectation. All other branching fractions, yields, and the continuum shape are allowed to vary in the fit. In the fit from $BABAR$, the following background categories are considered: continuum, $B$ decays into final states containing at least one charm meson, $B$ decays into charmless final states, $\rho^+ f_0$, $f_0 f_0$, $\rho^0 \pi^+ \pi^-$, non-resonant $4 \pi^\pm$, $a_1(1260)\pi^\mp$, $\rho^0 K^{*0}$, and $f_0 K^{*0}$. All yields are allowed to vary in the fit, except the last two which are fixed to the expected values.

The $p.d.f.$s are obtained from MC for all $B$ decay components. For the $m_{ES}$, $\Delta E$, and $F$ or $NN$ $p.d.f.$s, possible differences between real data and the MC modeling are calibrated using a large control sample of $B^0 \rightarrow D^- (K^+ \pi^- \pi^-) \pi^+$ decays. To obtain the continuum $p.d.f.$, $Belle$ uses off-resonance data and $BABAR$ on-resonance sideband data ($m_{ES} < 5.27 \text{ GeV}/c^2$), with parameters of most $p.d.f.$s left free in the final fit. For the time-dependent analysis, the tagging efficiencies, mistag fractions, and the parameters of the proper-time distributions for signal modes are obtained in dedicated fits to events with identified exclusive $B$ decays as discussed in Section ??.

When possible, the $p.d.f.$ for each component is taken to be the product of analytical one-dimensional functions, but correlations as small as 2% between the fit variables are also accounted for using different techniques such as multidimensional $p.d.f.$s or different $p.d.f.$s for different slices of an orthogonal discriminating variable. Since the shape of $F$ depends on the flavor tagging quality $r$, it is described in seven bins of the variable $r$.

Figure 17.7.7 shows the projections of the fit results onto $m_{ES}$ and $m_{\pi+\pi-}$ (where the peaks of the $\rho^0$ and $f_0$ can be seen). Table 17.7.1 summarizes the measurements of the branching fraction, $f_L$, and the time-dependent asymmetries for $B^0 \rightarrow \rho^0 \rho^0$. This mode is seen with $9\sigma$ significance of 3.1 $\sigma$ by $BABAR$ and 2.9 $\sigma$ by $Belle$, taking into account systematic uncertainties.

Table 17.7.1. Branching fraction, fraction of longitudinal $p.d.f.$ polarization, and coefficients of the time-dependent $CP$ asymmetry in $B^0 \rightarrow \rho^0 \rho^0$.

<table>
<thead>
<tr>
<th>$BABAR$</th>
<th>$Belle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{\rho^0 \rho^0}[10^{-6}]$</td>
<td>$0.92 \pm 0.32 \pm 0.14$</td>
</tr>
<tr>
<td>$f_L^{\rho^0 \rho^0}$</td>
<td>$0.75 ^{+0.011}_{-0.014}$</td>
</tr>
<tr>
<td>$S_L^{\rho^0 \rho^0}$</td>
<td>$0.3 \pm 0.7 \pm 0.2$</td>
</tr>
<tr>
<td>$C_L^{\rho^0 \rho^0}$</td>
<td>$0.2 \pm 0.8 \pm 0.3$</td>
</tr>
</tbody>
</table>

The branching fraction of longitudinally polarized $B^0 \rightarrow \rho^0 \rho^0$ is an input to the isospin analysis in $B \rightarrow \rho \rho$. The time-dependent analysis performed by $BABAR$ is a proof of this principle of this technique which, with significantly larger data sets, would help determine $\phi_2$ with high precision. The effect of using the measured values of $S$ and $C$ from the $BABAR$ analysis in the $\rho \rho$ isospin analysis is evident as the shoulder above $\phi_2 = \alpha \sim 100^\circ$ in Fig. 17.7.12. With larger statistics this input will help to resolve some of the discrete ambiguities inherent in the determination of $\phi_2$ using an isospin analysis.

