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THEORY OF THE ^{3}He ELECTRIC FORM FACTOR*

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ABSTRACT

The problem of the position of the zero of the electric form factor of ^{3}He and the large size of the observed secondary maximum, which are not obtainable with realistic two body forces in the Faddeev equations is discussed. Corrections to the single impulse approximation arising from vector dominance of the incident virtual photon are calculated. It is shown that such shadowing effects have neither the shape nor the magnitude to fit both the position of the zero and the size of the secondary maximum. The hyperspherical harmonic expansion for the trinucleon bound state wave function is reviewed. It is then argued that the same physical reasoning which can be used to support the two body boundary condition model of strong interactions can be used to predict a similar but explicitly three body effect for the three nucleon interaction. A three body boundary condition imposed on the lowest order hyperharmonic contribution to the three body bound state gives a good quantitative fit to the ^{3}He electric form factor in the region q^2 greater than 10 F^{-2}, when the position of the zero is fit by adjusting the boundary hyperradius. This result is shown not to be affected by the inclusion of S', D or other higher states. It is also not affected by taking into account the long range tails of external OBE potentials. By way of extension, a similar fit is presented for the ^{4}He form factor with similar results. We also discuss the possibilities for refinements of the model and its use in systems other than the trinucleon bound state.

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CHAPTER I

GENERAL INTRODUCTION

I. 1 Who Cares?

Some time before this report was written the author was asked by an elementary particle theorist, "Does anyone really care about the three body problem these days?".[1] For the particle physicist the few body problem can provide a test of the way in which a strong interaction theory should relate asymptotic events such as meson productIcn or resonance excitation and their role in the forces between two or more nucleons. The few nucleon problem, at its most refined, might give us a hint as to the structure of a renormalized theory involving these particles. The various theories of composite nucleon structure that are becoming prominent will have to account for the few nucleon forces in the same way that the electron theory of the atom gives molecular forces. Here nonpair behavior of the nucleon forces when three of them are brought together might be an important test of such theories as well as a test of how the usual ideas of elementary particle interactions build into potentials.[2]

Of course, for systems involving three or more nucleons a knowledge of the three body problem can have importance as part of the input of the theory. An example of this is the role of three body clusters in the structure of nuclear matter.[3] Finally we believe, that unless the theory of elementary particles reduces the two and three body problem to one of nothing more than computational tedium, that it is a system with much physical interest in its own right and one that can give us insight into the problem of systems with few constituents, in general.

The specific problem that we shall consider here is that of the ^3He electric form factor which is expected to be a reasonably good measure of the charge distribution in this nucleus.[4] The reason for considering this form factor is that it has recently been measured out to momentum transfers[5] of $q^2 = 20\ F^{-2}$ with some results that are rather puzzling in terms of calculations performed with potentials that fit the two nucleon data.[6-9] Also this form factor is expected, by Siegert's theorem, to be less complicated by exchange currents than the magnetic form factor.

The next section of this chapter will discuss the measurement of the ^3He electric form factor $F_{el}^{^3He}(q^2)$ and the difficulties that potential models have in fitting the data at large q^2. The third section will contain an outline of the rest of the report.

I. 2 Theoretical Difficulties with $F_{el}^{^3He}(q^2)$

The development of the Faddeev equations has greatly increased interest in the three nucleon problem over the past decade. The reason for this is that these equations make it theoretically possible to compute three body scattering and rearrangement amplitudes along with bound state properties if the forces in the three body system are given by the two body pair forces. (Three body forces may be included if they are known.) Even in the case of three body S waves generated from two body S wave interactions these equations are still two variable integral equations,[11] for the case of local potentials however, since the work of Osborn,[12] the Faddeev equations have started to become tractible. For the bound state case advances in electronic calculation have made elaborate variational computations possible.

These methods have been applied to computations of the ^{3}He electric form factor by several authors using "realistic" two nucleon potentials.[6-9] Where by "realistic" potentials we mean those potentials that provide good fits to the two nucleon phase shift sets below inelastic threshold as well as polarization data and the properties of the deuteron. These potentials tend to either have a soft core (Reid[7]) or a hard core (Hamada-Johnston) at $r_{12} \approx .5$ F. Another requirement is that for large distances they be compatible with the forms predicted by meson theory, in fact many of them, are derived this way in part.

The experimental form[5] of the form factor is a slightly depressed Gaussian for $q^2 \leq 8\ F^{-2}$ followed by a sharp drop and apparent passage through zero at $q^2 = 11.6\ F^{-2}$. There is then a strong (assumed negative) secondary maximum with a peak absolute value of the form factor of about 6×10^{-3} reached at about $q^2 = 16.25\ F^{-2}$. The kernels of the Faddeev equations involve the fully off-shell two body t-matrices; these differ for the various "realistic" two body potentials and it was hoped that their fits to the three body data might distinguish among them. Unfortunately none of them fit the ^{3}He electric form factor at large q^2. They all tend to predict the position of the zero at too large q^2 and the size of the secondary maximum they give is a factor of three to ten too small. Typical results are those shown for the truncated Reid potential as calculated by Harper, Kim and Tubis.[13] (This will also display the data in Fig. I. 1.) Simple phenomenological forms to simulate the effects of relativistic and three body corrections do little to remedy this.

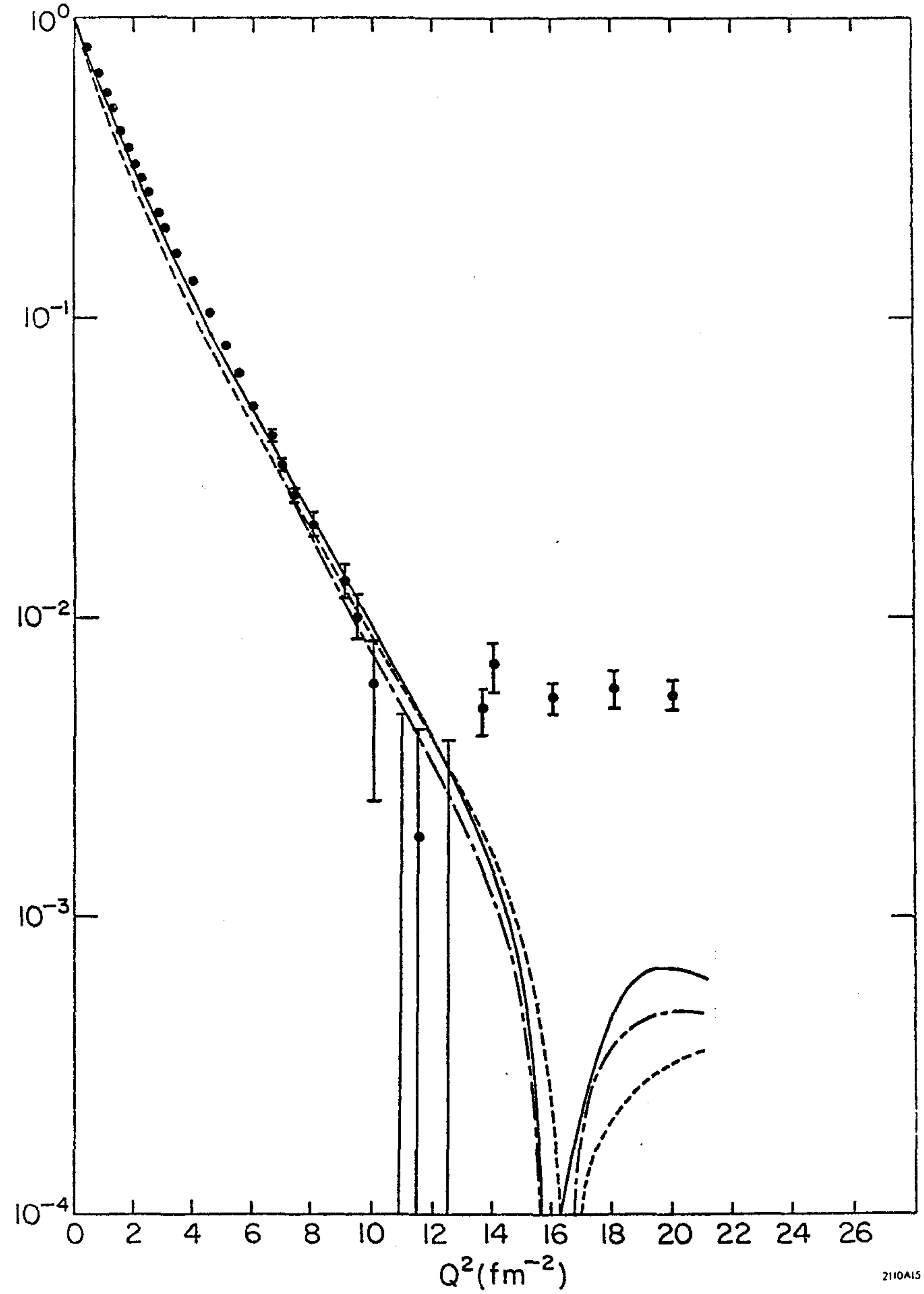

FIG. I. 1--A calculation of the ^{3}He electric form factor with a truncated Reid potential compared with experiment. Figure taken from E. Harper, Y. E. Kim, and A. Tubis, Purdue preprint (1972).

I.3 Synopsis of the Remainder of the Report

The calculation of the form factor is based on the single impulse approximation. The virtual photon is assumed to be absorbed by one nucleon. The electric form factor could then be considered, for point nucleons, as the probability that states with virtual momenta 2/3 q exist in the nucleus. In Chapter II we consider the validity of this approximation. Possible double impulse contributions are considered which come from the propagation of the incident photon as a strongly interacting vector meson through the nucleus. These contributions to the form factor are shown to be too small to account for the large secondary maximum. Even if the position of the zero is forced the resultant secondary maximum is still too small. While if the double impulse shadowing contributions are adjusted in size to fit the size of the secondary maximum, the shape of the form factor at lower q^2 is completely destroyed. This means that the ^{3}He electric form factor will have to be explained in terms of the structure of the nucleus and not by distortion caused by the photon acting as a fourth strongly interacting particle.

In Chapter III we give the expansion of the wave function in hyperspherical harmonics. In this method the Jacobi coordinates which describe the three body system in its center-of-mass:

$$\vec{\eta} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2)$$
$$\vec{\xi} = \sqrt{\frac{2}{3}} \left(\frac{\vec{r}_1 + \vec{r}_2}{2} - \vec{r}_3 \right) \tag{I. 1}$$

are combined into a six dimensional vector with one variable ρ to determine the scale and an angle Ω_6 defined on a five dimensional hyperspherical surface where

$$\rho^2 = \eta^2 + \xi^2 \tag{I. 2}$$

It is shown that a rapidly convergent expansion of the wave function can be written in terms of functions of the hyperradius ρ and a set of hyperspherical harmonics with definite permutation symmetry properties. The wave functions are determined by a set of one variable differential equations with well defined boundary conditions for the case of the bound state.

In the following chapter we describe the boundary condition model of strong interactions and give arguments that it should be extended in an explicitly three body way to give a boundary condition in the lowest hyperspherical wave functions. These arguments closely parallel those given for the two body version of this model and are based on both heuristic considerations of the opening of virtual channels at short distances and, more formal, arguments about the energy dependence of certain dispersion integrals.

Chapter V gives our results. Working in the lowest order hyperharmonic, we impose an exterior boundary condition on the three nucleon bound state wave function. This is done in such a way as to take account of the trinucleon binding energy as an input parameter. The boundary hyperradius is then chosen to reproduce the experimental position of the zero in the ^{3}He electric form factor. The resultant form factor then is shown to give a good fit to the data for q^2 greater than 10 F^{-2}. The goodness of this fit is unaltered when we take account of S' and D waves by means of higher hyperharmonics or when we add external two body potentials which are nonsingularly attractive at the origin.

We briefly extend our model to ^{4}He and, also, the possibility of generalization to other cases is discussed.

The final chapter contains a summary and analysis of our conclusions.

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CHAPTER II

THE SHADOWING OR STRONG INTERACTION CORRECTION

II. 1 Vector Dominance in the Form Factor

Blankenbecler and Gunion have noted that if we assume vector dominance for the interaction of a deuteron with a virtual photon, then the vector meson, which couples to the photon, can scatter strongly from one of the nucleons and be absorbed by the other. With the assumption that for the deuteron to remain bound each nucleon receives approximately equal momentum transfer, and assuming a simple phenomenological form for the momentum transfer dependence of the vector meson nuclear scattering amplitude, they are able to approximate the integrals involved in the calculation of this contribution to the electric and magnetic form factors of deuterium.

With a further assumption on the relative strength of the t=0 ρ and ω photoproduction amplitudes (see their paper for details, by this we mean their condition $f_1=f_2$), they obtain the normalization of the vector meson-nucleon scattering amplitude by requiring that this shadowing process account for the correction required to the value of the deuteron magnetic moment calculated using a Partovi wave function. They then note that the Partovi or other "good" wave functions may predict a deuteron electric form factor which is a bit too low in the region of momentum transfer; $\Delta^2 \geq 24\,F^{-2}$ and which falls off somewhat too rapidly in this region. The suggestion is then made that a slight relaxation of their requirement $f_1=f_2$ can give a less rapidly falling form factor which could be in which could be in better agreement with experiment than is that calculated by the single impulse approximation.

They finally suggest that similar double and triple impulse diagrams in ^{3}He (Fig. 2) could be an explanation for the dip and tail in the ^{3}He elastic form factor[2] above.

In Section II we will undertake to give an estimate of the size of such effects. Let us first make a few general remarks. First, we expect a substantial contribution to the ^{3}He magnetic form factor from mesonic exchange currents and the like. (See, for example, the review article of Delves and Philips.[2]) Therefore, we shall have no more to say about the magnetic form factor in this Chapter. Next, the single impulse approximation to the electric form factor; $F_1^{e\ell}(\Delta^2)$, can be expressed as a product of a sum of single nucleon form factors and a body form factor $F_B(\Delta^2)$,[4] where Δ is the momentum transfer and $F_B(\Delta^2)$ is the Fourier transform of the square of the ^{3}He wave function.[3] Now in the double impulse contribution; $F_2^{e\ell}(\Delta^2)$, in order to maintain a bound state each impulsed nucleon should receive a momentum transfer of about $\Delta/2$. This is equivalent to keeping the two struck nucleons fixed and giving the third an impulse of $-\Delta/2$. Therefore the body form factor that appears in $F_2^{e\ell}(\Delta^2)$ will be centered on $F_B(\Delta^2/4)$. As a consequence of this, in the tail region from $\Delta^2 = 12F^{-2}$, where $\Delta^2/4 \leq 5F^{-2}$ the wave function factor in $F_2^{e\ell}(\Delta^2)$ is from a region where several phenomenological models and most "realistic" potentials give a good fit to the form factor. We thus expect $F_2^{e\ell}(\Delta^2)$ to be almost model independent and, in particular, to be independent of any high momentum transfer structure that might appear in the model we use.

Our computation will be done with a Gaussian wave function[4,5] $\psi(r_{12}, r_{23}, r_{13}) = A \exp -\left(\frac{1}{2}\alpha^2\left(r_{12}^2 + r_{23}^2 + r_{23}^2\right)\right)$ where the r_{ij} are the internucleon distances and A and α are constants, A being determined by the nor-

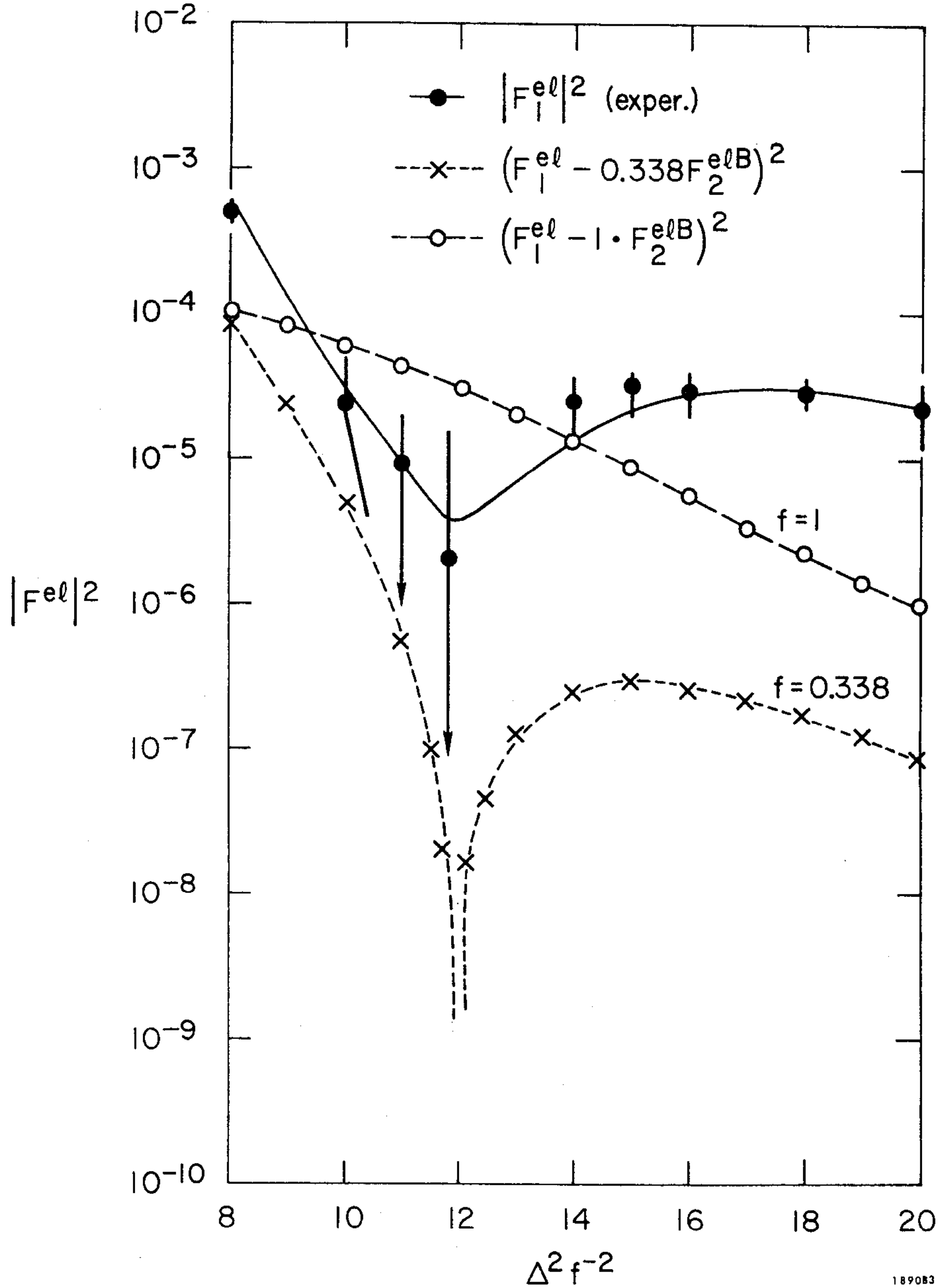

FIG. II.1--The square of the ^{3}He elastic form factor from McCarthy et al. With our calculation of $(F_1^{el} + F_2^{el} - f F_2^{elB})^2$ for f=1 and f=0.338.

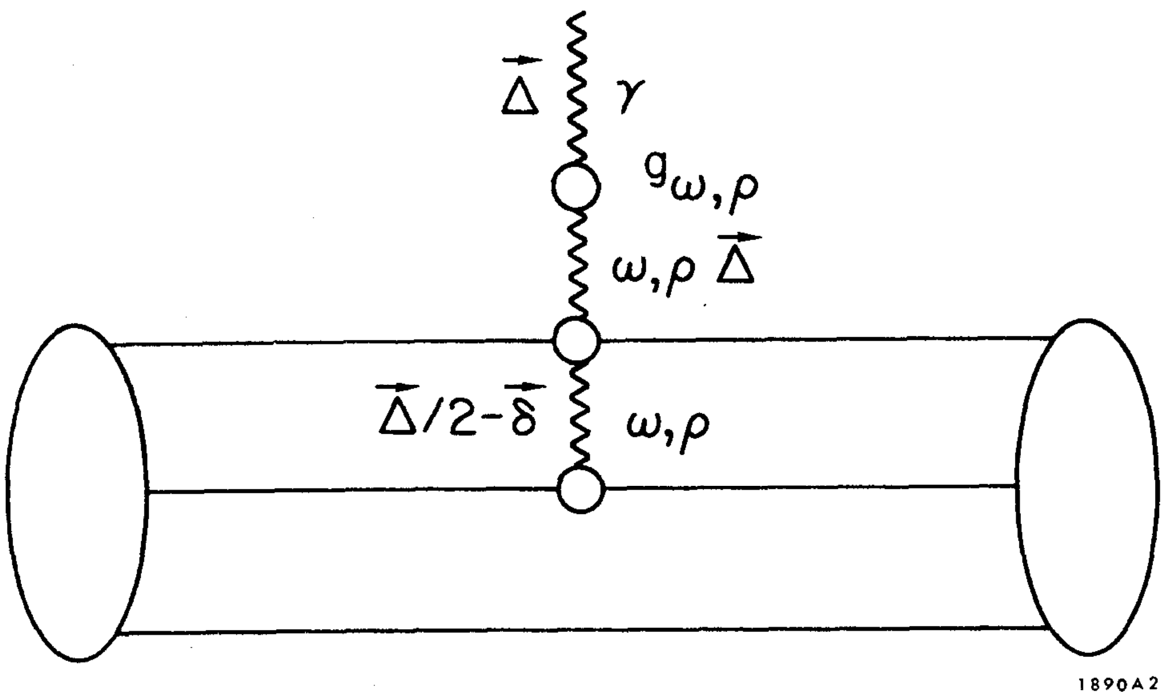

FIG. II.2--The vector meson scattering correction of Gunion and Blankenbecler in ^{3}He.

malization condition that the integral of $|\psi^2|$ over the two independent particle position vectors in a given coordinate system be unity. Since we are interested in the magnitude of a correction, we will not include the spacially antisymmetric S' state and the D state contributions to the ^{3}He wave function as the sum of these contributions is perhaps 10% of the total wave function normalization.[2]

In the next section possible criticisms of the Gunion-Blankenbecler contributions and further correction to $F_1^{e\ell}(\Delta^2)$ are discussed.

II. 2 Further Corrections

Consider the Born terms of Fig. 2 (Fig. 3). If we had a wave function that was fully relativistic and included exactly many-body effects then, for example, that part of Fig. 3a to the left of the broken line would already be included in the Bethe-Salpeter iteration of the wave function. Thus the Born terms would be included in the single impulse approximation and should thus be subtracted off from the full vector scattering amplitude of Fig. 2 to avoid double counting. Indeed, the difference between the full vector meson scattering amplitude and its Born terms includes such relativistic effects as production of resonances and multiparticle states along one of the nucleon lines, followed by their vector decay.

However, the wave functions that we use are solutions of the Schroedinger equation with static nonrelativistic potentials that describe low energy nucleon-nucleon scattering. In this view, we look at the Born term for vector meson-nucleon scattering and the other diagrams that contribute to the full scattering amplitude as additional pieces in the form factor, which are not included in the potential generated wave function. Instead they are due

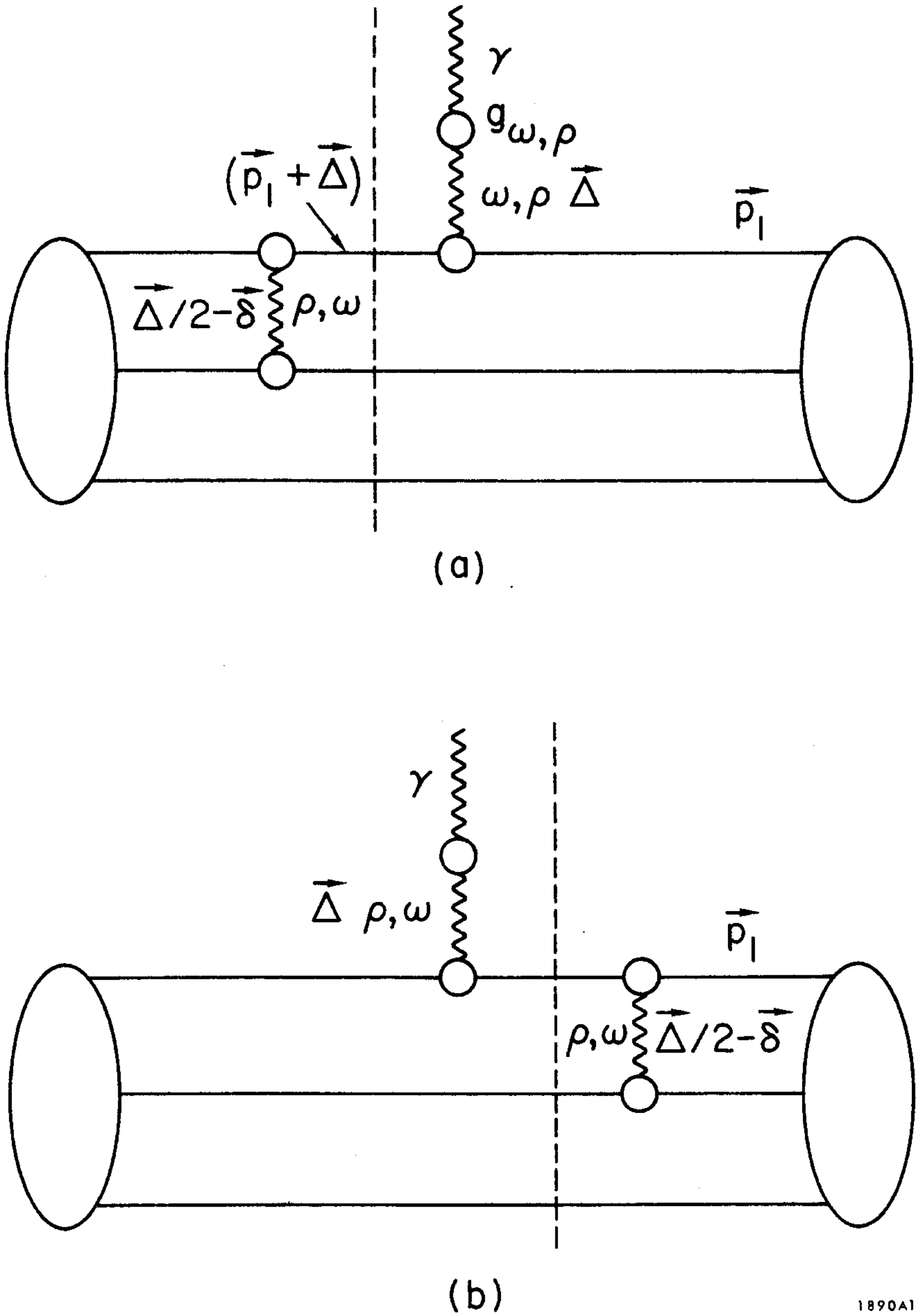

FIG. II. 3--The Born terms for the vector meson scattering correction in ^{3}He.

to many-body and other relativistic effects (i.e., distortion of the ^{3}He wave function by the vector meson field). This means that, depending on the details of the potential, there is, at most, some double-counting. To the extent that the potential contains nonrelativistic reductions of such effects as intermediate particle creation and absorption we may still have a degree of double-counting which, possibly, varies with energy.

In the latter part of the next section we calculate the contribution of the Born terms (Fig. 3) $F_2^{e\ell B}(\Delta^2)$ to the electric form factor in the momentum transfer region $\Delta^2 = 8F^{-2}$ to $\Delta^2 = 20F^{-2}$, again using a Gaussian wave function. We then multiply $F_2^{e\ell B}(\Delta^2)$ by a factor $-f$ representing the double-counting. This factor is taken to be constant throughout the momentum transfer range and is expected to lie in the range zero to one. An upper limit for f is estimated from the data. Calculations of pion photoproduction from nucleons indicate that we should not be surprised if $|F_2^{e\ell B}(\Delta^2)| \gg |F_2^{e\ell}(\Delta^2)|$.[5]

II. 3 Details of the Calculation

We will now explicitly compute the contributions to the elastic form factor of ^{3}He of the diagrams in Fig. 2. The energy transfer for which the helium nucleus remains bound is given by $\Delta_0 = \Delta^2/6M$; thus for $\Delta^2 \ll 27F^{-2}$ we can take $\Delta^2 = \vec{\Delta}^2$. Since we expect approximately half the incoming momentum Δ to be transferred to each impacted nucleon we represent the momentum of the exchanged vector meson as $\vec{\Delta}/2 - \vec{\delta}$ where we integrate over all $\vec{\delta}$ but anticipate important contributions only when $|\vec{\delta}|$ is small.

Next we give our approximation for the vertices (ignoring magnetic effects). The ρ-nucleon vertex is given by $G_p(q^2)\vec{\tau}_N$ where $\vec{\tau}_N$ is the nucleon isospin and the isovector form factor is

$$F_V(q^2) = g_\rho \frac{1}{m_\rho^2 + q^2} G_\rho(q^2) \qquad \text{(II. 1a)}$$

Here g_ρ is the photon-vector meson vertex constant. Likewise for the ω-nucleon vertex, $G_\omega(q^2)$ is defined by the isoscalar form factor

$$F_S(q^2) = g_\omega \frac{1}{m_\omega^2 + q^2} G_\omega(q^2) \qquad \text{(II. 1b)}$$

The vector meson-nucleon scattering amplitudes are assumed to be spin and energy independent and of the form[5]

$$A_{\omega\omega}(t) = 4\pi a e^{-.165\Delta^2} \qquad \text{(II. 2a)}$$

$$A_{\rho\omega}(t) = 4\pi b e^{-.165\Delta^2} \qquad \text{(II. 2b)}$$

$$A_{\rho\rho}(t) = 4\pi c e^{-.165\Delta^2} \qquad \text{(II. 2c)}$$

The results of Blankenbecler and Gunion are consistent with SU(3) D-coupling for the vector meson-nucleon scattering amplitudes. This gives $b = \frac{1}{\sqrt{3}} a$ and $c = \frac{1}{3} a$. We combine this with their result $a = -.12F$ to get our vertices. Finally, the photon-vector meson vertices are given by $g_\omega = m_\omega^2/2\gamma_\omega$ and $g_\rho = m_\rho^2/2\gamma_\rho$. From SU(3) $\gamma_\omega = \sqrt{3}\,\gamma_\rho$ and we take $\gamma_\rho^2/4\pi = \frac{1}{2}$.[7] In all our computations we will ignore the ρ-ω mass difference and take $m_\omega^2 = m_\rho^2 = m_V^2 = 14.9F^{-2}$.

Evaluating the isospin factors of the vector exchange matrix elements using the fully antisymmetric spin-isospin wave function of Schiff[4] we obtain the contribution of the double impulse diagram to the ^3He electric form factor:

$$F_2^{e\ell}(\Delta^2) = 4\pi a \int \frac{d^3\delta}{(2\pi)^3} \left(\frac{1}{\Delta^2 + m_V^2}\right) \times \exp\left(-.165\left(\frac{\vec{\Delta}}{2} + \vec{\delta}\right)^2\right) F_B(q_r, q_\rho)$$

$$\times\ 6\ F_S\left(\frac{\vec{\Delta}}{2} - \vec{\delta}\right)^2 \qquad \text{(II.3)}$$

where the Jacobi momenta are $\vec{q}_r = -\frac{\vec{\Delta}}{4} + \frac{\vec{\delta}}{2}$, $\vec{q}_\rho = \frac{\vec{\Delta}}{6} + \vec{\delta}$ and $F_B(\vec{q}_r, \vec{q}_\rho)$ the body form factor is determined in terms of the wave function $\psi(\vec{r}, \vec{\rho})$ (expressed as a function of the Jacobi coordinates) as:

$$F_B(\vec{q}_r, \vec{q}_\rho) = \int d^3\vec{r} \int d^3\vec{\rho}\ e^{-i\vec{q}_r \cdot \vec{r}}\ e^{-i\vec{q}_\rho \cdot \vec{\rho}}\ |\psi(\vec{r}, \vec{\rho})|^2 \qquad \text{(II. 4)}$$

We have chosen a Gaussian wave function which gives a good fit to the data out to $\Delta^2 = 8F^{-2}$. In this case (4.4) gives $F_B(\vec{q}_r, \vec{q}_\rho) = \exp -(q_r^2/6\alpha^2) + (q_\rho^2/8\alpha^2)$ with $\alpha^2 = .14F^{-2}$. F_S and F_V are taken from Ref. 8. The integral (4.3) can be evaluated if we approximate $F_{S,V}\left(\left(\frac{\Delta}{2} - \delta\right)^2\right) = F_{S,V}(\Delta^2/4)$. This yields

$$F_2^{e\ell} = -4.4 \times 10^{-3} \left(1/(1+\Delta^2/14.9)\right) \cdot \left(1/\left(1+\frac{\Delta^2}{+2}\right)^2\right) \cdot \exp(-.135\Delta^2) \qquad \text{(II. 5)}$$

We note that since $F_2^{e\ell} = -4.4 \times 10^{-3}$, in practice we need not renormalize the form factor to five $F_1^{e\ell}(0) + F_2^{e\ell}(0) = 1$. At $\Delta^2 = 5F^{-2}$, $F_2^{e\ell}(5) = -1.5 \times 10^{-3}$ compared with the Gaussian single impulse form factor of 10^{-1}. By $\Delta^2 = 10F^{-2}$ we have $F_2^{e\ell} = 7.9 \times 10^{-4}$ which is about 12% of the measured value of $F^{e\ell} = 6.5 \times 10^{-3}$. $|F_2^{e\ell}(\Delta^2)|$ then continues to fall off exponentially. It, therefore, cannot begin to explain the relatively flat tail in the data from $\Delta^2 = 14F^{-2}$ out to $\Delta^2 = 20F^{-2}$ where $|F^{e\ell}| \approx 6.5 \times 10^{-3}$. In fact, $F_1^{e\ell}(\Delta^2)$;

the single impulse contribution for the Gaussian is given by (see Table 1 for numerical values):

$$F_1^{e\ell}(\Delta^2) = \frac{1}{2}\left(\frac{3}{2} F_S(\Delta^2) + \frac{1}{2} F_V(\Delta^2)\right) \exp(-8\,\Delta^2/20) \qquad \text{(II. 6)}$$

Rough estimates show that the triple impulse contribution; $F_3^{e\ell}(\Delta^2)$ is about 10% of $F_2^{e\ell}(\Delta^2)$ and due to the rapid fall off of the two vector meson-nucleon scatterings present will fall off rapidly. Thus the corrections to the single impulse diagrams suggested by Blankenbecler and Gunion[1] can explain neither the tail nor the dip in the observed ^{3}He electric form factor.

