Groups in Physics

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SASS
September 15, 2010

Klein-Four Group

Definition of a Group

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1) Closure: If a and b are in G, then $a * b$ is in G.

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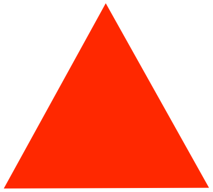
3) Existence of Identity: There exists an element e in G such that $a * e = e * a = a$ for all a in G.

4) Existence of Inverse: For all a in G, there exists an element a^{-1} such that $a * a^{-1} = a^{-1} * a = e$.

Example

The set of symmetries of an equilateral triangle:

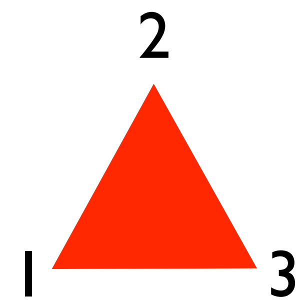


"What can I do to the triangle so that you can't tell it has been changed?"

-Rotate by 120°
-Reflect about vertical axis

These operations close and form a group called S_3 or D_6

Example, revisited

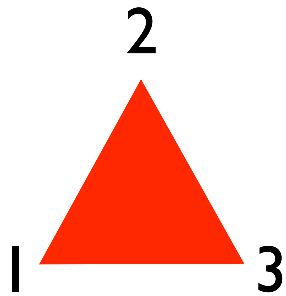


Represent corners by entries in a vector:

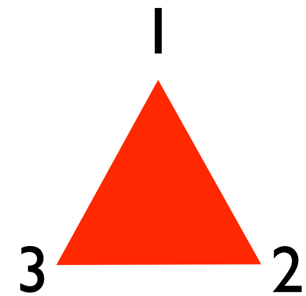
$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Rotation and reflection can be represented by matrix multiplication!

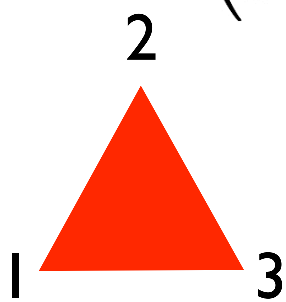
Example, revisited



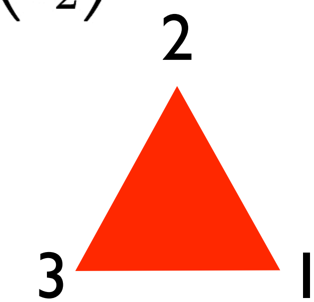
Rotate by 120°



$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_1 \\ v_2 \end{pmatrix}$$

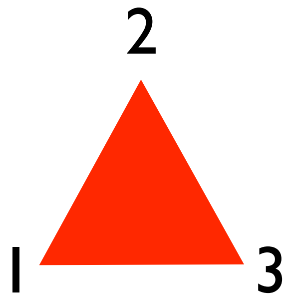


Reflect

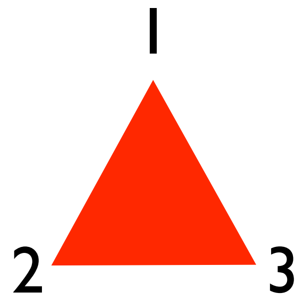


$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_2 \\ v_1 \end{pmatrix}$$

Example, revisited



Rotate by 120°
and Reflect



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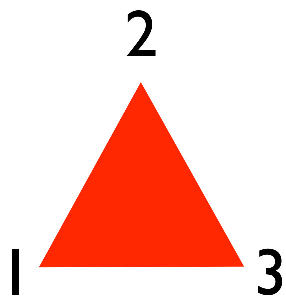
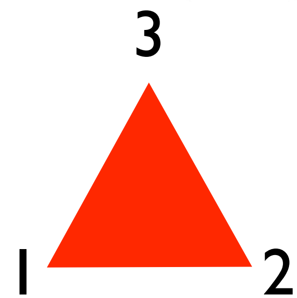