17.7.4 $B^0 \rightarrow (\rho \pi)^0$

The decays $B \rightarrow \rho \pi$ are similar to $B \rightarrow \pi \pi$ and $B \rightarrow \rho \rho$ in that they are also dominated by the $b \rightarrow u$ tree amplitude and contain $b \rightarrow d$ penguin contributions. The $CP$-violation in the $\rho \pi$ final state is related to the unitarity triangle angle $\phi_2$. The main complicating factor for measuring $\phi_2$ in $B \rightarrow \rho \pi$ is that it is not a $CP$-eigenstate; one can still measure $CP$-violation in $B^0 \rightarrow \rho^+ \rho^-$ but it is more challenging than the case for $B^0 \rightarrow \pi^+ \pi^-$ decays. A measurement of $\phi_2$ is possible in $\rho \pi$ but instead of a triangular isospin relationship between the amplitudes it is pentagonal, requiring measurements of all of the rate and $CP$-violation parameters of the decays $B^0 \rightarrow \rho^+ \rho^-$, $B^0 \rightarrow \rho^0 \pi^0$, $B^+ \rightarrow \rho^- \pi^0$, and $B^+ \rightarrow \rho^0 \pi^+$. Due to the number of parameters involved, extracting $\phi_2$ from $\rho \pi$ via the pentagon relationship yields poor results with many discrete ambiguities. Early attempts to constrain $\phi_2$ using these decays were based on a quasi-two-body analysis methodology with the intent of performing an isospin pentagon analysis to extract information on the weak phase. However, the $B^0 \rightarrow \rho^+ \pi^-$, $B^0 \rightarrow \rho^- \pi^+$, and $B^0 \rightarrow \rho^0 \pi^0$ all decay to the same final state, $\pi^+ \pi^- \pi^0$ and there are regions in the 3-body phase space where there is interference between decay amplitudes. Because of this interference, it is possible to perform an amplitude analysis of the Dalitz plot so that instead of simply extracting decay rates and quasi-two body $CP$ asymmetry parameters sensitive to $CP$ violating and $CP$ conserving observables $S$, $C$, $\Delta S$, $\Delta C$ and $A_{CP}$ (c.f. the $B \rightarrow a_1 \pi$ time-dependent anal-
ysis discussed in Section 17.7.5), one can extract the complex decay amplitudes. Snyder and Quinn (1993) pointed out that there are enough observables in a full time- and tag-dependent amplitude analysis of the $B^0 \to \pi^+ \pi^- \pi^0$ Dalitz plot to simultaneously extract the tree and penguin amplitudes along with the weak phase, $\phi_2$.

The time-dependent rate for $B^0 (\bar{B}^0)$ decays, $|A_{3\pi}^\pm (\Delta t)|^2$, is given by:

$$|A_{3\pi}^\pm (\Delta t)|^2 = \frac{e^{-|\Delta t|/\tau_{\rho^0}}}{4\tau_{\rho^0}} \left( |A_{3\pi}|^2 + |\bar{A}_{3\pi}|^2 + 2 \sum_{\kappa \in \{+,-,0\}} |f_{\kappa}|^2 \right) \left( |A_{3\pi}|^2 - |\bar{A}_{3\pi}|^2 \right) \cos(\Delta m_d \Delta t)$$

$$+ \sum_{\kappa \in \{+,-,0\}} \left( |f_{\kappa}|^2 \right) \cos(\Delta m_d \Delta t)$$

where $\tau_{\rho^0}$ is the mean neutral $B$ lifetime and $\Delta m_d$ is the $B^0 \bar{B}^0$ mass difference. The $B^0 \to \pi^+ \pi^- \pi^0$ Dalitz plot is dominated by the $\rho$ resonances; other contributions ($f_0(980)$, non-resonant, etc.) can safely be neglected because of the size of the current data samples. The amplitudes can be written as a sum of terms:

$$A_{3\pi} = f_1 A^+ + f_2 A^- + f_0 A^0,$$

$$\bar{A}_{3\pi} = f_1 \bar{A}^+ + f_2 \bar{A}^- + f_0 \bar{A}^0,$$

where the $A_{3\pi}$ and $\bar{A}_{3\pi}$ are Dalitz-plot position dependent $p$ line shapes and the $A^\pm, A^0$ are Dalitz-plot independent complex amplitudes which contain information on the strong and weak phases. Inserting the amplitudes of Eq. (17.25) and Eq. (17.26) into Eq. (17.24), one obtains the following for the terms appearing in Eq. (17.24):

$$|A_{3\pi}|^2 + |\bar{A}_{3\pi}|^2 = \sum_{\kappa \in \{+,-,0\}} |f_{\kappa}|^2 \sum_{\kappa \in \{+,-,0\}} \left( \frac{q}{p} A_{3\pi} A_{3\pi}^* \right)$$

$$\sum_{\kappa \in \{+,-,0\}} \left( \text{Re} [f_{\kappa} f_{\kappa}^*] U_{\kappa\kappa} + \text{Im} [f_{\kappa} f_{\kappa}^*] U_{\kappa\kappa}^* \right)$$

$$\text{Im} \left( \frac{q}{p} A_{3\pi} A_{3\pi}^* \right) = \sum_{\kappa \in \{+,-,0\}} \left( |f_{\kappa}|^2 I_{\kappa} + \right.$$

$$\left. \sum_{\kappa \in \{+,-,0\}} \left( \text{Re} [f_{\kappa} f_{\kappa}^*] I_{\kappa}^* + \text{Im} [f_{\kappa} f_{\kappa}^*] I_{\kappa} \right) \right)$$

(17.7.24)

where the coefficients of the bi-linear terms ("the $U$'s and $I$'s") are related to the amplitudes by:

$$U_{\kappa\kappa}^\pm = |A_{\kappa}|^2 \pm |\bar{A}_{\kappa}|^2,$$

$$U_{\kappa\kappa}^\pm \text{Re} = \text{Re} \left[ A_{\kappa} A_{\kappa}^* \pm \bar{A}_{\kappa} \bar{A}_{\kappa}^* \right],$$

$$U_{\kappa\kappa}^\pm \text{Im} = \text{Im} \left[ A_{\kappa} A_{\kappa}^* \pm \bar{A}_{\kappa} \bar{A}_{\kappa}^* \right],$$

$$I_\kappa = \text{Im} \left[ \bar{A}_{\kappa} A_{\kappa}^* \right],$$

$$I_{\kappa}^* = \text{Re} \left[ A_{\kappa} A_{\kappa}^* - \bar{A}_{\kappa} \bar{A}_{\kappa}^* \right],$$

$$I_{\kappa}^* = \text{Im} \left[ \bar{A}_{\kappa} A_{\kappa}^* + \bar{A}_{\kappa} A_{\kappa}^* \right].$$

(17.7.28)

The above coefficients are used as the fit parameters in the time-dependent maximum likelihood fit to the $B^0 \to \pi^+ \pi^- \pi^0$ Dalitz plot. In order to extract the amplitude-level information (e.g. the phase $\phi_2$) from the bi-linear coefficients, one can construct a $\chi^2$ quantity using the 27 measured $U$'s and $I$'s and the full correlation matrix and minimize to find the 12 best fit amplitude parameters.

