# TOLERABLE SYSTEMATIC ERRORS IN REALLY LARGE HADRON COLLIDER DIPOLES

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## ABSTRACT

Maximum allowable systematic harmonics for arc dipoles in a Really Large Hadron Collider are derived. The possibility of half cell lengths much greater than 100 meters is justified. A convenient analytical model evaluating horizontal tune shifts is developed, and tested against a sample high field collider.

## I. INTRODUCTION

Both "low field" and "high field" concepts of a future Really Large Hadron Collider (RLHC) were discussed at Snowmass 96. Both concepts invoke novel magnet designs. The goal of this paper is to establish semi-quantitative estimates of what would constitute good or bad field quality in arc dipoles in either machine, and to directly draw the connection between field quality and maximum (optimum) half cell length. It is implicitly assumed (after the discussion immediately below) that systematic errors dominate random errors, and that they therefore deserve the closest attention. It is fortunate that this appears to be true for contemporary superconducting magnets - if not for future magnets using high temperature superconductor technology - since it is far harder to make even semi-quantitative mathematical statements about random errors.

### A. Do systematic or random errors dominate?

In 1983, when the Ann Arbor SSC workshop was held, the SSC was little more than a gleam in the physicists eye. The proceedings of that workshop contain the first systematically documented attempts to predict SSC dipole harmonic errors [1]. These predictions rested heavily on extrapolations from the limited experience with superconducting magnets then available - from the Tevatron and Isabelle/CBA. It was judged that, in general, *random errors were expected to dominate systematic errors* in SSC magnets. From the time that the official lattice was established in 1986 - in the Conceptual Design Report (CDR) of the SSC [2] - until the demise of the project in 1993, the SSC half cell length was consistently in the range $L_{SSC} = 100 \pm 10$ meters. The tables of expected dipole harmonic errors that were used for tracking purposes did not change significantly in this period. However, an analysis of 10 or so of the last SSC dipoles built shows that the *as built* harmonics were, in most cases, 3 to 10 times smaller than the *expected* CDR harmonics [3]. This implies that the SSC half cell length could have been much longer than 100 meters, and/or that it might have been possible to remove some of the nonlinear correctors.

Considerable experience has been gained since then, and the state of the art has been significantly advanced, with the construction of superconducting magnets for HERA-p and RHIC.

RHIC experience is that, to the contrary of the SSC canon, *systematic errors dominate random errors*. Preliminarily, it also appears that systematic errors dominate random errors in LHC magnets [4]. RHIC demonstrated that systematic harmonic errors can be adjusted during industrial production, using mil-size adjustments of mid-plane caps and coil pole shims [5]. This was done without interrupting the production line schedule - without adjusting the coil/collar/yoke geometry, and with only negligible redistribution of stress patterns. As a result, it was possible to reduce systematic harmonics in standard RHIC dipoles and quadrupoles to such an extent that the octupole and decapole correctors installed in the arcs will not be powered - except, perhaps, for the purpose of Landau damping. The only nonlinear correctors that will be powered in the arcs are two families of chromatic sextupoles.

High field quality in arc dipoles is most important at injection, when the beams are at their largest. It may therefore seem irrelevant that "tuning shims" in RHIC interaction region quadrupoles have been discovered to significantly improve top energy performance. However, the same tuning shim technology can also be used in arc dipoles at injection for the same purpose - to easily adjust several harmonics in an individual magnet *after* that magnet has been constructed and measured. Tuning shims could be used on each and every RLHC dipole magnet, to remove both systematic and random errors. Or, they could be applied to a single dipole at one end of each half cell, and a single dipole in the middle, in a "pseudo Simpson Neuffer scheme" that would correct many harmonics - at a single excitation.

### B. Really Large Hadron Collider

It is fiscally imperative that RLHC designs stress simplicity, reliability, and economy - three virtues that are closely related. Complicated and copious magnet interconnects and spool pieces should be avoided wherever possible, in order to keep the average cost per meter low. One way to reduce the number of spools is to increase the half cell length as far as possible, beyond the conventional 53.4 meters of the LHC, and 100 meters of the SSC. Spool complexity can be reduced by eliminating most or all of the nonlinear correctors from the arcs. It may even be possible to correct the closed orbit and the chromaticity with sparse dipole and sextupole correctors - less than one of each per half cell [6].

