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June 1966
C.G.T.M. #9

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AN OUTLINE OF THE BASIC THEORY OF FORMAL LANGUAGES

I. INTRODUCTION

A. Two Classes of Languages

1. Natural languages -- example: English
   Attempts to define these languages use transformational grammars, composed of
   a. Phrase structures
   b. Transformational rules with a structural index and a structural change
   c. Lexicon

2. Artificial languages -- examples: Fortran, ALGOL
   Formal definition in terms of
   a. Phrase structure
   b. Semantic interpretation rules

B. Basic Problems in Linguistics

1. Formal definition of languages
2. Recognition (Decision) Problem: determine if a string is in the language.
3. Generation Problem: generate all strings in the language
4. Analysis Problem: determine all structures of a string in a language
   Examples of English sentences with more than one structure:
   1. The Danes like good food as well as Americans.
   2. Geo. Murphy is the one to watch.
   3. I just can't see flying kites.

II. GRAMMAR (PHRASE STRUCTURE SYNTAX)

<u>Definition</u>: The vocabulary $\mathcal{V}$ is the set of all symbols used in the definition of a grammar.

Examples:

1. ALGOL all visible characters, delimiters, word delimiters and Meta-linguistic variables.

   A, <u>Integer</u>, <identifier>

2. $\{\alpha | \alpha = a^n b^n, n \geq 1\}$ S, a, b

   $S \rightarrow aSb | ab$

<u>Definition</u>: $\mathcal{V}^*$ is the set of all strings generated by concatenation of the elements of $\mathcal{V}$. (This includes the null string Λ.)

<u>Nomenclature</u>: Denote all members of $\mathcal{V}$ by either upper or lower case latin letters and strings in $\mathcal{V}^*$ by lower case greek letters.

<u>Definition</u>: A syntax rule (production) is a rewriting rule $\alpha \rightarrow \beta$, $\alpha, \beta \in \mathcal{V}^*$ on a string.

Examples:

1. Backus Normal Form

   <bound pair list>::= <bound pair>

   <bound pair list>::= <bound pair list>,<bound pair>

   abbreviation for the above two rules

   <bound pair list>::= <bound pair>|
   <bound pair list>,<bound pair>

2. $A \rightarrow aaA$

3. $ABA \rightarrow A\alpha A$

4. <number>::= <digit> <number>|<digit>

   <digit>::= 0|1|2|3|4|5|6|7|8|9

A syntax rule can be interpreted as follows: if $\alpha \rightarrow \beta$ is a syntax rule and the string $\varphi\alpha\psi$ is under consideration, then rewrite $\varphi\alpha\psi$ as $\varphi\beta\psi$.

Denote the set of syntactic rules for a grammar as Φ.

Definition: ϑ_N is the nonterminal vocabulary defined by

$$\vartheta_N = \{A \mid \exists \varphi, \psi, \alpha, \alpha \in \vartheta^* \wedge \varphi\ A\ \psi \rightarrow \varphi\ \alpha\ \psi \in \Phi\}$$

Example:

ALGOL	all metalinguistic variables
$\{a^n b^n\}$	$\{S\}$
as defined above	

Definition: ϑ_T is the terminal vocabulary defined by $\vartheta_T = \vartheta - \vartheta_N$.

Definition: A Grammar (phrase structure syntax) G is the ordered quadruplet $G(\vartheta, \Phi, \vartheta_T, S)$ where

ϑ is the vocabulary

ϑ_T is the terminal vocabulary

Φ is the set of syntax rules

S is the initial symbol ($S \in \vartheta_N$).

The initial symbol $S \in \vartheta_N$ is chosen arbitrarily.

Examples:

1. $\vartheta = \{S, a, b\}$
   $\vartheta_T = \{a, b\}$
   $\Phi = \{S \rightarrow aSb,\ S \rightarrow ab\}$
   $G_1(\vartheta, \Phi, \vartheta_T, S)$.
2. $\vartheta = \{S, A, a, b\}$
   $\vartheta_T = \{a, b\}$
   $\Phi = \{S \rightarrow ASb,\ S \rightarrow ab,\ Aa \rightarrow aa\}$
   $G_2(\vartheta, \Phi, \vartheta_T, S)$

Definition: A string α directly generates a string β ($\alpha \dot{\rightarrow} \beta$) $\leftrightarrow$