Both BABAR (Aubert, 2007d) and Belle (Kusaka, 2008) have performed the time-dependent analysis to the $B^0 \to \pi^+ \pi^- \pi^0$ Dalitz plot using the above method. The resulting values, including correlation matrices, for the bi-linear coefficients can be found in the references. The combined power of the $B$ Factory results is sufficient to start providing important information in terms of our knowledge on $\phi_2$. A single solution appears with a value $\sim 120^\circ$ that can be used to suppress non-SM solutions when combined with the results from $\pi\pi$ and $\rho\rho$ isospin analyses. A discussion on the extraction of $\phi_2$ using these decays can be found in (Miyashita, 2013).

A significant increase in the available statistics for this mode, which is expected from the next generation of flavor factories, will result in the $B^0 \to \pi^+ \pi^- \pi^0$ time-dependent Dalitz plot analysis playing a more prominent role in the determination of $\phi_2$. Given the available level of precision it is not possible to make concrete predictions on the possible sensitivity that one might be able to achieve in the future as there are too many parameters that feed into the determination of $\phi_2$, and many of these are only weakly constrained by the current data.

17.7.5 $B^0 \to a_1^\pm (1260) \pi^\mp$

The following describes analyses of $B^0 (\bar{B}^0) \to a_1^+ (1260) \pi^-$ and $B^0 (\bar{B}^0) \to a_1^- (1260) \pi^+$ performed using the quasi-two-body approximation (which is directly analogous to the corresponding early analyses of $B \to \rho \pi$ decays). The $\Delta t$ distributions are given by (Gronau and Zupan, 2004)

$$F_q^{a_1^\pm \pi^\mp} (\Delta t) = \frac{1}{\tau_{\rho^0}} \frac{1 - \frac{q}{p} \Delta t}{\tau_{\rho^0}} \left[ 1 + \frac{q}{p} \Delta t \right]$$

$$\cdot \left( S \pm \Delta S \sin(\Delta m_d \Delta t) \right)$$

(17.7.29)

where $q = +1(-1)$ when the tagging meson $B_{\text{tag}}^0$ is a $B^0 (\bar{B}^0)$. The time- and flavor-integrated charge asymmetry $A_{\text{CP}}$ measures direct CP violation. The quantities $S$ and $C$ parameterize mixing-induced CP violation related to the angle $\phi_2$, and flavor-dependent direct CP violation, respectively. The parameter $\Delta C$ describes the asymmetry between the rates $\Gamma (B^0 \to a_1^+ (1260) \pi^-) + \Gamma (\bar{B}^0 \to a_1^- (1260) \pi^+) + \Gamma (B^0 \to a_1^- (1260) \pi^-) + \Gamma (\bar{B}^0 \to a_1^+ (1260) \pi^-)$, while $\Delta S$ is related to the strong phase difference between the amplitudes contributing to $B^0 \to a_1^\pm (1260) \pi^\mp$ decays.
The parameters $\Delta C$ and $\Delta S$ are insensitive to $CP$ violation.

The following effective value $\phi_2^{\text{eff}}$ of the weak phase $\phi_2$ is obtained (Gronau and Zupan, 2004)

$$\phi_2^{\text{eff}} = \frac{1}{4} \left[ \arcsin \left( \frac{S + \Delta S}{\sqrt{1 - (C + \Delta C)^2}} \right) + \arcsin \left( \frac{S - \Delta S}{\sqrt{1 - (C - \Delta C)^2}} \right) \right].$$  \hspace{1cm} (17.7.30)

A bound on $|\Delta \phi_2|$ is derived (Gronau and Zupan, 2004) from the $CP$ asymmetry parameters and from the ratios $R_0^a$ and $R_0^+\pi$ of $CP$-averaged rates

$$R_0^a \equiv \frac{\sum f_{2a}^2 B(K_{1A}\pi^-)}{f_{K1A}^2 B(a_1^+ K^-)} \quad \text{and} \quad R_0^+\pi \equiv \frac{\sum f_{2a}^2 B(K_{1A}\pi^+)}{f_{K1A}^2 B(a_1^+ K^0)}.$$  \hspace{1cm} (17.7.31)

The constant $\lambda$ is $|V_{ub}/V_{cd}| = |V_{cd}/V_{es}|$ while $f_K, f_\pi, f_{a_1(1260)}$ and $f_{K_{1A}}$ are the decay constants of $K, \pi, a_1(1260)$, and $K_{1A}$ mesons, respectively. The branching fraction measurements are described in detail in Section 17.4.