The busy or disinterested reader may wish to skip the next two sections of this paper, "TUNE SHIFTS" and "MAXIMUM TUNE SHIFTS", which develop the mathematical model and demonstrate its accuracy with a high field RLHC example. It should be possible to go directly to section IV, "MAXIMUM ALLOWABLE HARMONICS", and pick up the story when it focuses on practical consequences and real numbers.

## II. TUNE SHIFTS

The normal harmonic errors in a standard arc dipole are parameterized by the coefficients $b_n$ in the expression

$$B_y \;=\; B_0 \left[1 \;+\; \sum_n \frac{b_n}{r_0^n}\, x_t^n\right] \tag{1}$$

where $B_y$ is the vertical field at a horizontal displacement of $x_t$ from the design trajectory at the center of the dipole, and $r_0$ is the reference radius. As a test particle moves along a dipole with a single harmonic, the horizontal angle $x_t'$ that it makes with the design trajectory changes at the rate

$$\frac{dx_t'}{ds} \;=\; -\,\frac{B_0}{B\rho(1+\delta)}\;\frac{b_n}{r_0^n}\;x_t^n \tag{2}$$

where $B\rho$ is the on-momentum magnetic rigidity and $\delta = \Delta p/p$ is the relative momentum offset. Assuming a perfect closed orbit, the total horizontal displacement is given by

$$x_t \;=\; x \;+\; \eta\delta \tag{3}$$
$$x \;=\; A_x \cos(\phi_x) \tag{4}$$

where $x$ is the betatron displacement contribution, $A_x$ and $\phi_x$ are the betatron amplitude and phase, and $\eta$ is the dispersion function at that location. The rate of change of betatron angle is derived from Equation 2, after recognizing that the dispersion itself is modified by the error harmonic. This gives

$$\frac{dx'}{ds} \;=\; -\,\frac{B_0}{B\rho(1+\delta)}\;\frac{b_n}{r_0^n}\;\left[(x+\eta\delta)^n - (\eta\delta)^n\right] \tag{5}$$

To proceed to calculate the horizontal tune shift as a function of $A_x$ and $\delta$, it is next necessary to derive the additional betatron phase advance as dipoles are traversed.

Consider a single discrete angular kick $\delta x'$. The additional betatron phase advance is given by

$$\delta\phi_x \;=\; -\,\frac{\beta_x \cos\phi_x}{A_x}\;\delta x' \tag{6}$$

where $\beta_x$ is the horizontal beta function at that location. The total betatron phase advance in one turn, number $n$, is therefore given by an integral over all dipoles

$$\Delta\phi_x(n) \;=\; -\int_{dip} \frac{\beta_x \cos\phi_x}{A_x}\;\frac{dx'}{ds}\;ds \tag{7}$$

The one turn phase advance fluctuates from turn to turn, since it depends on the initial betatron phase at the beginning of the turn, while the betatron tune shift $\Delta Q_x$ is found by averaging the phase advance over many turns. That is,

$$\Delta Q_x \;=\; \frac{\langle\Delta\phi_x\rangle}{2\pi} \tag{8}$$

where the angle brackets denote an average over many turns, or equivalently (it is assumed), an average over the initial betatron phase. Putting all this together,

$$\Delta Q_x \;=\; \frac{b_n}{r_0^n(1+\delta)} \left\langle \frac{\beta_x \cos\phi_x}{A_x}\;\left(A_x\cos(\phi_x) \;+\; \eta\delta\right)^n \right\rangle \tag{9}$$

where the facts that

$$\int_{dip} \frac{B_0}{B\rho}\,ds \;\equiv\; 2\pi \tag{10}$$

and

$$\left\langle \frac{\beta_x \cos\phi_x}{A_x}\;(\eta\delta)^n \right\rangle \;=\; 0 \tag{11}$$

have been used. Angle brackets now denote a double average, over all the dipoles in the lattice and over the betatron phase.

The tune shift is a function of the betatron amplitude and the (constant) momentum offset, which are conveniently parameterized by $m_x$ and $m_\delta$ in writing

$$A_x \;=\; m_x\,\sigma_x \tag{12}$$
$$\eta\delta \;=\; m_\delta\,\sigma_\delta \tag{13}$$

where the root mean square betatron and momentum beam sizes are

$$\sigma_x \;=\; \sqrt{\frac{\epsilon_x\beta_x}{(\beta\gamma)}} \tag{14}$$
$$\sigma_\delta \;=\; \eta\,\frac{\sigma_p}{p} \tag{15}$$

Here $\epsilon_x$ is the horizontal normalized emittance, and $\sigma_p/p$ is the RMS relative momentum spread.