Next we estimate the effects due to the possible partial double-counting of the Born terms for intermediate vector meson exchange (Fig. 3). By taking a nonrelativistic limit for the nucleon spinors and the intermediate propagators we can derive the nucleon vector-meson scattering vertex in Born approximation. The kinematics are chosen so as to agree with the computation of the first part of this section. Consider the diagram of Fig. 3a. Ignoring magnetic effects, the amplitude in the static limit for the absorption of an isoscalar photon and the omission of a ρ is given by

$$A_{\gamma\rho}^{B_1}(\Delta^2) = u^+(p_1') \frac{(-i(\not{p}_1 + \not{\Delta}) + M_N)}{(p_1+\Delta)^2 + M_N^2} u(p_1)\, F_S(\Delta^2) G_V\left(\frac{\vec{\Delta}}{2} + \vec{\delta}\right) \vec{\tau}_N$$

(II. 7)

where p_1 is the four-momentum of the impacted nucleon, Δ is that of the photon, q_ρ is that of the ρ and $p_1' = p_1 + \Delta - q_\rho$. With the approximation $p_{10} = M_N$ and using $\Delta_0 = \Delta^2/6M_N$ and $p_1^2 = -M_N^2$ we obtain

$$A_{\gamma\rho}^{B_1}(\Delta^2) = \chi_1^+ \frac{2M_N}{2\Delta^2/3} \chi_1 F_S(\Delta^2) G_V\left(\frac{\vec{\Delta}}{2} + \vec{\delta}\right)\vec{\tau}_N \qquad \text{(II. 8)}$$

TABLE I

Results of Our Calculation of $F_1^{e\ell}$, $F_2^{e\ell}$ and $F_2^{e\ell B}$

Δ^2 (F^2)	$F_1^{e\ell}$	$F_2^{e\ell}$	$F_2^{e\ell B}$
8	1.86×10^{-2}	-1.5×10^{-3}	2.92×10^{-2}
10	7.69×10^{-3}	-5.3×10^{-4}	1.52×10^{-2}
12	2.73×10^{-3}	-3.6×10^{-4}	8.36×10^{-3}
14	1.06×10^{-3}	-2.4×10^{-4}	4.79×10^{-3}
16	4.14×10^{-4}	-1.6×10^{-4}	2.84×10^{-3}
18	1.63×10^{-4}	-1.2×10^{-4}	1.78×10^{-3}
19	1.03×10^{-4}	-9.4×10^{-5}	1.38×10^{-3}
20	6.47×10^{-5}	-7.7×10^{-5}	1.08×10^{-3}

with χ_1 the spinor part of the ^{3}He wave function for the first nucleon. Similar expressions hold for the diagrams of Fig. 3b and for the scatterings involving the other combinations of vector mesons.

Equation (II. 8) no longer has any p_1 dependence therefore the Born term vector meson-nucleon scattering amplitude is effectively local. Also, the exponential form of the full nucleon vector meson scattering amplitude is replaced by a rational function of Δ^2. Therefore, we expect this term $F_2^{e\ell B}(\Delta^2)$ to fall off like $F_1^{e\ell}(\Delta^2/4)$ which is more slowly than $F_2^{e\ell}(\Delta^2)$. The term (II. 8) diverges at $\Delta^2 = 0$. However, as in the case of photon bremsstrahlung,[9] the divergence is cancelled by vertex and self interactions (here the binding energy of the ^{3}He nucleus plays the same role as the finite energy resolution of the detector in bremsstrahlung in providing a cutoff which makes each term finite). The terms that cancel the divergence at $\Delta^2 = 0$ involve interactions with single nucleon line and thus fall off like $F_1^{e\ell}(\Delta^2)$. So in the Δ^2 region of interest they should be only a few percent of the Born terms and we will ignore them. The expression for the Born term contribution to the electric form form factor is then of the form

$$F_2^{e\ell B}(\Delta^2) = \sum_{\substack{\text{all vector}\\ \text{mesons}}} \int \frac{d^3\delta}{(2\pi)^3} \left\{ \left(\left(\frac{\vec{\Delta}}{2} + \vec{\delta} \right)^2 + M_N^2 \right) \cdot F_{S,V}(\Delta^2) \cdot F^2_{S,V}\left(\left(\frac{\vec{\Delta}}{2} - \vec{\delta} \right)^2 \right) \cdot \right.$$

$$\left. 2M_N \times \exp\left(-\Delta^2/72\alpha^2 - \delta^2/6\alpha^2\right) \times 1/\left(g^2_{\rho,\omega} \cdot \frac{5\Delta^2}{12} \right) \right\} \qquad \text{(II. 9)}$$

The handling of the isospin is as in the case of the full scattering amplitude. As above we ignore the δ dependence of F^V and F^S. This gives an approximation for $F_2^{e\ell B}(\Delta^2)$

$$F_2^{e\ell B}(\Delta^2) = \frac{16}{\sqrt{\pi}} \frac{M_N}{M_V^4} \left(6\alpha^2\right)^{3/2} \left(\frac{1}{4} + \frac{\left(M_V^2 + 9\,\alpha^2\right)}{\Delta^2}\right)$$

$$\times\left(\left(F_V(\Delta^2) - 3\,F_S(\Delta^2)\right)\cdot F_V^2\left(\frac{\Delta^2}{4}\right) + 3\left(F_V(\Delta^2) + 3\,F_S(\Delta^2)\right)\cdot F_S\left(\frac{\Delta^2}{4}\right)\right)$$

$$\times\ \exp\left(-\Delta^2/72\,\alpha^2\right) \qquad \text{(II. 10)}$$

First we note that $F_1^{e\ell} + F_2^{e\ell} - F_2^{e\ell B}$ fits the data at $\Delta^2 = 14F^{-2}$. However, in this case $F_1^{e\ell} - F_2^{e\ell B}$ shows no dip near $\Delta^2 = 12$. In addition at $\Delta^2 = 20$, $\left(F_1^{e\ell} - F_2^{e\ell B}\right)^2 = 10^{-6}$ which is much too small. Therefore, corrections of the form $F_2^{e\ell B}$ if adjusted to fit the magnitude of the tail of the electric form factor, cannot fit its shape and, in addition, move the dip much too far in. Next let us try to adjust f so as to fit the dip, which for computational convenience we take at $\Delta^2 = 12F^{-2}$. If we assume $F^{e\ell}(12) = 0$, we get $f\,F_2^{e\ell B} = F_1^{e\ell}(12)$. This gives f = .338. We summarize our numerical results for $F_1^{e\ell}$, $F_2^{e\ell}$ and $F_2^{e\ell B}$ in Table 1. With this crude attempt to fit the dip $|F_1^{e\ell} - f\,F_2^{e\ell B}|$ is no greater than a few percent of $|F^{e\ell}|$ in the tail region from $\Delta^2 = 14F^{-2}$ to $\Delta^2 = 20F^{-2}$. Thus to avoid too drastic a dip in the form factor we have, at most, about one-third double-counting of the Born term. On the other hand, with, say, one third double-counting the calculated tail of the electric form factor is at least an order of magnitude too small. Our results are given in Fig. 1 for f = 1 and for f = 0.338.

II. 4 Conclusions and Summary

The corrections to the ^3He electric form factor, calculated from phenomenological nonrelativistic nucleon pair potentials can be divided into two categories. The first is due to corrections to the internal dynamics of the three-body system.

This includes virtual elementary particle production resonance excitation etc.; also included here are any explicit three-body effects that are present. All these effects occur in the isolated three nuclear system regardless of how we probe it. The other category we shall call electromagnetic corrections. These come about as the incident virtual photon can induce processes in the target nucleus which are not present in the isolated three-nucleon system. In fact, to the extent that the (virtual) photon in hadronic electromagnetic interactions is dominated by vector meson poles, the wave function for ^{3}He will be distorted by the presence of a fourth particle the interactions of which with the constituent nucleons are as strong as those among the nucleons themselves. Let us discuss the meaning of the results of our computation of the simplest electromagnetic effects, namely the shadowing effect discussed in Section II.

In the region of interest $\Delta^2 = 10\,F^{-2}$ to $\Delta^2 = 20\,F^{-2}$ the electromagnetic correction is expected to have little dependence on the internal dynamics correction. The reason for this is that both the full scattering corrections and any Born term subtractions at momentum transfer Δ^2 show a dependence on the square of the ^{3}He wave function which is strongest in a region near $\Delta^2/4$, which is less than $5\,F^{-2}$ here. In this region the ^{3}He electric form factor is well described by a wave function calculated from phenomenological two nucleon potentials. Therefore we should be able to calculate the electromagnetic corrections using a simple uncorrected ^{3}He wave function such as the Gaussian we have used.

The contribution of processes similar to those described by Blankenbecler and Gunion for deuterium seem of insufficient magnitude to describe the structure of the ^{3}He electric form factor in the region $\Delta^2 = 10\,F^{-2}$ to

$\Delta^2 = 20F^{-2}$. In addition, $F_2^{e\ell}(\Delta^2)$ shows a too rapid fall off in Δ^2 to fit the tail of the form factor. This fall off of at least one quarter as fast as the single impulse contribution (on a semi-log plot) can be seen from the details of our calculation to apply to double impulse corrections in general with reasonable assumptions about the behavior of the vertex functions included, therefore, it is unlikely that the inclusion of other processes similar in form to those generating $F_2^{e\ell}(\Delta^2)$ can explain the high momentum transfer behavior of the form factor which is the most salient feature of the results of McCarthy et al.[2]

The possibility of double-counting the Born term of $F_2^{e\ell}$ led us to consider $F_2^{e\ell B}$. Indeed, it is much larger than $F_2^{e\ell}$. But if we attempt to fit the tail of the electric form factor by it, we find that first, the shape is not right. $F_2^{e\ell B}$ falls off too rapidly and second, the correction term is so large in the region $10\,F^{-2} \leq \Delta^2 \leq 14F^{-2}$ as to render the intermediate momentum transfer to behavior of the form factor completely incorrect. On the other hand, if we have about 34 percent double-counting, the subtraction of this percentage of $F_2^{e\ell B}$ gives a qualitatively reasonable fit to the diffraction minimum but yields much too small a form factor in the tail region with the wrong shape (even if we assume $F_2^{e\ell B}$ is the only contribution to the electric form factor here). This result is qualitatively similar to that of Ref. 6, which is not surprising as vector meson exchange can be used to generate a core. Therefore the most we can say for the Born term subtraction and electromagnetic corrections, in general, is that they may be important near the diffraction minimum.

Our main conclusion is then that the large amplitude of the high momentum transfer (small distance) part of the ^{3}He electric form factor and its constancy of shape will have to be explained by those features of the internal dynamics of the two and three nucleon systems which generate the large momentum components in the ^{3}He wave function and not in the nature of the interaction with the virtual photon.

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CHAPTER III

THE HYPERSPHERICAL EXPANSION

III. 1 Introduction and Motivation

We will now introduce the method of hyperspherical harmonics which will serve as the formal method we use to treat the three body problem in the rest of this report. The method will be presented in somewhat more generality than will actually be used in our calculations. This will allow the present chapter to serve as an introductory review which, to our knowledge, has not been provided in English.

Our hyperspherical formulation was pioneered by Simonov and Badalyan and collaborators.[1-3] The method has been modified and generalized by Fabre de la Ripelle[4] (in French) is an excellent, detailed and lengthy development of the formalism.

First we consider a system of two isolated interacting particles. In their center-of-mass, a motion of the two particles may be described by rotation, that is, an element of the group 0(3) whose generators, here, are defined on a two-dimensional angular space and a dilitation of the distance r between them.

The Casimir operator of 0(3) is the angular part of the Laplacian in the three-dimensional center-of-mass system. The eigenfunctions of the Casimir operator are the angular parts of the harmonic polynomials

$$P(\vec{x}) = C(\alpha, \beta, \gamma) x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \qquad \text{(III. 1)}$$

where

$$\alpha + \beta + \gamma = L \geq 0$$

$$\Delta P(r) = 0 \qquad \text{(III. 2)}$$

Δ is the laplacian and $\vec{r} = (x_1, x_2, x_3)$

$$r^2 \equiv x_1^2 + x_2^2 + x_3^2$$

and we define the angular variables by

$$\begin{aligned} x_1 &= r \sin\theta \cos\phi \\ x_2 &= r \sin\theta \sin\phi \\ x_3 &= r \cos\theta \end{aligned} \qquad \text{(III.3)}$$

then

$$\Delta = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{1}{\sin\theta} r^2 \frac{d}{d\theta} \sin\theta \frac{d}{d\theta} + \frac{1}{r^2 \sin\theta} \frac{d}{d\phi^2} \qquad \text{(III.4)}$$

and $P(\vec{r}) = r^L f_{\{L\}}(\Omega)$ (III.5)

where Ω is the two-dimensional solid angle defined by θ and ϕ and L indicates one of a set of possible harmonic functions.

Defining $-\Delta_\Omega = 1/r^2 \, L^2(\Omega)$ as the last two terms on the right side of (III.4) we get inserting (III.5) in (III.4) and invoking (III.2)

$$L^2(\Omega) f_{\{L\}}(\Omega) = L(L+1) f_{\{L\}}(\Omega) \qquad (L \geq 0) \qquad \text{(III.6)}$$

In this simple example $L^2(\Omega)$ is the square of the angular momentum operator and the $f_{\{L\}}(\Omega)$ can be taken as the spherical harmonics

$$Y_{LM}(\theta, \phi) \qquad M = -L, -L+1 \ldots L-1, L$$

The $Y_{LM}(\theta, \phi)$ form a complete orthonormal set in $\{L, M\}$.

In the mathematically (by this we mean we ignore angular momentum conservation and cylindrical symmetry) general case we can write the two particle wave function as

$$\psi(\vec{r}) = \sum_{L,M} \psi_L^M(r) \, Y_L^M(\Omega) \qquad \text{(III.7)}$$

the kinetic energy operator applied to ψ becomes

$$-\frac{\hbar^2}{2\mu}\Delta_r\ \psi(\vec{r}) = \sum_{L,M} -\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{L(L+1)}{r^2}\right)\psi_L^M(r)\ Y_L^M(\Omega) \qquad \text{(III. 8)}$$

with the reduced mass given by $\mu = m_1 m_2/(m_1+m_2)$. Therefore for states with $L > 0$ there is an effective repulsive potential $(\hbar^2/2\mu)L(L+1)/r^2$. This has immediate physical consequences, in the scattering problem; states with fixed c.m. energy and L will be kinematically suppressed in the impact parameter region $b_L < L/<p>$ where $<p> = \sqrt{2\mu E}$ provided the interaction potentials are sufficiently smooth. Likewise, by the virial theorem for the virtual motion of the particles in a bound state. The largest L we expect to have a significant coefficient in the expansion (III. 7) is given by

$$L = \sqrt{2\mu E_B}\ R$$

where E_B is the binding energy and R is a radius characteristic of the system.

With the full potential $V(\vec{r})$ we define;

$$V_{LL'}^{MM'}(r) = \iint d\Omega\ Y_L^M(\Omega)\ V(\vec{r})\ Y_{L'}^{M'}(\Omega)\ .$$

The three variable partial differential Schroedinger equation reduces to the coupled set of ordinary differential equations;

$$\left\{\left(-\frac{\hbar^2}{2\mu}\frac{d}{r^2 dr}r^2\frac{d}{dr} + \frac{L(L+1)}{r^2} - E\right)\delta_{LL'}\ \delta_{MM'} + V_{LL'}^{MM'}(r)\right\}\ \psi_{L'}^{M'}(r) = 0 \qquad \text{(III. 9)}$$

The observations that under the above conditions there exists an L_{max} for the system (III.9) will reduce the set (III.9) to a limited number of equations. Thus if L_{max} is small we effect a considerable simplification in the problem. What this really says is the normalizations

$$A_L = \sum_{M=-L}^{L} \int_0^{\infty} |\psi_L^M(r)|^2 r^2 \, dr$$

decrease rapidly for $L > L_{max}$. On the other hand, if $V(\vec{r})$ has sufficiently complex behavior in Ω and sharp behavior near some r then since $\vec{p}_{op} = \hbar \frac{\partial}{\partial \vec{r}}$ and $V_{LL'}^{MM'}$ may not be small for $L \neq L'$ the product $V_{LL'}^{MM'}(r) \, \psi_{L'}^{M'}(r)$ may be large for L or L' large compared with L_{max}. This means that it is possible that a convergent calculation of the wave function or binding energy (for bound states) may require many more terms than actually appear in the final result.

In the preceding paragraphs we have presented the familiar example of the angular momentum decomposition of the two body system in a rather abstract way and have not made the assumption that the Casimir operator corresponding to the angular part of the laplacian is a constant of the motion. Despite this, we have noted that, under certain physical conditions, there are great benefits to be derived from this procedure. In the next section of this chapter we will do a similar analysis of the N-body wave function. First, let us note some of the geometrical and physical effects that we expect in the system of more than two particles. The dilation symmetry is one-dimensional and we can thus define one hyperradial variable which carries dilitations of the system the other variables constitute a hyperangle Ω. For three bodies the configurations described by Ω contains the group of rotations 0(3) and gives the global angular momentum L. L is limited in exactly the same way as for two bodies. In addition, we can deform the triangle formed by the three particles. This group is given by SU(2).[5] This means that the dilation free group of motion of three particles reduces to SU(3).[5] The second order Casimir operator of SU(3) is given by the angular part of the laplacian but as deformations are not an external symmetry it is not conserved. Furthermore, for the scattering problem the system can undergo long range deformation (in momentum space) due to scattering arbitrarily far from the center of mass. So we expect no further simplification including deformations symmetry in the angular part of the expansion of the part of the wave function. On the other hand, for a compact bound state of three identical particles we expect that states that are deformations out of an equilateral triangle will be harder tobind or that for systems with small sizes and binding energies the wave function should be dominated by an expression containing angular functions which have the lowest few eigenvalues of the hyper-angular part of the laplacian.

III.2 K-Harmonic Formulation

In this section the hyperspherical formulation will be presented for the general case of N particles, although the actual harmonics will not be given. In the next section we will give the explicit construction of the Simonov[7] K harmonics for three particles.

Let the position vectors of the N particle be given by $\vec{x}_i$ $i = i, \ldots N$. We take out the center-of-mass motion and assume that all N particles have mass m then

$$\sum_{i=1}^{N} \vec{x}_i = 0 \qquad \text{(III.10)}$$

In the c.m. the particle's configuration is described by N-1 combinations of the interparticle vectors; $\vec{x}_i - \vec{x}_j$; $\vec{\xi}_\alpha$ $\alpha = 1, \ldots, N-1$. The $\vec{\xi}_\alpha$ will be combined into a 3N-3 dimensional Cartesian space with

$$\xi_i \equiv \left(\vec{\xi}\left[\frac{i-1}{3}\right]\right)_{i-3\left[\frac{i-1}{3}\right]} , \qquad i = 1, \ldots 3N-3 \qquad \text{(III.11)}$$

and the hyperradius ρ defined by

$$\rho^2 = \sum_{i=1}^{3N-3} \xi_i^2 \quad \sum_{i=1}^{N} (\vec{x}_i)^2 , \qquad \text{(III.12)}$$

We will take ρ as our single scale variable. The c.m. kinetic energy in terms of our 3N-3 dimensional laplacian is

$$T = -\frac{h^2}{2m} \nabla_\xi^2 \quad \text{with} \quad \nabla_\xi^2 = \sum_{i=1}^{3N-3} \frac{\partial^2}{\partial \xi_i^2} \qquad \text{(III.13)}$$

Our goal in this section is to define hyperspherical coordinates and thus reduce the N-body dynamics to a set of coupled equations in the single variable ρ. This will be done by the expansion of the wave function in terms of a complete set of harmonic polynomials of ∇_ξ^2 (III.13).

Following Vilenkin et al.[6] we define our hyperangular variables as follows: take a node and associate an angle θ with it

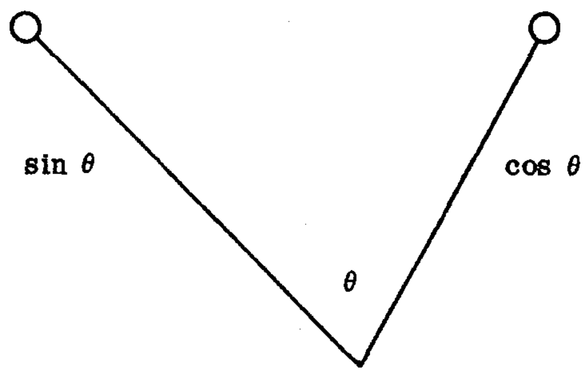
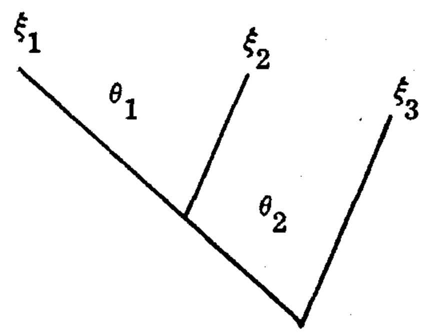


Fig. III.1

Let a line to the right be given by cos θ, one to the left by sin θ (Fig. III. 1) then starting from the left assign an angle θ_1, θ_2, etc., to each node until there are 3N-3 end points (in filling the last line it may be desirable to skip alternate nodes as this will group the ξ_i in groups of three corresponding to the vectors $\vec{\xi}_\alpha$ (III. 11). Then $\xi_i = \rho\, \pi_k \overset{}{\underset{k}{\text{———}}}\text{O}$ where $\underset{k}{\text{———}}\text{O} = \sin\theta_k$ or $\cos\theta_k$ depending on whether it is to the left or right of node k. The product on k is over all nodes passed through in connecting the branch to the origin. We give two examples in Example 1: N=2 (Fig. III. 2), we have



$$\xi_1 = \rho \sin\theta_1 \sin\theta$$

$$\xi_2 = \rho\cos\theta_1 \sin\theta$$

$$\xi_3 = \rho\cos\theta$$

Fig. III. 2

with $\theta_1 = \frac{\pi}{2} - \phi$ this is just the usual expression for the components of a three-dimensional vector in spherical coordinates. Example 2: N=3, Fig. III. 3 here the triplets (ξ_1, ξ_2, ξ_3) (ξ_4, ξ_5, ξ_6) are the Jacobi vectors $\vec{\xi} = \vec{\xi}$ and $\vec{\eta} = \vec{\xi}_2$ given in (I. 1) and $(\theta_1, \pi/2 - \theta_3)$ and $(\theta_2, \pi/2 - \theta_4)$ are the polar angles $\Omega_{\vec{\xi}}$ and $\Omega_{\vec{\eta}}$. We have $|\xi| = \rho\sin\theta$ and $|\eta| = \rho\cos\theta$.

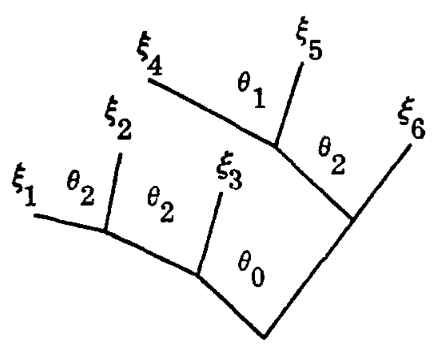


Fig. III. 3

We will construct the laplacian in our hyperspherical variables in terms of the matrix g^{ab} where

$$d\ell^2 \equiv \sum_{i=1}^{3N-3} d\xi_1^2 = \sum_{\rho,\theta_i,\theta_j} g_{\rho\rho}\,(d\rho)^2$$

$$+ g_{\rho i}\, d\rho d\theta_i \quad + \quad g_{ij}\, d\theta_i d\theta_j \qquad \text{(III. 14)}$$

From the relationship

$$(d(\cos\theta_i \sin\theta_j))^2 + d(\cos\theta_i \cos\theta_j)^2$$

$$= (-\sin\theta_i \cos\theta_j\, d\theta_i + \cos\theta_i \cos\theta_j\, d\theta_j)^2$$

$$+ (-\sin\theta_i \cos\theta_j\, d\theta_i - \cos\theta_i \sin\theta_j\, d\theta_j)^2$$

$$= \sin^2\theta_i (d\theta_i)^2 + \cos^2\theta_i\,(d\theta_j)^2 \quad + \quad d\theta_i\, d\theta_j$$

We obtain g_{ab} as diagonal with $g_{\rho\rho} = 1$ $g_{kk} = \rho^2 h_k^2$ where h_k^2 is the product of the $\sin\theta_i$ and $\cos\theta_j$ connecting the origin to node k. For example:

$$d\ell^2 = d\rho^2 + \rho^2 (d\theta_0)^2 + \rho^2 \sin^2\theta_0\,(d\theta_1)^2$$

$$+ \rho^2 \cos^2\theta_0\,(dp\,\theta_2)^2 + \rho^2 \sin^2\theta_0 \sin^2\theta_1\,(d\theta_3)^2$$

$$+ \rho^2 \cos^2\theta_0 \sin^2\theta_2\,(d\theta_4)^2$$

The range of the θ's for a single covering of the hypersphere is given by:

$0 < \theta < 2\pi$ if both lines from θ terminates in end points

$0 < \theta < \pi$ if only the line on the right is free

$0 < \theta < \pi/2$ if both lines lead to nodes

The volume element is given by[9]

$$dV = \sqrt{g}\; d\rho d\theta_1 \ldots d\theta_{3N-4} \qquad \text{(III. 15)}$$

where

$$g \equiv \det g_{ab} = \rho^{2\cdot(3N-4)} \prod_{k=0}^{3N-5} h_k^2 \qquad \text{(III. 16)}$$

The laplacian is given for g_{ab} diagonal by

$$\nabla_\xi^2 = \frac{1}{\sqrt{g}} \left\{ \frac{d}{d\rho} \left(\sqrt{g} \frac{\partial}{\partial\rho} \right) + \sum_{i=0}^{3N-5} \frac{\partial}{\partial\theta_i} \sqrt{g} \frac{1}{g_{ii}} \frac{2}{\theta_i} \right.$$

Or, in this case using (III. 16):

$$\nabla_\xi^2 = \frac{1}{\rho^{3N-4}} \frac{2}{\partial\rho} \rho^{3N-4} \frac{2}{\partial\rho} + \frac{1}{\rho^2} \sum_{i=0}^{3N-5} \frac{1}{\pi_j h_j} \frac{2}{2\theta_j} \frac{1}{h_i} \prod_{k\neq i} h_k \frac{2}{2\theta_i} \qquad \text{(III. 17)}$$

We give (III. 15) and (III. 17) for N=2 and N=3

N=2 $\quad dV = \rho^2 \sin\theta_0 \, d\rho \, d\theta_0 d\theta_1$

$$\nabla_\xi^2 = \frac{1}{\rho^2} \frac{\partial}{d\rho} \rho^2 \frac{d}{d\rho} + \frac{1}{\rho^2} \left(\frac{1}{\sin\theta_0} \frac{\partial}{\partial\theta_0} \sin\theta_0 \frac{\partial}{\partial\theta_0} + \frac{1}{\rho^2} \frac{1}{\sin\theta^2} \frac{\partial^2}{\partial\theta_1^2} \right)$$

as expected

N=3 $\quad dV = \rho^5 \sin^2\theta \cos^2\theta_0 \sin\theta_1 \sin\theta_2 \, d\rho \, d\theta \, d\theta_1 \, d\theta_2 \, d\theta_3 \, d\theta_4$

$$= \rho^5 \sin^2\theta \cos^2\theta \, d\Omega_\xi d\Omega_\eta \, d\rho \, d\theta$$

and with a regrouping of terms

$$\nabla^2 = \frac{1}{\rho^5}\frac{d}{d\rho}\ \rho^5\ \frac{d}{d\rho} + \frac{1}{\rho^2}\ \frac{1}{\sin^2\theta_0\cos^2\theta_0}\ \frac{\partial}{\partial\theta_0}\sin^2\theta_0\cos^2\theta_0\frac{\partial}{\partial\theta_0}$$

$$+ \frac{1}{\rho^2}\ \sin^2\theta_0\left(\frac{1}{\sin\theta_1}\ \frac{\partial}{\partial\theta_1}\ \sin\theta_1\ \frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\ \frac{\partial^2}{\partial\theta_3^2}\right)$$

$$+ \frac{1}{\rho^2}\cos^2\theta_0\left(\frac{1}{\sin\theta_2}\ \frac{\partial}{\partial\theta_2}\ \sin\theta_2\frac{\partial}{\partial\theta_2} + \frac{1}{\sin^2\theta_2}\ \frac{\partial^2}{\partial\theta_4^2}\right)$$

Writing, in general, (III.17) as

$$\nabla_\xi^2 = \frac{1}{\rho^{3N-4}}\ \frac{\partial}{\partial\rho}\ \rho^{3N-4}\ \frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\ K^2(\Omega) \qquad \text{(III.18)}$$

where $K^2(\Omega)$, the second term in (III.18) is the hyperangular momentum operator, from (III.17) we can write

$$K^2(\Omega) = \frac{1}{\left(\sin\theta_0^s\cos\theta_0^c\right)}\ \frac{\partial}{\partial\theta_0}\ \sin\theta_0^s\ \cos\theta_0^c\ \frac{\partial}{\partial\theta_0}$$
$$+ \frac{k^2(\Omega_R)}{\cos^c\theta_0} + \frac{k^2(\Omega_L)}{\sin^s\theta_0} \qquad \text{(III.19)}$$

where c(s) is the number of nodes to the right (left) of the origin and $\Omega_{R(L)}$ is the hyperangle defined starting at the first node to the right (left) of the origin. This is illustrated above for the case N=3.

It is clear that $K^2(\Omega_R)$ and $K^2(\Omega_L)$ can in turn be expressed in the form (III.19) and so on until we are down to $K^2(\Omega_{\vec{\xi}\alpha}) = -L^2(\Omega_{\vec{\xi}\alpha})$ the angular momentum operators of the vectors $\vec{\xi}_\alpha$. The eigenfunctions are the $Y_L^M(\Omega_{\vec{\xi}\alpha})$ with eigenvalues $-L(L+1)$. If $\vec{\xi}_\alpha$ and $\vec{\xi}_{\alpha+1}$ are joined at a node with angle θ_β

we can look for eigenfunctions of

$$K^2(\Omega_\beta) = \frac{1}{\sin^2\theta_\beta}\ \frac{1}{\cos^2\theta_\beta}\ \frac{d}{d\theta_\beta}\ \sin^2\theta_\beta \cos^2\theta_\beta\ \frac{2}{d\theta_\beta}$$

$$- \frac{L_\alpha(L_{\alpha+1})}{\sin^2\theta_\beta} - \frac{L_\alpha(L_{\alpha+1})}{\cos^2\theta_\beta}$$

in the form

$$F(\theta_\beta)\, Y_{L_\alpha}^{M_\alpha}(\Omega_{\xi\alpha})\, Y_{L_{\alpha+1}}^{M_{\alpha+1}}(\Omega_{\xi\alpha+1})$$

and so on until we reach the origin (θ_0). This is the method derived by Delves[10] and used by Fabre de la Ripello in his review article.[4] It has the advantages that completeness is manifest, the harmonics are expressed as simple products of functions of a single angle and the angular momentum of individual pairs is manifestly taken into account. On the other hand, the total angular momentum corresponding to rotations of the rigid system is not explicit and the properties of the harmonics under spacial permutation of two particles is complicated. In the next section we will give a development due to Simonov[1] for the case N=3 where these two defects are not present.

Returning to (III.18) we see that the harmonic condition $\nabla_\xi^2 \rho^K u_K(\Omega) = 0$ gives that

$$K^2(\Omega)\, u_K(\Omega_\xi) = -K(K+3N-5)\, u_K(\Omega_\xi) \qquad K = 0,\ 1,\ 2,\ \ldots \qquad \text{(III.20)}$$

for the eigenvalues of the harmonic functions of3N-4 angles. The kinetic energy operator, when applied to a state $f(\rho)\, u_K(\Omega)$, is given by

$$T = -\frac{\hbar^2}{2m}\ \nabla_\xi^2\ f(\rho)\, u_K = \Bigg\{ - \frac{\hbar^2}{2m}\ \frac{1}{\rho^{3N-4}}\ \frac{d}{d\rho}\quad 3N-4\ \frac{\partial f(\rho)}{\partial \rho}$$

$$+ \frac{\hbar^2}{2m}\ \frac{K(K+3N-5)}{\rho^2}\ f(\rho)\ \ u_K(\Omega_\xi) \qquad \text{(III.21)}$$

thus, just as in the case N=2 the second term on the right of (III.21) acts like an effective repulsive potential $\sim 1/\rho^2$ for $K \neq 0$.