The $a_1(1260) \to 3\pi$ decay proceeds mainly through the intermediate states $(\pi\pi)\rho$ and $(\pi\pi)\rho_\pi$ (Amsler et al., 2008). Because of the limited number of signal events, an attempt is made to separate the contributions of the dominant P-wave $(\pi\pi)\rho$ and the S-wave $(\pi\pi)\rho_\pi$. The $a_1(1260)$ meson is reconstructed in its $\rho\pi$ decay. A systematic uncertainty is then estimated due to the difference in the selection efficiency. The $B^0 \to a_1^+(1260)\pi^\mp$ candidates are reconstructed by combining an $a_1(1260) \, \pi$ candidate and a charged pion.

As for other charmless $B$ decays studied at the $B$ Factories, for this decay mode the dominant background is continuum. To suppress it, the angle $\theta_{\pi\pi}$ between the thrust axis of the $B$ candidate and that of the rest of the tracks and neutral clusters in the event, calculated in the CM frame is used. To suppress further combinatorial background in the modes containing an $a_1(1260)$ meson, it is required that the absolute value of the cosine of the angle between the direction of the $\pi$ meson from $a_1(1260) \to \rho\pi$ with respect to the flight direction of the $B$ in the $a_1(1260)$ meson rest frame is less than 0.85. Peaking backgrounds from $B$ decays to final states involving $D^+ (\to K^- \pi^+ \pi^\mp, K^0_{\pi^\mp})$, $D^0 (\to K^- \pi^+, K^+ \pi^0)$, $J/\psi (\to \mu^+\mu^-)$ and $K_{1A} \to \pi^+\pi^-$ mesons are removed by applying the corresponding mass vetoes.

The $CP$ asymmetry parameters are measured at BABAR using an unbinned extended ML fit using the selected sample of $B^0 \to a_1^+(1260)\pi^\mp$ with the input observables $\Delta E$, $m_{\text{ES}}$, a Fisher discriminant $F$ (described in Chapter 9), $m_{a_1(1260)}$, a helicity angle $\mathcal{H}$, and $\Delta t$ (Aubert, 2007e).

At Belle a two step procedure is used. Firstly the signal yield (thus the branching fraction measurement) in $a_1(1260)\pi$ decay mode is obtained from an extended ML fit to the selected sample $B \to a_1^+(1260)\pi^\mp$ with the input observables $\Delta E$, $m_{a_1(1260)}$, $\mathcal{H}$, and $F$. The $m_{\text{ES}}$ distribution is not included in the fit, as it is used to select the best $B$ candidate in cases of multiple candidates per event. After that a one-dimensional unbinned maximum likelihood fit is performed to the $\Delta t$ distribution of events in the signal region $|\Delta E| < 0.46$ GeV, using the branching fraction measurement from the first step to provide the event-dependent signal probability for each component (Dalseno, 2012).

In both experiments the $p.d.f.s$ for signal and $B\bar{B}$ backgrounds are determined from MC distributions in each observable. For the continuum background the functional forms and initial parameter values of the $p.d.f.s$ are established with off-resonance data. The $p.d.f.$ of the $a_1(1260)$ meson invariant mass distribution of signal events is parameterized as a relativistic Breit-Wigner line-shape with a mass dependent width (Armstrong et al., 1990).

At $\bar{B}\bar{B}$, the maximum likelihood fit to a sample of 29300 events results in a signal yield of $608 \pm 53$, of which $461 \pm 46$ have their flavor identified. Fig. 17.7.8 shows distributions of $m_{\text{ES}}$ and $\Delta E$, enhanced in signal content by requirements on the signal-to-continuum likelihood ratios using all discriminating variables other than the one plotted.

With a data sample of $304 \times 10^6 B\bar{B}$ pairs, the following results for the $CP$ asymmetries are obtained (Aubert, 2007e)

$$S = \quad 0.37 \pm 0.21 \pm 0.07,$$
$$C = -0.10 \pm 0.15 \pm 0.09,$$
$$A_{CP} = \quad -0.07 \pm 0.07 \pm 0.02,$$  \hspace{1cm} (17.7.31)

where the errors are statistical and systematic in nature, respectively. For the $CP$ conserving parameters one obtains

$$\Delta S = -0.14 \pm 0.21 \pm 0.06,$$
$$\Delta C = \quad 0.26 \pm 0.15 \pm 0.07.$$  \hspace{1cm} (17.7.32)
At Belle (Dalseno, 2012), the fit to 83799 $a_1^\pm(1260)\pi^\mp$ candidates in the signal region results in

\[
\begin{align*}
A_{CP} &= -0.06 \pm 0.05 \pm 0.07, \\
C &= -0.01 \pm 0.11 \pm 0.09, \\
S &= -0.51 \pm 0.14 \pm 0.08, \quad (17.7.33)
\end{align*}
\]

and the $CP$ conserving parameters obtained are

\[
\begin{align*}
\Delta C &= +0.54 \pm 0.11 \pm 0.07, \\
\Delta S &= -0.09 \pm 0.14 \pm 0.06. \quad (17.7.34)
\end{align*}
\]

Figure 17.7.9 shows the $\Delta t$ and $CP$ asymmetry distributions obtained for the Belle analysis. The BABAR and Belle values of $S$, while not in good agreement are considered to be compatible with each other.

\[
\phi_\text{eff}^0 = (107.3 \pm 6.6 (\text{stat}) \pm 4.8 (\text{syst}))^\circ.
\]

From a likelihood scan, the significance of mixing-induced $CP$ violation is found to be 3.1$\sigma$ including systematic uncertainties.