### A. Master equation

This allows the master equation to be written as

$$\Delta Q_x \;=\; \frac{b_n}{r_0^n(1+\delta)} \sum_{i=0}^{n-1-2i\geq 0} C_{n,i}A_{n,i}\;m_\delta^{n-1-2i}\;m_x^{2i} \tag{16}$$

where $A_{n,i}$ are optical averages over dipoles

$$A_{n,i} \;=\; \left\langle \beta_x\;\sigma_\delta^{n-1-2i}\,\sigma_x^{2i}\right\rangle \tag{17}$$

and $C_{n,i}$ are constant coefficients

$$C_{n,i} \;=\; \frac{1}{2^{2i+1}}\,\frac{n!(2i+2)!}{(n-2i-1)!(2i+1)!(i+1)!(i+1)!} \tag{18}$$

These coefficients result from betatron phase averages of $\cos^m(\phi)$ terms, multiplied by binomial coefficients generated when Equation 9 is expanded into a polynomial series. Their values, up to 14-pole, are displayed in Table I.

For illustration purposes, consider a simple lattice with a single short dipole in the middle of each half cell. The optical averages depend only on optical function values at the dipole. Tune shifts for different harmonic errors are given in Table II.

### B. Scaling with cell length, emittance, and energy

The optical averages $A_{n,i}$ depend on the lattice, the emittance, the momentum spread, and the energy. To see how the tune shift scales, assume that there is a standard FODO cell in the arcs, with a phase advance per cell of $\phi_c$. It is also necessary to

Table I: $C_{n,i}$ coefficients for harmonics up to 14-pole.

| n | Multipole | $i=0$ | 1 | 2 |
|---|---|---|---|---|
| 1 | Quadrupole | $1/2$ | | |
| 2 | Sextupole | 1 | | |
| 3 | Octupole | $3/2$ | $3/8$ | |
| 4 | Decapole | 2 | $3/2$ | |
| 5 | 12-pole | $5/2$ | $15/4$ | $5/16$ |
| 6 | 14-pole | 3 | $15/2$ | $15/8$ |

Table II: Tune shifts for the simple example of one thin dipole in the middle of each half cell. $b'_n = b_n/(1+\delta)$.

| Multipole | $\Delta Q_x$ |
|---|---|
| Quadrupole | $b'_1 \beta_x \frac{1}{2}$ |
| Sextupole | $b'_2 \beta_x (\eta\delta)$ |
| Octupole | $b'_3 \beta_x [3/2(\eta\delta)^2 + 3/8 A_x^2]$ |
| Decapole | $b'_4 \beta_x [2(\eta\delta)^3 + 3/2(\eta\delta) A_x^2]$ |
| 12-pole | $b'_5 \beta_x [5/2(\eta\delta)^4 + 15/4(\eta\delta)^2 A_x^2 + 5/16 A_x^4]$ |
| 14-pole | $b'_6 \beta_x [3(\eta\delta)^5 + 15/2(\eta\delta)^3 A_x^2 + 15/8(\eta\delta) A_x^4]$ |

assume some relationship between the betatron and momentum contributions to the total horizontal beam size. For example, suppose that the RMS momentum spread is manipulated with a fixed longitudinal emittance by adjusting the RF voltage, so that the two contributions are equal where they are largest

$$\widehat{\sigma_\delta} \;=\; \widehat{\sigma_x} \tag{19}$$

at the center of the horizontally focusing quadrupole. This is physically reasonable for a high field 30 TeV hadron collider [7]. It is then easy to show that

$$A_{n,i} \;=\; \alpha_{n,i}(\phi_c)\, L^{(n+1)/2} \left(\frac{\epsilon_x}{\beta\gamma}\right)^{(n-1)/2} \tag{20}$$

where $L$ is the half cell length, and $\alpha_{n,i}$ is a non-trivial function of (only) the phase advance per cell. This makes it possible (finally!) to write down how the tune shift scales with cell length, emittance, and energy. Substituting Equation 20 into the master equation, Equation 16, gives

$$\Delta Q_x \;=\; \frac{b_n}{r_0^n(1+\delta)}\, L^{(n+1)/2} \left(\frac{\epsilon_x}{\beta\gamma}\right)^{(n-1)/2} \times \sum_{i=0}^{n-1-2i\geq 0} C_{n,i}\alpha_{n,i}(\phi_c)\, m_\delta^{n-1-2i}\, m_x^{2i} \tag{21}$$

This relatively ugly expression has the virtue of laying bare the dependence of the tune shift on all the parameters of interest.