$\alpha = \gamma\delta\epsilon \quad \beta = \gamma\eta\epsilon \quad \exists \gamma, \epsilon \quad \delta \rightarrow \eta \in \Phi \quad \alpha, \beta, \gamma, \delta, \epsilon, \eta \in \vartheta^*$

Example:

Using G_2

$AAASbbb \dot{\rightarrow} AAAabbbb$

Definition: A string α generates a string β ($\alpha \overset{*}{\rightarrow} \beta$) $\leftrightarrow$

$\alpha = \alpha_0, \alpha_1 \ldots \alpha_n = \beta \quad \in \vartheta^*$

$\alpha_{i-1} \dot{\rightarrow} \alpha_i \quad (i=1, 2, \ldots n)$

Example:

Using G_2

1. $S \overset{*}{\Rightarrow} a^n b^n \ (n \geq 1)$
2. $A^{n-1}ab^n \overset{*}{\Rightarrow} a^n b^n$
3. <program> $\overset{*}{\Rightarrow}$ <u>Begin</u> X : = 1 <u>End</u>

<u>Definition</u>: A string α is terminal $\leftrightarrow \alpha = \Lambda$ or head $(\alpha) \in \vartheta_T$ and tail (α) is terminal. Head and tail are the usual string operators.

Examples:

	terminal strings	nonterminal
ALGOL:	<u>BEGIN</u> x:=y <u>END</u>	<u>BEGIN</u> <body> <u>END</u>
G_2 :	$a^n b^n$	$A^{n-1}ab^n$

<u>Definition</u>: A language $\mathcal{L}$ (simple phrase structure language) defined by G is

$$\mathcal{L}(G) = \{\alpha \mid S \overset{*}{\Rightarrow} \alpha \wedge \alpha \text{ is terminal}\}$$

Example:

1. $\mathcal{L}(G_1) = \{\alpha \mid \alpha = a^n b^n \ n \geq 1\}$
2. $\mathcal{L}(G_2) = \{\alpha \mid \alpha = a^n b^n \ n \geq 1\}$
3. All ALGOL programs generable from the syntax as specified in the revised report, i.e., all error free ALGOL programs.

III. EQUIVALENCE OF GRAMMARS

<u>Definition</u>: Two grammars G_1 and G_2 are weakly equivalent $\leftrightarrow \mathcal{L}(G_1) = \mathcal{L}(G_2)$. Clearly $G_1 \overset{W}{=} G_2$ in the above example.

IV. HIERARCHY OF GRAMMAR

<u>Definition</u>: A grammar $G(\vartheta, \Phi, \vartheta_T, S)$ is type i if every production of G satisfies restriction i.

<u>Definition</u>: A language is type i if it can be generated by a grammar of type i. In terms of weak equivalence

$$\mathcal{L} \text{ is type } i \leftrightarrow \exists G_1 \text{ which is type } i \wedge \mathcal{L} \overset{W}{=} \mathcal{L}_i(G_1)$$

<u>Restriction 0</u>: (unrestricted rewriting system) A production satisfies restriction 0 if it is of the form $\varphi A \psi \rightarrow \varphi \alpha \psi$

<u>Theorem</u>: If a language $\mathcal{L}$ is generated by an unrestricted grammar G, then $\mathcal{L}$ is also generated by a type 0 grammar G_1.

(Restriction 0 is not a restriction on the languages which may be formed, but on the structure of the languages.)

Restriction 1: (Context Sensitive) A production satisfies restriction 1 if it is of the form $\varphi A \psi \rightarrow \varphi \alpha \psi$ $\quad \ell(\alpha) \geq 1$ (no erasure)

(ℓ is the length function on strings.)