The main contributions to the systematic error on the signal parameters come from the p.d.f. parameterization, uncertainty due to $CP$ violation present in the $B\bar{B}$ background and uncertainty due to the interference between $B^0 \to a_1^\pm(1260)\pi^\mp$ and other $4\pi$ final states.

In BABAR a MC technique is used to estimate a probability region for the bound on $|\Delta\phi_2|$. The $CP$-averaged rates and $CP$ asymmetry parameters used in estimating the bounds are generated according to the experimental distributions. The input values of branching fractions are those presented in Section 17.4 while $CP$ asymmetry parameters are from this section (Eq. 17.7.31). For the decay constants the following values are used: $f_\pi = (130.4 \pm 0.2)$ MeV (Amsler et al., 2008), $f_K = 155.5 \pm 0.9$ MeV (Amsler et al., 2008), $f_{a_1} = 203 \pm 18$ MeV (Cheng and Yang, 2007), and $f_{K_{1\pi}} = 207$ MeV (Bloch, Kalinovsky, Roberts, and Schmidt, 1999). For $f_{K_{1\pi}}$ an uncertainty of 20 MeV is assumed. For the constant $\lambda$ the value 0.23 (Amsler et al., 2008) is used.

For each set of generated values, the bound on $|\Delta\phi_2|$ is evaluated. The limits on $|\Delta\phi_2|$ are obtained by counting the fraction of bounds within a given value and the results are $|\Delta\phi_2| < 11^\circ (13^\circ)$ at 68% (90%) probability (Aubert, 2010). Combining the solution near 90$^\circ$, consistent with the results of global CKM fits, with the bound on $|\phi_2 - \phi_2^\text{eff}|$ we measure the weak phase $\phi_2 = (79 \pm 7 \pm 11)^\circ$.

This solution is in agreement with the value of $\phi_2$ found in the analyses of $B \to \pi\pi$, $B \to \rho\rho$, and $B \to \rho\pi$ decays. This measurement is currently limited by statistics and a substantial improvement of its precision may come from a next generation Super Flavor Factory.

**Fig. 17.7.9.** Background subtracted time-dependent fit results for $B^0 \to a_1^\pm(1260)\pi^\mp$ (Dalseno, 2012). (a) and (b) show the $\Delta t$ distributions for the $B_{L(0)}$ flavor $q$ and the product of the $B_{L(0)}$ flavor and $a_1$ charge $q_c$, respectively. The left plots show the effect of flavor-dependent CP violation while the plots on the right show the effects of the $CP$ conserving parameters. The solid blue and dashed red curves represent the $\Delta t$ distributions for $q = +1$ and $q = -1$, respectively. (c) and (d) show the asymmetry of the plots immediately above them, $(N^{+\text{fit}}_q - N^{-\text{fit}}_q)/(N^{+\text{fit}}_q + N^{-\text{fit}}_q)$, where $N^{+\text{fit}}_q$ $(N^{-\text{fit}}_q)$ is the measured signal yield of positive (negative) quantities in bins of $\Delta t$.

$\phi_\text{eff}^0$ can be determined from Eq. (17.7.30) up to a two-fold discrete ambiguity in the range $[0^\circ, 180^\circ]$ with the assumption that the relative strong phase of the tree amplitudes of the $B^0$ decays to $a_1^\pm(1260)\pi^\mp$ and $a_1^\pm(1260)\pi^\mp$ is much smaller than 90$^\circ$ (Gronau and Zupan, 2004), as predicted by QCD factorization (Beneke and Neubert, 2003). This assumption is valid to leading order in $m_b$ (Bauer and Pirjol, 2004; Bauer, Pirjol, Rothstein, and Stewart, 2004). The two solutions at BABAR are $(78.6 \pm 7.3)^\circ$ and $(11.4 \pm 7.3)^\circ$. The first of these two solutions is compatible with the results of Standard Model based fits. At Belle from the four obtained solutions for $\phi_\text{eff}^0$, the one most compatible with the results of Standard Model based fits is $\phi_\text{eff}^0 = (107.3 \pm 6.6 (\text{stat}) \pm 4.8 (\text{syst}))^\circ$. From a likelihood scan, the significance of mixing-induced $CP$ violation is found to be 3.1$\sigma$ including systematic uncertainties.
The value of $\phi_2$ obtained using this method, the Belle data provide a more stringent constraint on $\delta_{PT}$ than BABAR. This can be seen in Figure 17.7.10 as the mirror solutions at $\phi_2 \sim 5^\circ$ is less probable than the other possible solutions. The ratio of penguin to tree amplitudes (see Section 17.7.1.4) obtained from the data is $r_{PT} = 0.10 \pm 0.04$.