Let us now assume that we have derived harmonics $u_{[K]}(\Omega)$ with the following properties

(1) [K] is a set of 3N-4 quantum numbers where K is the order of the harmonic polynomial $\rho^K u_{[K]}$.

(2) These additional quantum numbers in [K] specify first the total orbital angular momentum L and also classify the $u_K(\Omega)$ into representations of the permutation group of any two particles. This will allow trivial construction of states angular momentum J which are overall symmetric or antisymmetric under the interchange of two particles according to whether the particles be bosons or fermions.

(This will be clearer later in the section.)

(3) The $u_K(\Omega)$ are orthonormal

$$\int u^*_{[K]} u_{[K']} \, d\Omega_\xi = \delta_{[K],[K']} \tag{III.22}$$

where $\delta_{[K],[K']}$ is null unless the set $[K] = [K']$ in which case it equals unity and $d\Omega_\xi$ is $dV/\rho^{3N-4} d\rho$ where dV is given by (III.15) and (III.16). The number of independent K-harmonics is then given by[6]

$$\frac{(n+K-3)!}{(n-1)!\,K!} \left[n(n-3) + 2K(n-1) + 2\right] \tag{III.23}$$

where n is the dimensionality of the space we are working in (n=3N-3), N is the number of particles, and K is the eigenvalue from (III.20). For N=2 n=3 and (III.23) becomes (4K+2)/2 = 2K+1 as it should be. For n=3 n=6 and (III.23) is now

$$(K+3)\,(K+2)^2(K+1)/12 = 1(K=0),\ 6(K=1),\ 20(K=2),\ \text{etc.}$$

We note that, in general, that the number of equations in the N=3 analogy to (III.9) will increase quite rapidly with K as compared with the N=2 case. In fact, the number of independent harmonics with $K \leq K_{max}$ is for N=2.

$$(K_{max} + 1)^2 \text{ and for N=3}$$

$$\frac{1}{360}\left[6\,K_{max}^5 + 57\,K_{max}^4 + 300\,K_{max}^3 + 825\,K_{max}^2 + 894\,K_{max} + 360\right]$$

Thus even for N=3, it is important that our wave functions have significant components only for small K and/or the sum of the two body pair potentials and any three body potentials has a simple hyperangular structure that allows us to use symmetry properties of the interaction to reduce the number of coupled equations. For example, in the two body case the interaction term in (III.9) becomes $V_{LL'}^{MM'} = \delta_{MM'}\,\delta_{LL'}\,V_L(r)$ if $V(\vec{r})$ depends only on $|\vec{r}|$.

We now give the formal development of Schroedinger equation in hyperspherical harmonics. Assume that we have a system by hyperspherical harmonics obeying conditions 1-3 above. With the two particle potentials given by V_{ij} $i > j$ = $V_{ij}(\vec{x}_i, \vec{x}_j)$ where $\vec{x}_{i(j)}$ is the position (in the c.m.) of particle i(j). Components of $\vec{x}_i$ are linear combinations of the components of the 3N-3 dimensional vector $\vec{\xi}_\alpha$ so we can write in general.

$V_{ij} = V_{ij}(\vec{\xi})$ where we mean by this that V_{ij} is some function of those components of $\vec{\xi}$ which define $\vec{x}_i$ and $\vec{x}_j$. We have assumed here that V_{ij} is local but have not assumed that it is central. In a similar manner we can define $V_{ijk}(\vec{\xi})$ a three-body force $i < j < k$ and so on.

The N body Schroedinger equation is written

$$(T + V - E)\ \psi(\vec{\xi}) = 0$$

where

$$\psi(\vec{\xi}) = \psi(\xi_1, \xi_2, \ldots \xi_{3N-3}) \qquad \text{(III.24)}$$

$$T = \frac{\hbar^2}{2m} \nabla_\xi^2$$

and

$$V(\vec{\xi}) = \sum_{i<j} V_{ij}(\vec{\xi}) + \sum_{i<j<k} V_{ijk}(\vec{\xi}) + \ldots$$

Ignoring, for now, spin and isospin we have, using the completeness of the eigenfunctions in (III.20),

$$\psi(\vec{\xi}) = \sum_{[K]} \psi_{[K]}(\rho)\ u_{[K]}[\Omega_\xi] \qquad \text{(III.25)}$$

Inserting (III.25) into (III.24) and using (III.21) and (III.22) we obtain the N-body equivalent of (III.9):

$$-\frac{\hbar^2}{2m} \frac{1}{\rho^{3N-4}} \frac{d}{dp} \rho^{3N-4} \frac{d\psi_{[K]}}{d\rho}(\rho) + \frac{\hbar^2}{2m} \frac{(K+3N-5)}{\rho^2} \psi_{[K]}(\rho) - E\,\psi_{[K]}(\rho)$$

$$= -\sum_{[K]} V_{[K],\,[K]}(\rho)\ \psi_{[K]}(\rho) \qquad \text{(III.26)}$$

where

$$V_{[K][K']}(\rho) = \int d\Omega_\xi u_{[K]}(\Omega_\xi)\ V(\vec{\xi})\ u_{[K]}(\Omega_\xi) \qquad \text{(III.27)}$$

We note that (III.26) is a set of coupled second order ordinary differential equations. In fact letting $\psi_{[K]}(\rho) = \phi_{[K]}(\rho) / \rho^{\frac{3N-5}{2}}$ the left hand side of

(III. 26) becomes

$$- \frac{\hbar^2}{2m\rho^{\frac{3N-5}{2}}} \left[\frac{d^2\phi}{d^2\rho} + \frac{1}{\rho}\frac{d\phi}{d\rho} + \frac{2mE^2}{\hbar^2} \cdot \frac{-(K + \frac{3N-5}{2})^2}{\rho^2} \phi \right] \qquad \text{(III. 28)}$$

but the term inside the square brackets is just the operator that yields Bessel's equation of order K + (3N-5)/2 in the variable $x = (\sqrt{2mE}/\hbar)\rho$.[11] Furthermore if the potential V is made up of pair potentials of the form $V_{ij}(x_i - x_j)$ where

$$\lim_{|x_i - x_j| = \infty} |x_i - x_j| \; V_{ij} \; |x_i - x_j| = 0$$

then we can find unique solutions of (III. 26) with boundary conditions that $\psi_{[K]}(\rho)$ is finite at the origin and specific boundary conditions at $\rho = \infty$. This follows from the fact that

$$\lim_{\rho = \infty} \rho V_{[K][K']}(\rho) = 0$$

and the assumption that

$$\lim_{\rho = \infty} \rho \sum_{[K]} V_{[K],[K']}(\rho) \quad \phi_{[K']}(\rho) = 0$$

and will be true as long as the sum

$$\sum_{[K']} V_{[K][K']} \; \phi_{[K]}$$

is given by a finite number of terms such as in the case of a bound state. However, the kernal for the N(> 2) body Lippmann-Schwinger (L-S) equation is not compact. Therefore we cannot invert (III. 26) in the infinite dimensional (in K) case. Thus when the physical assumption of a finite size low energy bound state is made we invert an already truncated subset of (III. 26). The solution will then closely approximate a full solution to the L· S. equation. When this can be done we have no need to invoke coupled channel equations such as the

Faddeev equations for N=3 or their generalizations for $N > 3$.[12,13] Of course, for the full scattering problem, we can't truncate the set (impose a finite set of boundary conditions on a hypersphere). (Note (III.4).)

In the next section we will derive Simonov's[1,3] set of hyperspherical harmonics for N=3 and give some results that have been obtained with model potentials. First let us briefly consider two models which are soluble and useful for large N.[4]

If there are N identical particles connected by harmonic oscillator forces

$$V_{ab} = + A(\vec{x}_a - \vec{x}_b)^2$$

we have, since in the c.m. frame $N \cdot \sum_{a=1}^{N} (\vec{x}_a)^2 = \Sigma_{a>b}(\vec{x}_a - \vec{x}_b)^2$, that the potential (III.24) is for this case

$$V(\vec{\xi}) = \sum_{i<j} V_{ij}(\vec{\xi}) = + A\, N\rho^2 \qquad \text{(III.29)}$$

thus $V_{[K][K']}(\rho) = \delta_{[K][K']} A\, N\rho^2$ and there is no coupling of states with different $[K]$ in (III.26). For a given K we can solve (III.26) exactly. Let $\psi_{[K]}(\rho) \rightarrow \psi_{[K]}(\rho)/\rho^{3N-4}$ then (III.26) is written as:

$$d^2\psi_{[K]}(\rho)/d\rho^2 - \frac{1}{\rho^2}\left(\left(K + \frac{3N-3}{2}\right)^2 - \frac{1}{4}\right)\psi_{[K]}(\rho) + \frac{2m_\xi}{\hbar^2}\psi_{[K]}(\rho)$$

$$- \frac{2m}{\hbar^2} A\, N\rho^2\psi_{[K]}(\rho) = 0 \qquad \text{(III.30)}$$

letting $2A = m\pi^2\omega^2$ we get the solutions

$$\psi_\nu^n(x) = \frac{2}{x_0}\left[\frac{n}{\Gamma(n+v+1)}\right]^{1/2} x^{\nu+1/2} e^{-x^2/2} L_n^\nu(x^2) \qquad \text{(III.31)}$$

where $x = \rho/x_0$, $x_0^{-2} = \sqrt{N}\ \frac{m\pi\omega}{2}$ n=0, 1, ... , ∞_0 $\nu = K + \frac{3N-5}{2}$, $L_n^\nu(x)$ is a Laquerre polynomial and the eigenvalues of the energy are $E_{\nu,n} = \sqrt{N}\left(n + \frac{\nu+1}{2}\right)h\omega$. All this is precisely like the two body harmonic oscillator if we normalize the strength of the interaction by letting $\omega \rightarrow \sqrt{N}\omega$. This solubility coupled with the fact that for small displacements of the classical N body system from equilibrium the restoring forces are harmonic leads to the use of the solutions (III. 31) or some modification of them as the basis for a variational calculation in a hyperspherical formulation of the problem.[14]

Another exactly soluble N-body problem is the hyperspherical well[4] $V(\vec{\xi}) = \infty$ $\rho > \rho_0$. Here again $V_{[K][K']}(\rho) = \delta_{KK'}$ $V(\rho) = 0$ $\rho < \rho_0$. The equation for this case is (III. 28) set equal to zero. The solutions are

$$\phi_{\nu,n} = C_\nu^n\ J_\nu(\sqrt{2mE_n}/\hbar\,\rho)$$

where ν is again K+(3N-5)/2 and $\sqrt{2mE_n}/\hbar\,\rho_0$ is the nth zero of $J_\nu(\rho)$ and C_ν^n the normalization is given by $|C_\nu^n|^2 \int_0^{\rho_0} \rho d\rho \left| J_\nu(\sqrt{2mE_n}/\hbar\rho)\right|^2 = 1$. Fabré de la Ripelle[15] has suggested that the hyperspherical well might be a better description of nuclear matter than the spherical box usually employed. Each new particle adds three degrees of freedom say n_i, ℓ_i, m_i with ℓ_i its angular momentum m_i the projection of ℓ_i on the z axis and n_i a principle quantum number such that the degree of the overall harmonic polynomial is increased by $\Delta K = 2n_i + \ell_i$. Because of spin and isospin we can use each set (n_i, ℓ_i, m_i) four times. The procedure is to fill up the well by ordering the energies $E_{\nu,n}$ in order of increasing energy until all our A nucleons are used up. We can then calculate E/A the average kinetic energy per particle which is $25.47/\rho_0^2$ MeV/f^2 which is 88% of the energy in a spherical box with the same density of matter. This taking into account of high K-states means that it is energetically possible to have states with considerable deformations from spherical symmetry and produces an ambiguity in nuclear matter calculations.[4]

III. 3 The Case N=3, the Simonov Harmonics

In this section we will explicitly construct hypershperical or K-harmonics for the three body system following the development of Simonov[1] for L=0 and of Pustovalov and Simonov[3] for L > 0. A few calculations with simple model potentials which show the rapid convergence of the expansion of the wave function in K will be mentioned and finally a few general formulas of interest will be given.

Our plan is to, first, construct K harmonics with L=0 possessing definite permutation symmetry and then to define a differential operator which gives definite K and L when applied to the K=0 K harmonics. For L=0 then, K harmonics are scalar functions of the Jacobi coordinates

$$\vec{\xi} = \sqrt{\frac{2}{3}}\left(\frac{\vec{r}_1+\vec{r}_2}{2} - \vec{r}_3\right), \quad \vec{\eta} = \frac{1}{\sqrt{2}} (\vec{r}_1-\vec{r}_2) \qquad \text{(III.32)}$$

Thus we have K even, our work will be simplified if we reduce the six-dimensional real space to a three dimension complex one defined by a three vector $\vec{z}$.

$$\vec{z} = \vec{\xi} + i\,\vec{\eta}, \qquad \vec{z}^* = \vec{\xi} - i\,\vec{\eta} \qquad \text{(III.33)}$$

The properties of $\vec{z}$ and z^* under permutations; p_{ij}: $r_i \leftrightarrow r_j$, $i, j = 1, 2, 3$ are simple $p_{12}\, z = z^*$, $p_{13}\, z = z^*\, e^{-i2/3\pi}$, $p_{23}\, z = z^*\, e^{i2\pi/3}$

$$p_{12}\, z^* = z,\; p_{13}\, z^* = z\, e^{i2/3\pi} \quad p_{23}\, z^* = z e^{-i2\pi/3} \qquad \text{(III.34)}$$

The line element is given by $d(\rho^2) = (d\,\vec{\xi}\,)^2 + (d\,\vec{\eta}\,)^2$

$$= d(zz^*) = d\,z_1\, z_1^* + d\,z_2\, z_2^* + d z_3\, z_3^*$$

this corresponds to a metric $g_{ii*} = 1$ in all other g's = 0. From (9) we get the laplacian for terms of z and z*:

$$\Delta = \frac{\partial^2}{\partial\xi_i\,\partial\xi_i^*} + \frac{\partial^2}{\partial\xi_2\,\partial\xi_2^*} + \frac{\partial^2}{\partial\xi_3\partial\xi_3^*} \qquad \text{(III.35)}$$

The S-wave harmonic polynomials are functions of the three complex scalars $zz^* = \rho^2$, $z^2 = \vec{\xi}^{\,2} - \vec{\eta}^{\,2} + 2i\,\vec{\xi}\cdot\vec{\eta}$, $z^* = \vec{\xi}^{\,2} - \vec{\eta}^{\,2} - 2i\,\vec{\xi}\cdot\vec{\eta}$. The angular part of the volume element is given by (III.14) and (III.15) as

$$d\Omega_6 = \sin^2\theta\,\cos^2\theta\;d\theta\;d\Omega_\xi\;d\Omega_\eta \qquad \text{(III.36)}$$

where in Fig. III.3 we take $\xi = \rho\sin\theta$ $\eta = \rho\cos\theta$. For L=0 we can eliminate three of the five angles in (III.36) by defining A and λ with

$$z^2\,z^{*2} \equiv \rho^4 A^2 = \rho^4(\cos^2 2\theta + \sin^2 2\theta\,\cos^2(\vec{\xi}\;\vec{\eta}))$$

$$\rho^2\cos 2\theta = \vec{m}^2 - \vec{\xi}^2 = A\cos\lambda \qquad \text{(III.37)}$$

then

$$d\Omega_6 = \pi^2\,A\,dA\,d\lambda \qquad 0 < A < 1,\quad 0 < \lambda < 2\pi \qquad \text{(III.38)}$$

From the preceding remarks it follows that the most general harmonic polynomial that we can write for L=0 is of the form

$$\rho^K a_K^v(A,\lambda) = \sum_{\substack{\alpha+\beta+\gamma=K/2\\ \alpha,\beta,\gamma\geq 0}} a_{\alpha\beta\gamma}(z^2)^\alpha(z^{*2})^\beta(zz^*)^\gamma \qquad \text{(III.39)}$$

Since the condition L=0 reduces the angular space to a two dimensional one, we only need one quantum number; ν in addition to K. "ν" is defined so as to specify the permutation symmetry of the functions (III.39) and is given by

$$\nu = (\alpha - \beta) \qquad \text{(III.40)}$$

Applying p_{12} to (III.39) we have for u_k^ν $\binom{\text{symmetric}}{\text{antisymmetric}}$ under p_{12} $a_{\alpha\beta\gamma} = \pm a_{\alpha\beta\gamma}$ under p_{123} we have from (III.34)

$$a_{\alpha\beta\gamma} = \pm\, a_{\beta\alpha\gamma} \quad \text{and} \quad \exp \frac{4\pi\nu i}{3} = 1 \tag{III.41}$$

or

$$\nu = \frac{3}{2} n \qquad n = 0, \pm 1, \pm\ 2, \ldots \text{ etc.}$$

The condition $\Delta_\xi\, \rho^K u_k^\nu = 0$ gives using (III.35) and (III.40)

$$\sum_{\alpha+\beta+\gamma = K/2} a_{\alpha\beta\gamma}\, 4\, \alpha\beta(zz^*)^2 + \gamma\, (K+2-\gamma)\; z^2(z^*)^2$$

$$(z^2)^{\alpha-1}\, (z^{*2})^{\beta-1}\, (zz^*)^{\gamma-1} = 0 \tag{III.42}$$

The coefficients of the same term in $(z \cdot z^*)^2\, (\vec{z}^*)^2\, (z)^2$ are set equal to zero yielding

$$4\alpha_{\alpha\beta\gamma} + (\gamma+2)(K-\gamma)\, a_{\alpha-1,\beta-1,\,\gamma+2} = 0 \tag{III.43}$$

Note that if $\gamma = -1$ and $\alpha, \beta \geq 1$ we have

$$a_{\alpha-1\;\beta-1\;\gamma} = 0\,, \qquad \alpha-1,\ \beta-1 \geq\ 0 \tag{III.44}$$

Thus since (III.43) relates terms with $\gamma \rightarrow \gamma + 2$ we take $a_{\alpha\beta\gamma}$ in terms of $a_{\alpha\beta 0}$ but then in terms of K and ν we can express all the coefficients in (III.39) in terms of

$$a_{1/2\,(K/2+\nu),\ \ 1/2\,(K/2-\nu),\ 0}$$

since α and β are integral we get $-\frac{K}{2} \leq\ \nu \leq\ \frac{K}{2}$ and it increases in steps of two.

From (III.43) we now have with $\alpha = \frac{1}{2}(K/2+\nu)$, $\beta = \frac{1}{2}(K/2-\nu)$,

$$\rho^K u_K^\nu = C \sum_{\gamma=0,2,\ldots}^{K/2-\nu} (-1)^{\gamma/2} \frac{4^{\gamma/2}(\alpha-\gamma/2+1)\ldots\alpha(\beta-\gamma/2+1)\ldots\beta}{\gamma(\gamma-2)\ldots 2(K-\gamma+2)\ldots K}$$

$$\times (z^2)^{\alpha-\gamma/2}(z^{*2})^{\beta-\gamma/2}(zz^*)^\gamma \qquad \text{(III.45)}$$

where C is chosen such that $\int u^{*\nu}_K u^\nu_K d\Omega_6 = \delta_{K'K'}\ \delta_{\nu'\nu}$. This series is summable and in terms of the variables A and λ defined in (III.37)

$$u_K^\nu(A,\lambda) = \sqrt{\frac{K+2}{2\pi^3}}\ e^{-i\lambda\nu}\ A^{|\nu|}\ P^{|\nu|,0}_{\frac{1}{2}\cdot\left(\frac{1}{2}-|\nu|\right)}(1-2A^2) \qquad \text{(III.46)}$$

where $\quad p_c^{ab}(y)$ is a Jacobi polynomial

under $\quad P_{12i}\ \lambda \to -\lambda, P_{13}$ gives $\lambda \to -\lambda\ \frac{-2\pi}{3}$

and $\quad P_{23i}\ \lambda \to -\lambda + \frac{2\pi}{3}$

therefore, recalling that K is even, we define the new orthornormal set, the functions with $\nu \geq 0$

$$v_K^\nu = \sqrt{\frac{K+2}{\pi^3}}\ (1+P_{12})/2\ u_K^\nu(\delta_{\nu 0}\frac{1}{\sqrt{2}} + \tilde{\delta}_{\nu 0}\cdot 1)$$

$$= \sqrt{\frac{K+2}{\pi^3}}\ \cos\lambda\nu\ A^\nu\ P^{\nu,0}_{\frac{1}{2}(K/2-\nu)}(1-2A^2)\times\left[\delta_{\nu 0}\ \frac{1}{\sqrt{2}} - \tilde{\delta}\ _{\nu 0}\right]$$

$$w_K^\nu = -\sqrt{\frac{K+2}{\pi^3}}\quad 1/2\,(1-P_{12})/2\,u_K^\nu$$

$$= \sqrt{\frac{K+2}{\pi^3}}\ \sin\lambda\nu\ A^\nu\ P^{\nu,0}_{\frac{1}{2}\left(\frac{1}{2}K-\nu\right)} \qquad (=0 \text{ if } \nu=0) \qquad \text{(III.47)}$$

form a two dimensional representation of the permutation group if $\nu \neq 3n$. If $\nu = 3n$, n=0,1,2,3, v_K^ν is fully symmetric under permutations and w_K^ν is fully antisymmetric. Since $w_K^o = 0$ and ν=K/2, K/2-2 ... $\geq$ 0 there are $\frac{K}{2} + 1$ independent functions (III. 46) for fixed K and the first fully antisymmetric one is w_6^3.

At this point we note that for L=0 we have generated harmonics satisfying i - iii of the preceding section. We now look at the case L $\geq$ 0. The procedure is to take the functions $\rho^K u_K^\nu(A,\lambda)$ (or v and w) of (III. 46 or 47) and apply to them a completely symmetric traceless tensor operator which is a polynomial in $\partial/\partial z$ and $\partial/\partial z^*$ and carries angular momentum L. Since $\partial/\partial z^*$ commutes with Δ_ξ (see III. 35) the resultant polynomials are still harmonic but now with angular momentum L instead of zero. Since the u_K^ν are functions of the scalars z^2, z^{*2} and zz^* differentiation with respect to the spherical components $(\partial/\partial_{z\pm0})$ (*) will leave us with a product of $z_{\pm0}$ or $z^*_{\pm0}$ allowing the writing of the harmonics with L≠0 in terms of tensors formed from the spherical components of z and z*. Here we define the spherical components of a complex vector $\vec{a}$ by

$$a_+ = (a_1 + i\,a_2)/\sqrt{2}$$

$$a_- = -(a_1 - i\,a_2)/\sqrt{2} \qquad \text{(III. 48)}$$

$$a_0 = a_3$$

The metric tensor g^{ab} is then $g^{00} = 1$, $g^{+-} = g^{-+} = -1$ and $g^{0+} = g^{+0} = g^{-0} = g^{0-} = g^{--} = g^{++} = 0$ (note the last two). We remark that $a_{\pm,0}$ carries $m_L = \pm1, 0$ as, for example, in three dimensions $\frac{1}{\sqrt{2}}(x_1 + i x_2) = \frac{1}{\sqrt{2}} \sin\theta\, \epsilon^{i\phi} \sim Y_1^1(\theta, \phi)$. Additionally, in the case L+K is odd we shall need pseudotensor operators. We define a pseudovector A from a by[3]

$$A_k = \epsilon_{ijk}\, a_i a_j^* \qquad \text{(III. 49)}$$

and from this the differential operators $\partial/\partial A_k$. Since, by parity alone, the product of two pseudovectors is a tensor. The pseudotensor operator can be formed from the tensor operator by multiplication by ∂A_i.

The functions that we seek will be eigenfunctions of the following operators K^2, or the global angular momentum, given by

$$K^2 = \left(z \frac{\partial}{\partial z} + z^* \frac{\partial}{\partial z^*}\right) \left(z \frac{\partial}{\partial z} + z^* \frac{\partial}{\partial z^*} + 4\right) - (zz^*)\Delta \qquad \text{(III.50)}$$

The total angular momentum L^2 with

$$L = \hbar/i \left[\left[z, \frac{\partial}{\partial z}\right] + \left[z^*, \frac{\partial}{\partial z^*}\right]\right. \qquad \text{(III.51)}$$

M = the projection of L on some z-axis,

$$(M = L_z) \qquad \text{(III.52)}$$

ν the operator that gives the permutation symmetry of the state. We saw above that for K=0, $\nu=\alpha-\beta$ in general ν = (number of z's - number of z*'s)/2 or

$$\nu = \frac{1}{2} \quad z \frac{d}{dz} - z^* \frac{d}{dz^*} \qquad \text{(III.53)}$$

Finally we can choose the fifth operator as

$$\Omega = L \cdot z\left(\frac{\partial}{\partial z} \cdot L\right) - \sum_{ij} L_i \frac{\partial}{\partial z_j^*} \; z_i \, L_j \qquad \text{(III.54)}$$

In general it is not difficult to find a set of harmonic polynomials which are eigenfunctions of (III.50-III.53). On the other hand, (III.54) has no clear physical contact. Pustavalov and Smorodinskii[16] have shown that for $K \leq 5$ (or $K \leq 6$ for pseudotensors). The eigenfunctions of Ω have the same spectrum as an operator ω. Where the eigenvalue ω is half the difference of the number of ∂'s and the number of ∂^*'s in the term that acts on a harmonic with L=0 to give one of the angular momentum L. This means the hyperspherical harmonics will be a complete orthonormal set for $K \leq 5$ (or 6 as above). For arbitrary K Pustavalov and Simonov[3] show that the functions specified by K, L, M, ν and ω are independent by demonstrating that the total number of states is $(K+3)(K+2)^2(K+1)/12$ in accordance with (III.23) for n=6.

The differential operator which generates harmonics of angular momentum L can be written as

$$D^{LN\omega}(\partial, \partial^*) \equiv P^L \partial_a \partial_b \cdots \partial_c^* \partial_d^* \cdots \quad \text{(L terms)} \tag{III.55}$$

where ∂_a^* is the derivative with respect to the component z_a^*, N is the difference of the number of indices + and - in D and is clearly equal to -M, ω is defined above and P^L is an operator which renders an arbitrary tensor traceless and symmetric in all its indices (see (3)). Note that from (III.34):

$$P_{12} D^{LN\omega} = D^{LN-\omega}, \qquad P_{13} D^{LN\omega} = e^{4\pi i\omega/3} D^{LN-\omega}$$

and

$$P_{23} D^{LN\omega} = e^{-4\pi i\omega/3} D^{LN-\omega} \tag{III.56}$$

The $D^{LN\omega}$ can be written in the spherical coordinates ρ, A, λ (or $\sigma \equiv e^{-i\lambda}$) (III.37) by using

$$\begin{aligned} \partial_a &= g^{ab}(z_b \Lambda_1 + z_b^* \Lambda_2) \\ \partial_a^* &= g^{ab}(z_b \Lambda_1^* + z_b \Lambda_2) \end{aligned} \tag{III.57}$$

where

$$\begin{aligned} \Lambda_1 &= \frac{1}{\rho^2} \frac{\partial}{\partial \Lambda} - \frac{1}{\rho 2 A} \frac{\partial}{\partial \sigma} \\ \Lambda_1^* &= \frac{\sigma}{\rho^2} \frac{\partial}{\partial A} + \frac{\sigma^2}{\rho^2 A} \frac{\partial}{\partial \sigma} \\ \Lambda_2^* &= \Lambda_2 = \frac{1}{\partial \rho} \frac{\partial}{\partial \rho} - \frac{A}{\rho 2} \frac{\partial}{\partial A} \end{aligned} \tag{III.58}$$

Whence with some manipulation

$$D^{LN\omega} = g^{ab} g^{cd} \dots g^{fg} \sum_{k=-L/2}^{L/2} T^{Lk}_{bd \dots g} \sum_{m=0}^{(L+2\omega)/2} \binom{(L+2\omega)/2}{m}$$

$$\binom{(L-2\omega)/2}{k-\omega+m} \Lambda_1^{L/2+\omega-m} \Lambda_1^{*\,L/2-k-m} \Lambda_2^{2m+k-\omega} \qquad \text{(III.59)}$$

where M = -N (follows from $g^{+-} = g^{-+} = -1$, $g^{++} = g^{--} = 0$ in (III.59)) and $T^{LNk}_{bd \cdots g}$ is a symmetric traceless tensor with angular momentum = L.

The pseudotensor operators given by

$$\partial_{bA} \cdot D^{LN\omega} \equiv D^{LN\omega}_A .$$

where

$$\partial_{bA} = - g^{bc} A_c(\Lambda_1 \Lambda_1^* - \Lambda_2^2) \qquad \text{(III.60)}$$

For the tensor case the K harmonics are given by

$$u^{\nu,\omega}_{KLM} = D^{L-M\omega} P^{\nu+\omega}_{L+K} \qquad \text{(III.61)}$$

where

$$P^{\nu+\omega}_{L+K} = u^{\nu+\omega}_{L+K}(A,\lambda)$$

of (III.45) and from (III.56) we can take $\omega \geq 0$, $\omega \leq L/2$, by definition, and ν is again $0 \leq \nu \leq K/2$ where $\nu+\omega$ goes in units of two.

In analogy with (III.47) we have

$$v^{\nu,\omega}_{KLm} \propto (1+P_{12})/\sqrt{2}\; v^{\nu,\omega}_{KLm} (\delta \nu_0 \frac{1}{\sqrt{2}} \tilde{\delta} \nu_0)$$

$$w^{\nu,\omega}_{KLm} \propto -(1-P_{12})/\sqrt{2}i\; (1-P_{12}) u^{\nu,\omega}_{KLm} \qquad \text{(III.62)}$$

if ν is a multiple of 3/2 the v are symmetric and the w antisymmetric under particle interchange otherwise v and w form a two dimensional representation of the permutation group.

For pseudotensors we have since $P_{12}A_a = -A_a$,

$$w^{A\nu,\omega}_{KLM} \propto (1-P_{12})/\sqrt{2}\ D^{L-M\omega}_{A} P^{\nu+\omega}_{K+L+1} (K+L \text{ odd})$$

$$v^{A\nu,\omega}_{KLM} \propto (1+P_{12})/\sqrt{2}\ D^{L-M\omega}_{K+L+1} \times \begin{cases} 1 : \nu = 3/2n \\ \frac{2}{\sqrt{3}} \sin 4\pi\nu/3 & \nu \neq 3/2n \end{cases} \qquad \text{(III. 63)}$$

We define

$$u^{A\nu,\omega}_{KLm} \equiv D^{L-M\omega}_{A} P^{\nu+\omega}_{K+L+1} \qquad \text{(III. 64)}$$

as for $u^{\nu,}_{KLm}$ in (III. 61). Then from (III. 46), (III. 58) and (III. 59) we can derive the K-harmonics of Pustavalov and Simonov[13]

Tensor case:

K+L even

L=0, u = -K/2 ... K/2

$$u^{v}_{K} = (K+2)/2\pi^3)^{1/2} H^{\nu,0}_{n}$$

L=1 $\quad \xi = -(K-1) - \frac{K-1}{2} + 2 \ldots \frac{K-1}{2}$ $\quad$ K odd

$$u^{\xi+1/2,0}_{K1m} = \left[\frac{3}{4}\frac{(K+2)(\xi+3)}{\pi^3}\right]^{1/2} \left[\frac{z_m}{\rho} H^{(\xi,1)}_{n} \quad \frac{z^*_m}{\rho} H^{(\xi+1,1)}_{n-1}\right]$$

where $\quad M = 1, 0, -1 \qquad z^{(*)}_{M} = z^{(*)}_{+},\ z^{(*)}_{0},\ z^{(*)}_{-}$

and $\quad \nu = \xi + 1/2$

L=2 $\quad \omega = 1 \quad \xi = -\frac{(K-2)}{2}, \frac{K-2}{2} + 2, \ldots \frac{(K-2)}{2} \qquad \nu = +1$

$$\nu^{\xi+1,1}_{K2M} = C\left\{\left(\frac{K-2}{2} + \xi + 2\right) T^{2M1} H^{(\xi,2)}_{n} + - 2\left(\frac{K-2}{2} + \xi + 2\right) T^{2M0} H^{(\xi+1,1)}_{n-1} + T^{2M-1} H^{(\xi+2,1)}_{n-1} + \left(\frac{K-2}{2} + \xi + 2\right) H^{(\xi+2,2)}_{n-2}\right\}$$

$$\omega = 0: \; \xi = 0,\ 2,\ \frac{K-2}{2}, \frac{K-2}{2} \quad \text{even}$$

$$\xi = 1,\ 3\ \ \frac{K-2}{2}, \frac{K-2}{2} \quad \text{odd}$$

$$u_{K2M}^{\xi,0} = C \left\{ -\left(\frac{K-2}{2} + \xi\right) T^{2M1} H_n^{(\xi-1,2)} \right.$$

$$+ T^{2M0} \left[H_n^{(\xi,1)} + \left(\frac{K-2}{2} + \xi\right) H_{n-1}^{(\xi,2)} \right.$$

$$\left. - \left(\frac{K-2}{2} + \xi + 4\right) H_n^{(\xi,2)} \right]$$

$$\left. + \left(\frac{K-2}{2} + \xi + 4\right) T^{2M-1} H_{n-1}^{(\xi+1,2)} \right\}$$

The terms with $-\nu$ are derived by letting $z \rightarrow z^*$ and $\sigma \rightarrow \sigma^*$.