Restriction 2: (Context Free) A production satisfies restriction 2 if it is of the form $A \rightarrow \alpha$ $\quad \ell(\alpha) \geq 1$

Restriction 3: (Finite State) A production satisfies restriction 3 if it is of the form $A \rightarrow aB$ or $A \rightarrow a$

Examples:

1. FS $\{(ab)^n\}$ $\quad \vartheta$: S,X $\quad \vartheta_T$: a,b

   Φ : $S \rightarrow aX$, $X \rightarrow bS$, $X \rightarrow b$

2. CF $\{a^n b^n\}$ $\quad \vartheta$: S $\quad \vartheta_T$: a,b

   Φ : $S \rightarrow aSb$, $S \rightarrow ab$

3. CS $\{a^n b^n a^n\}$ $\quad \vartheta$: S,U,X,E,A,B,C

   Φ :

$S \rightarrow UXE$	$BA \rightarrow CA$
$W \rightarrow AXE$	$CA \rightarrow CB$
$X \rightarrow BW$	$CB \rightarrow AB$
$X \rightarrow B$	
(generate loop)	(reverse symbols)
$BE \rightarrow Ba$	$Bb \rightarrow bb$
$Ba \rightarrow ba$	$aA \rightarrow aa$
$UA \rightarrow aA$	$UBE \rightarrow aBE$
$aA \rightarrow aa$	$aBE \rightarrow aBa$
$aE \rightarrow aa$	$aBa \rightarrow aba$
(start terminals)	(propagate terminals)

Theorem: For both grammars and languages type 0 $\supseteq$ type 1 $\supseteq$ type 2 $\supseteq$ type 3.

Theorem: There are type i languages which are not type i + 1 (i = 0,1,2).

Example:

1. $\{a^n b^n a^n\}$ is CS but not CF
2. $\{a^n b^n\}$ is CF but not FS
3. The set of all theorems in first order predicate calculus (RE) is type 0 but not type 1 (R).

V. EQUIVALENT RESTRICTIONS ON GRAMMARS (Note: $\Lambda \notin \mathcal{V}_N$)

Type 0

1. $\varphi \to \psi \qquad \varphi \neq \Lambda$
2. $\varphi A \psi \to \varphi \alpha \psi$

Type 1 Context Sensitive

1. $\varphi A \psi \to \varphi \alpha \psi \quad \ell(\alpha) \geq 1$
2. $\varphi A \psi \to \varphi \alpha \psi \quad \alpha \neq \Lambda$
3. $\varphi \to \psi \quad \ell(\varphi) \leq \ell(\psi) \quad \varphi \neq \Lambda$
4. $\varphi A \to \varphi \alpha$
   $A \varphi \to \alpha \varphi \quad \ell(\alpha) \geq 1$

Type 2 Context Free

1. $A \to \psi \qquad \psi \neq \Lambda$
2. $A \to \psi \qquad \ell(\psi) \geq 1$
3. $A \to aB_1 \ldots B_n$
   $A \to a$
4. $A \to aB_1B_2$
   $A \to aB_1$
   $A \to a$
5. $A \to B_1B_2$
   $A \to a$

Type 3 Finite State

1. $A \to aB$
   $A \to a$
2. $A \to Ba$
   $A \to a$
3. $A \to aB$
   $A \to B$
   $A \to a$

Definition: A grammar is Left Context Sensitive if all productions are of the form $\varphi A \to \varphi \alpha \quad \ell(\alpha) \geq 1$.

Theorem: There are LCS languages which are not CF.

Open Question: Is LCS an equivalent notation to CS?

VI. AUTOMATA, LANGUAGES AND COMPUTABILITY

<u>Definition</u>: A set of strings Γ is <u>K-Decidable</u> $\leftrightarrow$ there exists an automaton $\mathfrak{A}$ satisfying restriction K such that

$$\mathfrak{A} : \vartheta^* \to \{0,1\} \text{ and}$$
$$\Gamma = \{\alpha \in \vartheta^* \mid \mathfrak{A}(\alpha) = 1\}$$

<u>Definition</u>: A set of strings Γ is K-Generable $\leftrightarrow$ there exists an automaton $\mathfrak{A}$ satisfying restriction K such that

$$\mathfrak{A} : \vartheta^* \to \Gamma \cup \{\Lambda\}$$

<u>Theorem</u>: K-Decidable $\subseteq$ K-Generable

Languages and Automata equivalent

	<u>language</u>	<u>strings</u>
1.	type 0	Recursive Enumerable Recursive Generable (Turing Machine)
2.	type 1	Recursive Recursive Decidable
3.	type 2	Push Down Store
4.	type 3	Finite State

<u>Theorem</u>: Finite State Decidable = Finite State Generable

Push Down Store Decidable $\subset$ Push Down Store Generable

Push Down Store Generable = Recursive Decidable

Recursive Decidable $\subset$ Recursive Generable

VII. PARSING AND AMBIGUITIES

<u>Definition</u>: β <u>directly reduces</u> to $\alpha \leftrightarrow \alpha \xrightarrow{\cdot} \beta$.

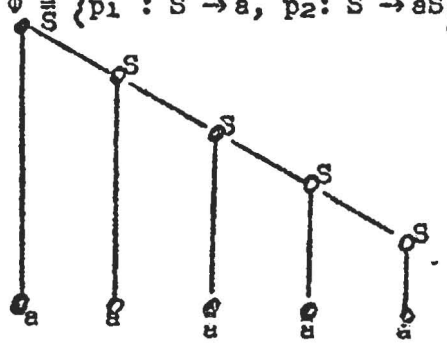
<u>Definition</u>: β <u>reduces</u> to $\alpha \leftrightarrow \alpha \xrightarrow{*} \beta$.

<u>Definition</u>: A <u>parse</u> of a string α into the symbol A is a sequence of syntactic rules $p_1, p_2 \ldots p_n \in \Phi$ such that p_j directly reduces α_{j-1} into α_j $j = 1 \ldots n$ and $\alpha = \alpha_0$, $\alpha_n = A$.

Examples: Parse of α into S.

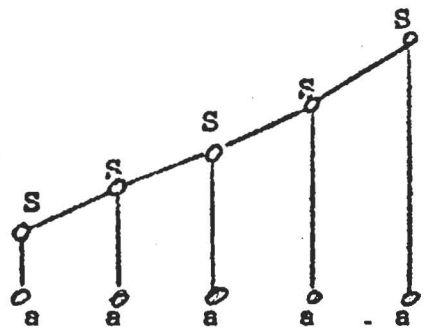
α = aaaaa, $\vartheta = \{S\}$, $\vartheta_T = \{a\}$

1. $\Phi \equiv \{p_1 : S \to a,\ p_2 : S \to aS\}$



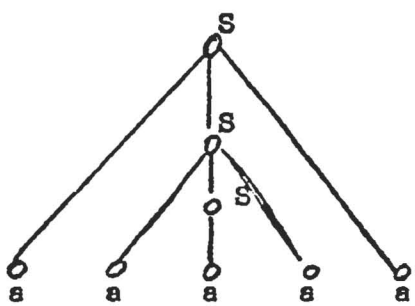
p_1, p_2, p_2, p_2, p_2

2. $\Phi = \{p_1 : S \to a,\ p_2 : S \to Sa\}$



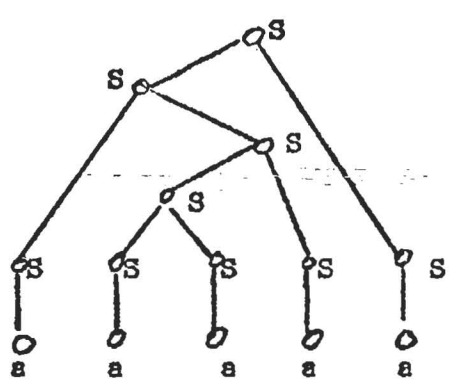
p1,p2,p2,p2,p2

3. $\Phi = \{p_1 : S \to a,\ p_2 : S \to aSa\}$



p1,p2,p2

4. $\Phi = \{p_1 : S \to a,\ p_2 : S \to SS\}$



p1,p1,p1,p2,p1,p2,p2,p1,p2
(one parse)

Definition: The Canonical form of the section of the parse reducing α_i into α_k where

$$\alpha_k = A_1 A_2 \ldots A_m$$

$$\alpha_i = \beta_1 \beta_2 \ldots \beta_m$$

$$A_\ell \overset{*}{\underset{p}{\rightarrow}} \beta_\ell \qquad (\ell = 1 \ldots m)$$

is $\{p_1\}\ \{p_2\} \ldots \{p_m\}$ where the sequence $\{p_\ell\}$ is the canonical form of the section reducing β_ℓ into A_ℓ, and

1. $\{p_\ell\}$ is empty if $\beta_\ell = A_\ell$
2. $\{p_\ell\}$ is canonical if $A_\ell \dot{\rightarrow} \beta_\ell$.

Example: The canonical form of the parse in 4. with $\Phi = \{p_1\colon S \rightarrow a,\ p_2\colon S \rightarrow SS\}$

$$\{\{\{\{p_1\}\ \{\{\{p_1\}\ \{p_1\}\}\ \{p_1\}\}\}\ \{p_1\}\}$$

Definition: The canonical parse is a parse which proceeds strictly from left to right in a sentence and reduces a leftmost part of the sentence as far as possible* before proceeding further to the right.

Example:

$$\{\{\{\{p_1\}\ \{\{\{p_1\}\ \{p_1\}\}\ \{p_1\}\}\}\ \{p_1\}\}$$

Definition: A language $\mathcal{L}$ is ambiguous if for any terminal string $\alpha \in \mathcal{L}$, there exists more than one canonical parse.

Theorem: There is no algorithm which decides for an arbitrary syntax, whether it is unambiguous.

*without blocking

VIII. PROGRAMMING LANGUAGES

Definition: The ordered sextuple $\mathcal{L}_p\ (\mathcal{V}, \Phi, \mathcal{V}_T, S, \Psi, \mathcal{E})$ is the programming language defined by

1. $\mathcal{V}$ the vocabulary
2. Φ the set of syntax rules
3. $\mathcal{V}_T$ terminal vocabulary
4. S initial symbol
5. Ψ semantic interpretation rules
6. $\mathcal{E}$ environment

Example:

$\mathcal{V}$: {<program>,<statement>,<expression>,<term>,<factor>, a}

$\mathcal{V}_T$: {a} S = <program>

Φ : {p1: <program>::= <statement>,

p2: <statement>::= a ← <expression>,

p3: <expression>::= <expression> + <term>,

p4: <expression>::= <term>,

p5: <term>::= <term>× <factor>,

p6: <term>::= <factor>,

p7: <factor>::= a,

p8: <factor>::= (<expression>)}.

Ψ : {S_1: NOP

S_2: STO a

S_3: ADD

S_4: NOP

S_5: MUL

S_6: NOP

S_7: STK a

S_8: NOP }.

$\mathcal{E}$: Memory of stack computer

<u>Definition</u>: The <u>meaning</u> of a sentence α in a language $\mathcal{L}_p$ is $S_1\ S_2 \ldots S_m$ where a parse of α into S is $p_1\ p_2 \ldots p_m$ and S_i corresponds to p_i $(i = 1 \ldots m)$.

Example: a ← a + a

{{{{{{{p7}6}4}{{{p7}6}5}2}1} (canonical form)

p7 p6 p4 p7 p6 p5 p2 p1 (parse)

S_7 S_6 S_4 S_7 S_6 S_5 S_2 S_1 (meaning)

NOP NOP NOP NOP

STK a STK a ADD STO a

IX. PRECEDENCE RELATIONS AND PRECEDENCE GRAMMARS

<u>Definition</u>: The <u>Left set</u> L(<u>Right set</u> R) of $A \in \mathcal{V}_N$ is

$$L(A) = \{S \mid \exists\, \alpha (A \overset{*}{\Rightarrow} S\alpha)\},$$
$$R(A) = \{S \mid \exists\, \alpha (A \overset{*}{\Rightarrow} \alpha S)\}.$$

<u>Definition</u>: For S_i and $S_j \in \mathcal{V}$

$$S_i \doteq S_j \leftrightarrow \exists_p \in \Phi \ni \; p: U \rightarrow \alpha\, S_i S_j \beta$$

$$S_i < S_j \leftrightarrow \exists_p \in \Phi \ni \; p: U \rightarrow \alpha\, S_i U_\ell \beta \;\; S_j \in L(U_\ell)$$

$$S_i > S_j \leftrightarrow \exists_p \in \Phi \ni$$

$$(p: U \rightarrow \alpha U_r S_j \beta \wedge S_i \in R\,(U_r))$$

$$\vee (p: U \rightarrow \alpha U_r U_\ell \beta \wedge S_i \in R\,(U_r) \wedge S_j \in L(U_\ell))$$

<u>Definition</u>: A grammar G is a simple precedence grammar $\leftrightarrow$ at most one relation holds between ordered pairs of symbols of $\mathcal{V}_N$.

<u>Definition</u>: A language is a <u>simple precedence phrase structure programming language</u> $\leftrightarrow$ it is defined by a precedence grammar.

<u>Theorem</u>: There exists a simple algorithm for simple precedence phrase structure programming languages which obtains the unique canonical parse of a string in the language.

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