![Fig. 17.7.10.](image)

**Fig. 17.7.10.** The 1 – C.L. distribution obtained on $\phi_2 = \alpha$ from the B Factories (solid blue) using $B^0 \rightarrow \rho^+\rho^-$, and $B^+ \rightarrow K^{*0}\rho^+$ decays and the BGRS method discussed in the text. The constraints from BABAR (dashed black) and Belle (dotted red) are also shown. The distributions shown include solutions for all possible values of $\delta_{PT}$.

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17.7.7 Summary

First we discuss the constraints on $\phi_2$ obtained by the B Factories mode-by-mode and then on the ‘average’ of the $B \rightarrow \pi\pi/\rho\rho$ and the $B \rightarrow \rho\gamma \rightarrow 3\pi$ Dalitz results. The mode $B \rightarrow a_1(1260)\pi$ can play an important role in constraining $\phi_2$; however, at the time of writing this book the SU(3) constraint from $B \rightarrow a_1(1260)\pi$ is typically not included in global averages together with the SU(2) constraints, as it is believed that the former measurements have to account for SU(3) breaking effects which may not be straightforward to compute, whereas the latter are agreed to have small theoretical uncertainties. A ‘naive’ weighted average of the two sets of measurements is given at the end. The SU(3) constraint from $B^0 \rightarrow \rho^+\rho^-$ and $B^+ \rightarrow K^{*0}\rho^+$ discussed in Section 17.7.6 is completely correlated with inputs for the isospin analysis and this result is used only as a cross check of the underlying theoretical interpretation of $S$ and $C$. The averaging procedure adopted here is to combine the $\chi^2(\phi_2)$ distributions obtained from each individual constraint ($B \rightarrow \pi\pi/\rho\rho$, $\pi^+\pi^-\pi^0$ and $\rho\rho$), and from this determine the distribution of $1 – C.L.$ as a function of $\phi_2$. The value of $\phi_2$ obtained near $90^\circ$ is reported as the SM value of this angle.

The naive average including $B \rightarrow a_1(1260)\pi$ is simply the weighted average obtained from the $\chi^2(\phi_2)$ combination described in this section with the result of the SU(3) constraint from Section 17.7.5.

The constraint on $\phi_2$ obtained from $B \rightarrow \pi\pi$ decays from Belle and BABAR is shown in Figure 17.7.11. The zero value is suppressed by physical constraints on possible magnitudes of penguin contributions within the SM (Bona et al., 2007). Furthermore one can exclude allowed values around $\phi_2 \sim 40 – 50^\circ$ as can be seen from the figure.

![Fig. 17.7.11.](image)

**Fig. 17.7.11.** The solid (blue) curve shows 1 – C.L. distribution obtained on $\phi_2 = \alpha$ from the B Factories using the Gronau-London isospin analysis for $B \rightarrow \pi\pi$ decays. The dashed (black) curve shows the BABAR only constraint, the dotted (red) curve shows the Belle only constraint. As discussed in the text, the solutions near $\phi_2 = 0$ are unphysical and can be removed by extending the isospin analysis.

The constraint on $\phi_2$ obtained from the BABAR and Belle analyses of $B \rightarrow \rho\rho$ is shown in Figure 17.7.12. One gets two values, one near zero and the other near $90^\circ$. The underlying eight-fold ambiguity expected for the isospin analysis is not visible, since the triangles are essentially flat as a result of the equally large branching fractions of the $\rho^+\rho^-$ and $\rho^+\rho^0$ modes, and the small value of $\rho^0\rho^0$.

The value obtained for the solution consistent with SM expectation is $\phi_2 = (89.5^{+6.0}_{-6.3})^\circ$. The accurate value obtained using isospin symmetry with $\rho\rho$ decays is similar to that attained using the BGRS SU(3) flavor based approach discussed in Section 17.7.6. The individual BABAR and Belle results are $(92.4_{-6.4}^{+6.4})^\circ$ and $(84.4_{-15.8}^{+15.8})^\circ$, respectively. The better accuracy of BABAR with respect to Belle is partly due to the time-dependent CP asymmetry measurement of $\rho^0\rho^0$ final states, which plays a role in resolving ambiguities and partly due to the accuracy on the $\rho^+\rho^0$ branching fraction from BABAR. One can see from the figure that the $\rho\rho$ modes exclude values of $\phi_2$ around $\sim 50^\circ$ and $\sim 140^\circ$.

These decays help to discard unphysical solutions resulting from the $B \rightarrow \pi\pi$ isospin analysis. The constraint on $\phi_2$ from $B^0 \rightarrow \pi^+\pi^-\pi^0$ decays obtained by the B Factories is shown in Figure 17.7.13. While these results do not completely exclude any interval, they do strongly suppress unphysical values. It is the abil-
identify underlying non-SM contributions from three body final states is ongoing.