Pseudotensor version: $K+L = \text{odd} \qquad L \geq 1$

$$L=1 \quad \xi = -\frac{K-2}{2}, -\frac{K-2}{2} + 2, \ldots \frac{K-2}{2}$$

$$u_{KM}^{A\xi,0} = \left[3\left(\frac{K-2}{2} + \xi\right)\left(\frac{K-2}{2} + \xi + 2\right)\right]^{1/2} \left[\left(\frac{K-2}{2} - \xi + 2\right) \pi^3\right]^{-1/2}$$

$$\left[\frac{z}{\rho}, \frac{z^*}{\rho}\right]_M H_n^{(\xi,1)}$$

$$L=2 \quad \xi = -\frac{K-2}{2}, -\frac{K-2}{2} + 2, \ldots \frac{K-3}{2}$$

$$\nu = \xi + \frac{1}{2}$$

$$u_{K2M}^{A\xi+1/2,0} = -\frac{3}{2}\left[\left(\frac{K-3}{2} + \xi + 2\right)\left(\frac{K-3}{2} + \xi + 4\right)\right]^{1/2}$$

$$\left[\left(\frac{K-3}{2} - \xi + 2\right)\pi^3\right]^{-1/2} \times \left\{ \tilde{T}^{2M1/2} H_{\tilde{n}}^{(\xi,2)} - \tilde{T}^{2M\,1/2} H_{\tilde{n}-1}^{(\xi+1,2)} \right\}$$

Again to get $\nu \to \nu$ let $z \to z^*$.

The notation employed in the above harmonics is:

$$n = \left(\frac{K-L}{2} - \xi\right)/2 \qquad \tilde{n} = \left(\frac{K-L-1}{2} - \xi\right)/2$$

$$H^{(\alpha,\beta)}_{n-s} = \sigma^{\alpha} A^{\alpha} P^{(\alpha,\beta)}_{n-s} (1-2A)^{2}; \qquad (=0 \;\; n-s < 0)$$

$$T^{2M1} = \rho^{-2} (z_a z_b - \tfrac{1}{2} g^{ab} z^2) \quad M = M_a + M_b$$

$$T^{2M0} = \rho^{-2} \left\{ \left(\frac{z_a z_b^* + z_a z_b^*}{2} \right) + \frac{1}{3} g^{ab} (zz^*) \right\}$$

$$T^{2M-1} = P_{12} T^{2M1}$$

$$T^{2M\frac{1}{2}} \quad \frac{z_a}{\rho} \left[\frac{z}{\rho}, \frac{z^*}{\rho}\right]_b + \frac{z_b}{\rho} \left[\frac{z}{\rho}, \frac{z^*}{\rho}\right]_a$$

$$\tilde{T}^{2M\frac{1}{2}} = - P_{12} \tilde{T}^{2M\frac{1}{2}}$$

and C is a normalization constant.

The above harominces which will be referred to collectively as (III. 65) with extensions for L > 2, form a complete set of independent hyperspherical harmonics which are orthogonal with respect to K, L, M, ν and ω for $K \leq 5$ ($K \leq 6$ for pseudotensors) with definite properties under permutation of the particles see (III. 62) and (III. 63). Other than the troublesome question of the operator Ω or quantum number ω for large K,[10] we have constructed a suitable basis for the hyperspherical part of three particles with equal mass obeying an SU(4) or SU(6) symmetry.

III. 4 The Development of the Wave Function

In this section we will give the expansion of the three nucleon bound state wave function in terms of the Simonov harmonics of the preceding section.

A priori estimates for the convergence of the resultant series are given and finally the results of several authors on the relative weight of the various hyperspherical harmonics in the wave function calculated with simple model potentials will be quoted. Along the way, some general and useful formulas will be exhibited.

The trinucleon bound state wave function has a total angular momentum $J = \frac{1}{2}$. As long as we are not interested in polarization measurements we will fix $J_z = \frac{1}{2}$. Since there seems to be no three neutron bound state and certainly none with a binding energy near 8 MeV, the three nucleon bound state can be taken to have isospin $T = \frac{1}{2}$ (a $T = 3/2$ contribution of from .001 to .01 percent of the wave function normalization is expected to arise from Coulomb forces).[18] The three nucleon spinors can form states of $S = \frac{1}{2}$ or $S = \frac{3}{2}$ the former couple to spatial S and P waves the latter to the P and D waves. The partial wave decomposition of the trinucleon ground state shows, in models, that the main contribution is from a spatially symmetric S-state. There is a mixed symmetry S state called the S' which contributes, probably, from less than one to, at best, four percent of the wave function normalization.[18] The fully antisymmetric S state has K=6 or greater (see above) and thus is expected to be suppressed. The P states are due to the L-S force and are small. The D states are generated by the tensor force and are probably between four and twelve percent of the wave function.[18]

We now consider the spin-isospin function $\eta^{SorT}_{1/2MS(T)}$ and $\eta^{S}_{3/2MS}$ constructed from three spin (isospin) 1/2 particles.[19]

For S or T=1/2 we take $M = 1/2$ ($M_T = 1/2$ for ^3He) then there are two spin functions which form a two dimensional representation of the permutation group

$$\eta^{1S(T)}_{\frac{1}{2}\frac{1}{2}} = \left\{ \frac{1}{\sqrt{6}} \; (|+>|->+|->|+>) \quad |+> - 2\; |+>\; |+>\; |->\right\}$$

$$\eta^{-1S(T)}_{\frac{1}{2}\frac{1}{2}} = \frac{1}{\sqrt{2}} \; (|+>|->-|->|+>)\; |+> \tag{III.66}$$

where $|+>$ as a particle with spin (isospin) given by $S_z(T_z) = +\frac{1}{2}$ and $|->$ one with $S_z(T_z) = -\frac{1}{2}$. Here $P_{12}\,\eta^1 = \eta^1$ and $P_{12}\,\eta^{-1} = -\,\eta^{-1}$ (as in (III.47)).

For S = 3/2 three is one set of fully symmetric spin function

$$\eta_{\frac{3}{2}\frac{3}{2}} = |+>\;|+>\;|+>$$

$$\eta_{\frac{3}{2}\frac{1}{2}} = \frac{1}{\sqrt{3}} \left\{(|+>\;|+> + |->\;|+>)\;|+> + \;|+>|+>|->\right\} \tag{III.67}$$

The functions with $M_s = -1/2$ and $-3/2$ are given by interchanging + and - in (III.67).

Since ρ is fully symmetric in the particle coordinates, the hyperspherical harmonics (III.62) carry the spatial symmetry of the wave function. Writing

$$\psi(\vec{\xi}, \vec{\eta}) = \frac{1}{\rho^2} \sum_{K,\,L,\,\omega,\nu,\,M_J=1/2} \chi^{\nu,\omega}_{KL}(\rho)\; \Phi^{\nu,\omega}_{KL(J=1/2)S,\,T}(\Omega_6) \tag{III.68}$$

where the Φ's are normalized by

$$\int d\Omega_6\; \Phi^*_{[K']}\Phi_{[K]} = \delta_{[K'][K]}$$

where $[K]$ is the set of quantum numbers in (III.68). We note that the Φ's are fully antisymmetric under particle interchange. The χ's are then normalized by

$$\sum_{K,\,L,\,\nu,\omega} \int |\,\chi^{\nu,\omega}_{KL}(\rho)\,|^2\; \rho\, d\rho = 1$$

For the principle S-state we have

$$\Phi^{\nu,\omega}_{K0,J=1/2,S=1/2,T=1/2}(\Omega_6) = v^{\nu}_{K0}(\Omega_6)\,\frac{1}{\sqrt{2}}\left(\eta^{-1S}_{\frac{1}{2}\frac{1}{2}}\,\eta^{1T}_{\frac{1}{2}\frac{1}{2}} - \eta^{1S}_{\frac{1}{2}\frac{1}{2}}\,\eta^{-1T}_{\frac{1}{2}\frac{1}{2}}\right)$$

$$\nu = \frac{3}{2}n,\ n = 0,\ 1,\ 2,\ \ldots$$

$$\nu = K/2;\ \ K/2-2\ \ldots \geq 0 \qquad \text{(III.69)}$$

For L=0 K is even and the leading terms are

$$v^{0}_{00},\ v^{0}_{40},\ v^{3}_{60}\ \ldots$$

Note that there is no term between K=0 and K=4 in (III.69).

For the S' state

$$\phi^{\nu,\omega}_{K0,J=1/2,S=1/2,T=1/2} = \frac{1}{2}\ v^{\nu}_{K0}(\Omega_6)\left(\eta^{-1S}_{\frac{1}{2}\frac{1}{2}}\,\eta^{-1T}_{\frac{1}{2}\frac{1}{2}} - \eta^{1S}_{\frac{1}{2}\frac{1}{2}}\,\eta^{1T}_{\frac{1}{2}\frac{1}{2}}\right)$$

$$- v^{\nu}_{K0}(\Omega_6)\left(\eta^{-1S}_{\frac{1}{2}\frac{1}{2}}\,\eta^{+1T}_{\frac{1}{2}\frac{1}{2}} + \eta^{1S}_{\frac{1}{2}\frac{1}{2}}\,\eta^{-1T}_{\frac{1}{2}\frac{1}{2}}\right)$$

here $\nu \neq \frac{3}{2}M$ $\qquad \nu = K/2, \frac{K}{2} - 2, \ldots > 0$ $\quad$ K even (III.70)

the first terms are

K=2 $\qquad \nu=1$

K=4 $\qquad \nu=2\ldots$

For the D-states: the $\eta^{S}_{3/2}$ are fully symmetric. Since the $\eta^{T}_{1/2}$'s form a mixed representation only w^{ν}_{K2} and v^{ν}_{K2} forming a mixed representation contribute to the D-wave part of the wave function which in terms of the $v(w)^{\nu}_{K2M_L}$

is given by:

$$\Phi^{\nu,\omega}_{K2,S=3/2,T=1/2} = \sqrt{\frac{4}{20}}\left(w^{\nu}_{K22}\,\eta^{1T}_{\frac{1}{2}\frac{1}{2}} - v^{\nu}_{K22}\,\eta^{-1T}_{\frac{1}{2}\frac{1}{2}}\right)\eta^{S}_{\frac{3}{2}-\frac{3}{2}}$$

$$+\sqrt{\frac{3}{20}}\left(w^{\nu}_{K21}\,\eta^{1T}_{\frac{1}{2}\frac{1}{2}} - v^{\nu}_{K21}\,\eta^{-1T}_{\frac{1}{2}\frac{1}{2}}\right)\eta^{S}_{\frac{3}{2}-\frac{1}{2}}$$

$$+\sqrt{\frac{2}{20}}\left(w^{\nu}_{K20}\,\eta^{1T}_{\frac{1}{2}\frac{1}{2}} - v^{\nu}_{K20}\,\eta^{-1T}_{\frac{1}{2}\frac{1}{2}}\right)\eta^{S}_{\frac{3}{2}+\frac{1}{2}}$$

$$-\sqrt{\frac{1}{20}}\left(w^{\nu}_{K2-1}\,\eta^{1T}_{\frac{1}{2}\frac{1}{2}} - v^{\nu}_{K2-1}\,\eta^{-1T}_{\frac{1}{2}\frac{1}{2}}\right)\eta^{S}_{\frac{3}{2}\,\frac{3}{2}} \qquad \text{(III. 71)}$$

by parity since K is even for L=0, K is always even. Thus $K \geq 2$ and $\nu \neq 3/2n$ to get a mixed representation of the permutation group. The lowest states are K=2, ν=1 and K=4, ν=2.

The coupled equations that describe the functions $\chi^{\nu,}_{KL}(\rho)$ are given by (III. 26) and (III. 28):

$$\frac{d^2\chi^{\nu,\omega}_{KL}(\rho)}{d\rho^2} + \frac{1}{\rho}\,\frac{d\,\chi^{\nu,\omega}_{KL}(\rho)}{d\rho} + \left(2m\,\frac{E}{\hbar^2} - \frac{(K+2)^2}{\rho^2}\right)\chi^{\nu,}_{KL}(\rho)$$

$$= \frac{2m}{\hbar^2}\sum_{K'L'\nu'\omega'} V^{\nu,\omega;\nu'\omega'}_{KL;K'L'}(\rho)\;\chi^{\nu'\omega'}_{K'L'}(\rho) \qquad \text{(III. 72)}$$

L' = 0, 2 here

where we now have

$$V^{\nu,\omega;\nu'\omega'}_{KL;K'L'}(\rho) = \int d\Omega_6\;\Phi^{\nu,\omega}_{KLST}(\Omega_6)\;V(\vec{\xi})\;\Phi^{\nu,\omega}_{K'L'S'T'} \qquad \text{(III. 73)}$$

The $V(\vec{\xi})$ in (III. 73), in general, includes tensor forces.

If

$$V(\vec{\xi}) = \sum_{i>j} V_{ij}(\vec{r}_{ij}\ \tau_i,\ \tau_j,\ S_i,\ S_j) \qquad \text{(III.74)}$$

then since the Φ's are completely antisymmetric we have[20]

$$V^{\nu,\omega;\nu'\omega'}_{KL;K'L'}(\rho) = 3 \int d\Omega_6 \quad \Phi^{*\nu,\omega}_{K,L} V_{12}(\vec{r}_{12},\ \tau_1, \tau_2,\ S_1,\ S_2)$$

$$\times \ \Phi^{\nu'\omega'}_{K'L'} \qquad \text{(III.75)}$$

To obtain estimates of the $V_{[K]\,,\,[K']}(\rho)$ and the partial waves $\chi^{\nu,\omega}_{KL}$ we follow the treatment of Badalyan and Simonov[21] and appeal to the case of central potentials with L=0. Since $|r_{12}| = |\sqrt{2}\,\eta| = \sqrt{2}\,\rho\cos\theta$, from (III.46) we see that v^{ν}_K or w^{ν}_K is a polynomial in $\cos\theta$ and/or $\sin\theta$ with terms up to power K. Thus we can express V as a series;

$$V(\sqrt{2}\eta) = \sum_{K=0,2\dots} V_K(\rho)\, C_{K/2}(\cos 2\theta) \qquad \text{(III.76)}$$

where $C_n(x)$ is a Chebyshev polynomial (of the second type) with

$$\int_{-1}^{1} \sqrt{1-x^2}\, C_n(x)\, C_{n'}(x)\, dx = \frac{\pi}{2}\,\delta_{nn'}$$

Equation (III.76) implies that since $V(r_{12})$ is expandable in terms of a single angular series then the moments $V^{\nu';\nu}_{K;K}$ should be given by a recursion formula in terms of $V^{0;\nu}_{0K}$. This is of the form[14]

$$V^{\nu'';\nu'}_{K'';K'} = \sum_{K,\nu} \left\langle \begin{matrix} K''/2 & K'/2 & K/2 \\ \nu'' & \nu' & \nu \end{matrix} \right\rangle V^{0;\nu}_{0;K} \qquad \text{(III.77)}$$

where in terms of 3j symbols

$$\left\langle \begin{matrix} K''/2 & K'/2 & K/2 \\ \nu'' & \nu' & \nu \end{matrix} \right\rangle = \left(\frac{(K''/2+1)(K'/2+1)(K/2+1)(1+\delta_{00})}{2\pi^3(1+\delta_{\nu'0})(1+\delta_{\nu''0})} \right)^{1/2}$$

$$\times \left(\begin{pmatrix} K''/4 & K'/4 & K/4 \\ \nu''/2 & \nu'/2 & -\nu/2 \end{pmatrix}^2 + \begin{pmatrix} K''/4 & K'/4 & K/4 \\ \nu''/2 & -\nu'/2 & -\nu/2 \end{pmatrix}^2 \right) \qquad \text{(III.78)}$$

we can take $v'' \geq v'$ (by redundancy of the sum) and thus $\nu = \nu'' \pm \nu'$

$$K = |K'' - K'|; \qquad |K'' - K' + 4|; \ldots |K' + K''|$$

The $V_{0;K}^{0;\nu}$ are given in terms of the $V_K(\rho)$ of (III.76) by:

$$V_{0K}^{0;\nu} = V_{K;0}^{\nu;0} = \frac{2(-1)^{K/4-\nu/2}}{\sqrt{K+2}} V_K(\rho) \cdot \left(1+2\cos\frac{2\pi\nu}{3}\right) \begin{cases} 1 & \nu > 0 \\ \frac{1}{\sqrt{2}} & \nu = 0 \end{cases}$$

(III.79)

where we note that only the $v_K^{\nu}(\Omega_6)$ contribute to (III.79). (In fact, only transitions into symmetric spatial states survive, this is because we have, for simplicity, assumed all the pair potentials identical. If instead, we were to include isospin and spin dependent forces like ρ and ω exchange. Then we could mix in the S' state.) It will turn our that the $V_{0K}^{0,\nu}$ determine the magnitude of χ_K^{ν} for large K in the trinuclear bound states.

We consider separately the cases of ρ small and of ρ large. For small ρ most of the potentials employed in nuclear physics can be expanded as

$$V(r) = \sum_{n=-1}^{\infty} a_n r^n/n! \qquad ((-1)! = 1 \text{ here}) \qquad \text{(III.80)}$$

where by taking moments:

$$V_K(\rho) = V_K^{-1}/\rho + V_K^0 + V_K^1\rho + V_K^2\rho^2/2! \qquad \text{(III.81)}$$

and

$$V_K^n = a_n \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2}\, C_{K/2}(x)(1+x)^{n/2}\, dx \qquad n\neq 0$$

$$= a_{-1} \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2}\, C_{K/2}(x)\, dx \qquad n=0 \qquad (\text{III}.82)$$

This yields for n even and $n > K$

$V_K^n = 0.$ As K increases we have

$V_K^n \sim 1/K^{n+2}$ and in particular

$V_K^{-1} \sim \frac{1}{K}$ but from (III.79) for ρ small

we have at worst,

$$V_{0;K}^{0;\nu}(\rho) \sim V_{eff} \frac{1}{K^{3/2}}$$

Now for large ρ we let $r = \rho\sqrt{1+x}$ and invert (III.76)

$$V_K(\rho) = \frac{1}{\rho^3} \frac{4}{\pi} \int_0^{\sqrt{2}} \sqrt{2-r^2/\rho^2}\, C_{K/2}\left(\frac{r^2}{\rho^2} - 1\right) V(r)\, r^2\, dr \qquad (\text{III}.83)$$

letting $V(r) << r^{-3}$ as $r \to n\infty$ and taking $r^2/\rho^2 << 1$ $V_K(\rho) \sim (\frac{K}{2} + 1)/\rho^3$. But if the range of the potential is a then $r \le a$, so, if $\rho >> Ka/2$ we have

$$r/\rho << 1 \quad \text{and} \quad V_K(\rho) << V_{eff}/K^2 \qquad (\rho >> Ka/2)$$

For $\rho << Ka/2$ we use

$$C_n(\cos\phi) = \sin((n+1)\phi)/\sin\phi$$

and with $x=\cos\phi$ we get:

$$V_K(\rho) = \frac{2}{\pi} \int_0^{\pi} d\phi \sin\phi \sin\left(\left(\frac{k}{2} + 1\right)\phi\right) V\left(\rho\sqrt{2}\cos(\phi/2)\right). \quad (\text{III}.84)$$

The integral receives contributions only from $\phi \leq 1/K$. Then from (III. 84) $V_K(\rho) \sim V_{eff}/K$ ($\rho \ll Ka/2$). Finally, $V_{K,0}^{K,0}$ is estimated as:

$$V_{K,0}^{K,0} \sim V_{eff} \times \begin{cases} 1/K^{3/2} & \rho \ll Ka/2 \\ K^{1/2}\, a^3/\rho^3 \sim \frac{1}{K^{5/2}} & \rho \gg Ka/2 \end{cases} \qquad \text{(III. 85)}$$

From (III. 78) and (III. 79) we can obtain estimates for $V_{KK'}^{\nu\nu'}(\rho)$ in particular $V_{KK}^{\nu\nu} \sim V_{0,0}^{0,0}$.

To obtain estimates of the size of the χ_K^{ν} we write (III. 72) as an integral equation

$$\chi_{KL}^{\nu,\omega} = -\frac{2m}{h^2}\int_0^{\infty} \rho'\, d\rho'\; G_K(\rho,\rho',\chi) \quad V_{K,L;K'L'}^{\nu,\omega;\nu',\omega'}(\rho')\, \chi_{K'L'}^{\nu'\omega'} \qquad \text{(III. 86)}$$

where the Green's function $G_K(\rho, \rho', \chi)$ is given by

$$G_K(\rho,\rho',\chi) = I_{K+2}(\chi\rho_<)\, K_{K+2}(\chi\rho_>) \qquad \text{(III. 87)}$$

with

$$\rho_{\gtrless} = \rho' \gtrless \rho$$

$$= \rho \gtrless \rho'$$

I and K are modified Bessel functions and $\chi^2 \equiv -2m/\hbar^2 E \equiv 2m/\hbar^2\, E_B$ ($E<0$).

Defining:

$$G_K(\rho,\rho',\chi) \equiv I_{K+2}(\chi\rho)K_{K+2}(\chi\rho)\, f_K(\rho,\rho',\chi)$$

$$F_K^{\nu}(\rho) \equiv \sum_{[K]\neq[K']} V_{[K];[K']}(\rho)\, \chi_{[K']}$$

so $f_K \sim (z_</z_>)^{K+2}$; z, z' << K and decreases strongly for z, z' >> K, and (III. 86) becomes

$$\chi^{\nu,\omega}_{KL} + \frac{2m}{\hbar^2}\int_0^\infty \rho' d\rho' I_{K+2}(\chi\rho)\, K_{K+2}(\chi\rho)\, f_K(\rho,\rho',\chi) V^{\nu,\omega;\nu,\omega}_{KL;KL} \;\chi^{\nu,\omega}_K$$

$$= -\frac{2m}{\hbar^2}\int_0^\infty \rho' d\rho' I_{K+2}(\chi\rho)\, K_{K+2}(\chi\rho)\, f_K(\rho,\rho',\chi)\; F^V_K(\rho') \qquad \text{(III. 88)}$$

Now assume that there exists a bound state with E_B near E_K, the energy for which (III. 88) (for some K) has a solution for $F_K = 0$ then $\|\chi_{K,L}\|$ is determined only by the requirement that the wave function be normalized.

From the "centrifugal" barrier term $(K+2)^2/\rho^2$ in (III. 72). We expect

$$E_{B0} > E_{B1} > \ldots > 0$$

For the trinucleon bound state there exists only one energy level. Let us assume that this is due to E_{B0} and that the wave function is dominated by χ^o_0 we will now show that this assumption is self-consistent.

Assume that we have solved (III. 88) for χ^o_0 with $F_0 = 0$. Then, to first approximation, $\chi^\nu_{K=0}$ is given by

$$\chi^\nu_K = -\frac{2m}{\hbar^2} I_{K+2}(\chi\rho) K_{K+2}(\chi\rho)\int_0^R dp' p' f_K(\rho,\rho',\chi)\; V^{\nu,o}_{K,0}(\rho')\; \chi^o_0(\rho')$$

(III. 89)

where R is the size of the three nucleon system. From our discussion in Section II if we take K+2 >>χR, then $I_{K+2}(\chi\rho)\, K_{K+2}(\chi\rho) \sim 1/2(K+2)$ and the integral by our estimate for $V^{\nu,o}_{K,0}$, above, is $\sim V^{oo}_{00}\chi^o_0\; 1/K^{3/2}\;\; 1/(K+2)$ where the 1/(K+2) factor comes from integrating $f(\rho,\rho',\chi) \sim (\rho^2_</\rho_>)^{K+2}$.

Thus overall:

$$\chi_K^{\nu} \sim \frac{1}{K^{7/2}} \chi_0^o \quad \text{(III. 90a)}$$

$$\int \rho \, d\rho \; |\chi_K^o|^2 / \int \rho d\rho \, |\chi_0^o|^2 \sim 1/K^7 \quad \text{(III. 90b)}$$

So the contribution of the K-harmonics for $K > 0$ falls off rapidly. (From (III. 89) when $\chi R \sim 1$ such as the trinucleon system (III. 90a) is, more accurately $\chi_K^o \sim \frac{1}{K^{3/2}(K+2)^2} \chi_0^o$.)

Before we quote numerical results from model calculations for the relative size of the χ_K^{ν} we will look at the formalism of the ^3He electric form factor.[19] The charge density of ^3He where the three nucleons are centered at $\vec{r}_i$, $i = 1, 2, 3$ is

$$\rho_c(\vec{r}, \vec{r}_i) = \sum_{i=1}^{3} \frac{1}{2} (1 + \tau_{i3}) f_{ch}^p(\vec{r} - \vec{r}_i) + \frac{1}{2}(1 - \tau_{i3}) f_{ch}^n(\vec{r} - \vec{r}_i) \quad \text{(III. 91)}$$

Then

$$F_{el}(q^2) = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{3} \int d^3\vec{r} \int d^3\vec{r} \; e^{i\vec{q}\cdot\vec{r}_i} \psi^* \rho_c(\vec{r}, \vec{r}_i)\psi \quad \text{(III. 92)}$$

Noting that between <u>identical</u> isospin wave functions

$$\sum_i \langle | \tau_{i3} | \rangle = 1 \qquad \text{(for } ^3He\text{)}$$

$$\sum_i \langle | 1 | \rangle = 3$$

we obtain the contribution of the term in $\psi^+\psi$ with identical isospin wave functions to the form factor is $(F_{el}^p(q^2) + \frac{1}{2} F_{el}^n(q^2)) \, F_1(q^2)$ with:

$$F_1(q^2) = \sum_{[K][K']} \int \rho \, d\rho \int d\Omega_6 \; e^{i\vec{q}\cdot\vec{r}_3} \Phi^*_{[K]} \Phi_{[K]} \chi^*_{[K']}(\rho) \chi_{[K]}(\rho) \quad \text{(III. 93)}$$

where $\Phi_{[K]}, \Phi^*_{[K']}$ both belong to the S, S' or D waves. The D and S states do not interfere as their spinor wave functions have S = 3/2 and S = 1/2 respectively and are thus orthogonal. The interference term is given by evaluating isospin matrix elements and is: $-\frac{1}{3}(F^p_{el}(q^2) - F^n_{el}(q^2))F_2(q^2)$ with

$$F_2(q^2) = \sum_{[K],[K']} \int d\Omega_6 \int \rho d\rho \left(e^{i\vec{q}\cdot\vec{r}_1} - e^{i\vec{q}\cdot\vec{r}_2}\right)\frac{1}{\sqrt{2}}\left(v_{[K]}w_{[K']}\right.$$

$$\left. + \sqrt{3}\, v_{[K]}w_{[K']}\right)\chi^*_{[K']}\chi_{[K]} \qquad \text{(III. 94)}$$

where small v_K is completely symmetric and $v_{[K]}, w_{[K]}$ are a mixed representation in the expression of the S' state.

The ^{3}He electric form factor is thus

$$F^{^3He}_{el}(q^2) = F^p_{el}(q^2) + \frac{1}{2}F^n_{el}(q^2)\, F_1(q^2) - \frac{1}{3}\, F^p_{el}(q^2) - F^n_{el}(q^2)\, F_2(q^2)$$

(III. 95a)

likewise:

$$F^{^3H}_{el}(q^2) = F^p_{el}(q^2) + 2F^n_{el}(q^2)\;\; F_1(q^2) - \frac{2}{3}\, F^n_{el}(q^2) - F^p_{el}(q^2)\;\; F_2(q^2)$$

(III. 95b)

Simonov and Badalyan have given $F_1(q^2)$ for the first few K-harmonics

$$F_1^{S+S'}(q^2) = 8\int \frac{\rho d\rho}{a^2}\left\{(\chi^0_{00})^2\, J_2(0) - 2\sqrt{3}\,\chi^0_{00}\chi^0_{40}\, J_6(0)\right.$$

$$\left. + (\chi^1_{00})^2\left[J_2(0) + J_6(0)\right]\right\} \qquad \text{(III. 96)}$$

where our normalization of χ^1_{20} differs from theirs and $a = \sqrt{2/3}\; q\rho$.

By explicit integration over $d\Omega_6$ we have that for the lowest D-wave function

$$F_1^D(q^2) = 8\int \frac{\rho d\rho}{a^2} \, |\chi_{22}^1(\rho)|^2 \left[J_2(0) + J_6(0)\right] \tag{III.97}$$

Badalyan (using our normalization) has evaluated $F_2^{S,S'}$ using the lowest contributing states[21]:

$$F_2^{S,S'}(q^2) = 24\sqrt{2}\int \frac{\rho d\rho}{a^2} \, \chi_{00}^{o}(\rho) \, \chi_{20}^{1}(\rho) \, J_4(0) \tag{III.98}$$

Using a square well she obtained that if $F_{el}^n(q^2) = 0$ then, with a simple two body square well, for $q^2 \sim 8\text{-}10 \ F^{-2}$ we get $F_2/F_1 \sim .3$ or as in this case $F_{el}^{^3He}(q^2) = F_{el}^p(q^2)\left(F_1 - \frac{1}{3}F_2\right)$ we get about a ten percent correction from S'S interference in the form factor.

An interesting application of the K-harmonic analysis is the sum rule for the Coulomb energy of ^3He of Fabré de la Ripelle[22] the Coulomb energy is the limit as $q \to 0$ of

$$E_c(q) = \sum_{i>j} -e^2 \iiint d^3r \, d^3r_i \, d^3r_j \, \rho_i(r_i - r) \, \rho_j(r - r_j) \frac{1}{r_{ij}} \, |\psi^2(r)| e^{i\vec{q}\cdot\vec{r}_{ij}} \tag{III.99}$$

while $F_{el}(q^2)$ is given by (III.92). We note that the ρ_i's give factors $F^{p(n)}(q^2)$ and that

$$\frac{d}{dq} E_c(q) \sim \sum_{i>j} e^2 \int \rho(r - r_i) \, \rho(r - r_j) \, d^3r_i \, d^3r_j \, d^3r \, |\psi^2(r)| \, e^{i\vec{q}\cdot\vec{r}_{ij}} \cos\theta r_{ij}$$

therefore if we can relate integrals over $e^{i\vec{r}_{ij}\cdot q} \sim e^{\sqrt{2}\eta\cdot q \cos\theta_\eta}$ to integrals over $e^{iq\cdot r_3} = e^{i\vec{q}\cdot\vec{\xi}\sqrt{2/3}}$ (and noting that $E_c(\infty) = 0$ due to the rapid oscillation of $e^{i\vec{q}\cdot\vec{r}_{ij}}$) we can get a sum rule relating E_c and $F_{el}(q^2)$.

Expansions of $\psi, e^{i\vec{q}\cdot\vec{\eta}}, e^{i\vec{k}\cdot\vec{\xi}}$ in terms of K-harmonics and Bessel functions of $q\rho$ and $k\rho$ respectively enable us to do this. In fact, retaining terms up to K=2 in ψ for L=0 Fabre de la Ripelle gets

$$\Delta E_c = E_{c\,^3He} - E_{c\,^3H} = \frac{2e^2}{\pi\sqrt{3}} \int_0^\infty 4F_E^v\left(\frac{q^2}{3}\right) F_E^S\left(\frac{q^2}{3}\right)$$

$$\times \left\{ \frac{\left(2F^{^3He}_{el\,^3He}(q^2) + F^{^3He}_{^3He\,el}(q^2)\right)}{6F_E^S(q^2)} - \frac{1}{3F_E^v(q^2)} \left(F_{el}^{^3H}(q^2) - F_{el}^{^3He}(q^2)\right)\right.$$

$$\left. - F_E^S(q^2) - F_E^v(q^2)/4F_E^S(q^2) \quad 2F_{el}^{^3He}(q^2) + F_{el}^{^3H}(q^2) \right\} dq \qquad \text{(III. 100)}$$

with

$$F^v = F^p - F^n; \qquad F^S = F^p + F^n \quad .$$

Presumably, if we knew the charge form factors out to q^2 large enough so that we could cut off the integral in (III. 100) with confidence that what we ignore is small then (III. 95) becomes model independent and we can insert the experimental data. The sum rule yields ΔE_c = .65 MeV compared with the experimental value of .764 MeV. The difference may be a result of charge dependent nuclear forces. However, the size of the secondary maximum in the ^{3}He electric form factor indicates a little caution.

Finally, we quote some results in the contributions of the various K-harmonics to the normalization of the trinuclear wave function Dzyuba, Pustavalov, Rybachenko, Sadovoi and Efros[20] have considered eight different potentials with tensor forces and included K-harmonics up to K=4. They included three exponentials, four Yukawa's (one purely central) and a Gaussian with a square well. For five of these potentials they give percentage contributions to the normalization of the wave function from each of the

K-harmonics they included. They found:

In all cases the Φ_0^o term contributed between 93.35 and 95.67 percent of the normalization and was clearly dominant. The next contribution to the principle S state $\Phi_{4,0}^{o,o}$ contributes a minimum of .17 percent and a maximum of .375 percent and is thus quite small. The S' state is suppressed compared to what is needed to account for the difference between the ^3H and ^3He electric form factors (in Ref. 21, $P_{S'} \sim 1.7\%$ to do this) and ranges from .02 percent up to .44 percent for the Gaussian with square well. $P_{S'}$ is about 75 percent from K=2 and the rest from K=4. However, in one exponential potential the two contributions are much closer in size. The D-wave probability P_D ranges from 3.977 percent to 6.327 percent. The contributions from K=4 states is about one third that from K=2 states except for the Gaussian where it is half as big.

We mention a variational calculation of Erens, Vissechers and Von Wageningen, using a slightly different hyperspherical basis and considering only S-waves they obtain between 98.6 and 99.2 percent K=0 in their wave function. However the convergence of the binding energy can require up to K=8 in the case of a potential with a hard core. This reflects the fact that, locally, terms with large K may be important for potentials with large slopes.

In practice, it seems reasonable to assume that to a good approximation we need only retain the K=0 contribution in the principle S states in any a priori description of the wave function. For the D-wave the K=2 terms should provide, at least, an estimate of their gross effects.

In the next chapter we will consider the boundary condition model for two nucleon interactions and apply an extension of the idea to the three nucleon system, suggested by our consideration of K-harmonics, in the following chapter.

Notes

III. 1 The idea that problems in an N-dimensional space can be simplified by representing vectors by one radial coordinate and N-1 angular coordinates is not new. Two early attempts are: L. M. Delves, Nucl. Phys. 9, 392 (1959) and F. T. Smith, Phys. Rev. 120, 1058 (1960).

However, these attempts either do not explicitly separate states of definite permutation symmetry or are extremely complicated. Some more recent expansions are Werner Zichendraht, Ann. Phys. (N. Y.) 35, 181 (1965) and J. A. Castillo Alcaras and J. Lealferrelra, Revista Brasileira de Fisica 1, 63 (1971).

III. 2 For the development, see Simonov papers 1 - 3.

III. 3 A related example is found in variational calculations where a much larger number of terms is required to get a convergent binding energy than are important in the wave function. See Beiner and Fabré de la Ripelle[5] for a three body case in point.

III. 4 This is not totally true; in momentum space there are no deformations of the configuration of the particles when they are in free motion.

If interactions occur only when the system has a small hyperradius (hyperimpact parameter) the description might still be good. For short range forces this will be the region where at least two pairs are within the range of their pair potentials. The exterior-interior separation of Noyes tells us that the wave function outside of this interior can easily be solved if we know it in the interior. Thus the hyperspherical can provide a simple description of the wave function in terms of a few hyperspherical harmonics and functions of the hyperradius in the interior and the interior exterior method used to solve the entire problem. See Smith and Delves in Note IV. 1 and Refs. 7 and 8 for details.

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CHAPTER IV

THE THREE BODY B. C. M.

In the last chapter we discussed one ingredient of our analysis of the trinucleon wave function; the K harmonic expansion. In this chapter we will discuss the other principle component of our model, the boundary condition model (b. c. m.).

What we will do is to discuss the theoretical basis of this model for the two nucleon interaction, indicating some of the difficulties and ambiguities encountered in applying this model to systems of three or more nucleons. The external local potential to be used in conjunction with the two body b. c. m. will also be briefly discussed. Arguments parallel to those of Feshbach and Lomon will be advanced to show that there should exist an explicitly three nucleon b. c. m. These considerations will be applied to a calculation of the ^{3}He electric form factor in Chapter V.

The b. c. m. originated in the work of Breit and Bouricius[1] who used it to provide a pseudopotential which could reproduce nucleon-nucleon $^1S_0$ effective range parameters. Since then extensive work on the b. c. m. has been carried out by the school of Feshbach and Lomon.[2-6] The b. c. m. has been coupled with external energy independent potentials based on meson exchange to give a set of phase shifts and deuteron parameters which is competitive with results obtained with "realistic" local potentials.[5,6]

In the first section of this chapter we will describe the b. c. m. and give heuristic arguments for it which will also be used in the three nucleon case. The next section will briefly outline the work of Feshbach and Lomon[3] in relating the b. c. m. to certain parts of integrals appearing in the Mandelstam representation of the scattering amplitude. The applicability of these ideas to the three body case is discussed next. The final section of the chapter will be a short discussion of the local potentials to be used with the two body b.c.m.[5]

IV. 1 The Boundary Condition Model

It is well known that for large interparticle distances and low energies the nucleon-nucleon interaction may be described by energy independent local potentials. On the other hand, for short distances and/or large energies this is no longer true. However, if we believe that for some r_0 the interaction is local and given by some model, the phase shifts can be calculated if we know $f_\ell(E)$ for each ℓ in question; where (for spinless particles)

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\ u_\ell(r)/r$$

and

$$r_0 d\, V_\ell(r)/dr\Big|_{r_0} / V_\ell(r) \equiv f_\ell(E) + 1 \qquad \text{(IV. 1)}$$

(i.e., $r_0 d\,\psi/dr\big|_{r_0} / \psi = f_\ell$). This is clear as the boundary conditions at r_0 and at ∞ allow us to integrate the second order differential equation from r_0 to r to get a unique solution satisfying (IV. 1). Therefore, the logarithmic derivatives $f_\ell(E)$ specify the problem and take into account all the asymptotic effects of the short distance behavior of the wave function. The b.c.m. is the further assumption that that the $f_\ell(E)$ are independent of E at some r_0. The usual hard core is an example of the b.c.m. with $f_\ell = \infty$. In fact, the energy independence of the $f_\ell(E)$ is closely linked with the vanishing of the wave function in the interior ($r < r_0$). Hoenig and Lomon,[4] assuming that the partial wave potential $V_{\ell\ell}$ (E, r, r') is local ($\sim \delta(r-r')$) for $r > r_0 + \epsilon$ ($\epsilon > 0$), subtract the once integrated Schrodinger equation at E from that at E + ΔE, to obtain (uncoupled partial waves):

$$\frac{1}{r_0} (u_\ell(r_0+\epsilon))^2 \, df_\ell/dE = -\int_0^{r_0+\epsilon} dr \; |u_\ell(r)|^2$$

$$+ \int_0^{r_0+\epsilon}\int_0^{r_0+\epsilon} dr dr' \, u_\ell^*(r') u_\ell(r) \frac{\partial V_{\ell\ell}(E, r, r')}{dE}$$

(IV. 2)

For E below inelastic threshold and $V_{\ell\ell}(E, r, r')$ a two fragment pseudo-potential derived from an energy independent many body (nucleons, mesons, etc.) potential $V(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n)$,[7] it can be proved that $\partial V(E, r, r')/\partial E$ is a negative semi-definite operator. In that case, both terms of the right side of (IV. 2) are negative. If $df_1/dE = 0$ then we must have $\int_0^{r_0+\epsilon} |u_\ell(r)|^2 = 0$, this implies $u_\ell(r) = 0$ almost everywhere in the interior $r < r_0 - \epsilon_{\epsilon\to 0}$. Since $u_1(r_0+\epsilon) \not\to 0$, generally, and $u_1(r)$ and $\partial V/\partial E$ have no δ function singularities, then the condition that $u_\ell(r) = 0 \;\; r < r_0 - \epsilon$ implies that $df_\ell/dE = 0$. We can easily generalize these results to the case of coupled channels using the fact that $df_\ell/dE \leq 0$ which follows from causality of the S-matrix.

Since there is a strong relation between the vanishing of the two nucleon wave function and the b. c. m., let us see what heuristic arguments we can give for the vanishing of $u_\ell(r)$ when $r < r_0$ and also to estimate r_0. Consider two nucleons approaching each other. For low energies and large distances they interact via local energy independent potentials. By the uncertainty principle alone, when the nucleons approach each other they produce virtual states containing meson pairs. The condition that we have a pion pair is, using the relativistic limit $v=c=1=\hbar$,

$$\Delta p \; \Delta x = \Delta E \; \Delta x = 2\mu_\pi \; \Delta x = 1 \quad . \qquad \text{(IV. 3)}$$

Therefore, the largest distance at which inelastic virtual channels enter is given by

$$r_1 \cdot 2\mu_\pi = 1 \; ; \qquad r_1 = \frac{1}{2}\mu_\pi^{-1} \tag{IV.4}$$

where μ_π is the pion mass and μ_π^{-1} the corresponding Compton wave length. (The creation and absorbtion of a single pion presumably is included in the potential, also virtual states of one pion can't be created out of the vacuum by the uncertainty principle energy of the two nucleons.) Physically the onset of inelasticity at short distances can be expected to rapidly diminish the incident channel amplitude in the interior due to the opening of inelastic channels. Assuming that the incident channel is destroyed over a short distance the integrals on the right side of (IV. 2) become small and df_ℓ/dE is then also small so we approach the b.c.m.[3] This means that $r_0 = r_1 \sim \frac{1}{2}\mu_\pi^{-1} =$ .7F which is considerably larger than the core in models which assume that the short range repulsion is due to vector meson exchange. In these models (e.g., Hamada-Johnston[8]) $r_0 = .4 - .5$F which is considerably smaller than the b.c.m. r_0.

At this point we should comment that the b.c.m. does not say that there is no hadronic matter when r is less than r_0, but rather, that the incident channel is destroyed by the opening of competing ones. This means that scattering processes in which we are interested in the elastic phase shift, binding energy of a bound state, and the like are expected to be well described. This may well include electric form factors which are expected to be relatively insensitive to virtual meson pairs, at least for low q^2, by Siegert's[9] theorem. On the other hand, magnetic form factors and matrix elements of the axial current could be quite sensitive to hadronic matter in the core region. For example, the neutron-proton four point weak interaction is usually considered to be suppressed due to the presence of the core in the two nucleon interaction.[10] The next diagram that contributes goes through a strange intermediate state and hence gives a rate that is proportional to $\sin^4 \theta_c$ for parity nonallowed electromagnetic nuclear transitions, as these arise because the nuclear levels are not eigenstates of parity in the presence of weak interactions. Unfortunately these rates are well given by theory except for the factor $\sin^4 \theta_c$ which seems not to be present. If there is considerable hadronic matter in the core which couples to the weak interaction, then the boundary matrix $\underline{F}$ ($\underline{F}$ generalizes the f_ℓ of (IV. 1)) will contain parity violating components $\sim G_W$ and there will be no $\sin^4 \theta_c$ in the rate.

Another case where there is ambiguity is in the predictions of magnetic form factors, for much the same reason, and we will not discuss these in our three body generalization of the b. c. m. Finally, the proof that $df_\ell/dE = 0$ gives us a vanishing interior wave function is valid for the Schrodinger equation and not for the Bethe-Goldstone equation[4] thus there may be an interior wave function in the off-shell case which occurs in systems of three or more strongly interacting particles.[11] Physically this arises because the off-shell nucleon contains virtual nucleon plus meson excitation states; these latter are just the states that compete with the two nucleon channel in the interior region. This considerably weakens the effect of the destruction of the normalization of the incident two nucleon channel in the interior, when the nucleons are off-shell. Thus the b. c. m. becomes ambiguous here (see Chapter V).

We end this section by mentioning a static equivalent to the b. c. m. which gives the same phase shift in a partial wave. Kim and Tubis[12] have shown that the partial wave phase shifts computed from the b. c. m. are the same as those from a potential,

$$V(r) = g_\ell \theta(r_0 - r) + g'_\ell r_0 \, \delta(r - r_0)$$
$$g_\ell > 0 \qquad g'_\ell < 0 \tag{IV.5}$$

in the limit: $|g_\ell|$, $|g'_\ell| \to \infty$ with the condition: $f_\ell / r_0 = \sqrt{g_\ell} + r_0 g'_\ell - 1/r_0$. This shows that the b. c. m. is the limit of a repulsive core with a strong singular attraction directly outside it. We can even make the attraction more singular by noting that since, $u(r_0) \sim u'(r_0)$ in the b. c. m. we can replace $\delta(r-r_0)$ by a term proportional to $\delta'(r-r_0)$ in (IV.5). We will look at the effects of these two types of singularities in the three body problem in the next chapter.

IV. 2 Energy Dependence and Dispersive Properties of the B. C. M.

The usual potentials that one takes in nonrelativistic potential scattering are energy independent. While for the b. c. m. the interior potential that produces an energy independent f_ℓ is $V(r) + E$.[3] As E increases this potential becomes strongly repulsive and obviously has a large energy dependence.

The energy dependence of the amplitude and phase shift for the b. c. m. is markedly different than that of potential models. We follow the lead of Feshbach and Lomon[3] here. The normalization of the amplitude in terms of the S-matrix and phase shift will be:

$$A_\ell(s) = \frac{1}{2ik} (k^2 + M^2)^{1/2} (S_\ell - 1)$$

or (IV.6)

$$A_\ell(s) = \frac{1}{k} (k^2 + M^2)^{1/2} \sin \delta_\ell(s) \cdot e^{i\delta_\ell(s)}$$

where $s = k^2$.

At infinite E or s in potential scattering, $A_\ell(s) \rightarrow V_\ell$ where V_ℓ is the matrix element of the potential in momentum space in the ℓth partial wave. Here, we expect from the above, $A_\ell \rightarrow E$ and $\delta_\ell(s) \rightarrow -kr_0 + \text{const}$ as $k \rightarrow \infty$, directly from the model, as the relation (IV.6) is not well defined in this limit due to the oscillatory nature of sin $\delta_\ell(s)$. From the physical input to the b.c.m. however, it is expected that as s increases resonances are excited in the interior which in Ref. 3 are shown to lead energy dependence in f_ℓ like $f_\ell = a/(E-E_{res})$. With further increase in E it is possible to create states other than the incident two nucleon channel which can exist at $r = \infty$. The result of this is to shrink the r_0 at which incident wave is destroyed by competing states which do not exist asymptotically. Equivalently keeping the same $r_0 \approx \frac{1}{2}\, 2u_\pi^{-1}$ the f_ℓ's acquire an energy dependence. In fact, with the condition for a dispersion relation to exist $S_\ell \rightarrow 1$ as $s \rightarrow \infty$ we get $f_\ell + 1 \rightarrow -kr_0 \tan kr_0$ and $\delta_\ell(k) \rightarrow -C/s$ after having become negative with $-kr_0$ for low k as k increased.

For moderate k^2 it can be proved from the Wigner condition $(df_\ell/dE \leq 0)$ that both $\partial\delta_\ell(s)/\partial s$ and $\partial(\text{Re}\, A_\ell)/\partial s$ are smaller for $df_\ell/dE = 0$ than for $df_\ell/dE < 0$ (the condition on Re A_ℓ holds for $s < 0$ and $|n\pi - \delta_\ell(s)| < \pi/4$ where n is the number of bound states). Since for models with short range repulsion $\partial\delta_\ell/\partial s$ or $\partial\text{Re}\, A_\ell/\partial s < 0$ (as s increases) we have that $\delta_\ell(s)$ or Re $A_\ell(s)$ varies more rapidly for the b.c.m. than for a potential model for the interior.

Brayshaw[13] has shown that for $\lim_{s \rightarrow \infty} A_\ell(s)/s < \infty$ an off-shell partial wave b.c.m. t-matrix can be defined from (IV.5) which has correct analyticity properties in the s plane. The pure b.c.m. t-matrix only has a right hand, or unitarity, cut extending from zero to infinity. This is in contrast to potential scattering where the partial wave amplitudes have a left hand cut running from $-\mu_\pi^2$ to $-\infty$ (in the case of OPE potentials) corresponding to physical intermediate states (mesons) in the t channel or, more generally, to the singularities of the momentum space potential.

We can now compare the b. c. m. with the Mandelstam representation (which for simplicity we take here for $\pi - \pi$ scattering quantum numbers);

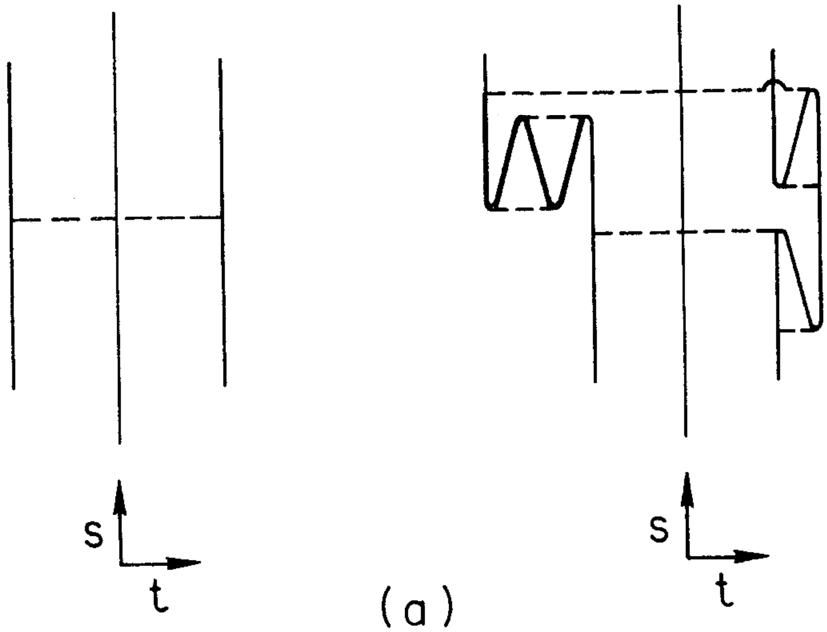
$$A(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \int_{4\mu^2}^{\infty} ds' \; \rho(s',t')/(s'-s)\,(t'-t)$$

$$+ \text{poles} + \text{single integrals} \qquad \text{(IV. 7)}$$

The strip approximation[14] relates the s and t dependence of (IV. 7) to the inclusion of certain intermediate states in the double dispersion integral. The α strip arises from states with low t' and large s'. Clearly this is the structure of a single boson exchange diagram in the t channel. Chew and Frautschi[15] have shown that, in this case, the effective relativistic elastic potential is given by

$$V^{(\alpha)}(s,t) = -\frac{1}{\pi}\int_{>16\mu^2} \frac{\rho(s',t)\,ds'}{(s'-s)} \qquad 4\mu^2 < t < 16\mu^2 \qquad \text{(IV. 8)}$$

Now the α strip is formally the region $s^\alpha > 16\mu^2$, $4\mu^2 < t' < 16\mu^2$ while the β strip is $4\mu^2 < s' < 16\mu^2$; $t' > 16\mu^2$. So this effective potential comes from the α strip and has as weak an energy dependence as possible (Fig. IV. 1a). Feshbach and Lomon[3] argue that the difference between the iterated form of (IV. 8) and the amplitude arising from the box diagram (Fig. IV. 1b) is not totally cancelled by the effects of higher order corrections. Since such diagrams have a cut starting at threshold ($s = 4\mu^2$) in the s channel, it follows that they can contribute to the amplitude through the β strip. This will result



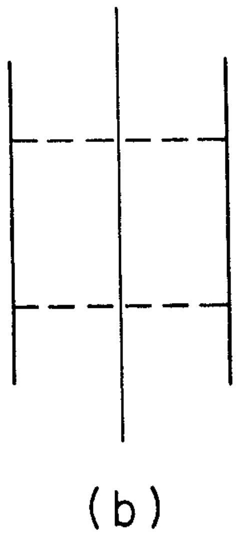


FIG. IV.1--(a) Potential like diagrams in the α strip.
(b) A diagram with some β strip contribution.

in a "potential"

$$V^{(\beta)}(s,t) = -\frac{1}{\pi}\, P\int_{4\mu^2}^{16\mu^2} \frac{ds'}{s'-s}\, \sigma_{st}(s',t) \qquad t > 16\mu^2 \tag{IV.9}$$

$\sigma_{st}(s',t)$ is defined as $\rho(s',t) - \Delta(s',t)$ where $\Delta(s',t)$ gives the iteration of $V^{(\alpha)}$ and the cancellation effects above. $V^{(\beta)}$ has a rapid dependence on s and if, as the authors of Ref. 3 assume, $\sigma_{st}(s',t)$ is positive definite over the range of the variables in (IV.9) then we have $V^{(\beta)}(s,t)$ becomes more repulsive as s increases from $4\mu^2$ and then finally approaches zero from above. This is precisely the behavior of the b.c.m. modified to account for inelasticity at high energy. This means that dispersion theory can give us an idea of what sort of intermediate states contribute to the b.c.m. and also, the condition of two pion exchange predicts $r_0 \approx \frac{1}{2}\mu_\pi^{-1}$, as before.

Now let $s = +4(k^2+M^2)$ (k is the wave number). Then for nucleon-nucleon scattering we can write the partial wave dispersion relation for angular momentum ℓ (no bound states):

$$A_\ell(k^2) = \frac{1}{\pi}\int_0^\infty \frac{\text{Im}\, A_\ell(v)dv}{v-k^2-i\epsilon} + \frac{1}{\pi}\int_\infty^{\mu^2} \frac{h_\ell(v)dv}{v-k^2} \tag{IV.10}$$

Again, the β strip which we take as $0 < v < 3M^2$ in the integral on the right hand cut and as $-v > 16\mu^2$ on the left hand cut, gives the correct behavior of $\delta_\ell(k^2)$ for the b.c.m. This includes $k^2 \to \infty$ providing we can ignore $h_\ell(v)$ when k^2 is large or, $\int h_\ell(v)\,dv < \int \text{Im}\, A_\ell(v)dv$ this gives $\delta_\ell \to -C/v$, $v \to \infty$ as expected. It is the β strip contribution to the integral over the right hand cut that gives the rapid variation in energy. In contrast, potential scattering achieves its energy variation from the left hand cut and its iterations. As $t=-2k^2(1-\cos\theta)$, the rapid variation of the potential with t (α strip) means that on integrating over $P_\ell(\cos\theta)\, d\cos(\theta)$ to get A_ℓ, there will be rapid variation of A_ℓ with k^2 from the left hand cut, in the potential scattering case, but slow variation from the right hand cut.

What we have done in this section is to compare the energy dependence of the b. c. m. amplitude and phase shifts with that arising from certain integrals in the Mandelstam representation. Their similarity is an argument for the b. c. m. for nucleon-nucleon scattering with $r_0 \approx \frac{1}{2}\mu_\pi^{-1}$ but it is not a derivation.

IV. 3 Extension to the Three Nucleon Problem

In the previous sections we described the effective interaction when two nucleons were brought very near each other. In this section it will be argued that very similar, but explicitly, three body, behavior can be expected when three nucleons come near each other. The key to our analysis will be the K harmonic method of Chapter III. We assume that the physical system or that part of the physical system that we are considering has the properties outlined in Chapter III so that the wave function is dominated by the first few hyperspherical harmonics and, that for a given K the potentials $V_{[K],[K']}(\rho)$ are limited to a few nearby values of K'. The reason for this is, first, that we suppress states that do not, on the average, have all three particles in a relatively symmetric configuration. This means that the energy available for coupling to virtual states will depend on the scale ρ. Second, in each state of given K the Schrodinger equation (III. 72) formally corresponds to a two body interaction with "orbital" angular momentum ℓ =K+3/2. In each partial wave the potentials can be expected to fall-off faster than $1/\rho$ (by integrating two body Yukawas over $d\Omega_6$ we can see this). This will allow us to take over formal results for two body partial wave scattering from sections IV. 1 and IV. 2

We immediately have that the relation between the condition $df_{\{K\}}/dE = 0$ at $\rho=\rho_0$ and vanishing of $\chi_{[K]}(\rho)$ for $\rho < \rho_0 - \epsilon$ holds provided $\partial V_{[K],[K']}/\partial E \leq 0$ for K' coupled to K (where $\rho_0 \frac{d\chi(\rho)}{d\rho} |\chi(\rho_0)|^{-1}_{\rho=\rho_0} \equiv f_{[K]} + 2$). In Appendix IV. A we show that this is the case for a truncated series of K harmonics in the three body wave function.[16]

Now, following the lead of section IV. 1, we will give heuristic arguments that there exists a ρ_0 inside of which the low K wave functions are expected to vanish. Consider three nucleons in an equilateral triangle then the side of the triangle = ρ. Now from the arguments in section IV. 1 if ρ were as small as $\frac{1}{2}\mu_\pi^{-1}$ then a core would be expected to develop because of the opening of competing channels due to the pair interaction along the legs. However, if one of the legs is fixed at a distance $r_{12} > \frac{1}{2}\mu_\pi^{-1}$, and the third particle is brought in, there will be enough energy from the uncertainty principle to create virtual pion pairs due to the localization of the third particle relative to the other two. Physically this means that we should get a three body boundary condition at a hyperradius somewhat larger than $\frac{1}{2}\mu_\pi^{-1} = .7$ F.

We required the relativistic form of the uncertainty principle in order to obtain the two body boundary radius. The corresponding generalization in each K harmonic wave is not clear. An extreme view is that ρ corresponds to two particle degrees of freedom in the c. m. while r_{12} corresponds to one. Then we have $\Delta E_\rho = \frac{2}{\Delta\rho}$ or $\frac{2}{\Delta\rho} = 2\mu_\pi$ or $\rho_0 \sim \mu_\pi^{-1} = 1.4$ F. So we expect $\rho_0 \leq 1.4$ F. Later we will estimate ρ_0 in a different way.

Since the individual K harmonics can not have a two body bound state, and we can define outgoing waves in ρ, we can use (III. 72) to postulate the existence of three body dispersion relations in the partial K waves

$$A_K(E) = \frac{1}{\pi}\int_0^{\infty} \frac{d\nu \, \text{Im}\, A_K(\nu)\, d\nu}{\nu - E - i\epsilon} + \frac{1}{\pi}\int_{-\infty}^{-\nu'} \frac{d\nu \, h_K(\nu)}{\nu - E} \qquad \text{(IV. 11)}$$

(see (IV. 10)) the value of ν' depends on the range of the potentials (for OPE pair potentials we get $\nu' = 2\mu_\pi^2$. This is derived in Appendix IV. B.). If the scattering process is $a+b+c \to a'+b'+c'$ then it is clear that, if we postulate a Mandelstam representation (with no two body bound states) that the two particle cuts with $k_{ij}^2 > 0$ contribute to the right hand cut. While diagrams with physical states in a crossed channel $\bar{a}'+a+b+c \to b'+c'$ contribute to the left hand cut.[17] (See Appendix IV. B again.) Again the three body b. c. m. has only the right hand cut and the partial K wave amplitudes have the same properties as the partial wave amplitudes for the two body b. c. m. described in section two of this chapter.

The lowest order three body potential is simply the sum of the channel OPE's $\sum_{i<j} V_{ij}(r_i - r_j) = V_{123}(\xi)$ (Fig. IV. 2a). This has a high threshold on the right hand cut and correspondingly slow energy variation. Again we assume that the iteration of the equivalent potential from OPE in each of two channels is not the same as the iteration term $V_{ij}G_0(E)V_{jk}$ where $G_0(E)$ is the three particle free Green's function (Fig. IV. 2b). Then Fig. IV. 2b will contain a piece that has $\nu = 0$ threshold in the right hand cut. Just as in section IV. 2 this leads to an energy dependence in a K wave amplitude similar to that of the three body b. c. m. Since the range $r_i - r_j \sim \mu_\pi^{-1}$ and $\sqrt{2}\rho \geq r_i - r_j$, by definition of ρ, we obtain

$$\sqrt{2}\rho > r_i - r_j = \mu_\pi^{-1} = 1.4 \text{ F} \quad \therefore \quad \rho_0 \geq 1 \text{ F} \qquad \text{(IV. 12)}$$

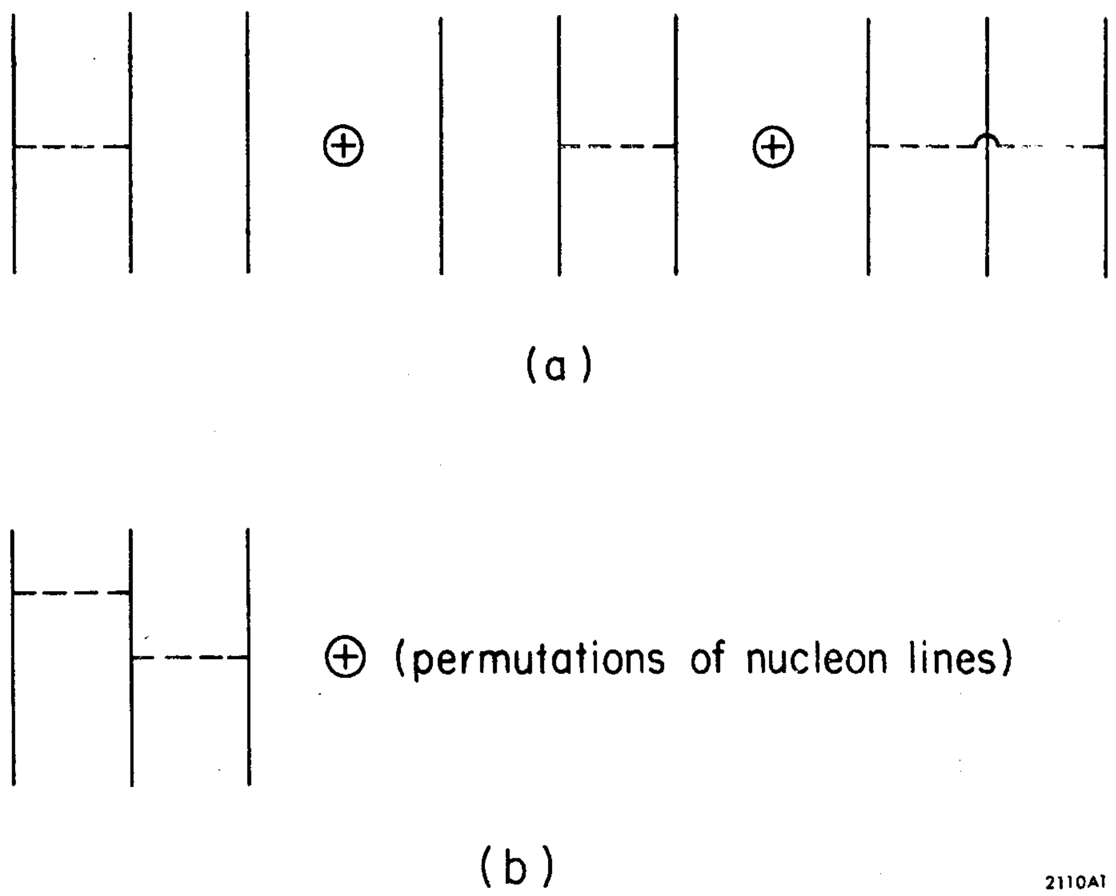


FIG. IV. 2--(a) Diagrams that contribute to $V_{00}(\rho)$.
(b) A diagram with energy dependence like the three body b. c. m.

Combining this with the estimate of ρ_0 from the uncertainty principle, we conclude that the arguments that led us to a two body b. c. m. also apply in the case of the lowest order hyperharmonic components of the three nucleon wave function, to suggest a three body b. c. m. with $1\ F < \rho_0 < 1.4\ F$. We note that since $\langle\rho^2\rangle_{av} = \frac{1}{3}\langle r^2\rangle_{av}$ for the three body wave function if $\rho_0 = 1.4$ F the core is at radius .82 F (or at .58 F if $\rho_0 = 1$ F). This compares with $R_{ch} = 1.68$ F for the ^{3}He body form factor, the core is thus seen to be well within the system.

We should also repeat that our comments in section IV. 2 about the physical meaning of the core region still hold for the three body b. c. m. However for the low K harmonic states our incident and final channel is three on-shell nucleons, we suppress terms where a pair and the third nucleon are far off shell. In the case of the three body bound state the off-shell behavior is symmetrized among the nucleons so it is of order E_B in the "incident" three nucleon state and is small compared to the nucleon mass. Therefore we don't expect the ambiguities mentioned above to be important in the hyperradial b. c. m.

IV. 4 The External Potential in the B. C. M.

In the discussion of the b. c. m. it has been mentioned that there is a long range local potential arising from the strip. The actual procedure for obtaining these forces from (IV. 8) is based on the work of Charap and Fubini.[18] The values of the meson nucleon coupling constants are adjustable parameters in $\rho(s,t)$ and hence in the potential. Lomon and Feshbach[5] then take π, ρ, ω, and η mesons and compute a potential which turns out to be

$$V = V_\pi + V_s + V_\omega + V_\eta + V_{2\pi}$$

$$V_\pi = g^2/12\ (\mu\ / M_N)^2\ \sigma_1 \cdot \sigma_2 \left[\sigma_1 \cdot \sigma_2 + S_{12}\left(1+3/\mu r + 3/(\mu r)^2\right)\right] \epsilon^{-\mu r}/r$$

$$V_\rho = 3/2\ N^2 (\sigma_1 \sigma_2) \left[1 + (1+2g_v)^2/12 (m_\rho/M_N)^2 \left\{2\sigma_1 \cdot \sigma_2 - \left(1 + \frac{3}{m_\rho r} + \frac{3}{(m_\rho r)^2}\right) S_{12}\right\}\right] \epsilon^{-m_\rho r}/r$$

$$V_\omega = 9/4\ (N')^2 \left[1 + (1+2g_v)^2/12\ (m_\omega/M_N)^2 \left\{2\sigma_1 \cdot \sigma_2 - \left(1 + \frac{3}{m_\omega r} + \frac{3}{(m_\omega r)^2}\right) S_{12}\right\}\right] \epsilon^{-m_\omega r}/r \qquad \text{(IV. 13)}$$

$$V_\eta = g_\eta^2/\ 12\ (m_\eta/ M_N)^2 \left[\sigma_1 \cdot \sigma_2 + S_{12}\left(1 + \frac{3}{m_\eta r} + \frac{3}{(m_\eta r)^2}\right)\right] \epsilon^{-m_\eta r}/r$$

For $V_{2\pi}$ we refer to Ref. 3 . In (IV. 12) $\mu = m_\pi$, σ_i and τ_i are the spin and isospin spinors for nucleon i. S_{12} is the tensor operator: $S_{12} = \frac{1}{3r^2}\left[3(\sigma_1 \cdot r)\ (\sigma_2 \cdot r) - r^2 \sigma_1 \cdot \sigma_2\right]$. The meson masses and the couplings g^2, N^2, $N2$, g_v, and g_s are regarded as free parameters although they should be consistent with experimental values determined from direct meson nucleon data (the meson masses can vary with the charge state as they do experimentally). The best fit to the S and P wave N-N phase shifts that Lomon and Feshbach obtain has $N^2 = .65$, m_ρ = 765.0 MeV, $g_v = 1.87$, m_ω = 782.8 MeV, $g_s = 0.06$, m_η = 548.7 MeV, $g_\eta^2 = 1$ taken from experiment then μ_π ranges from 135 to 139 MeV depending on the reaction. Finally $r_0 = .51373\ \mu_\pi^{-1}$ quite close to the expected value. These parameters give excellent fits to scattering in higher partial waves and polarization in N-N scattering at various energies. This hold provided the two pion potential includes such diagrams as IV.3a and IV.3b where a factor of λ is inserted

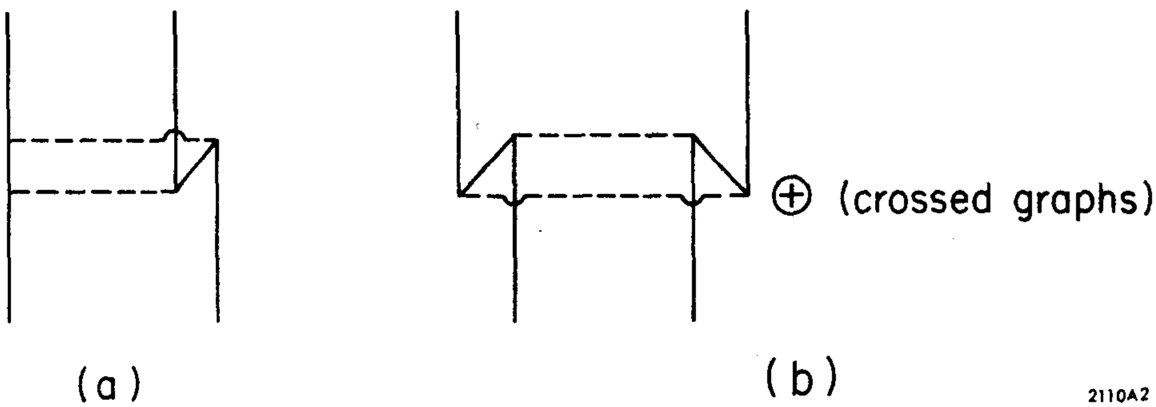


FIG. IV. 3--(a) Single nucleon pair diagram in $V_{2\pi}$.
(b) Double nucleon pair diagram in $V_{2\pi}$.

for each $N\bar{N}$ pair in the intermediate s channel state. Lomon and Feshbach find the best fit is with $\lambda = .934 \neq 1$. It is interesting to compare the value of r_0 with that which best fits the low ℓ phase shifts with no external potentials. For P waves and higher $r_0 = 1.32$ F,[2] for the $^1S_0$ interaction $r_0 = 1.32 \exp(-.03E^{\frac{1}{2}})$ F or, if r_0 is constant, $r_0 = 1.095$ F. The reason for this is that the singular attraction in the b. c. m. moves out to compensate for the absence of the long range OPE potential.

In fact, the two pion potential is ambiguous, for example, it may be necessary to include σ meson exchange also there are phenomenological parameters present not derived from field theory. In the three body problem diagrams such as Fig. IV.4a and IV.4b should be comparable to the two pion potential terms but are explicitly three body effects. A final comment is that we should note that the potential V is attractive in the T=1, S=0 state and less strong but repulsive in the T=0, S=1 state for $r \gtrsim r_0$.[5] This means that our results could be altered by our inability include the two body core in the K harmonic method (see Chapter V). (This attraction is lessened by decreasing the ρ contribution relative to the ω in the potential. This is the case when the higher partial waves are fit with π exchange and the lower ones with successively heavier meson exchange. The reason for this is that the potential in this method is from the α strip with no core effects from the β strip.)

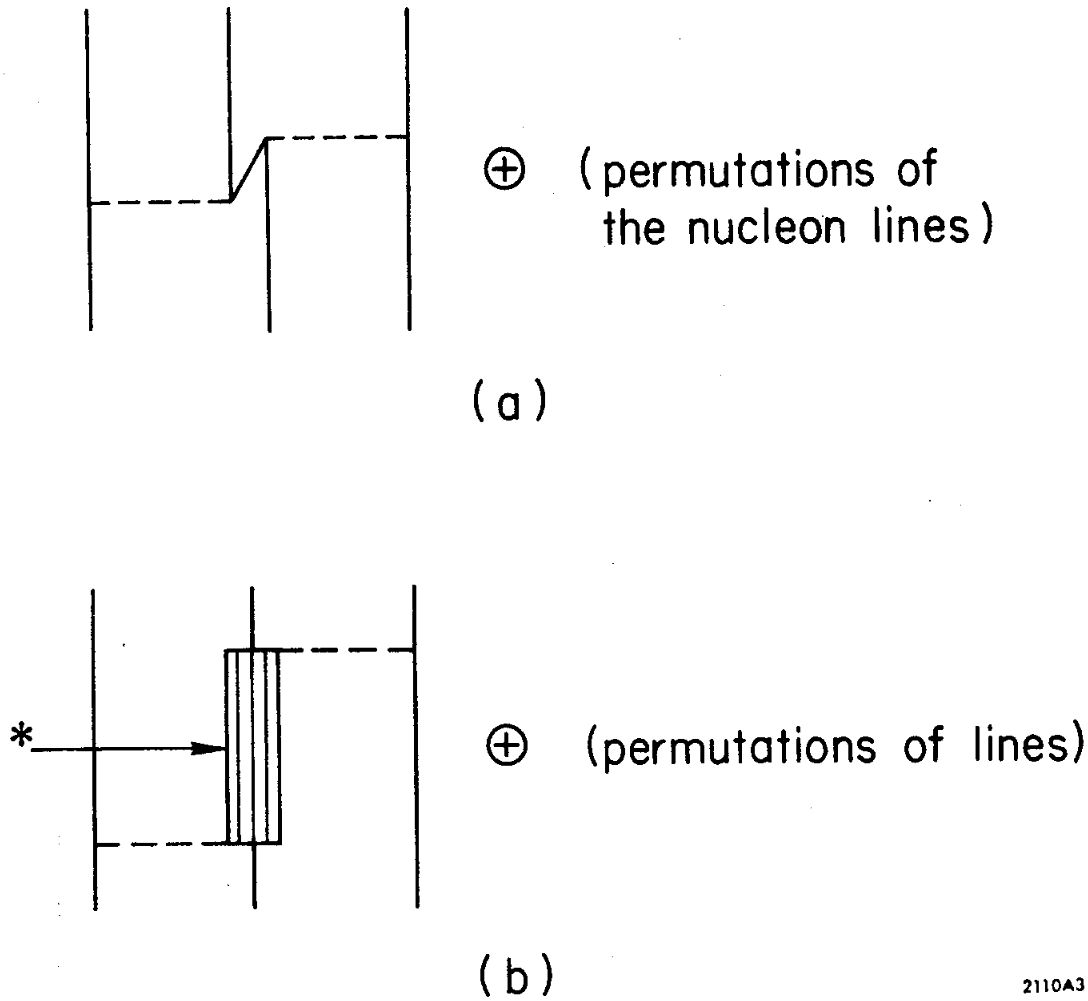


FIG. IV.4--(a) Diagram that can contribute three body forces of the order of Fig. 3a and Fig. 3b.
(b) Three body force via resonance excitation.

APPENDIX IV.A

In this appendix we will investigate the energy dependence of the potential $V_{123}(\vec{\xi})$ defined in Chapter III. We will closely follow the development of Newton[16] for the two body case. First, assume that there are no two body bound states. Additionally assume that we can define a three body fragment channel by considering only a limited number of hyperharmonics. More precisely, we assume that we are in a physical situation such as three body bound states, clusters or the interior of an interior-exterior separation. Then in a many body theory the total Hamiltonian H is written as[16]:

$$H = H_{3N} + H'_{3N} \qquad \text{(IV.A-1)}$$

where $H'_{3N} \to 0$ as $\rho \to \infty$ and H_{3N} describes the channel composed of three nucleons. Next we define projection operators P_{3N} and P_β such that:

$$P_{3N} + P_\beta = \text{"1"} \qquad \text{(IV.A-2)}$$

where P_β is a projection operator onto all channels except the three nucleon one and we have assumed no transitions to states of high K. P_{3N} projects onto three nucleon states. Transition operators are defined as:

$$V_{3N\beta} = P_{3N}\, H'_{3N}\, P_\beta \qquad \text{(IV.A-3)}$$

The channel Hamiltonians are:

$$\begin{aligned} H_{3N} &= P_{3N}\, H P_{3N} \\ H_\beta &= P_\beta\, H P_\beta \end{aligned} \qquad \text{(IV.A-4)}$$

Define the Green's functions with outgoing waves (in ρ) as:

$$\begin{aligned} (\mathscr{E}_{3N} - H_{3N})\, G(\mathscr{E}_{3N}) &= P_{3N} \\ (\mathscr{E}_\beta - H_\beta)\, G_\beta(\mathscr{E}_\beta) &= P_\beta \end{aligned} \qquad \text{(IV.A-5)}$$

Then since $P_{3N}P_\beta = P_\beta P_{3N} = 0$ we have:

$$G(E) \equiv G_{3N}(E) + G_\beta (E) = (E-H_{3N}-H_\beta)^{-1} \qquad \text{(IV.A-6)}$$

The full Green's function $g(E) \equiv (E-H)^{-1}$ is then:

$$g = G+G(V_{3N\beta} + V_{\beta\, 3N})g \qquad \text{(IV.A-7a)}$$

and

$$P_{3N}g = G_{3N} + G_{3N}V_{3N\beta}P_\beta\, g \qquad \text{(IV.A-7b)}$$

$$P_\beta\, g = G \quad G_\beta\, V_{\beta 3N}P_{3N}g \qquad \text{(IV.A-7c)}$$

where we have used $V_{3N\beta} = V_{3N\beta}P$, say. Equations (IV.5) and (IV.7) can be combined to yield

$$(\mathscr{E}_{3N} - H_{3N} - V_{3N\beta}\, G_\beta\, V_{\beta 3N})\, P_{3N}g = P_{3N} + V_{3N\beta}\, G_\beta \qquad \text{(IV.A-8)}$$

This suggests that the total effective Hamiltonian for channel 3N is

$$\mathscr{H}_{3N}(\mathscr{E}_\beta) = H_{3N} + V_{3N\beta}G_\beta(E_\beta)\, V_{\beta 3N} \qquad \text{(IV.A-9)}$$

whence the total Green's function in this channel is (using IV.A-7)

$$(\mathscr{E}_{3N}-\mathscr{H}_{3N}(\mathscr{E}_\beta))g_{3N} = P_{3N} \qquad \text{(IV.A-10)}$$

with

$$g_{3N} = P_{3N}g\, P_{3N} = g_{3N}(\mathscr{E}_\beta) \qquad \text{(IV.A-11)}$$

Thus from (IV.A-9) and (IV.A-11) we see that the energy dependence of the effective potential in 3N-3N scattering (in the lowest K harmonics) arises from virtual transitions to many body channels, β. Finally setting $E_{3N} = E_\beta = E$ we have, using the outgoing wave boundary condition:

$$\mathscr{H}_{3N}(E) = H_{3N} + V_{3N\beta}\mathscr{P}(E-H_\beta)^{-1}V_{\beta 3N} - i\pi\, V_{3N\beta}\, \delta(E-H_\beta)\, V_{\beta 3N}$$

(IV.A-12)

The $\mathscr{P}$ in (IV. A-12) refers to a principle value not a projection operator. Since by hermiticity $V_{3N\beta} V_{\beta 3N} \geq 0$ we get (for E below the threshold for channel β) $\partial \mathscr{H}_{3N}(E)/\partial E \leq 0$. Since H_{3N} is the three nucleon kinetic energy we get our desired result as $\partial H_{3N}/\partial E = 0$. If there are two body bound states they give cuts in (IV. A-12) starting when $E = -E_B$ where E_B is the two body binding energy. However, the transition to these bound states is equivalent to being in a situation where we can not impose outgoing boundary conditions in ρ, but we assumed that these situations differ physically from those that we are considering here.

APPENDIX IV. B

For potential scattering v' is the position of the singularity in k^2 of $V_{00}(k)$ where:

$$V_{00}(\rho) \propto \int k^2 dk\, J_2(k\rho)/(k\rho)^2\, V_{00}(k) \qquad \text{(IV.B-1)}$$

but

$$V_{00}(\rho) \propto \int d\Omega_6\, V(r_{12}) \qquad \text{(IV.B-2)}$$

and

$$V(r_{12}) \propto \int d^3q/(q^2+u^2)\, e^{i\vec{q}\cdot\vec{r}_{12}} \qquad \text{(IV.B-3)}$$

For OPE potentials using $r_{12} \equiv \sqrt{2}\,\rho\,\cos\,\theta$ and inserting (IV.B-3) in (IV.B-2) we get

$$V_{00}(\rho) \propto \int d^3q/(q^2+u^2)\; e^{iq\sqrt{2}\,\rho\,\cos\theta\,\cos(\hat{q}\cdot\hat{r}_{12})}\, d\Omega_6 \qquad \text{(IV.B-4)}$$

but letting $\sqrt{2}\,q = k$ (IV.B-4) gives:

$$V_{00}(\rho) \propto \int J_2(k\rho)/(k\rho)^2\, \frac{k^2 dk}{k^2+2\mu^2} \qquad \text{(IV.B-5)}$$

From (IV.B-5) we obtain the threshold if the left hand cut

$$v' = 2\mu_\pi^2 \qquad \text{(IV.B-6)}$$

This estimate is for the longest range OPE pair potentials.

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CHAPTER V

RESULTS

In this chapter the theoretical considerations of Chapter IV will be applied to the development of the bound state wave function in K harmonics.

The plan of the chapter is to, first, discuss the lowest order approximation; that the wave function is entirely composed of the principle S state and that that state is given by the lowest (K = 0) hyperharmonic. A boundary condition will be imposed on this leading hyperharmonic and the resultant properties of the ^3He electric form factor, in consequence of the variation of the input parameters of the model, will be discussed. The inclusion of S' and D waves will be considered and then the inclusion of external meson exchange potentials. After which comparisons will be made with other models. Finally, for dessert, we will give some possible extension to systems other than the trinucleon bound state.

V. 1 The Pure Three Body Boundary Condition Model

We consider only the L = 0, K = 0 fully spatially symmetric state. The trinucleon binding energy is a fixed input parameter $E_B = 8.5$ MeV or equivalently $\chi = \sqrt{(2M/\hbar^2)E_B} = .636F^{-1}$. The equation for the wave function $\chi_0^o(\rho)$ becomes from (III. 72):

$$\frac{d^2\chi_0^o}{d\rho^2} + \frac{1}{\rho}\frac{d\chi_0^o}{d\rho} - \left(\chi^2 + \frac{4}{\rho^2}\right)\chi_0^o = 0 \qquad \rho>\rho_0 \qquad \text{(V. 1)}$$

$$\chi_0^o(\rho) = 0, \quad \rho<\rho_0$$

The solution to V. 1 which is regular at $\rho = \infty$ is:

$$\chi_0^o(\rho) = CK_2(\chi\rho) \qquad \rho>\rho_0 \tag{V. 2}$$

$$\chi_0^o(\rho) = 0 \qquad \rho>\rho_0$$

Here ρ_0, the boundary hyperradius is a free parameter and χ is fixed. For a given ρ_0 the fixing of χ is obviously equivalent to the fixing of the logarithmic derivative

$$\left(\rho_0 \left.\frac{d\chi_0(\rho)}{d\rho}\right|_{\rho=\rho_0}\right)\left(\frac{1}{\chi_0^o(\rho_0)}\right)$$

The normalization condition on $\chi_0^o(\rho)$ gives:

$$C^{-2} = \int_{\rho_0}^{\infty} \rho d\rho \left| K_2(\chi\rho)\right|^2 \tag{V.3}$$

From (III. 95) the ^{3}He electric form factor is simply:

$$F_{e\ell}^{^3He}(q^2) = C^2/4(3F_S(q^2) + F_V(q^2)) \; 8\int_{\rho_0}^{\infty} \rho d\rho \left|K_2(\chi\rho)\right|^2 J_2(a)/a^2 \tag{V. 4}$$

$$a = \sqrt{2/3}\; q\rho$$

F_S and F_V are calculated from the analytic expressions quoted by Gasiorowicz (although any fit to the data will work here).

In Fig. V. 1 $\left|F_{e\ell}^{^3He}(q^2)\right|$ is plotted for a range of ρ_0's. The behavior of the form factor is reasonably close to a Gaussian for small q^2; it shows a sharp passage through zero and then a large secondary maximum. At very high q^2 this oscillary behavior continues with the maxima decreasing by about

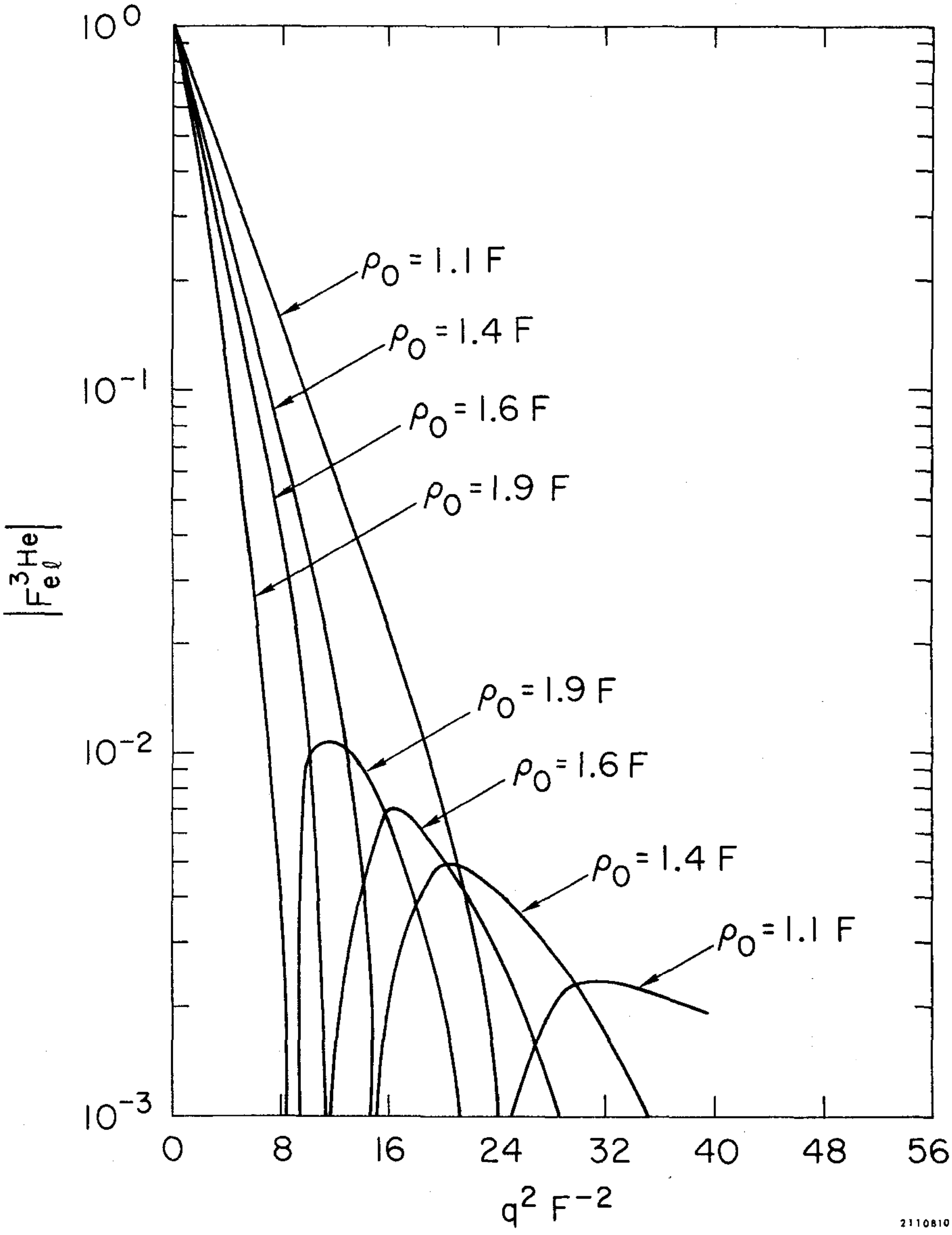

FIG. V.1--The ^{3}He electric form factor for the pure three body b.c.m. K=0 state for a selection of ρ_0's.

one order of magnitude from peak to peak. As ρ_0 increases:

(i) The position of the zero in q^2 decreases.

(ii) The size of the secondary maximum increases.

(iii) The width of the secondary maximum increases.

In Figure V. 2 we have plotted the two most striking features of the observed ^{3}He electric form factor, from the point of view of the difficulties of two body potential models; the height of the secondary maximum and the position of zero against each other, as parametrized by ρ_0 . The experimental point is taken as $q^2_{dip} = 11.6 \pm .2F^{-2}$ and $|F^{^3He}_{e\ell}|_{max}$ $6.5 \pm .25 \times 10^{-3}$.[1] The position of this maximum is at about $q^2 = 16.25\ F^{-2}$. However there is an ambiguity in the data reported by McCarthy et. al.[1] Their experimental points show a spoike at $q^2 = 14\ F^{-2}$ with magnitude 8×10^{-3}. The data point at $q^2 = 13\ F^{-2}$ is consistent with this. McCarthy et. al. have chosen to de-emphasize these points as have most of the fits quoted in theoretical derivations of the form factor. We have dealt with this problem by extending the $|F^{^3He}_{e\ell}|_{max}$ for the experimental point to include a range centered about 8×10^{-3}. (See note V. 1.)

Figure V. 3 shows our fit to the data. We have adjusted the boundary hyperradius to fit the position of the experimental zero in q^2. The charge radius is somewhat too small $R_{ch} = 1.55$ F compared with the experimental value of $R_{ch} = 1.88 \pm .5$ F.[4] There is a "bulge" in the calculated form factor near $q^2 = 4$ to $q^2 = 8\ F^{-2}$. With a maximum discrepancy from experiment of a factor of about 1.8. On the other hand, the basic shape over several orders of magnitude of the form factor is in rough agreement with the experimentally observed electric form factor.

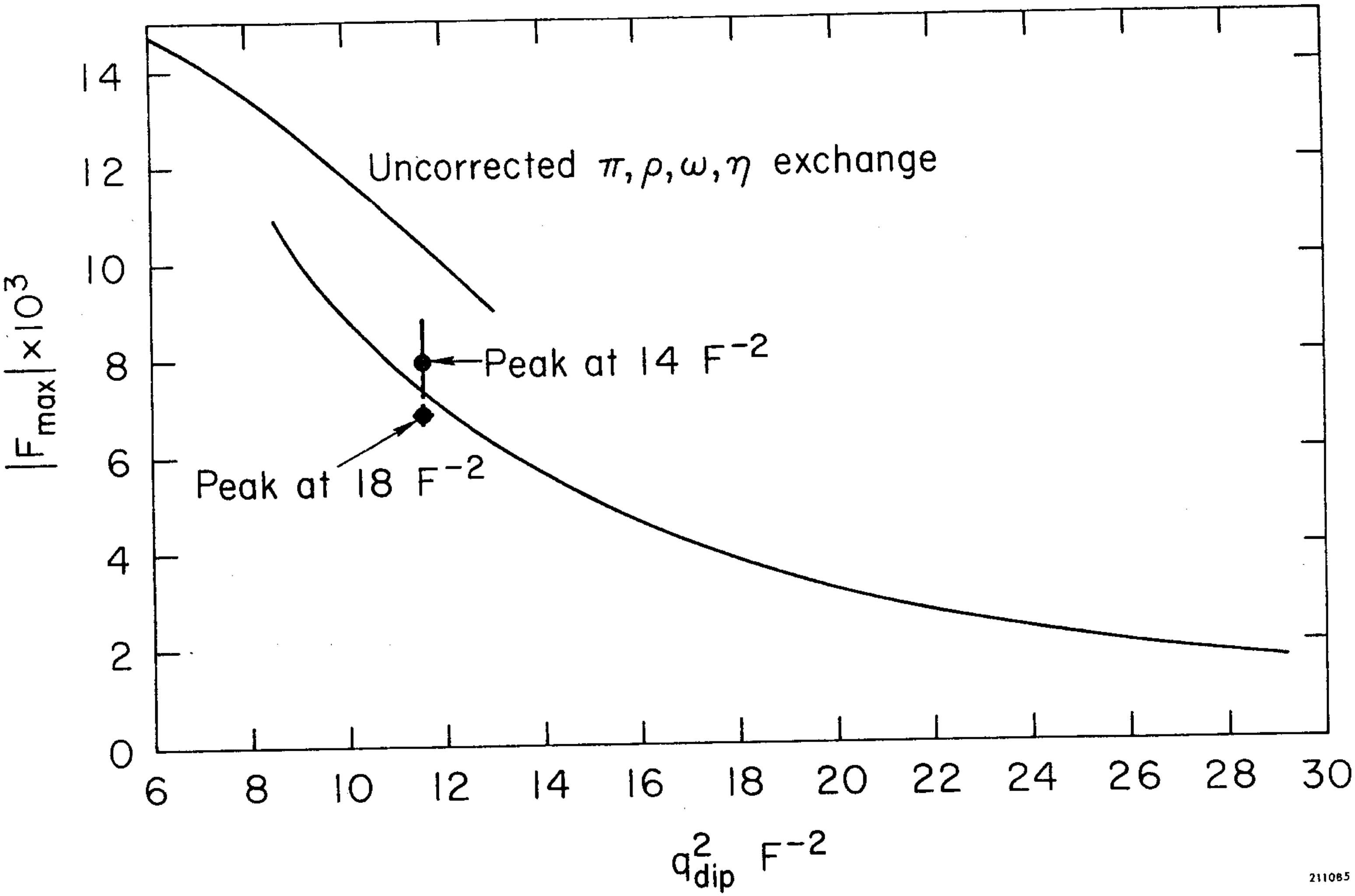

FIG. V.2--The position of the first zero of $F_e^{^3He}(q^2)$ plotted against the size of the secondary maximum $|F_e^{^3He}|_{max}$.

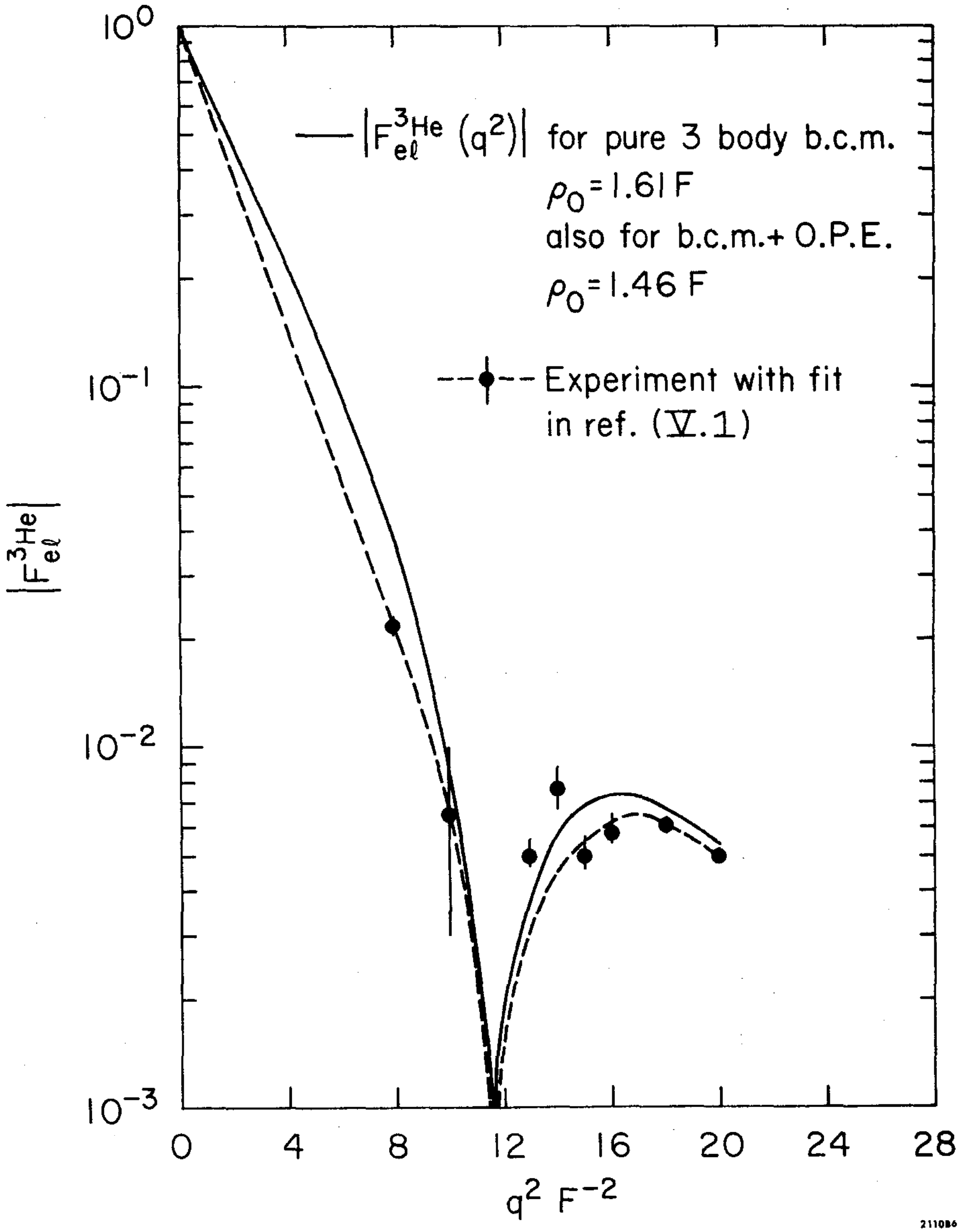

FIG. V.3--The ^{3}He electric form factor for K=0, $\rho_0 = 1.61$ F vs. experiment.

In the region beyond the zero (i. e. $q^2 > 12 \; F^{-2}$0) the fit is remarkably good, being within the error brackets of all the points out to $q^2 = 20 \; F^{-2}$ with the exception of those at $q^2 = 15$ and $q^2 = 16 \; F^{-2}$. The value of the peak is $|F_{e\ell}^{^3He}|_{max} = 7.3 \times 10^{-3}$, which is reached at $q^2 = 15.75 \; F^{-2}$. This value is about fifteen percent higher than the fit of McCarthy et. al. ignoring the points near $q^2 = 14 \; F^{-2}$ but it is lower than the value of the peak if they are included.[4] The boundary hyperradius for this fit is $\rho_0 = 1.61$ F. In short, the gross features of the ^{3}He electric form factor are reproduced at least qualitatively by this lowest order model.

Before considering corrections to the model we comment on the reasonableness of the parameters involved. There is little dependence on the binding energy. This is a consequence of the large boundary hyperradius so that the oscillations of the Bessel function $J_2\left(\sqrt{2/3}\; q\rho\right)$ damps out the wave function except near the boundary hyperradius. This greatly suppresses dependence on the exponential tail in $K_2(\chi\rho)$. Doubling the binding energy produces less than a five percent change in any of the major features of the form factor. In fact, in several computer runs the binding energy was inadvertently increased by a factor of 89, the boundary hyperradius was moved out to 1.7 F and the height of the secondary meximum merely doubled! The boundary hyperradius $\rho_0 = 1.61$ F is not unreasonable when we recall that for the equilateral triangle configuration ρ is equal to the side of the triangle. The considerations of the last chapter precit ρ_0 between 1 and 1.4 F, in not too bad agreement with the above value. For the pure two body b.c.m. the boundary radius is 1.32 F (1.095 F for the $^1S_0$ state alone). (See Chapter IV.) In the last chapter we saw that the boundary radius for the two body b.c.m. was reduced to near its predicted value when external potentials were added so that the b.c.m. no longer had to try to account for the effects of long range potential tails. We expect a similar situation to hold for our model and we shall see in Section 3 of the chapter that this is indeed the case.

V. 2 <u>The Inclusion of States with $K \neq 0$</u>

It will be assumed that the dominant S state (symmetric) is well accounted for by the state with K , 0. In order to include S' and D states we will assume that these states are dominated by the K = 2 components (note V. 2).

Physically (assuming pair interactions) the S' and the D states are generated by the difference between the isosinglet and isotriplet force and the tensor force respectively. For the two body b.c.m. with no potentials, D waves are generated by rewriting the boundary matrix given by $r_0(d\psi/dr)|_{r_0} \,/\psi(r_0)$ as a matrix with off-diagonal elements coupling different partial waves.[2] Likewise, we can write for our three body b.c.m.

$$\frac{\rho_0 d\chi_{[K]}(\rho)}{d\rho}\bigg|_{\rho=0} = \sum_{[K^o]} f_{[K][K']}\, \chi_{[K']}(\rho_0) \qquad \text{(V. 5)}$$

where $[K]$ is the set $\left| K,\ L,\ M_L,\ \nu,\ \omega \right|$. Equation (V. 5) can be further generalized by letting ρ_0 depend on $[K]$. For the pure b.c.m. $\chi^1_{20}(\rho)$ and $\chi^1_{22}(\rho)$ are just free solutions of (III. 72) for $\rho > \rho_0$ and are zero elsewhere. Since we have fixed E_B, the binding energy, under the assumption of K = 2 dominance of the S' and D waves, the matrix $\tilde{f}_{[K],[K']}$ is equivalent to the specification of the relative signs of the $\chi_{[K]}$ and their normalizations

P_S, $P_{S'}$, and P_D. The wave functions are given by:

$$\chi^{0}_{00} = C_S K_2(\chi\rho) \qquad (V.\ 6)$$

$$\chi^{1}_{20} = C_{S'} K_4(\chi\rho)$$

$$\chi^{1}_{22} = C_D K_4(\chi\rho)$$

Since the D waves are expected to be of longer range than the S waves we let ρ_{0D} be different than ρ_0 to give this effect, then

$$C_S = \sqrt{P_S}\left[\int_{\rho_0}^{\infty} \rho d\rho\, |K_2(\chi\rho)|^2\right]^{-1/2}$$

$$C_{S'} = \sqrt{P_{S'}}\left[\int_{\rho_0}^{\infty} \rho d\rho\, |K_4(\chi\rho)|^2\right]^{-1/2} \qquad (V.\ 7)$$

$$C_D = \sqrt{P_D}\left[\int_{\rho_{0D}}^{\infty} \rho d\rho\, |K_4(\chi\rho)|^2\right]^{-1/2}$$

From (III. 98) we see that, since the S and D waves don't interfere, the sign of C_D is immaterial for the trinucleon electric form factors. If C_S is taken as positive then comparison of (III. 95) and (III. 98) along with the hermiticity requirement that the form factor be real and the experimental fact that $F_{e\ell}^{^3H}$ is greater than than $F_{e\ell}^{^3He}$ for $0 \leq q^2 \leq 8$ ^{3} fixes $C_{S'}$ as real and positive. (Provided, of course, that this experimental difference is not primarily due to the violation of charge symmetry in the nuclear forces, which, in our approximation, means that ρ_0, ρ_{0D}, P_S, $P_{S'}$ and P_D can differ between ^3H and ^3He by more than the effects of the coulomb interaction.)

The result of all of this is that the ^{3}He electric form factor is given by:

$$F_{e\ell}^{^3He(q^2)} = \left(F_{e\ell}^p(q^2) + F_{e\ell}^n(q^2)\right) \times \int_{\rho_0}^{\infty} 8\rho d\rho \, |K_2(\chi\rho)|^2 \, J_2(a)/a^2$$

$$\times \quad P_S \Big/ \left(\int_{\rho_0}^{\infty} \rho d\rho \, |K_2(\chi\rho)|^2\right) + \left(P_{S'} \Big/ \int_{\rho_0}^{\infty} \rho d\rho \, |K_1(\chi\rho)|^2\right)$$

$$\times \int_{\rho_0}^{\infty} 8\rho d\rho/a^2 \left(J_2(a) + J_6(a)\right) |K_4(\chi\rho)|^2 + \left(P_D \Big/ \int_{\rho_{0D}}^{\infty} \rho d\rho \, |K_4(\chi\rho)|^2\right)$$

$$\times \int_{\rho_0}^{\infty} 8\rho d\rho/a^2 \left(J_2(a) + J_6(a)\right) |K_4(\chi\rho)|^2 - \frac{1}{3}\left(F_{e\ell}^p(q^2) - F_{e\ell}^n(q^2)\right)$$

$$\times \left[P_S \, P_{S'} \Big/ \left(\int_{\rho_0}^{\infty} \rho d\rho \, |K_2(\chi\rho)|^2 \times \int_{\rho_0}^{\infty} \rho d\rho \, |K_4(\chi\rho)|^2\right)^{-1/2}\right]$$

$$\times \; 24\sqrt{2} \int_{\rho_0}^{\infty} \rho d\rho/a^2 \, J_4(a) \, K_2(\chi\rho) K_4(\chi\rho) \qquad \text{(V. 8)}$$

as usual $a = \sqrt{2/3}\, q\rho$.

In order to evaluate (V. 8) we need $P_{S'}$ and P_D. Unfortunately, these are not available from experiment in an unambiguous way. The reason for this is that the processes that naively measure these such as tritium β decay, nucleon capture on deuterium, and the trinucleon magnetic charge radii are expected to be quite sensitive to magnetic exchange current contributions.[4] The S' state probability presumably accounts for much of the difference between the ^{3}H and the ^{3}He form factors. However, a fit to this difference over a wide range in q^2 requires a model or form for the various wave functions.

One common assumption is that the S and S' hyperradial wave functions are proportional as a function of ρ.[5] Using this assumption or a weaker form that the root mean square hyperradii are the same, we can derive $P_{S'}$ in terms of $R_{ch}^{^3H}$ and $R_{ch}^{^3He}$ (see appendix V. A).

$$\left(\frac{\left(4R_{ch}^{^3He^2} - R_{ch}^{^3H^2}\right)}{\sqrt{2}\left(2R_{ch}^{^3He^2} + R_{ch}^{^3H^2} - 3R_{ch}^{Proton^2}\right)} \quad \frac{1 + (\beta-1)P_D}{\sqrt{P_S - P_D}} \right)^2 = P_{S'} \qquad (V.\ 9)$$

where $\beta = \langle\rho^2\rangle_D / \langle\rho^2\rangle_S$.

For $P_D = 0$, $P_{S'} = 1.18\%$. With $\beta = 1$ and $P_D = 10\%$ we obtain $P_{S'} = 1.27\%$ and for $\beta = 4$, $P_D = 10\%$, we have $P_{S'} = 2.2\%$. Thus it is likely that $P_{S'}$ is between about 1.2 and 2.5%. In the literature there exist computations of $P_{S'}$ between .4 and 4%, however with $P_{S'}$ equal .4% it is unlikely that we could fit the difference between the ^3H and the ^3He electric form factors. While for $P_{S'} = 4\%$ we would require $\beta = 8.4$ which is again unlikely.

Besides phenomenological wave functions where the S' and D wave percentages are fit to the data, the other method of determining these percentages has been to perform a calculation with some model nucleon-nucleon potential which, presumably, fits the two nucleon data in all states to some degree of accuracy.[6] These tend to give about 1 to 4 percent S' and from 4 to 12 percent D waves in the trinucleon bound state although variational computations usually give considerably lessS' (about .5%)[7] than do direct solutions of the Faddeev equations (see note V. 3).

We have evaluated (V. 8) for various combinations of $P_S{}'$, P_D and β within the above ranges. Figure V. 4 shows the ^3He electric form factor for $\beta \sim 4$, $P_D = 10\%$ and $P_{S'} = 1.27\%$. Comparison with Figure V. 1 shous that the qualitative features of the form factor are unchanged from the S state only case. The size of the backward maximum is slightly increased by about five percent.

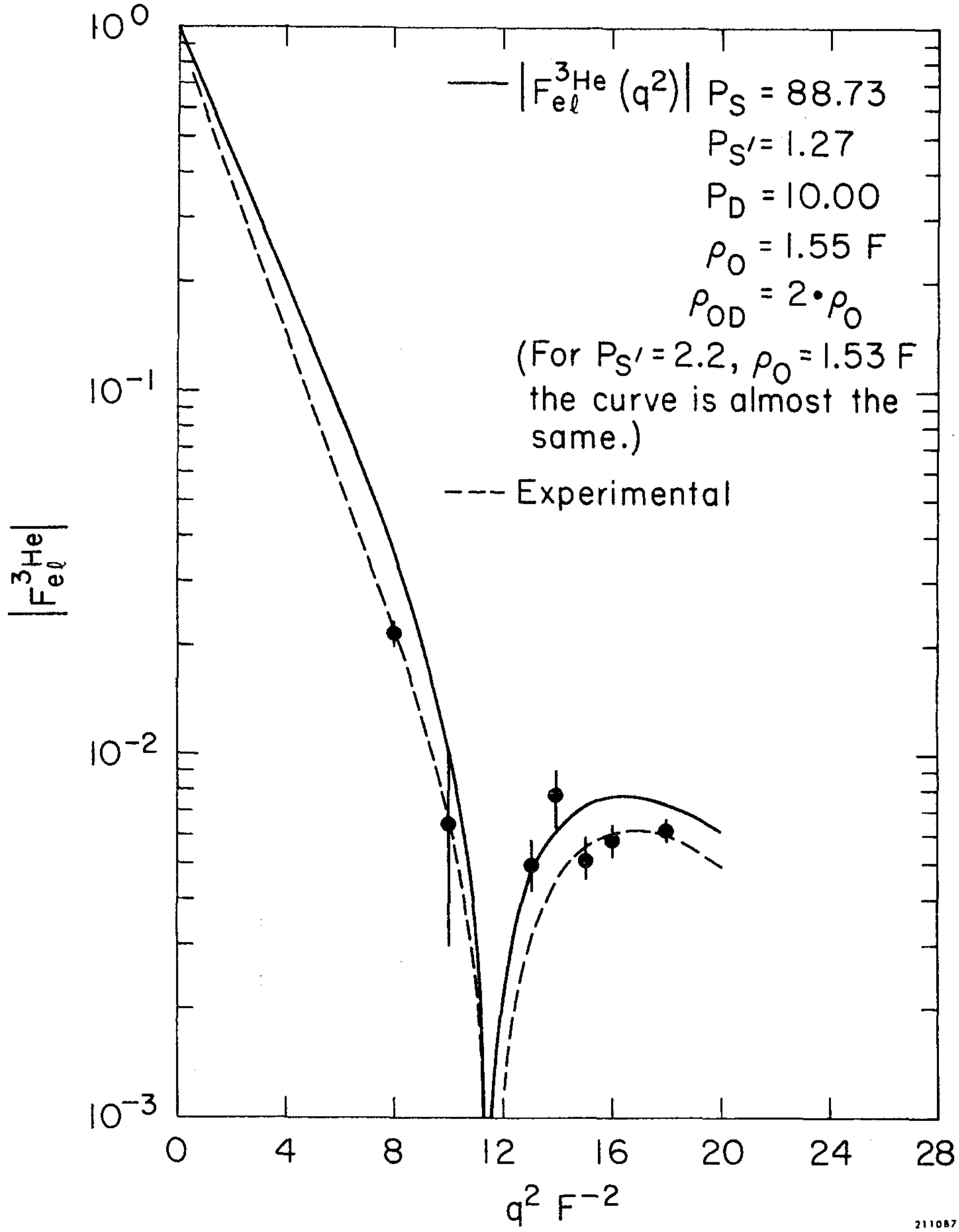

FIG. V.4--The same as Fig. 3 with S' and D states included to lowest order.

The charge radius is within the experimental range (since the form factors now agree with experiment for $q^2 \leq 1 F^{-2}$). However, the bulge at intermediate q^2 persists and is reduced by less than 10 percent. Adjustment of $P_{S'}$ to 2.2% in accord with (V. 9) changed little except increasing the size of the secondary maximum to about 8×10^{-3}. The boundary hyperradius is $\rho_0 = 1.55$ F for $P_{S'} = 1.27\%$ and $\rho_0 = 1.53$ F for $P_{S'} = 2.2\%$. This variation in ρ_0 is as expected as the negative contribution of F_2 in the form factor will, for fixed ρ_0, decrease the value of q^2 at which the form factor is zero. At the same time while the negative contribution will increase the size of the secondary maximum, this increase will be tempered by the decrease of ρ_0 (see Fig. V. 2).

For other reasonable values of $P_{S'}$ and P_D there is little quantitative and no qualitative difference in the form factor from the fit when only the principal S state is included, when ρ_0 is adjusted to give the experimentally observed position of the zero of the form factor in q^2. Therefore, the quality of the fit in the pure b.c.m., or its generalizations with potentials in the large q^2 region of the experimental range is not expected to be significantly altered by the inclusion of S' and D states. Neither should we expect to correct the difficulties of the model near $q^2 = 4$ to $8\ F^{-2}$. (We can trivially adjust the charge radius by including the S' state.) Since the sensitivity to the S' and D waves is not that great, on the level to which we are working, we need not worry about corrections to the assumption of K = 2 dominance of these states. In the following section where we introduce the external two body potentials into the calculation, the results of this section justify the retention of only the principal S state in order to obtain the qualitative features of the model.

V. 3 The Three Body B.C.M. with Potentials

We remarked in Chapter IV that, for consistency, the short range behavior of a nucleon pair when the third particle is well removed should be given by the two body b.c.m. with external potentials. In terms of a limited set of K harmonics we can not take the two body b.c.m. directly into account. However, we can determine the effect of external two body potentials on the fit to the form factor given in Section V. 1.

Using the results of Section V. 2 we will consider only the K = 0 principal S state. The Green's function for this problem is from (III. 8).

$$g_0(\rho,\rho',\chi) = \begin{cases} +I_2(\chi\rho')K_2(\chi\rho) & \rho>\rho' \\ +I_2(\chi\rho)K_2(\chi\rho') & \rho'>\rho \end{cases}$$

In the notation of Chapter III we can write:

$$\chi^{o}_{00}(\rho) = AK_2(\chi\rho) - 2M/\hbar^2\int_{\rho_0}^{\infty} V_{00}(\rho')\,g_0(\rho,\rho',\chi)\rho' d\rho' = 0 \qquad \text{(V. 10)}$$

$$\rho>\rho_0 \qquad\qquad \rho<\rho_0$$

The logarithmic derivative at $\rho = \rho_0$ is again fixed by the value of the binding energy, $\hbar^2\chi^2/2M$. A is simply a normalization constant chosen so that $\int_{\rho_A}^{\infty}\rho d\rho\,|\chi^{o}_{0}(\rho)|^2 = 1$. And ρ_0 is fixed by the requirement that the position of the zero of the form factor in q^2 be fit. For a given two body external potential, $V_{00}(\rho)$ is computed by an integration over $d\Omega_6$ and (V. 10) is solved numerically.

The form factor is calculated just as in (V. 4):

$$F_{e\ell}^{^3He}(q^2) = \frac{1}{4}\left(3F_e(q^2) + F_\nu(q^2)\right) 8\int_{\rho_0}^{\infty} \rho d\rho \, |\chi^{o}_{00}(\rho)|^2 \, J_2(a)/a^2 \qquad (V.\ 11)$$

with $a = \sqrt{2/3}\ q\rho$ again.

The potentials used were the meson exchange potentials of Lomon and Feshbach[8] quoted in Chapter IV. For the case of the principal S state the evaluation of the dot products $\sigma_1 \cdot \sigma_2$, $\tau_1 \cdot \tau_2$, $\sigma_1 \cdot \sigma_2$ and $\tau_1 \cdot \tau_2$ between the completely antisymmetric spin-isospin state $\Phi^{[S]}_{S=\frac{1}{2},\, T=\frac{1}{2}}$ defined in Chapter III is quite simple. $\tau_1 \cdot \tau_2 \cdot \sigma_1 \cdot \sigma_2 = -3$, always, as the nucleons are in a relative S state. Since two nucleons have an equal probability of being in a spin or isospin singlet or triplet state in the principal S state of the tri-nucleon bound state, the expectation value of $\sigma_1 \cdot \sigma_2$ or $\tau_1 \cdot \tau_2$ alone is just : $1/2\,(1 + (-3)) = -1$.
By symmetry (see Chapter III),

$$V_{00}(\rho) = \frac{3}{\pi^3}\int_3 d\,\Omega_6\ V_{12}\ \left(\vec{r}_{12}, \vec{\sigma}_1, \cdot \vec{\sigma}_2, \cdot \vec{\tau}_1, \cdot \vec{\tau}_2\right).$$

The first potential to be included was that from one pion exchange (OPE). The form factor was virtually identical to that shown in Fig. V. 3. However, now, the boundary hyperradius moved in to $\rho_0 = 1.46$ F which is very close to the predicted range of the last chapter of from 1 to 1.41 F. This supports the contention that the boundary hyperradius is moved out to account for the long range effects of the potential tail when the potential is omitted. The charge radius remains about 1.6 F as in the three body b.c.m. for the K = 0 state. Additionally, the coulomb energy of ^{3}He for point nucleons is .836 MeV compared with the experimental value of .764 MeV and the result of

Fabre de la Ripelle[11] and most "realistic" potential models of .65 MeV. So far, we have showed that the addition of an attractive long range OPE potential does not destroy the quality of the fit to the secondary maximum. On the other hand, the bulge at intermediate q^2 is not corrected. We have increased the strength of the OPE by a factor of three from that given in Ref. 8. The value of ρ_0 moves in a bit but otherwise the quantitative results for the form factor are only changed by a few percent. The ability of the boundary condition to reproduce the secondary maximum once the position of the zero of the form factor is fit is shown by reversing the sign of the OPE potential, again, there is no qualitative change in the resultant form factor.

The short range one boson exchange (OPE) potential is given largely by ρ and ω exchange. Using the rules given above for evaluating $\tau_1 \cdot \tau_2 \, \sigma_1 \cdot \sigma_2$, $\sigma_1 \cdot \sigma_2$ and $\tau_1 \cdot \tau_2$, we find that (with $m_\omega = m_\rho$) the sum of the two vector meson exchange potentials is attractive in the K=0 three body state. (See Chapter IV for a discussion of this.) This is a common feature of many models that impose hard cores. As we pointed out in Chapter IV, models based on fitting decreasing partial waves by meson exchange potentials serially, give a larger ratio of ω to ρ,g which decreases the OBE attraction in the trinucleon bound state.

When the ρ, ω and η exchange potentials are included there is a significant change in the ^{3}He electric form factor. The boundary hyperradius is increased to $\rho_0 = 1.7$ F and the value of the backward maximum goes to 1.14×10^{-2} which is a factor of two greater than the experimental value. (See Fig. V.5.)

As we noted above, the sum of ρ and ω exchange is attractive in the K=0 state. However, since we have not included short range repulsion, there is much too much attraction below $r_{12} = .7$ F. To partially correct this we have

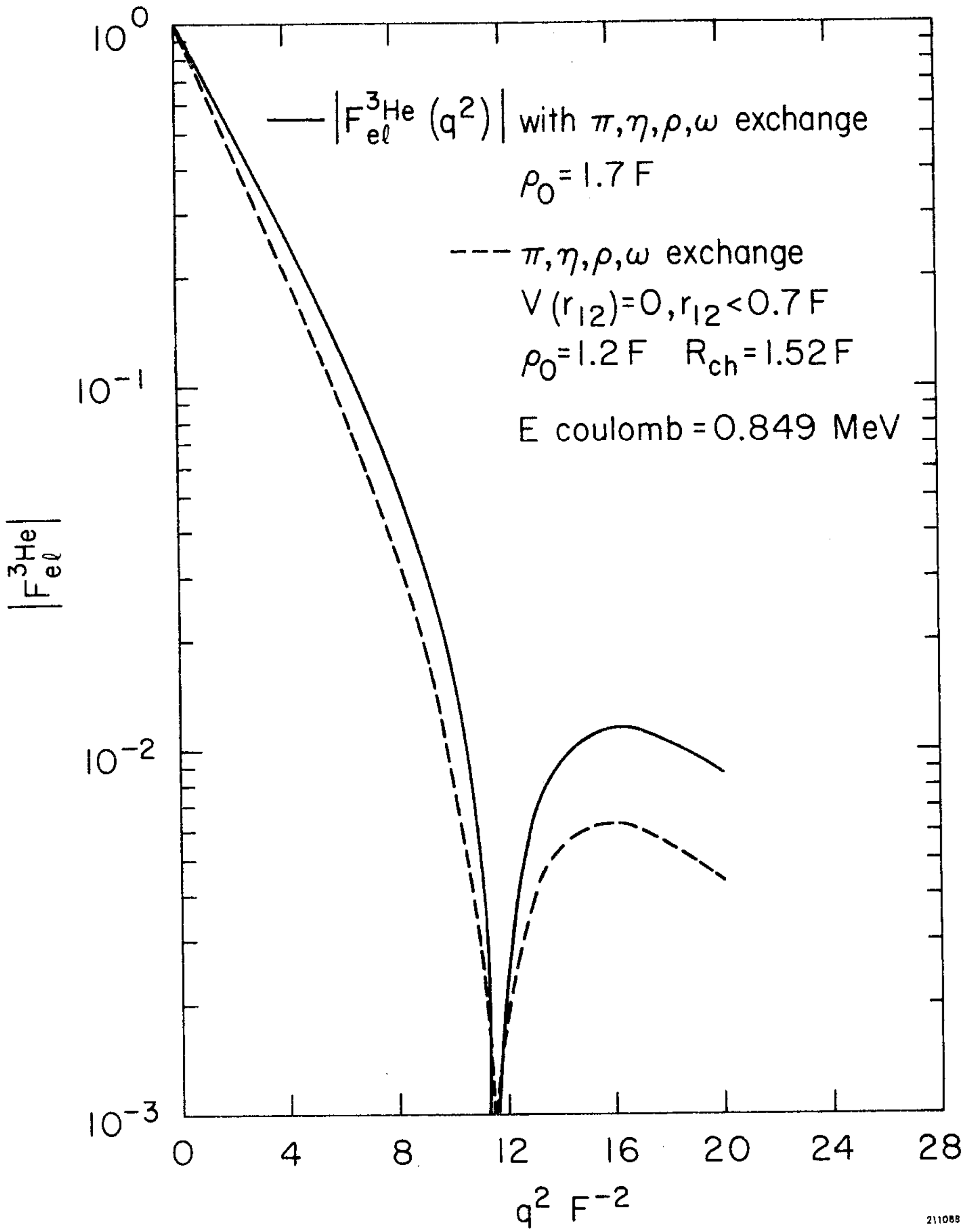

FIG. V.5--The ^{3}He electric form factor with OBE two body potentials and a K=0 three body b.c.m.

(i) $\rho_0 = 1.7$ F with no two body soft core.
(ii) $\rho_0 = 1.2$ F, $V_{12}(r_{12}) = 0$ when r_{12} is less than .7 F.

set $V_{12}(r_{12}) = 0$ for $r_{12} < .7$ F. The form factor then shows a secondary peak equal to 6.3×10^{-3} with $\rho_0 = 1.2$ F. The overall form factor is quite close to that for the pure three body b.c.m. in the K=0 state. This result is not qualitatively changed when V_{12} is constant for r_{12} less than .7 F when this constant is either + or − half the coefficient of the OPE potential in V_{00}. This supports the contention that once the highly singular attraction at short distances is removed, the backward maximum is well reproduced. We tried reversing the sign of the OBE potential with no short range cutoff and found that the charge radius was drastically increased to 2.49 F with $\rho_0 = 1.8$ F. However, by about $q^2 = 6\ F^{-2}$ the bulge at intermediate q^2 was present and the backward maximum was again about 6×10^{-3}. It is, therefore, unlikely that further refinements to the external potentials will be able to correct the intermediate q^2 behavior of the form factor. For this reason we have not pursued the question of two pion exchange potentials (which are ambiguous anyway) and σ-meson (ϵ-meson) exchange. We believe that these should be treated when we are able to include the full two body b.c.m. in our nucleon nucleon interaction.

What we have shown is that potentials which are not singularly attractive for small internucleon distances do not change the qualitative features of our fit. Additionally, we don't expect any major effects from S' and D waves, in accord with the results of the previous section.

V.4 Comparisons with Other Models

The difficulties that models using "realistic" potentials have in reproducing the ^3He electric form factor have already been discussed. Although, it should be pointed out that these difficulties are limited to the region $q^2 \geq 8\ F^{-2}$. For lower q^2 these models give a quite satisfactory form factor.

This is in contrast to the three body b.c.m. where the difficulties are at medium q^2 but the high q^2 behavior is well reproduced.

We will now compare our results with attempts, other than those using "realistic" two body potentials, to explain the form factor. The obvious starting point is the computation of Kim and Tubis[10] employing the pure two body b.c.m. in a solution of the Faddeev equations. They limited themselves to S wave two nucleon interactions and to the principle S state of the trinucleon bound state. Their computation made use of their off-shell partial wave b.c.m. t-matrix which assumes a strict hard core even for the off-shell two body interaction. With the $^1S_0$ boundary radius of 1.095 F and the boundary parameter fixed by the deuteron binding energy. They find a trinucleon binding energy of 12.69 MeV, however, the position of the zero of the form factor is at $q^2 = 6.5\ F^{-2}$ which is quite small. They then varied the boundary radius, keeping the deuteron binding energy fixed at its experimental value. With $r_0 = .825$ F they fit the position of the zero and obtained a secondary maximum of 7×10^{-3} at $q^2 = 18\ F^{-2}$. Their intermediate q^2 form factor shows almost precisely the bulge that we found in our three body b.c.m. The binding energy was now increased to 16.73 MeV, about twice the experimental value. Of course, with r_0 reduced to .825 F the fit of Feshbach and Lomon[2] to the nucleon-nucleon phase shifts is destroyed. In our model we use the ^3H binding energy as an input parameter (although there is a very weak dependence of the form factor on it) so it is not problem, additionally, we are not constrained by the two body phase shifts. Kim and Tubis[12] are now attempting to solve the Faddeev equations with the full two body b.c.m. including meson exchange potentials. If they are successful in fitting the large q^2 form factor and the binding energy, there would be little cause to support our model from the form factor alone. On the other hand, their approach and ours both show that a large core with boundary condition type strong attraction right outside is capable of generating enough diffraction to give the size of the secondary maximum.

Hoenig and Lomon[13] have suggested that the two body b. c. m. is ambiguous for off-shell nucleons (see Chapter IV). Hoenig[14] has computed the triton form factor for a modified b. c. m. in which an interior wave function is also assumed with a boundary condition at $r_0 - \epsilon$. This will not couple, for on-shell scattering, across any radius with energy independent boundary parameters but will affect the off-shell t-matrix which appears in the Faddeev equations. Fixing the boundary radius and exterior logarithmic derivative to fit the nucleon-nucleon scattering data, Hoenig adjusts the interior derivative to fit the triton binding energy. The resultant form factor is considerably larger than experiment, by $q^2 = 5\ F^{-2}$, the discrepancy is nearly a factor of 4. Hoenig does not give results for large q^2 so we can not compare his results to ours in the region of the secondary maximum. We should point out, again, that in our model the off-shell behavior is primarily due to the trinucleon binding rather than configurations in which a pair and the third particle are each far off-shell. So this ambiguity in the b. c. m. does not occur in the pure three body b. c. m. (additionally, Hoenig's model has unphysical poles on the physical sheet and thus has unacceptable analycity properties).

An other class of models which is quite successful is the modification of phenomenological wave functions,[15] such as Gaussians, by short range correlations of a Jastrow type:

$$f_{ij} = 1 - \exp\left(- \beta^2 (r_{ij})^2\right) \tag{V. 12}$$

Along with Gaussian basis functions this provides a two parameter fit to the data which is moderately successful. A fit to the low q^2 data which has β adjusted to fit the position of the zero of the form factor in q^2 reproduces the high q^2 form factor moderately well, the backward maximum being too low by a factor of 1.5 to 2. Khanna has also performed similar calculations with exponential forms for the basis functions and correlations. The results are qualitatively similar to the Gaussian case but not quite so good.

Lim[16] has calculated the ^{3}He electric form factor with Gaussian basis functions and performing perturbations with the Gaussian Eikemeier-Hackenbroich[17] potential, finds agreement with the form factor. Provided a small amount of S' state is admixed, the backward maximum is well fit. This is similar to the Gaussian fit of Samaranyake and Wilk.[18]

These calculations share with our model the property of relatively large anticorrelations in the nucleus. The range of the anticorrelations is considerably larger than in the "realistic" potential models. Furthermore, the Gaussian potentials have sharper attractive regions than the exponential tails of the OBE potentials. In this property there is a resemblance to the singular attraction at a large core radius of the various forms of the b.c.m. Khanna[15] has fit the form factors of several light nuclei with Gaussians and Jastrow type correlations. He finds that the value of β in (V.12) decreases as A increases, this might be evidence for explicit many body effects such as the three body b.c.m. We believe that there is a good deal of similarity, on a descriptive level, between the physics of our three body b.c.m. and the method of introducing anticorrelations into phenomenological wave functions.

An extremely interesting computation has recently been done by Brayshaw.[19] Assuming that the K=0 hyperspherical harmonic dominates the S state trinucleon bound state wave function, he computes by performing a Bessel transform of the form factor $F_1(q^2)$:

$$F_1(q^2) = 8 \int \frac{\rho\, d\rho}{a^2} J_2(a) \left|\chi^{o}_{00}(\rho)\right|^2 \qquad \text{(V. 13)}$$

$$a = \sqrt{2/3}\, q\rho$$

where $F_1(q^2)$ is obtained from $F^{^3He}_{el}(q^2)$ and $F^{^3H}_{el}(q^2)$ with $F_2(q^2)$ taken to account for the difference of the two over the experimentally measured range in q^2. From $\chi^{o}_{00}(\rho)$ the effective three body potential $V_{00}(\rho)$ is obtained by using:

$$-\hbar^2/2M \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{4}{\rho^2}\right) \chi^{o}_{00}(\rho) + |E_B| \chi^{o}_{00}(\rho) = -V_{00}(\rho)\, \chi^{o}_{00}(\rho) \qquad \text{(V. 14)}$$

$V_{00}(\rho)$, for large ρ, shows the expected attractive behavior. It then becomes repulsive as ρ decreases. For $\rho \approx 1$ F there is strong singular attraction with a small attractive core into the origin. The singular attraction is required only when the zero in the form factor occurs for q^2 less than 13.6 F^{-2}. The qualitative features of $V_{00}(\rho)$ are unchanged by variations in the behavior of the ^3He electric form factor for $q^2 > 20\ F^{-2}$. $V_{00}(\rho)$ is given by $3/\pi^3 \int d\Omega_6\, V_{12}(r_{12})$ so we may ask whether singular behavior of V_{12} as a function of r_{12} can account for the attraction in $V_{00}(\rho)$ at $\rho \approx 1$ F. We consider three cases:

$$-V_{12}(r_{12}) = a\, \theta\left(r_{12} - (r_0 - \epsilon)\right) \theta(r_0 - r_{12}) \quad (a\epsilon = c) \qquad \text{(V. 15a)}$$

$$-V_{12}(r_{12}) = \delta(r_{12} - r_0) \qquad \text{(V. 15b)}$$

and

$$-V_{12}(r_{12}) = C\,\delta'\,(r_{12}-r_0) \tag{V. 15c}$$

On integration over $d\Omega_6$ we obtain for the three cases:

$$-V_{00}(\rho) = 6/\pi\ C\ \frac{1}{\sqrt{2\rho^2-r_0^2}}\left(1 - \cos 4 \cos^{-1} r_0/\sqrt{2\rho}\right) \tag{V. 16a}$$

$$-V_{00}(\rho) = 24/\pi\ C\ r_0^2/\sqrt{2\rho}^{\,3}\ \sqrt{1 - \frac{r_0^2}{2\rho^2}} \tag{V. 16b}$$

$$-V_{00}(\rho) = -\,48/\pi\ C\ r_0/\sqrt{2\rho}^{\,3}\left(\sqrt{1 - r_0^2/\rho^2} - r_0^2/2\rho^2\ \frac{1}{\sqrt{1-r_0^2/2\rho^2}}\right). \tag{V. 16c}$$

where $V_{00}(\rho) = 0$ for $\rho < r_0/\sqrt{2}$ and C, in each case, has the appropriate dimensions so that V_{00} has the dimensions of an energy. Equations (V. 16a) and (V. 16b) are nonsingular at $\sqrt{2}\rho = r_0$, while (V. 16c) $\sim r_0^3/\rho^{5/2}\ 1/\sqrt{1-r_0^2/2\rho^2}$ which is a rather weak singularity. From the above it is unlikely that even a two body b. c. m. with the sharpest possible singularity can generate the sharp attraction in $V_{00}(\rho)$. Also the singularity occurs in $V_{00}(\rho)$ when $\rho = r_0/\sqrt{2}$, bit if $\rho_0 = 1$ F then $r_0 = 1.4$ F which is much larger than the boundary radius expected in the b. c. m. with external potentials.

Unfortunately Brayshaw's inversion of the form factor leads to negative values for $|\chi_{00}^{o}(\rho)|^2$ at small ρ. Brayshaw corrects this by introducing some interference with a K=4 state. This state has a probability of less than .1%. The potential now has the usual attractive tail for large ρ, at ρ of about 2.8 F there is a small repulsive "blip" followed by a shallow attractive plateau. At $\rho = .6$ F there is a large but finite repulsive core. (Brayshaw assumes that for small ρ the potential is given by the centrifugal term in (III. 72).) This behavior is not obtainable from finite two body potentials and would seem to support our contention that the backward maximum and position of the minimum in the ^{3}He electric form factor arises from a strong central three body repulsion with attraction immediately outside. Brayshaw's result gives a maximum in the wave function $\chi^{o}_{00}(\rho)$ at $\rho = 1$ F. This is similar to our model which cuts off the modified Bessel function $K_2(\chi\rho)$ at $\rho = 1.2$ F. If these results of Brayshaw's hold as the only possibility, it would mean that the reasoning that led to our model contained abstractions that are too extreme.

V.5 Some Extensions to Other Systems

The most natural extension of this model is to the ^{4}He form factor. The same sort of reasoning that we used in Chapter IV predicts a four body b.c.m. for the A=4, K=0 hyperspherical wave function.

Let us, first, briefly review the experimental and theoretical situation with regard to this form factor. The ^{4}He form factor is given by:

$$F^{^4He}(q^2) = \frac{1}{2} \sum_{i=1}^{4} \frac{1}{4} \int d^3r\, d^3r_i\, e^{i\vec{q}\cdot\vec{r}_i}\, \psi^x \rho_c(\vec{r},\vec{r}_i)\psi \qquad \text{(V.17)}$$

(see (III.91) and (III.92) for notation). $F^{^4He}(q^2)$ was measured out to $q^2 = 20$ F^{-2} by Frosch, McCarthy, Rand and Yearian.[20] They found a zero at $q^2 = 10$ F^{-2} with a secondary maximum at $q^2 = 16$ F^{-2} of magnitude $9.8 \pm .8 \times 10^{-3}$. The charge radius was about 1.7 F. This structure is remarkably similar to the ^{3}He electric form factor.

Khanna[15] has used the same method he employed for ^{3}He (see the above) for the ^{4}He form factor. He is able to reproduce the form factor with the zero at $q^2 = 10$ F^{-2}. In the region of the backward maximum the fit is too low by a factor of about three. The calculation excludes virtual quadrupole excitations, etc. Khanna believes that these states (based on a simple model calculation) can raise the secondary maximum to near its experimental value. This corresponds to including states with K greater than 0. Dzhibuti[21] and Mamasakhilov have given a solution to the Bethe-Goldstone equation with plane wave intermediate states and an oscillator basis for the inhomogeneous term. They find that with a Yukawa type interaction V(r) in a separable interaction $V(r, r') = \frac{1}{2}\left[V(r)\,\delta(r') + V(r)\,\delta(r')\right]$, they fit the binding energy. When two body correlations are included the position of the diffraction zero is fit and the size of the secondary maximum of the form factor is about 6×10^{-3}. These results are qualitatively quite similar to Khanna's.[15] On the other hand, Fink, Hebach, Schlitter and Kümmel[22] have used several "realistic" potentials in the generalized Bethe-Goldstone equation with a shell model basis in a calculation of the ^{4}He form factor. They find, for the Yale potential, that they reproduce the position of the zero and the low q^2 form factor, but the size of the secondary maximum is a factor of 8 too small (a remarkable similarity to the use of "realistic" potentials in ^{3}He).

Turning to a generalized b. c. m., we note that the ^{4}He nucleus is the heaviest one that can exist in a spacially degenerate K=0 state. For heavier nuclei the Pauli principle requires that higher K be included (see Chapter III). The ideas of Chapter IV which led us to the three body b. c. m. can be used to predict that the ^{4}He form factor is given by a boundary on condition on the lowest order (K=0) four body hyperharmonic.

The generalization of the Jacobi coordinates is given by:

$$\vec{\eta}_1 = \frac{1}{\sqrt{2}}\,(r_1 - r_2) \qquad \text{(V. 18a)}$$

$$\vec{\eta}_2 = \sqrt{2/3}\left(\frac{\vec{r}_1 + \vec{r}_2}{2} - \vec{r}_3\right) \qquad \text{(V. 18b)}$$

and

$$\vec{\eta}_3 = \sqrt{3/4}\left(\frac{\vec{r}_1+\vec{r}_2+\vec{r}_3}{3} - \vec{r}_4\right)$$

$$\left(\vec{r}_i \equiv \vec{r}_i - \vec{R}_{cm}\right) \qquad \text{(V. 18c)}$$

then

$$\rho^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2$$

$$= \frac{1}{4}\sum_{i\,j}^{4} (r_i - r_j)^2 \qquad \text{(V. 19)}$$

letting $\eta_1^2 + \eta_2^2 = \rho_3^2$ we have $\eta_3 = \rho \cos\theta$, $\rho_3 = \rho \sin\theta$. Then in the c.m., $r_4 = -(r_1 + r_2 + r_3)$ or

$$\vec{q}\cdot\vec{r}_4 = \sqrt{3/4}\; q\cdot\rho \;\; \cos(\hat{\eta}_3 - \hat{q}) \qquad \text{(V. 20)}$$

From the general considerations of Chapter III it follows that for the K=0 hyperspherical harmonic the ^{4}He form factor is given by:

$$F^{^4He}(q^2) = F_s(q^2)\; 105\sqrt{\pi}\int_0^\infty \rho\, d\rho\; |\chi_0(\rho)|^2\; J_{7/2}(a)/a^{7/2} \qquad \text{(V. 21)}$$

where $a = \sqrt{3/4}\, q\rho$. The ^{4}He binding energy is given by $E_B = 28$ MeV. This, in turn, gives $K_{max} = R\sqrt{2E_B M/\hbar^2} \approx 3$, which is twice K_{max} of ^{3}He. However, since ^{4}He is a closed shell nucleus, it is quite symmetric spacially, so we may be able to adequately describe the wave function by the K=0 hyperharmonic.

If the nucleons are arranged in an equilateral tetrahedron then from (V. 19) ρ is $\sqrt{3/2}$ times the length of a side. From the arguments of Chapter IV and the preceding analysis of ^{3}He we predicted that a three body boundary surface occurs when $\rho_3 \sim 1$-1.4 F. When the fourth particle is added, again, virtual pion production starts at a somewhat greater ρ. Our estimate of Chapter IV becomes: $\rho_0 \leq \frac{3}{2}\mu_\pi^{-1} = 2.1$ F. (Any further particles would have to be added in single particle states with $L>0$ so the argument can not be carried further.)

For the four body b.c.m. the K=0 wave function is given by:

$$\begin{aligned} \chi_0(\rho) &= C\, K_{7/2}(\chi\rho) & \rho > \rho_0 \\ \chi_0(\rho) &= 0 & \rho < \rho_0 \end{aligned} \tag{V.22}$$

where C is given by the normalization condition: $C^2\int_{\rho_0}^\infty \rho\, d\rho\, |K_{7/2}(\chi\rho)|^2 = 1$ and $\chi = \sqrt{2ME_B/\hbar^2}$.

The ^{4}He form factor is now

$$F^{^4He}(q^2) = C^2 F_s(q^2)\, 105\sqrt{\pi}\int_{\rho_0}^\infty \rho\, d\rho\, |K_{7/2}(\chi\rho)|^2 J_{7/2}(a)/a^{7/2}$$

$$\left(a = \sqrt{\tfrac{3}{4}}\, q\rho\right) \tag{V.23}$$

In Fig. V.6 we have plotted $F^{^4He}(q^2)$ given by (V.23) for $\rho_0 = 2.3$ F the agreement with experiment is quite good for q^2 up to the position of the zero at $q^2 = 10.5\ F^{-2}$. For greater q^2 the form factor is too small by a factor of about two. This is reasonably close to the result of Khanna[15] and presumably comes from ignoring components of the wave function with $K \neq 0$. Again we see that the simplest four body K=0 b.c.m. provides a good qualitative fit to the form factor. The value of ρ_0 is close to its physically predicted value of less than 2.1 F and could, it is assumed, be decreased by the inclusion of OBE potential tails.

Another possible extension, which is almost whimsical, is to the proton form factor itself. It is well known that baryon spectroscopy is well described

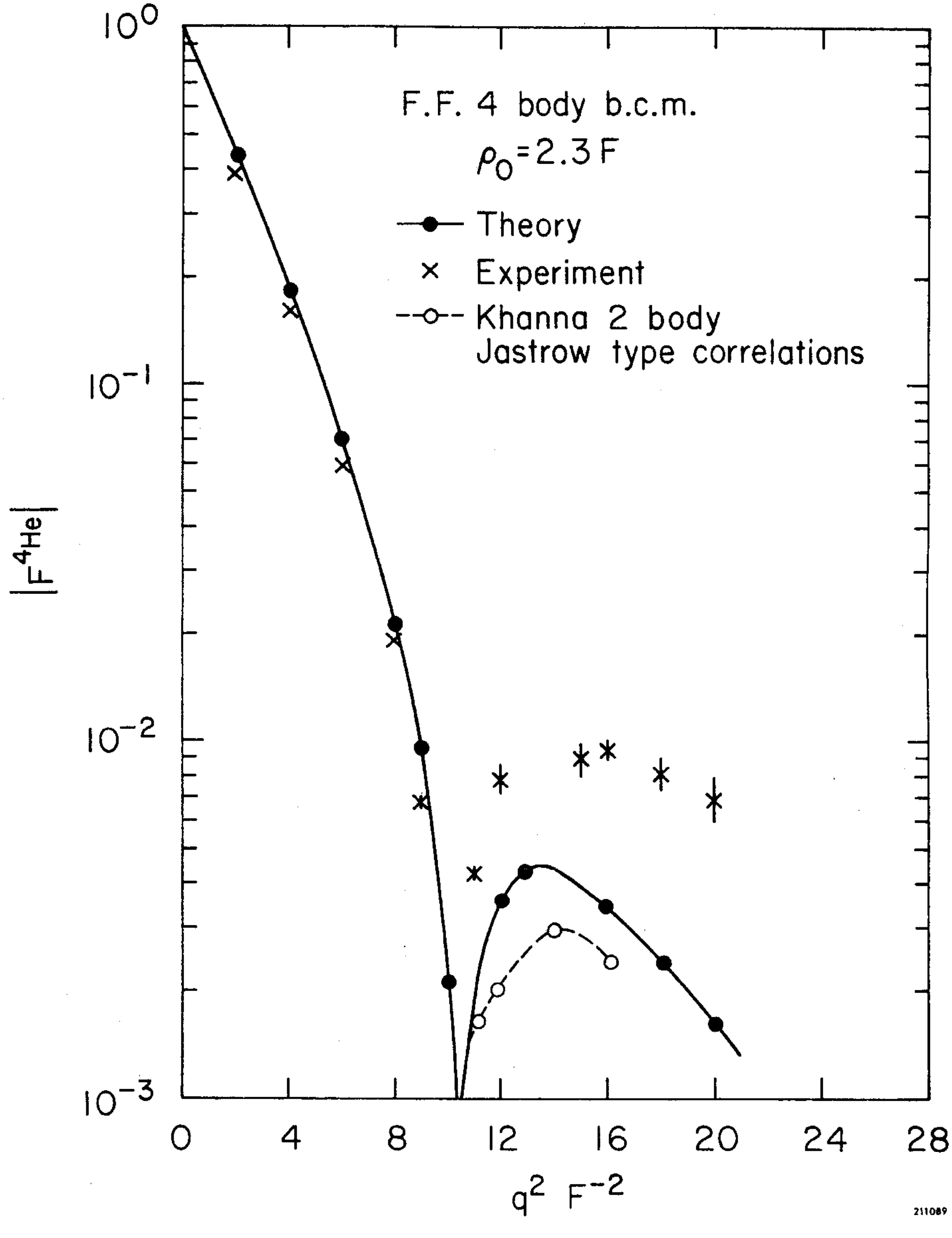

FIG. V.6--The ^{4}He form factor for a pure four body b.c.m. in the K=0 state vs. experiment.

by a model in which the nucleons are made up of three fractionally charged quarks. Since the spin of the proton is $\frac{1}{2}$ it follows that the spin of the quark is half-integral, provided that we assume the usual spin-statistics theorem holds. The lowest mass baryons seem to lie in a 56 (completely symmetric SU(6) state. Therefore, the spacial state should be antisymmetric. The difficulty with this is that an antisymmetric state must vanish at the origin thus the form factor should have a zero (of course, nothing tells us the position of the zero) but none has been observed. Also, it is difficult to conceive of the ground state of the system as being antisymmetric. We will not discuss the usual "outs" such as adding a core, parastatistics, etc., but rather, address ourselves to the last point.

The condition that a three particle system has an approximate ground state in some hyperspherical K component is $\sqrt{2M_E}\,\hbar^2\langle R\rangle \geq K$. We saw in Chapter III that for K larger than this, it is difficult to support a bound state with radius as small as $\langle R\rangle$. For a three quark system we have, additionally, $M_p = 3M_q - E_B$ where M_p is the proton mass and M_q is the quark mass. The lowest fully antisymmetric wave function is one with K=6 (w_6^3, see Chapter III). With $\langle R\rangle = .84$ F this yields $M_q > .7$ GeV. Assuming this numerical condition, we have calculated the proton form factor fitting the low q^2 data in a three body b.c.m. Our best fit was with a boundary hyperradius of 1.2 F (see Fig. V.7). The form factor is given by

$$F^p(q^2) = 8\,C^2 \int_{\rho_0}^{\infty} \frac{\rho d\rho}{a^2} \left\{ J_2(a) + \frac{9}{5} J_6(a) + J_{10}(a) - \frac{19}{5} J_M(a) \right\} |K_8(\chi\rho)|^2 \tag{V.24}$$

where $a = \sqrt{2/3}\, q\rho$ and $\chi = \sqrt{2M_q E_B}$.

Unfortunately there is a zero in our $F^p(q^2)$ at about $q^2 = 25$ F^{-2} whereas none is seen out to several hundred F^{-2}. On the other hand, if quarks exist

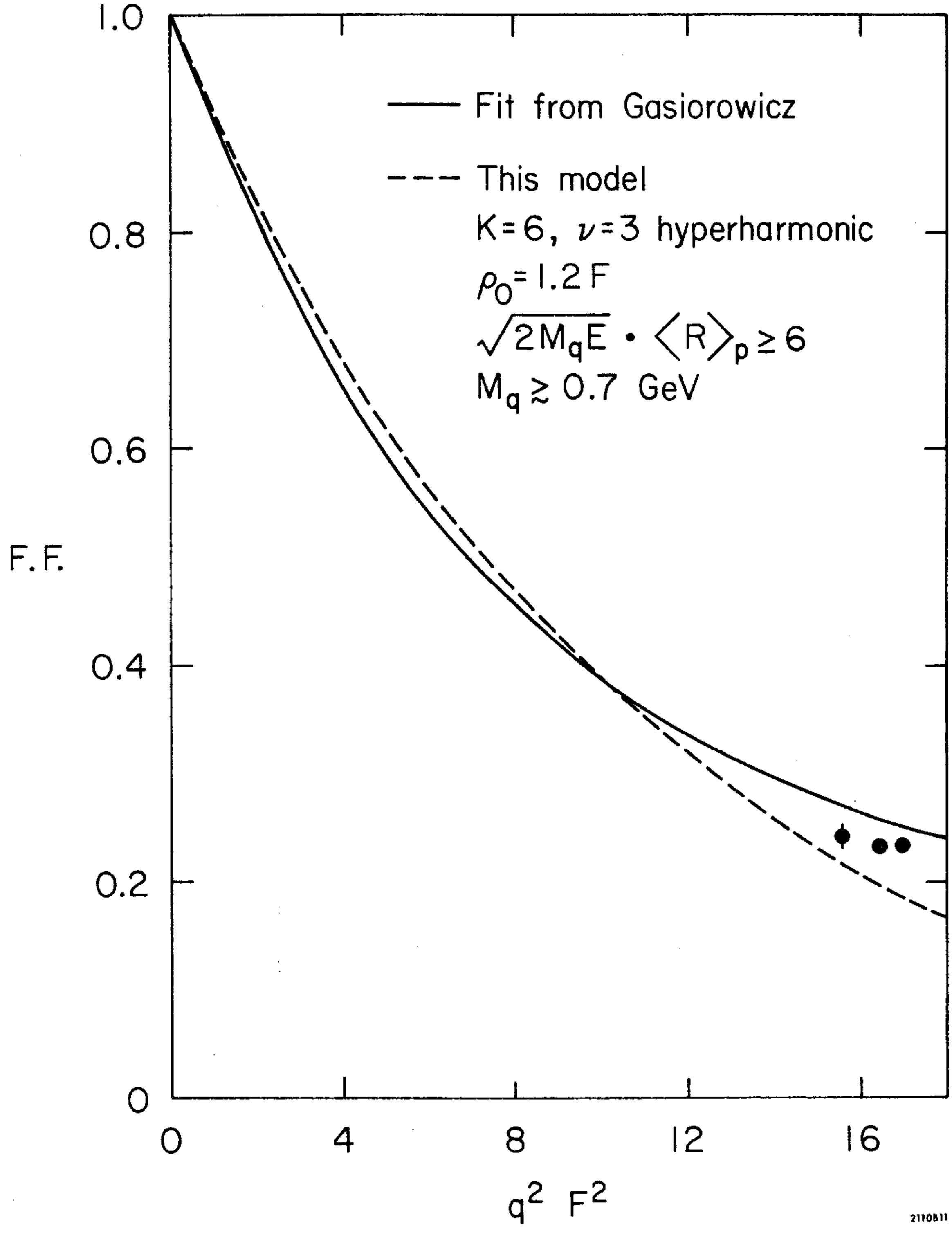

FIG. V.7--The proton form factor vs. experiment in a pure three body b.c.m. with K=6 vs. experiment for $q^2 < 18\ F^{-2}$.

with so low a mass as .7 F, we do not know their dynamics in terms of particle exchange mechanisms (avoiding tachyons) which will allow them to be kept in. Johnson's model[23] is hard to understand by this mechanism. In this case, the b.c.m. may really be little more than a phenomenology for low q^2. On the other hand, external potentials may allow us to drastically reduce ρ_0 and thus move the diffraction zero in the form factor to larger q^2. What this does show is that, out to 1 GeV/c^2, we can fit the data with antisymmetric spacial wave functions. Indeed, if $\sqrt{2ME_B/\hbar^2}\ \langle R\rangle \simeq 6$ and the potentials are SU(6) dependent so as to be most attractive in the 56 state, there is no reason for this not to be the ground state.

V.6 <u>Note on Numerics, Etc.</u>

In this brief note we will mention the numerical techniques that we used to evaluate the form factors of the previous sections of this chapter.

The numerical integrations were done by simple Gaussian quadrature which is expected to be rather rapidly convergent for nonsingular, reasonably smooth integrals.[24] The integrals for the three body b.c.m. are one dimensional integrals over a range from ρ_0 to ∞, where $\rho_0 \geq 1$ F. The integral involves $J_M(\sqrt{2/3}\, q\rho/(q\rho)^2$ which undergoes damped oscillation for large ρ and $K_M(\chi\rho)$ where since $\chi \approx .64\ F^{-1}$, $\chi\rho \geq .64$ thus the $K_M(\chi\rho)$ are already falling off exponentially fast and effectively limit the integration to one over a finite range. For this reason a simple mapping of $[-1, 1]$ to $[\rho_0, \infty]$ is expected to give good numerical results with the points near the singularity at x=1 being exponentially damped in the integral (our mapping is $2\rho_0/(1-x) = \rho$). For the pure K=0 b.c.m. an increase in the number of quadrature points from 6 to 10 changed such features as the position of the minimum and the size of the secondary maximum by only about 1%. The form factors, as plotted, were indistinguishable. The computations reported here for this case were carried out with the ten point quadratures.

The calculation with external potentials is trickier. However, since the Green's function is effectively $\theta(\rho-\rho_0)I_2(\chi\rho_<)\,K_2(\chi\rho_>)$ $(\rho_> > \rho_<)$ and $V_{00}(\rho)$ falls off rapidly for large ρ in the OBE case. The kernel of (V. 10) is compact and continuous. Thus the equation is Fredholm (we can clearly define a uniformly convergent set of kernels on a sequence of intervals $[\rho_0, \rho_n]$ $\rho_n \to \infty$). This means that we expect the integral equation for $\chi^o_{00}(\rho)$ can be rewritten as a matrix equation for $\chi^o_{00}(\rho_i)$ where the ρ_i correspond to the quadrature points in the variable x. These ρ_i are all that we need to evaluate the form factor numerically. The difficulty here is that the kernel has a discontinuous derivative at $\rho=\rho'$. However, since $\chi\rho_0$ is relatively big, the kernel itself is not particularly large at $\rho=\rho'$. So that the function with continuous nth derivative, which the quadrature effectively replaces the kernel with, should be a reasonably good approximation (provided that we have a sufficient number of points for fixed ρ in any interval $\rho \gtrless \rho'$ which is weighted heavily in the integral).

The form factors were calculated with both 10 and 12 point quadratures. If we compare the form factor for the same ρ_0 with our most singular potential, the OBE without core, the gross features such as the position of the zero and of the secondary maximum are changed by about 3% when the number of quadrature points is increased. The charge radius, Coulomb energy and size of the secondary maximum also show a similar change. This indicates sufficient convergence for our purpose. The only difficulty is in the value of the form factor near the zero. However, here the slope of log $F_{el}^{^3He}(q^2)$ becomes infinite so that a small change in the position of the zero can cause a large change in the form factor. This is similarly reflected in the large experimental errors associated with this region as the finite energy resolution of the final and initial electrons[7] and the large background will cause large errors in the form factor. So the region our quadrature method is subject to convergence problems in is also not amenable to precision in the experimental values of $F_{el}^{^3He}(q^2)$ and for exactly the same reason. Finally since the secondary maximum of the form factor is seen to be a smooth function of ρ_0 in the range considered (except possibly when the OBE potentials are reversed in sign with no core) we are well away from the situation where (V.10) has a homogeneous solution.

APPENDIX V.A

The charge radius is defined as:

$$R_{ch}^{3} = -6 \, dF_{el}(q^2)/dq^2 \big|_{q^2=0} \tag{V.A-1}$$

From (III.93) we have immediately (letting $F_{el}^{n}=0$)

$$R_{ch}^{2\ ^3H} = R_{ch}^{2\,Proton} + \left(-6 \, dF_1/dq^2\big|_{q^2=0} - 4 \, dF_2/dq^2\big|_{q^2=0}\right)$$

$$R_{ch}^{2\ ^3He} = R_{ch}^{2\,Proton} + 1\text{-}6 \, dF_1/dq^2\big|_{q^2=0} + 2 \, dF_2/dq^2\big|_{q^2=0} \tag{V.A-2}$$

Expanding the Bessel functions in (III.98) and (III.99) we obtain

$$F_1(q^2) = 1 - q^2/6 \quad \frac{1}{3}\int_0^\infty \rho^3 d\rho \left[|\chi_0^o(\rho)|^2 + |\chi_{20}^1(\rho^2)|^2 + |\chi_{22}^1(\rho)|^2\right] + 0(q^4)$$

$$F_2(q^2) = -q^2/6 \quad \left(-\sqrt{2}/4 \int_0^\infty \rho^3 d\rho |\chi_0^2(\rho) \quad \chi_{20}^1(\rho)|\right) + 0(q^4) \tag{V.A-3}$$

(Note that the contribution from the K=0, K=4 interference term $J_6(a)/a^2 = 0(q^4)$ does not affect the charge radius.) In terms of states normalized by $\int \rho d\rho |\chi^2(\rho)| = 1$, (V.A-3) is written as

$$F_1(q^2) = 1 - q^2/6 \left(\frac{1}{3} P_S I_S + P_{S'} I_{S'} + P_0 I_0\right)$$

$$F_2(q^2) = -q^2/6 \left(-\frac{\sqrt{2}}{4} \sqrt{P_S}\sqrt{P_{S'}} \, I_{SS'}\right) , \tag{V.A-4}$$

where

$$I_S = \int_0^\infty |\chi_{00}^o(\rho)|^2 \rho^3 d\rho \Big/ \int_0^\infty |\chi_{00}^o(\rho)|^2 \rho d\rho \tag{V.A-5}$$

and likewise for $I_{S'}$, I_D, and $I_{SS'}$. The assumption that the S and S' waves are proportional as a function of ρ means:

$$I_S = I_{S'} = I_{SS'} \equiv I \qquad \text{(V.A-6)}$$

We let $\beta = I_D/I$ then from (III. 95) and the preceding we have:

$$\begin{aligned} R_{ch}^{2\ {}^3H} &= R_{ch}^{2\ \mathrm{Proton}} + 1/3\left(P_S + P_{S'} + \beta P_D - \sqrt{P_S P_{S'}/2}\right) \\ R_{ch}^{2\ {}^3He} &= R_{ch}^{2\ \mathrm{Proton}} + 1/3\left(P_S + P_{S'} + \beta P_D + \tfrac{1}{2}\sqrt{P_S P_{S'}/2}\right) \end{aligned} \qquad \text{(V.A-7)}$$

Noting that $P_S + P_{S'} + P_D = 1$, $P_S + P_{S'} + \beta P_D = 1 + (\beta-1)\cdot P_D$, the system (V.A-7) can be solved,

$$\frac{R_{ch}^{2\ {}^3He} - R_{ch}^{2\ {}^3H}}{2R_{ch}^{2\ {}^3He} + R_{ch}^{2\ {}^3H} - 3\,R_{ch}^{2\ \mathrm{Proton}}} = \sqrt{2}/4\ \sqrt{P_S \text{-} P_{S'}}/(1 + (\beta-1)\,P_D) \qquad \text{(V.A-8)}$$

Since $P_{S'}$ is small, approximating $P_S = I\text{-}P_D$ and solving we get (V. 9).

Notes

1. There is marked lack of enthusiasm in the literature for admitting the possibility of a doubly peaked secondary maximum. Nobody, including us, wants it and their analytic fits to the experimental points have ignored it.

2. From Erens, Visschers and Van Wageningen[25] this may not be that good an assumption. However, since the normalizations of the S' and D' states amount to, perhaps, 10% of the total for the wave function, the contributions of states with K>2 to these are a correction to a correction and will be ignored. They can be included if the effects of the S' and D states turn out to be large.

3. Harper, Kim and Tubis[26] solve the Faddeev equations for a truncated Reid soft core potential and get $P_S=89.79\%$, $P_D=8.56\%$ and $P_{S'}=1.68\%$. Yang and Jackson[7] using a variational technique with the same potential find $P_S=90.56\%$, $P_D=8.92\%$ and $P_{S'}=.52\%$. It is a common feature of variational computations that they give less S' than do direct solutions of the Faddeev equations but the reason for this is not understood.

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CHAPTER VI

SUMMARY AND CONCLUSIONS

In this chapter we will first summarize our results and criticize our method. Then we will discuss possibilities for future work.

The purpose of the second chapter was to investigate whether the single impulse approximation is valid for elastic electron ^3He scattering for $q^2 > 10F^{-2}$. Indications that Glauber type multiple impulse effects in the form factor fit the backward maximum would mean that these corrections would have to be carefully accounted for before we could distinguish between competing models of the nucleon nucleon force. The conclusion of Chapter II is that such corrections do not have either the correct magnitude or functional form (when their magnitude is adjustable) to correct the ^3He electric form factor to the shape given by experiment. From Table II.1 and the results of Harper, Kim and Tubis[1] for the "realistic" Reid potential it can be seen by inspection that this statement is true for this potential as well as for the simple phenomenological fits to the low q^2 form factor used in Chapter II.

If shadowing corrections are not the explanation for the position of the zero in the ^3He electric form factor, the structure of the region from $q^2 = 10$ to $q^2 = 20\ F^{-2}$ strongly suggests that there is considerably more diffractive behavior in the electric scattering than would be obtained with three nucleon wave functions calculated with "realistic" two body potentials. Diffraction is associated with peaks or depressions in the spacial wave function. Since the nucleus is expected to have a core with little wave function, the size of which is inversely proportional to the position of the q^2 of the zero in the form factor, we were led to a generalization of the two nucleon boundary condition model with its large core. We argued that since in a hyperspherical basis the three nucleon system seems well described by a few hyperharmonics, the arguments that are put forth for the b.c.m. can be extended to give a similar three body effect in the lowest K harmonics.

Using this model for the $K = 0$ state only, with no external potentials, and the boundary hyperradius as the only adjustable parameter, it was found that the size of the secondary maximum was fit when the position in q^2 of the zero was. This good fit to the large q^2 form factor was insensitive to the introduction of external two body potentials consistent with the b.c.m. (except for the case of potentials singularly attractive at the origin). In fact, the introduction of potentials moved the boundary hyperradius that gives the correct zero in the form factor to within its theoretically predicted range of between 1 and 1.4 F. The quality of the large q^2 fit was also insensitive to the inclusion of S' and D states in the pure three body b.c.m.. Even when the S' state was included so as to fit the charge radius, the calculated form factor was too large in the region around $q^2 = 4$ to $8\ F^{-2}$. The inclusion of some $K = 4$ symmetric state was unable to remedy this. In Chapter V we noted that the three body b.c.m. is probably not a phenomenological representation of singular two body behavior. This means, since the three body b.c.m. seems to give a good representation or approximation of the small radius structure of the three nucleon bound state, that three body forces are important in this system.

The difficulties that our model has at q^2 below the position of the zero of the form factor may be an indication that the model is too extreme. The arguments given for the model in Chapter IV never actually derived the three body b.c.m.. Rather they suggested that the assymptotic three nucleon state should vanish at fixed energy at small enough hyperradius but they never told us how complete or sharp this vanishing ought to be. Likewise, the dispersion integral argument really only tells us that we find certain contributions to the spectral function which lead to energy dependence of the amplitude similar to that of the b.c.m.. This view of the three body b.c.m. is supported by the recent work of Brayshaw discussed in Chapter V. Another possibility is that when the hard core behavior of the two body b.c.m. can be included in the model (which is impossible directly in a K harmonic expansion of the wave function) both the low and high q^2 behavior of the ^{3}He electric form factor can be fit simultaneously.

An intriguing question is raised by our fit to the ^{4}He form factor. In Chapter V this was considered favorable to the consistency of our model. However, it should be noted that the four body b.c.m. fits the data when q^2 is below the position of the zero of the form factor and is less successful for large q^2 than the three body b.c.m. is in ^{3}He. The reason for this is likely to be our prediction that higher hyperharmonics are relatively more important in ^{4}He than in ^{3}He. This means that interference terms in the ^{4}He form factor, which are negative in order to increase the size of the secondary maximum, will move the position of the zero in from its position when they are not included. In other words, ρ_0 is too big when we omit the higher hyperharmonics. This, in turn, pulls down the low q^2 form factor, although the large four body boundary hyperradius that we predict may make the external two body forces relatively less important.

The most direct future extensions of the experimental work on the trinucleon form factors are measurement of the ^{3}He magnetic form factor in the region of q^2 greater than 12 F^{-2}, measurement of the electric, or both, form factors of ^3H in this region, and the determination of the ^3He electric form factor beyond $q^2 = 20\ F^{-2}$. One of the reasons that we have not discussed the ^3He magnetic form factor is that is not measured in the region of q^2 where our model is mostly likely to be good.[3] If it does show markedly similar behavior to that of the electric form factor it would mean that either mesonic corrections were considerably smaller than has been expected or that, within the context of our model, the core region really does exclude hadronic matter. This last possibility is difficult to understand from the theoretical considerations of Chapter IV, although we repeat; our model really makes no predictions about the magnetic form factor.

When we subtract out the effects of the S' state ($F_2(q^2)$) it is expected that any difference between the ^3He and ^3H electric form factors is due to charge assymmetry in the nuclear forces. Such assymmetry is probably present and may account for a difference of .8 F between the neutron neutron and proton proton S wave scattering lengths which is not explained by coulomb forces. If the two form factors were radically different for large q^2 it would embarrass our model which predicts similar diffractive behavior in both (see Fig. V. 2).

The three body b.c.m. predicts the existence of a second zero in the ^3He electric form factor at $q^2 \approx 33\ F^{-2}$ although this value is somewhat modified by the inclusion of states with K not equal to zero. The existence and position of this second zero as well as the size of the resulting tertiary maximum would be a good test of the claim that the three body b.c.m. is a good representation of the short distance behavior of the wave function.

One direction of future theoretical work on the model is its use to describe systems other than the three nucleon bound state. In Chapter III it was noted that the hyperharmonic expansion is not convergent for positive total energy because of two body pair interactions at large ρ. However, the work of Noyes[4] shows that knowledge of the wave function in the region where the two body pair forces overlap (or, presumably, where three body forces are present) allows a simple calculation of the wave function in the space exterior to this. Furthermore, the physically observable long range behavior of the three body system is parametrizable in terms of functions of the Jacobi momenta which are related to the interior wave function by boundary conditions on the hypersurfaces where the pair forces intersect. One way of parametrizing the interior in such an interior-exterior separation is by the K harmonic method (this has been discussed in the literature[5]). We could impose our three body b.c.m. on the interior wave function and thus use it in processes like nucleon deuteron scattering, n-d breakup, photodisintegration of ^{3}He etc. This would also allow us to check the consistency of the boundary parameters with those we used in the fit of the trinucleon electric form factor. This method might also enable us to incorporate the two body b.c.m. into our model for the form factor which was an important difficulty in Chapter V.

The work of Larsen and Mascheroni[6] in applying the hyperspherical harmonic method to the calculation of the third virial coefficient of a quantum mechanical gas suggests that the method and hence our model could be used to compute three body correlation effects in nuclear matter. In particular, it could have effects on the binding energy and charge distribution of intermediate and heavy nuclei. (A definitive review of the nuclear matter problem has recently been given by Bethe.[2])

One possibility for the future that we should not ignore is the use of the hyperharmonic partial wave dispersion relations of Chapter IV to discuss three body potentials. The input here would be the experimentally measured hadronic vertices, lifetimes and masses.

While we have not fully succeeded in relating the rather puzzling structure of the trinucleon bound state to the observed properties of the strong interaction, we believe that we have developed a model which has, in some approximation, many of the features of the way in which it will ultimately be done.

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