On Wakefields With Two-Dimensional Planar Geometry*

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I. INTRODUCTION

In order to reach higher acceleration gradients in linear accelerators, it is advantageous to use a higher accelerating RF frequency, which in turn requires smaller accelerating structures. As the structure size becomes smaller, rectangular structures become increasingly interesting because they are easier to construct than cylindrically symmetric ones. [1,2] One drawback of small structures, however, is that the wakefields generated by the beam in such structures tend to be strong. Recently, it has been suggested that one way of ameliorating this problem is to use rectangular structures that are very flat and to use flat beams. [3] In the limiting case of a very flat planar geometry, the problem resembles a purely two-dimensional (2-D) problem, the wakefields of which have been studied in Ref. [4].

In this work we consider the purely 2-D problem that is sketched in Fig. 1. The beam is considered to be infinitely long in the horizontal x-direction; it propagates with the speed of light c in the longitudinal z-direction from $z = -\infty$ to $z = +\infty$. The beam distribution in the y-z plane is arbitrary. The environment consists of boundaries which are independent of x, but are otherwise unrestricted; for example, in the y-z plane they can be of arbitrary shape, and they can be made of metal, dielectric or plasma material. We do assume, however, that the beam trajectory is entirely in free space and that it nowhere intersects the boundaries. A test charge e in the beam (or trailing the beam) also moving in the z-direction at the speed of light samples the force due to the wakefield generated by the beam. For these conditions, a theorem that we call the "planar wake theorem" was proven in Ref. [4]. The theorem states that the total transverse wake kick received by the test charge is independent of the y-positions of the beam and the test charge, and is also independent of D, the longitudinal separation between the beam and the test charge (see Fig. 1). In addition, the theorem states that the *longitudinal* wake kick is also independent of y, though it does not say anything about its D-dependence.

In this report, in Section II, we rederive the planar wake theorem. In Section III, we add a corollary to the theorem, applicable to the case when, in addition to the above conditions, the boundaries also have up-down symmetry. For this case, we will prove that the transverse wake kick not only is constant, but in fact is equal to zero. The proof consists of a simple application of the planar wake theorem. However, it was Ref. [3] which triggered the present extension to the theorem. Finally, in Section IV, we make additional observations concerning the wakefields in very flat 3-D accelerating structures.

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II. THE PLANAR WAKE THEOREM

The planar wake theorem was proven in Ref. [4], but it is reproduced here in more detail, starting with the Maxwell equations. From the symmetry of the problem, we know that the only nonvanishing field components are E_y, E_z and B_x, and all components depend only on y, z and t (and are independent of x). The charge density ρ and current density $\vec{j}$ are related by $\vec{j} = c\rho\hat{z}$. The Maxwell equations then become

$$\begin{aligned}
\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 4\pi\rho \\
\frac{\partial B_x}{\partial z} - \frac{1}{c}\frac{\partial E_y}{\partial t} &= 0 \\
\frac{\partial B_x}{\partial y} + \frac{1}{c}\frac{\partial E_z}{\partial t} &= -4\pi\rho \\
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial B_x}{\partial t} &= 0 \quad .
\end{aligned} \tag{1}$$

The instantaneous Lorentz force components are given by

$$\begin{aligned}
F_y(y,z,t) &= eE_y + eB_x \\
F_z(y,z,t) &= eE_z \quad .
\end{aligned} \tag{2}$$

Combining Eqs. (1) and (2), we obtain

$$\begin{aligned}
\frac{\partial F_y}{\partial y} &= -e\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) E_z \\
\frac{\partial F_y}{\partial z} &= e\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) E_y \\
\frac{\partial F_z}{\partial y} &= e\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) (E_y - B_x) \quad .
\end{aligned} \tag{3}$$

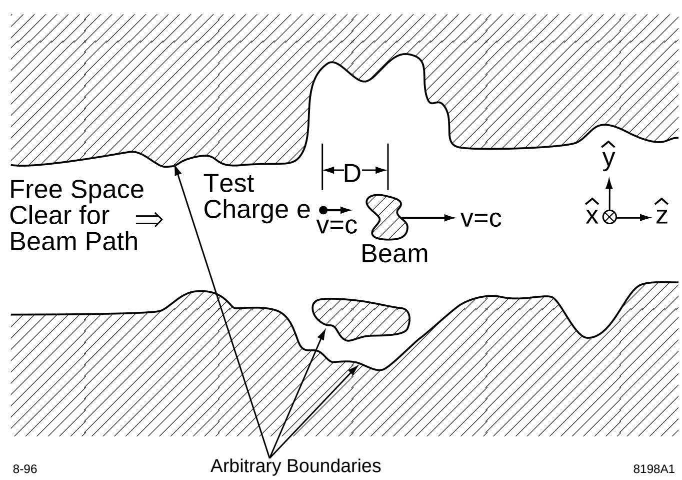

Figure 1: A sketch of our 2-D problem, showing the beam and its environment.

Note that the right hand sides of Eq. (3) all contain the operator $\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}$.

We are interested in the wake kick $c\Delta\vec{p}$, which is obtained by integrating the wake force along the path of the test charge from $z = -\infty$ to $z = +\infty$, *i.e.*

$$c\Delta p_{y,z}(y, D) = \int_{-\infty}^{\infty} F_{y,z}(y, D + ct, t)\, dt \quad , \tag{4}$$

where y is the vertical coordinate of the test charge and D is the longitudinal separation between the driving beam and the test charge. (Our convention is such that, for a test charge trailing the driving beam, $D < 0$.) Substituting Eqs. (3) into Eq. (4) we obtain integrals of the form

$$\int_{-\infty}^{\infty} dt \, \left[\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} \right) G(z, t) \right]_{z=D+ct} \quad , \tag{5}$$

which in all cases equals zero, since the integrand is proportional to the total derivative $dG(D + ct, t)/dt$ and since $G(D + ct, t)$ approaches zero as $|t| \to \infty$. It therefore follows that

$$\frac{\partial \Delta p_y}{\partial y} = \frac{\partial \Delta p_y}{\partial D} = \frac{\partial \Delta p_z}{\partial y} = 0 \quad , \tag{6}$$

which proves the planar wake theorem.

The proof of the planar wake theorem parallels a similar, perhaps more familiar, result for the case of cylindrically symmetric boundaries. In the cylindrically symmetric case, among the results that are obtained are that the wake kick of a monopole beam is independent of the transverse position of both the beam and the test charge, that the transverse wake kick of a dipole beam depends linearly on the radial position of the driving beam and is independent of the transverse position of the test charge, and that the longitudinal and transverse wake kicks of any multipole obey the Panofsky-Wenzel theorem. [5]

Note that the boundary properties never enter into our proof of the planar wake theorem; the theorem is valid independent of the shape and material of the boundary. For example, the boundary can be only above or below the beam trajectory, or on both sides of the beam. It can also consist of several separate parts, as indicated in Fig. 1. It can be made of a fuzzy material, such as a gradually fading plasma. The boundary can even separate parts of the beam, as long as the boundary is 2-D planar and as long as by changing y the beam path does not cross any boundary.

Also note that for our problem, the wake kick is also independent of the y position of the driving beam. This is because the wakefield is a response to the primary field carried by the driving beam, and the primary field at the boundaries is independent of the y-position of the beam in a 2-D planar geometry.

In Ref. [4] two examples of 2-D planar wakefields with open geometry were explicitly found by analytical methods. The boundaries in both cases are wedge-shaped, made of perfectly conducting metal, and are only on one side of the beam path (see Figs. 2(a) and 2(b)). By solving the Maxwell equations it was found that the transverse wake kick received by the test charge due to the wakefield generated by a rod beam (*i.e.* one that is infinitely long in x and a delta function in the y- and z-directions) of line charge density λ_0 is given by

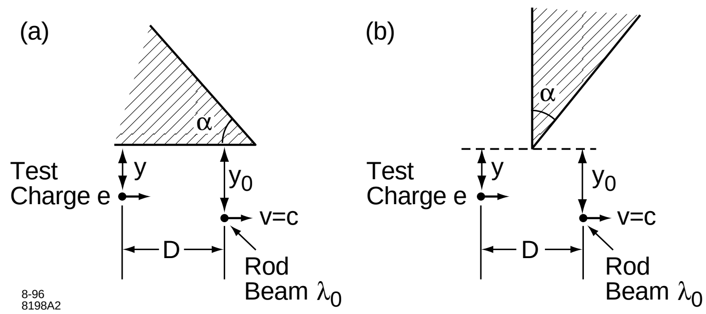

Figure 2: Two exactly soluble cases.

$$c\Delta p_y\,(y, y_0, D < 0) \;=\; \begin{cases} 2\pi e\lambda_0 & \text{for Fig. 2(a)} \\ \pi e\lambda_0 & \text{for Fig. 2(b)} \end{cases} \quad , \tag{7}$$

where y_0 is the y-offset of the rod beam. We note that the transverse wake kick is independent of y, y_0 and D, as the planar wake theorem states. It is also interesting that the result, Eq. (7), does not depend on α, the wedge angle. Although Eq. (7) applies only to a rod beam, the wake kick for a more general y-z beam distribution can be obtained by simple superposition.

III. A COROLLARY TO THE THEOREM

The planar wake theorem has an interesting corollary when the 2-D boundaries have the additional property of up-down symmetry, as is sketched in Fig. 3(a). In this case, the transverse wake kicks due to the upper and the lower halves of the boundaries cancel each other and the net transverse kick becomes zero. Note that the beam does not need to observe up-down symmetry and that it can have a y off-set.

To prove this corollary, let us first consider a rod beam. Since the wake kick does not depend on the y-positions of the beam and the test charge, we can choose to locate both along the symmetry axis $y = 0$ without changing the result. But for such a configuration the transverse wake kick must vanish due to the symmetry of the problem. Finally, we can extend this result to an arbitrary y-z beam distribution by applying superposition, and the corollary follows.

Note that the corollary applies to the total *integrated* wake kick received by the test charge. The *instantaneous* wake force is not necessarily zero. In case the boundary has translational symmetry, *i.e.*, is independent of the z-position, the instantaneous transverse wake force would, of course, also vanish. Note also that the up-down symmetry of the boundary is required for the corollary to hold; the transverse wake kick is not zero due to the 2-D planar geometry alone. It vanishes only when the additional requirement of up-down symmetry is applied.

Another application of the planar wake theorem is when the boundary on one side of the beam path is a perfectly conducting plate, as sketched in Fig. 3(b). In this case, one must have

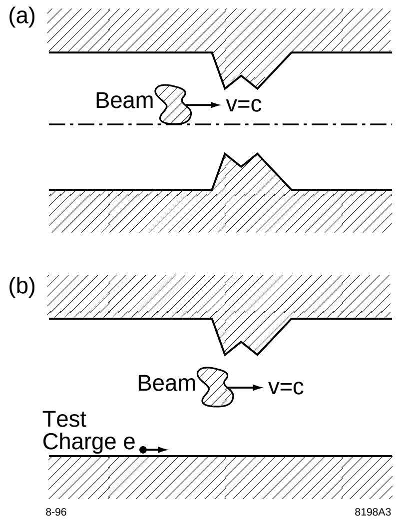

Figure 3: (a) An example of up-down symmetric boundaries, for which the corollary to the planar wake theorem applies. (b) Another example with a perfectly conducting plate on one side of the beam path.

$\Delta p_z \propto \delta(D)$. To demonstrate this, consider a test charge that travels immediately next to the surface of the plate. For this test charge, F_z necessarily vanishes and thus $\Delta p_z = 0$. An application of the planar wake theorem then predicts $\Delta p_z = 0$ for a test charge with any vertical position y. The only way this can happen is when $\Delta p_z(D) \propto \delta(D)$.

IV. DISCUSSIONS

We have shown that in a purely 2-D planar geometry, if the boundaries have up-down symmetry, then the transverse wakefields are zero. With the accelerating structures for future linear colliders continuing to be made smaller, one might ask if the use of very flat rectangular structures and flat beams can aid in eliminating the wakefield problem. This question obviously requires more study. Below we make only a few preliminary observations:

1. A very flat 3-D structure is not the same as a 2-D planar structure. In a 3-D structure there are effects associated with the edges of the structure and the edges of the beam. These effects may degrade the quality of the beam, in a process that begins at the edges and gradually moves toward the interior of the beam. A possible cure is to design the structure in such a way that the internal part of the structure sees “impedance-matched” edges in the sense that the wakefield seen by particles sufficiently away from the edges behaves as if the structure is truly 2-D (see Fig. 4). [6] Or one might consider a higher-order-mode damped structure designed with RF absorbers installed along the structure, perhaps in the edge region (see Fig. 4). This should at least help the long range wake effects associated with the edges.

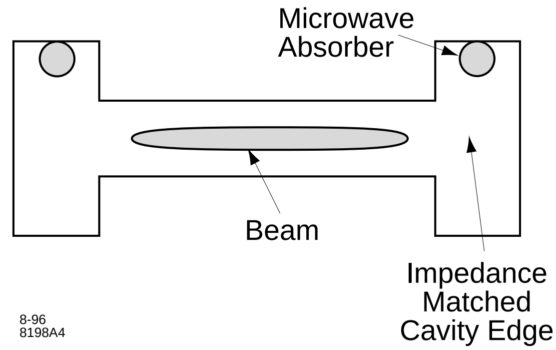

Figure 4: A very flat rectangular structure which may behave as 2-D planar. The beam moves in the direction into the page.

2. Even in a purely 2-D structure it may turn out that an initial, slight un-evenness of the beam distribution in the x-direction, or a tilt of the beam in the x-y plane, leads to unstable growth as the beam propagates down the linear accelerator. This question requires a study of collective instabilities. [7,8]
3. Although the net transverse wake kick will tend to vanish in a flat, rectangular structure, it may turn out that the longitudinal wake kick is larger than desirable. (For the purely 2-D examples of Fig. 2, for example, the longitudinal kick actually diverges. [4])
4. For the case of cylindrical symmetry a so-called “single mode” rf cavity has been proposed. [9] It is a cavity that features few or no higher order modes at the cost of a significant reduction in shunt impedance. With a very flat, rectangular structure it is conceivable that we can avoid the transverse wakefield problem with little sacrifice in shunt impedance.

V. ACKNOWLEDGMENTS

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VI. REFERENCES

[1] D. Yu, S. Ben-Menahem, P. Wilson, R. Miller, R. Ruth, and A. Nassiri, *AIP Proc.* 335, Adv. Accel. Concepts, Wisconsin, 1994, p.800.

[2] H. Henke, *Proc. IEEE Part. Accel. Conf., Dallas, 1995*, p. 1750.

[3] J. Rosenzweig, these Snowmass proceedings, 1996.

[4] A.W. Chao and H. Henke, *Proc. IEEE Part. Accel. Conf., Dallas, 1995*, p.3043.

[5] See, for example, A.W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, J. Wiley, New York, 1993.

[6] D. Yu and P. Wilson, *Proc. IEEE Part. Accel. Conf., Washington D.C., 1993*, p. 2681.

[7] H.S. Uhm, S. Ben-Menahem, and D. Yu, Phys. *Plasmas*, Vol.1, No.11, 3686 (1994).

[8] D.H. Whittum, Phys. Fluids B, Vol. 5, No. 12, 4432 (1993).

[9] T. Weiland, DESY Report 83-073 (1982).