The $B$ Factories results for $\phi_2$ from $B \to \pi \pi$, $\rho \rho$, and $\pi^+ \pi^- \pi^0$ decays are summarized in Figure 17.7.14. Initial measurements from $B \to \pi \pi$ decays excluded the range of $40 - 50^\circ$ while at the same time highlighted the issue of understanding penguin contributions in detail. As can be seen from the combined $B$ Factories result (the dashed line in the figure), it was not possible to measure the angle precisely with this channel alone even with full data sample obtained by both experiments. The measurement of $\phi_2$ using $B \to \rho \rho$ decays (the solid line in the figure) provides the most accurate value; it also suppresses the unphysical solution near $\sim 140^\circ$ as well as those in the range $40^\circ - 50^\circ$. Early quasi-two-body analyses of $B \to \rho \pi \to \pi^+ \pi^- \pi^0$ provided an interesting third alternative to constrain this angle. However, as the data sets of $B$ Factories increased,

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**Table 17.7.2.** A summary of experimental inputs used for the BGRS method, along with the constraints on $\phi_2$ derived using this method. ‡The SM solution includes a requirement that $|\delta_{T\tau}| < 90^\circ$ (consistent with theoretical expectations (Beneke, Buchalla, Neubert, and Sachrajda, 1999, 2000, 2001)).

<table>
<thead>
<tr>
<th></th>
<th>$\bar{B}$ &amp; $B$</th>
<th>Belle</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(\rho^+\rho^-)$</td>
<td>$-0.17\pm0.20^{+0.05}_{-0.06}$</td>
<td>$+0.19\pm0.30\pm0.08$</td>
<td>$-0.05\pm0.17$</td>
</tr>
<tr>
<td>$C(\rho^+\rho^-)$</td>
<td>$+0.01\pm0.15\pm0.06$</td>
<td>$-0.16\pm0.21\pm0.08$</td>
<td>$-0.06\pm0.13$</td>
</tr>
<tr>
<td>$\rho_{S,C}$</td>
<td>$-0.035$</td>
<td>$0.10$</td>
<td>$0.009$</td>
</tr>
<tr>
<td>$B(\rho^+\rho^-)$</td>
<td>$[10^{-6}]$</td>
<td>$25.5\pm2.1^{+3.6}_{-3.9}$</td>
<td>$22.8\pm3.8^{+2.3}_{-2.6}$</td>
</tr>
<tr>
<td>$B(K^{*0}\rho^+)$</td>
<td>$[10^{-6}]$</td>
<td>$9.6\pm1.7\pm1.5$</td>
<td>$8.9\pm1.7\pm1.2$</td>
</tr>
<tr>
<td>$f_L(\rho^+\rho^-)$</td>
<td>$0.992\pm0.024^{+0.026}_{-0.013}$</td>
<td>$0.941^{+0.034}_{-0.040} \pm0.030$</td>
<td>$0.978\pm0.023$</td>
</tr>
<tr>
<td>$f_L(K^{*0}\rho^+)$</td>
<td>$0.52\pm0.10\pm0.04$</td>
<td>$0.43\pm0.11^{+0.05}_{-0.02}$</td>
<td>$0.48\pm0.08$</td>
</tr>
<tr>
<td>$\phi_2 = \alpha$ (SM solution) ($^\circ$)</td>
<td>$89.5^{+6.7}_{-6.5}$</td>
<td>$81.7^{+11.4}_{-11.3}$</td>
<td>$86.5^{+7.3}_{-5.4}$</td>
</tr>
</tbody>
</table>

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**Fig. 17.7.12.** The solid (blue) curve shows the $1 - \text{C.L.}$ distribution obtained on $\phi_2 = \alpha$ from the $B$ Factories using the Gronau-London isospin analysis for $B \to \rho \rho$ decays. The dashed (black) curve shows the $\bar{B}$ & $B$ only constraint, the dotted (red) curve shows the Belle only constraint.

**Fig. 17.7.13.** The solid (blue) curve shows $1 - \text{C.L.}$ distribution obtained on $\phi_2 = \alpha$ from the $B$ Factories using the Dalitz plot analysis of $B \to \pi^+ \pi^- \pi^0$ decays. The dashed (black) curve shows the $\bar{B}$ & $B$ only constraint, the dotted curve (red) shows the Belle only constraint.
it became apparent that a time-dependent Dalitz analysis was required. The value of $\phi_2$ from $\pi^+\pi^-\pi^0$ is also shown (the dotted line). This result plays an important role in suppressing those values inconsistent with the SM solution for the unitarity triangle. Noting that physical penguin contributions to $B \to \pi\pi$ suppress $\phi_2 \sim 0$ leaves

$$\phi_2 = \alpha = (88 \pm 5)^\circ,$$  \hspace{1cm} (17.7.35)

which is consistent with the SM expectations as discussed in Section 25.1. While $B \to pp$ dominates the measured value of the angle, there is a theoretical uncertainty from isospin breaking currently of the order of $\sim 1^\circ$. It is worth noting that $B \to \pi\pi$ decays also have a theoretical limit from isospin breaking at a similar level. A high statistics study is called for in order to fully utilize interference between the various intermediate states and remove experimental limitations of the quasi-two-body approximation.

The $B \to a_1(1260)\pi$ SU(3) value of $(79 \pm 7 \pm 11)^\circ$ (from BABAR). This gives an average of

$$\phi_2 = \alpha = (87 \pm 5)^\circ.$$  \hspace{1cm} (17.7.36)

The measurement of this angle of the unitarity triangle is a direct constraint to test the CKM matrix description of quark mixing and thus the KM description of CP violation in the SM. The use of $\phi_2$ in this context is discussed in Chapter 25. The presence of penguin contributions, while a nuisance in terms of the determination of $\phi_2$, means that these decays are also sensitive to NP effects manifest in loops. High precision measurements of $\phi_2$ in the different modes can be compared in order to search for NP in analogy with the ongoing searches via penguin measurements of $\phi_1$ discussed in Section 17.6.