Reflect and
Rotate by 120°



$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Example, revisited

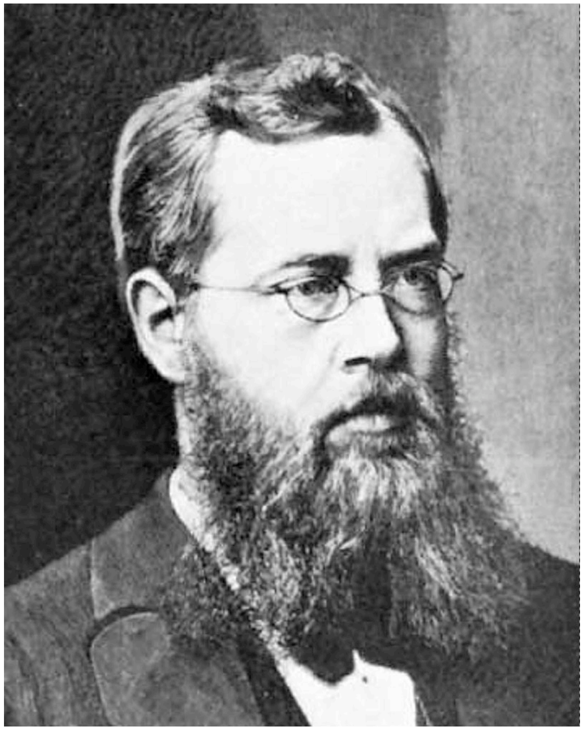
Rotate by 120° and Reflect	$\neq$	Reflect and Rotate by 120°

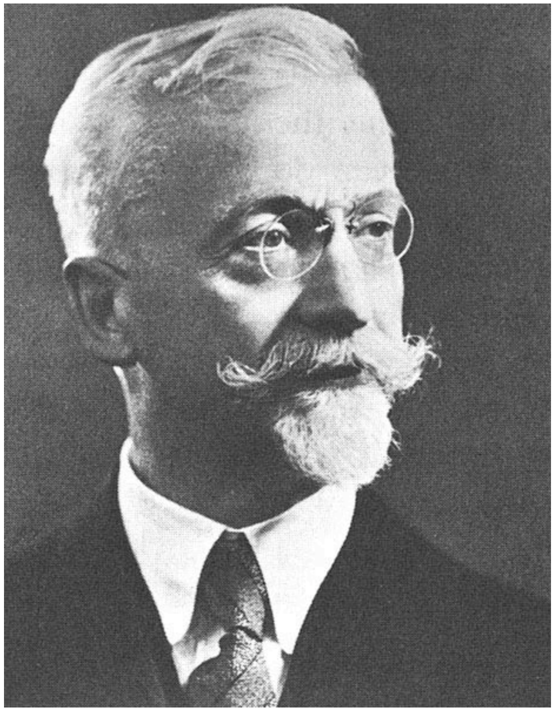
S_3 is a non-Abelian or non-commutative group!

S_3 can be represented by 3 x 3 matrices

S_3 has order or cardinality 6

S_3 is generated by reflection and 120° rotation

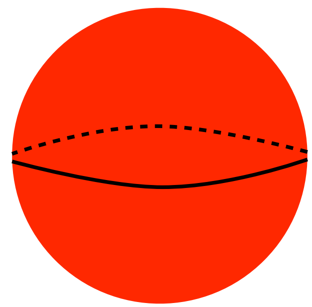




Continuous Groups

A continuous group has a continuously infinite cardinality

Take a sphere:



Symmetric under rotations about any axis and any angle

Group of symmetries called SO(3)

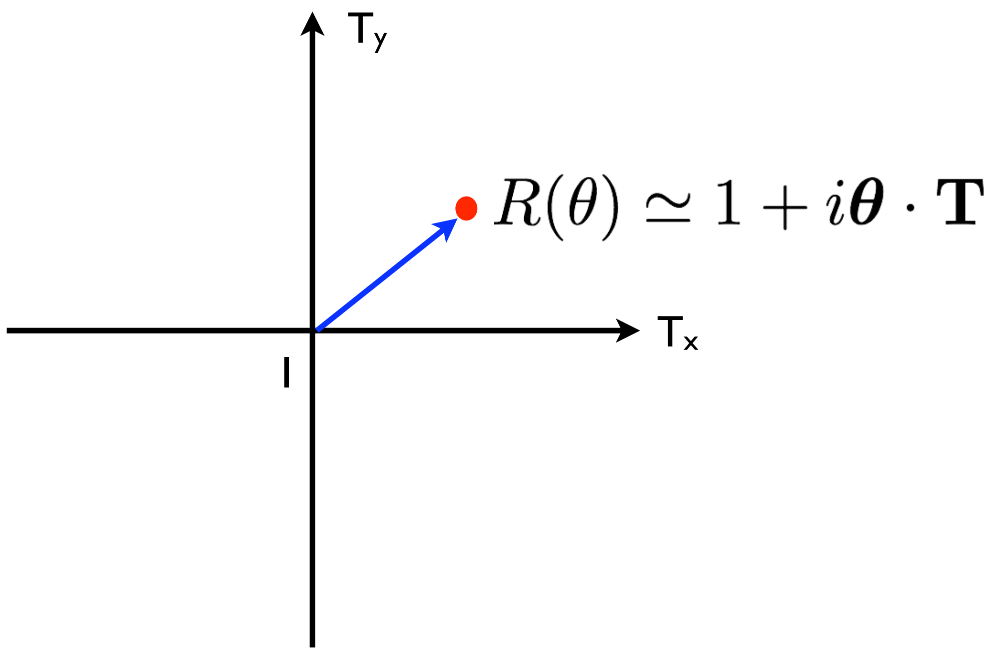
Three matrices that generate rotations about each axis:

$$\mathbf{T} = (T_x, T_y, T_z)$$

Continuous Groups

How to think about continuous groups:

Zoom in near the identity of the group ("the origin")



Continuous Groups

How to think about continuous groups:

What about a large angle?

$$R(\Omega) = \left(1 + i\frac{\mathbf{\Omega} \cdot \mathbf{T}}{N}\right)^N$$
$$= e^{i\mathbf{\Omega}\cdot\mathbf{T}}$$

Any group element of SO(3) can be written as

$$R(\Omega) = e^{i\mathbf{\Omega}\cdot\mathbf{T}}$$

The matrices **T** are in the Lie Algebra

Continuous Groups

How to think about continuous groups:

How do they compose?

$$R(\Omega_1)R(\Omega_2) = e^{i\mathbf{\Omega}_1\cdot\mathbf{T}}e^{i\mathbf{\Omega}_2\cdot\mathbf{T}}$$

$$= e^{i(\mathbf{\Omega}_1+\mathbf{\Omega}_2)\cdot\mathbf{T}+\frac{i}{2}\Omega_1^i\Omega_2^j[T_i,T_j]+\cdots}$$

The commutation relations of the generators define the group!

For this operation to close, we need

$$[T_i, T_j] = c_{ij}{}^k T_k$$

This is the Lie algebra

Continuous Groups

Compare to Taylor expansion:

$$
\begin{aligned}
f(\epsilon) &= f(0) + \epsilon \frac{d}{dx} f(0) \\
&= \left(1 + \epsilon \frac{d}{dx}\right) f(0)
\end{aligned}
$$

$$
\begin{aligned}
f(a) &= \left(1 + \frac{a}{N} \frac{d}{dx}\right)^N f(0) \\
&= e^{a \frac{d}{dx}} f(0) \\
&= f(0) + a f'(0) + \frac{a^2}{2} f''(0) + \cdots
\end{aligned}
$$

Simple Lie Groups

Cartan enumerated all simple Lie groups:

Classical Lie Groups

SU(N): Unitary matrices with unit determinant

SO(N): Orthogonal matrices with unit determinant

Sp(N): Symplectic matrices with unit determinant

Exceptional Lie Groups

F_4, G_2, E_6, E_7, E_8

Noether's Theorem

Symmetries of the action correspond to conserved currents

$$S = \int dt \frac{m}{2}\dot{x}^2 - U(x)$$

Symmetries:	Conserved Currents:
Time Translation	Energy

Proof:

Energy is $$H = \frac{m}{2}\dot{x}^2 + U(x)$$

Equation of motion for *x* is $$m\ddot{x} + \frac{dU}{dx} = 0$$

$$\frac{dH}{dt} = m\ddot{x} + \frac{dU}{dx} = 0$$

Noether's Theorem

Other examples

Symmetries:	Conserved Currents:
Time Translation	Energy
Translation	Momentum
Rotation	Angular Momentum

Spacetime Symmetries

U(1) rotation	Electric Charge
Isospin rotation	Isospin (in strong interactions)

Internal Symmetries

Isospin

Heisenberg: (Near) Symmetry between Proton and Neutron

Modern: (Near) Symmetry between Up and Down quarks

$$\begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix} \xrightarrow{\text{Isospin}} M \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix}$$

What is *M*?

M must conserve probability:

$$\begin{pmatrix} \Psi_p^\dagger & \Psi_n^\dagger \end{pmatrix} M^\dagger M \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix} = \begin{pmatrix} \Psi_p^\dagger & \Psi_n^\dagger \end{pmatrix} \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix}$$

$$M^\dagger M = 1 \qquad M \in \mathrm{SU}(2)$$

Isospin group is SU(2)!


LIFE

Quantum Numbers

Defining what a particle **is**

Rule:

If there exists an unbroken symmetry relating two particles, those particles are identical and indistinguishable

Examples:

Rotation

$\uparrow$ 180° $\downarrow$

Time Translation

Yesterday 2 days Tomorrow

Quantum Numbers

Defining what a particle **is**

A particle is defined by the representations of the known symmetry groups it possesses

Electron:

Spin: **2**

Electric Charge: -1

Color Charge: **1**

Gluon:

Spin: **3**

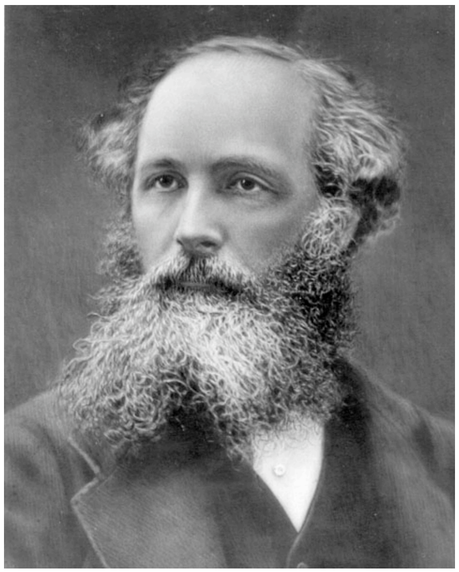
Electric Charge: 0

Color Charge: **8**

C. N. Yang (1922 -) and Robert Mills (1927 - 1999)
at Stony Brook in 1999.

Non-Abelian Gauge Theory

Reminder: Abelian Gauge Theory



$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$S = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu}$$

$$\partial_\mu F^{\mu\nu} = 0$$

Why "abelian gauge theory"?

Action is invariant under U(1) gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

Non-Abelian Gauge Theory

Extend this to non-Abelian groups

Vector potential is now a *matrix*:

$$A_\mu = A_\mu^a T^a$$

Field strength is now a *matrix*:

$$\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu] \\
&= \partial^\mu A^{\nu a} T^a - \partial^\nu A^{\mu a} T^a + A^{\mu b} A^{\nu c} [T^b, T^c] \\
&= (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + c^{bc}{}_a A^{\mu b} A^{\nu c}) T^a \\
&= F^{\mu\nu a} T^a
\end{aligned}$$

Action is now:

$$S = -\frac{1}{4} \int d^4x \; d_{ab} F^{\mu\nu a} F^b_{\mu\nu}$$

Non-Abelian Gauge Theory

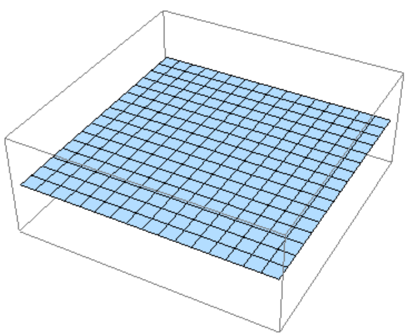
Question: What groups are possible?

Answer: Only simple (compact) Lie groups!

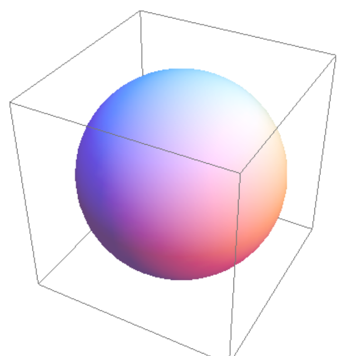
Another Question: Why?

$$S = -\frac{1}{4}\int d^4x \; d_{ab} F^{\mu\nu a} F^{b}_{\mu\nu}$$

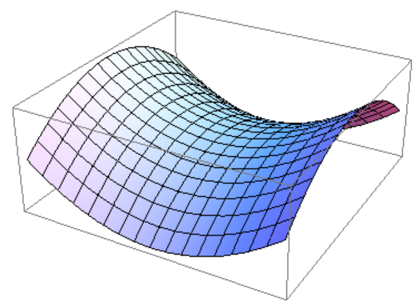
Note: Kinetic energy of all dynamical degrees of freedom must be positive!



Non-compact



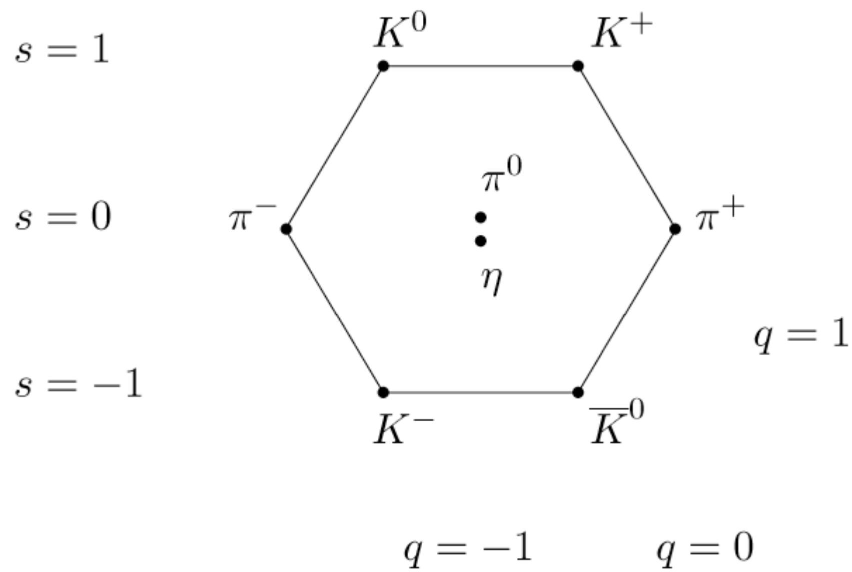
Compact!



Non-compact

Defining Quarks

The Hadron Problem

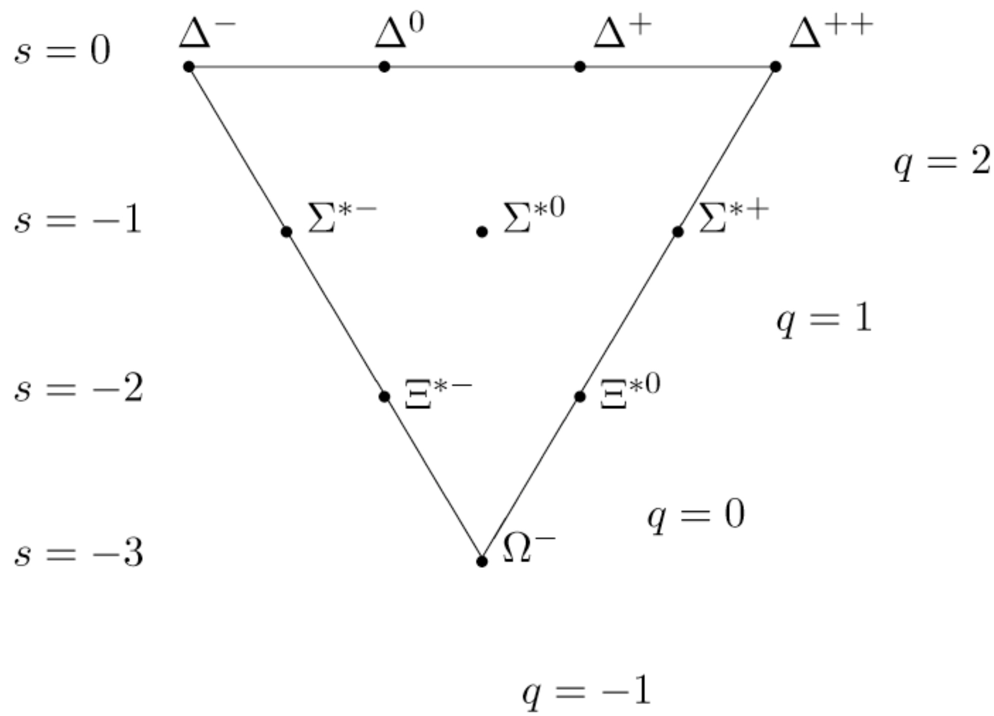


An 8 dimensional representation of $SU(3)_F$

"octet"

Defining Quarks

The Hadron Problem



An 10 dimensional representation of $SU(3)_F$
"decuplet"

Defining Quarks

The Hadron Problem

No fundamental (triplet) representations were ever found!

All higher dimensional representations can be constructed from fundamental!

Solution: Call fundamental objects quarks

Three quarks can explain proliferation of hadrons!

-Three quarks for Muster Mark!
Sure he hasn't got much of a bark
And sure any he has it's all beside the mark.
-Finnegans Wake

Defining Quarks

Octet Example:

$$K^0 = d\bar{s}$$

$$K^+ = u\bar{s}$$

$$\pi^+ = u\bar{d}$$

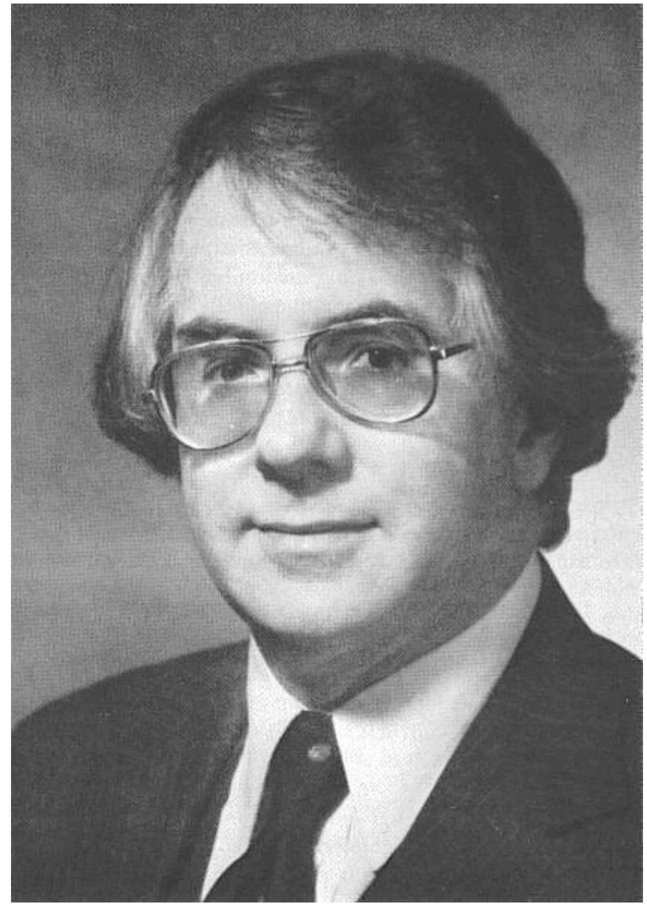
$$\bar{K}^0 = \bar{d}s$$

$$K^- = \bar{u}s$$

$$\pi^- = \bar{u}d$$

$$\pi^0 = \frac{u\bar{u} - d\bar{d}}{\sqrt{2}}$$

$$\eta = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}$$



The Standard Model

The culmination of everything before:

Elementary matter degrees of freedom:

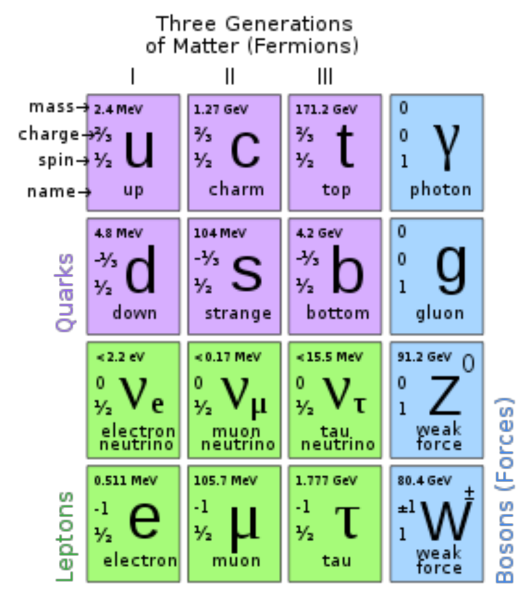
Quarks
Leptons

Elementary force carriers:

Non-abelian gauge bosons

Gauge Group of the Standard Model:

$SU(3)_C \times SU(2)_L \times U(1)_Y$



Things I didn't discuss

Lorentz symmetry

General Relativity (diffeomorphisms)

Supersymmetry

(Super)Conformal groups

Discrete lattice groups

Infinite dimensional algebras (Virasoro, Kac-Moody, Yangian)

Anomalies

Topology