Table III lists the $\alpha_{n,i}$ values for a lattice with thin quadrupoles in which the FODO cells are fully packed with dipoles - a fair approximation for an RLHC - with a 90 degree phase advance per cell. The application of Tables I and III to Equation 21 is then straightforward, if messy.

Table III: Numerically calculated values for $\alpha_{n,i}$ for fully packed FODO cells with $\phi_c = 90$ degrees per cell.

| n | Multipole | $i=0$ | 1 | 2 |
|---|---|---|---|---|
| 1 | Quadrupole | 1.667 | | |
| 2 | Sextupole | 2.412 | | |
| 3 | Octupole | 3.608 | 3.467 | |
| 4 | Decapole | 5.555 | 5.381 | |
| 5 | 12-pole | 8.753 | 8.536 | 8.340 |
| 6 | 14-pole | 14.06 | 13.78 | 13.53 |

## III. MAXIMUM TUNE SHIFTS

A numerical study of two high field RLHC designs has been performed, in order to verify the accuracy of the mathematical model, and to establish an approximate value for the maximum tolerable horizontal tune shift. Tables IV and V summarize the common primary parameters, and the different lattice parameters, for SHORT and LONG cell high field machines that are described in more detail elsewhere in these proceedings [7].

### A. Tracking results

Figure 1 shows the tune shift versus momentum in the SHORT machine for various values of $m_x$, with a systematic octupole harmonic of $b_3 = 5\times10^{-4}$ in the top plot, and a de-

Table IV: Primary parameters for a high field RLHC.

| Parameter | units | value |
|---|---|---|
| Storage energy | [TeV] | 30.0 |
| Injection energy | [TeV] | 1.0 |
| Dipole field (store) | [T] | 12.5 |
| Dipole coil ID | [mm] | 50 - 60 |
| Transverse RMS emittance, $\epsilon$ | [$\mu$m] | 1.0 |

Table V: Lattice parameters for SHORT and LONG cell high field machines, at injection.

| Parameter | units | SHORT | LONG |
|---|---|---|---|
| Half cell length, $L$ | [m] | 110 | 260 |
| Max. cell beta, $\widehat{\beta}$ | [m] | 376 | 898 |
| Max. cell dispersion, $\widehat{\eta}$ | [m] | 3.85 | 22.9 |
| Max. betatron size, $\widehat{\sigma}_\beta$ | [mm] | .594 | .918 |
| Circumference, $C$ | [km] | 55.44 | 54.08 |
| Horizontal tune, $Q_x$ | | 65.195 | 28.195 |
| Vertical tune, $Q_y$ | | 66.185 | 29.185 |
| Number of dipoles | | 2888 | 2900 |
| Number of sextupoles | | 456 | 168 |
| Mmtm. width, $\sigma_p/p$ | $[10^{-3}]$ | .1545 | .0401 |

capole systematic of $b_4 = 30 \times 10^{-4}$ in the bottom plot. A reference radius of $r_0 = 16$ mm is used throughout. Solid lines in the figure show the predictions of the model developed above, while data points represent the tune shifts measured using the tracking code TEAPOT.

The most striking general feature of these plots is that a systematic octupole (decapole) harmonic generates curves with an even (odd) symmetry. Agreement between prediction and measurement is quite good at small $m_x$ and small $m_\delta$, but not perfect. This discrepancy is mostly due to the presence of dispersion supressors, and the fact that the dipole packing fraction is only 81.8%, and not the 100% assumed in the model. Both of these factors throw the predicted optical averages $A_{n,i}$ into error. The packing fraction in the LONG machine is 86.5%, leading to a significant reduction of the total circumference by 1.36 km, or 2.5%.

The horizontal base tune was lowered to $Q_x = 65.145$ for this exercise, in order to place it approximately midway between the integer and fourth order resonances at 65.0 and 65.25, respectively. In principle, a perfectly smoothly distributed systematic octupole harmonic does not drive the fourth order resonance, due to vector cancellation. In practice, the cancellation is not perfect, and so the top plot clearly saturates at a tune shift of approximately $+0.1$, when the fourth order resonance is approached. The bottom plot shows minimum tune shifts of approximately $-0.1$, when the integer resonance is approached.

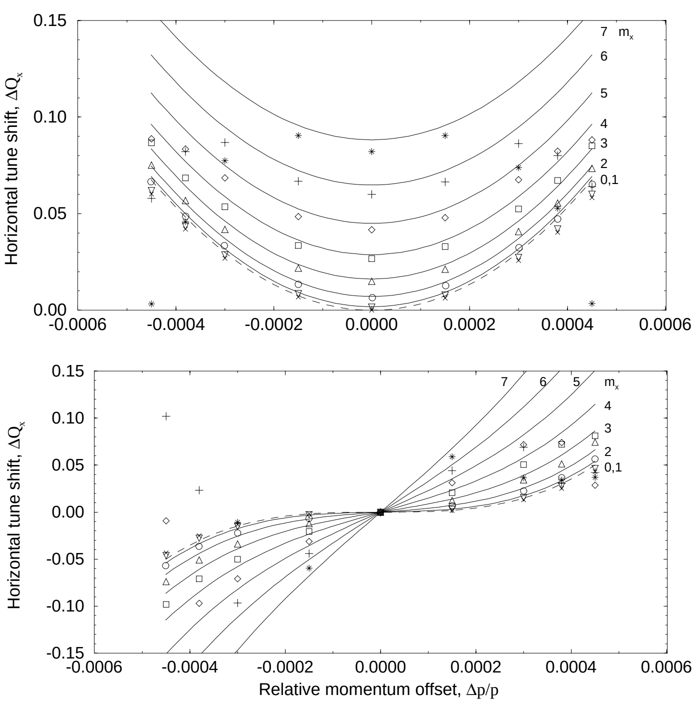


Figure 1: Tune shifts due to systematic octupole (top) and decapole (bottom) harmonics in SHORT machine dipoles. Solid lines are predictions, while data points are measured results.

It is entirely within the semi-quantitative spirit of this paper that the model and the analysis only discuss 1-D motion, in the horizontal. A more rigorous discussion would also include vertical betatron motion - and would also include synchrotron oscillations, and a whole host of realistic effects. As RLHC designs become more refined, so too must the simulations. At this point, when the RLHC is hardly even a gleam in the physicists eye, clarity and simplicity are more important than rigor.

# IV. MAXIMUM ALLOWABLE HARMONICS

While the previous section focused on the particular example of a 30 TeV high field collider, the conclusion that the maximum tolerable tune shift is

$$\widehat{\Delta Q_x} \approx 0.1 \tag{22}$$

is expected to hold in general - for any low or high field collider, at low or high energy, that conforms with the physical assumptions made so far:
1) systematic errors dominate random errors
2) the collider has many fully packed FODO cells
3) momentum and betatron beam sizes at F quads are equal
4) $\phi_c = 90$ degrees phase advance per cell
5) chromaticity sextupoles are not pathologically strong
It is relatively straightforward to derive the semi-quantitative results, below, for phase advances per cell other than 90 degrees. For example, while maximum allowable harmonics are smaller at 60 degrees per cell, there is not much advantage in increasing $\phi_c$ beyond 90 degrees per cell.

What is the necessary field quality in such a machine? How large can the half cell length $L$ be? Suppose, for example, that the horizontal tune shift must be guaranteed to be less than $\widehat{\Delta Q_x}$ for all test particles in the betatron amplitude and momentum distribution range

$$m_x \leq m \tag{23}$$
$$|m_\delta| \leq m \tag{24}$$

The extreme tune shift occurs when $m_x = m_\delta = m$, and is given by

$$\Delta Q_x(m) = \frac{b_n}{r_0^n} D_n \, m^{n-1} \, L^{(n+1)/2} \left(\frac{\epsilon_x}{\beta\gamma}\right)^{(n-1)/2} \tag{25}$$

An irritating and negligible term $(1+\delta)$ has been unceremoniously dropped from the denominator of this equation, in order to make it as simple as possible in comparison with the more general result of Equation 21, from which it is derived. The sum in Equation 21 has been replaced by $D_n$, a function of the phase advance per cell, which is given by

$$D_n(\phi_c) = \sum_{i=0}^{n-1-2i\geq 0} C_{n,i} \, \alpha_{n,i}(\phi_c) \tag{26}$$

Numerical values for $D_n$, derived from Tables I and III, are listed in Table VI.

Table VI: Lowest order $D_n$ values, with a phase advance of $\phi_c = 90$ degrees per FODO cell.

| n | Multipole | $D_n$ |
|---|---|---|
| 1 | Quadrupole | $.8333$ |
| 2 | Sextupole | $2.412$ |
| 3 | Octupole | $6.712$ |
| 4 | Decapole | $19.18$ |
| 5 | 12-pole | $56.49$ |
| 6 | 14-pole | $170.9$ |

Equation 25 is readily inverted, to give the maximum allowed systematic harmonic

$$\frac{b_n}{r_0^n} \leq \widehat{\Delta Q}_x \frac{1}{D_n} L^{-(n+1)/2} \left(\frac{\beta\gamma}{m^2\,\epsilon_x}\right)^{(n-1)/2} \qquad (27)$$

For example, with $\Delta Q_x(m) = \widehat{\Delta Q}_x = 0.1$, an injection energy of 1 TeV, $\epsilon_x = 1$ micron, and defining the edge of the particle distribution of interest by $m = 3$, then the maximum systematic harmonics are plotted for octupole through 14-pole harmonics in Figure 2. The lowest allowed harmonic, sextupole, is not shown in the Figure, since chromatic sextupoles are naturally available to correct $b_2$, and a proper analysis of its maximum tolerable value goes beyond the scope of this paper. The harmonics of most concern are the unallowed octupole $b_3$, which has the tightest tolerances but which is naturally relatively small, and the allowed decapole $b_4$, which is probably the most critical harmonic in practice.

It is worth inspecting the scaling in Equation 27 with a critical eye. The allowable systematic errors increase rapidly as the injection energy is increased - from 1 TeV to 3 TeV, for example - and as the injection emittance is decreased. Similarly, the chosen value of $m$ is very important, and needs more discussion than the simple assertion, in this paper, that a value of $m = 3$ is reasonable.

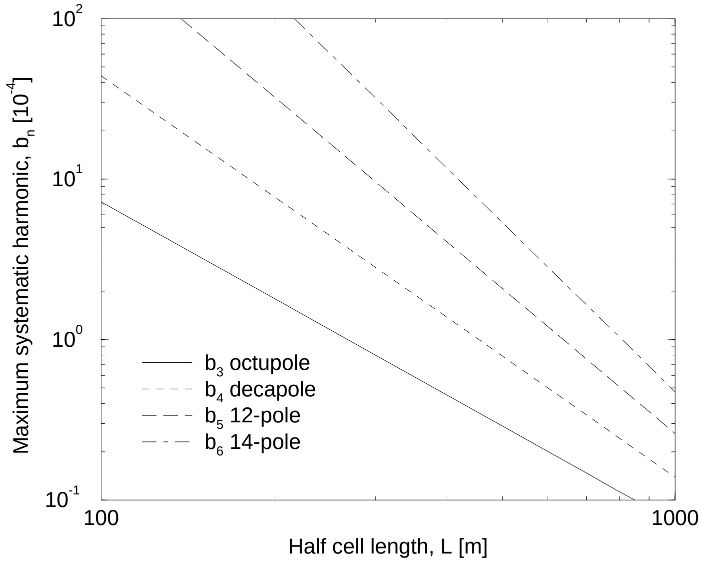


Figure 2: Maximum allowable systematic harmonics versus half cell length, when $\widehat{\Delta Q}_x = 0.1$, $\epsilon_x = 1$ micron, and $m = 3$, at an energy of 1 TeV.

## V. CONCLUSIONS

Lattices with relatively long arc cells have potential advantages, including significant cost savings, in a Really Large Hadron Collider. However, the susceptibility of the beam dynamics to systematic arc dipole errors increases as the cell gets longer. Therefore, reasonable expectations for the achievable dipole field quality at injection play a strong role in determining the cell length - or vice versa.

For example, if beam with a normalized emittance of 1 micron is injected at 1 TeV into a lattice with half cells $L = 300$ meters long, then dipoles with a systematic decapole of $b_4 \simeq 3 \times 10^{-4}$ (at a reference radius of 16 mm) will provide barely adequate performance. Higher values of this allowed harmonic would increase the horizontal tune shift beyond the rule-of-thumb physical maximum of $\widehat{\Delta Q}_x \approx 0.1$. The systematic tolerance at the same half cell length for the next allowed harmonic, the 14-pole, is $b_6 \simeq 30 \times 10^{-4}$. For the unallowed octupole and 12-pole harmonics the equivalent tolerances are $b_3 \simeq 0.8 \times 10^{-4}$ and $b_5 \simeq 10 \times 10^{-4}$, respectively.

Future hadron colliders with half cell lengths of a few hundred meters are cost effective, with adequate beam dynamics performance. This is especially true for high field colliders, in which the radiation damping forgivingly allows less stringent field quality tolerances.

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