One of the great successes of the BABAR and Belle B Factories has been the confirmation that the CKM matrix provides the leading order description of CP violation in the quark sector. The next generation of experiments will focus on searches for possible second order CP violation effects beyond the SM. This will require detailed studies of the interference patterns manifest in three and four body final states.

In the future it may be possible to include constraints on $\phi_2$ obtained using additional modes such as $B^0 \to K\bar{K}\pi$ and $B \to \alpha \rho \rho$ decays. The precision of the next generation of Super Flavor Factories will approach theoretical limits on isospin breaking for $B \to \pi\pi$ and $pp$, so it will be important for experimentalists and theorists alike to continue to explore all possible avenues to improve the precision on this weak angle.

**Bibliography: BaBar Publications**

Aubert 2001:

Aubert 2002:
B. Aubert et al. “Measurements of branching fractions and CP-violating asymmetries in $B^0 \to \pi^+\pi^-\pi^0$ Dalitz plot analysis peaks around $120^\circ$ which is somewhat larger than the SM expectation. The difference is not significant, however a higher statistics study is called for in order to improve the precision of the $B^0 \to \pi^+\pi^-\pi^0$ Dalitz result.

The final constraint on $\phi_2$ comes from time-dependent measurements of $B \to a_1(1260)\pi$ decays with input from SU(3) related modes in order to constrain penguins. The accuracy obtained using this method on $\phi_2^{\text{eff}}$ is $\pm 7^\circ$ for each experiment, with an additional $\pm 11^\circ$ uncertainty from penguin contributions. Usually the community does not include $a_1(1260)\pi$ results in the global average of $\phi_2$ to avoid the complexity of the SU(3) analysis. On the other hand one can compute a ‘naive’ weighted average for $\phi_2$ from the combination of $B \to \pi\pi, pp, \pi^+\pi^-\pi^0$ decays with

\(\frac{1}{C.L.}\)

$$0.1 - 0.8$$

\(\phi_1 = \alpha$$

Fig. 17.7.14. The $1 - C.L.$ distribution obtained on $\phi_2 = \alpha$ from the B Factories (dash-dotted thick black curve). Constraints obtained using the Gronau-London isospin analysis for (dashed blue) $B \to \pi\pi$ and (solid thin black) $pp$ decays as well as the Dalitz analysis for (dotted red) $\pi^+\pi^-\pi^0$ decays are shown.


Aubert 2008b: Observation of $B^+ $ Meson Decays to $a_1(1260)^+ K^0$ and $B^0 $ to $a_1(1260)^- K^{+\pi^0}$. *Phys. Rev. Lett.* 100, 051803 (2008). doi:10.1103/PhysRevLett.100.051803. 0709.4165.


Bibliography: Belle Publications


Adachi 2012: I. Adachi, K. Adamczyk, H. Aihara, K. Arinstein, Y. Arita et al. “Study of $B^0 \rightarrow \rho^0 \rho^0$ decays, implications for the CKM angle $\phi_2$ and search for other four pion final states” 1212.4015.

Adachi 2013: I. Adachi et al. “Measurement of the CP Violation Parameters in $B^0 \rightarrow \pi^+ \pi^-$ Decays” 1302.0551.

Chiang 2008: C. C. Chiang et al. “Measurement of $B^0 \rightarrow \pi^+ \pi^- \pi^0$ Decays and Search for $B^0 \rightarrow \rho^0 \rho^0$”, *Phys. Rev. D78*, 111102 (2008). doi:10.1103/PhysRevD.78.111102. 0808.2576.


Somov 2006:

Somov 2007:

Bibliography

Aleksan et al. 1995:

Aleksan et al. 1996:

Aleksan et al. 2008:

Armstrong et al. 1996:

Bauer and Pirjol 2004:

Bauer et al. 2004:

Beneke et al. 2006:

Beneke et al. 2009:

Beneke et al. 2000:


Beneke and Neubert 2003:

Bigi, Khoze, Uraltsev, and Sanda 1989:

Blok, Kalinovsky, Roberts, and Schmidt 1999:

Bona et al. 2007:

Charles 1999:

Cheng and Yang 2007:

Falk et al. 2004:

Gardner 1999:

Gronau 1991:

Gronau and London 1990:

Gronau, London, Sinha, and Sinha 2001:
Gronau and Zupan 2004:
M. Gronau and J. Zupan. “On measuring $\alpha$ in $B(t) \to \rho^\pm \pi^\mp \nu$. Phys. Rev. D 70, 074031 (2004).

Gronau and Zupan 2006:

Grossman and Quinn 1998:

Lipkin, Nir, Quinn, and Snyder 1991:

Miyashita 2013:
T. S. Miyashita. “$B\overline{B}$ Results For $\alpha$: Measurement of $CP$-Violating Asymmetries in $B^0 \to (\rho\pi)^0$ Using a Time-Dependent Dalitz Plot Analysis” 1301.3186.

Quinn and Silva 2000:

Shifman, Vainshtein, and Zakharov 1977:

Snyder and Quinn 1993:

Suzuki 2002: