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QUANTUM ELECTRODYNAMICS AT INFINITE MOMENTUM:
APPLICATIONS TO HIGH ENERGY SCATTERING

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ABSTRACT

This report explores the application of the infinite momentum limit to quantum electrodynamics. Our major goal is the systematic development of an approximate ultrarelativistic theory of QED which provides a simple description of extremely high energy phenomena.

We interpret the infinite-momentum frame as a change of coordinate system to a new "time" axis $\tau = \frac{1}{\sqrt{2}}(t+z)$, and a new "space" axis $\mathfrak{z} = \frac{1}{\sqrt{2}}(t-z)$, leaving the "transverse" variables x and y unchanged. Using this basis it is not difficult to prove that there exists a subgroup of the full Poincare group which is isomorphic to the Galilean group in two dimensions. Working with this isomorphism and traditional nonrelativistic quantum mechanics in two dimensions, the fully relativistic theory of QED is developed from the infinite-momentum point of view. In order to study high energy processes, we next include an external field in the formalism and obtain the high energy limit of the scattering operator describing the scattering of electrons and photons off the external field. A vivid physical picture emerges for these processes which proves to be a realization of Feynman's parton ideas.

We apply our physical picture to electron scattering, bremsstrahlung, pair production and Delbrück scattering. Electroproduction of μ pairs and π pairs are also discussed as potential models for scaling in the deep inelastic region. On a more speculative level, we question whether the popular Regge-eikonal approximation scheme presents a compelling picture of diffraction scattering. The investigation leads to a negative conclusion and a detailed calculation is presented to substantiate this point.

Preface and Acknowledgement

The research reported in this report was done in collaboration with Davison E. Soper and with the guidance of Professor James D. Bjorken. I have benefitted greatly from my contact with both of them and extend to them my warmest thanks.

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I. Introduction

The infinite-momentum frame has found application in several branches of high energy physics. Perhaps its most famous contribution has appeared in current algebra where Fubini[1] and others used it as a tool to construct sum rules. However, it has become apparent in the intervening years that the limit itself is of considerable importance and worth more attention. Weinberg[2], for example, has pointed out that the infinite-momentum limit of old-fashioned perturbation theory diagrams in $\lambda\phi^3$ field theory exist and possess certain simple and interesting features. Susskind[3], and later Bardakci and Halpern[4], have approached the problem more systematically and discussed relativistic kinematics and equations of motion from this new point of view. All of these workers have been impressed with the fact that dynamics at infinite-momentum possesses a rigorous nonrelativistic interpretation and is, in many ways, simpler and more familiar to deal with than the traditional manifestly covariant presentation of relativistic dynamics.

In this thesis we shall explore these questions within the framework of quantum electrodynamics. Our major goal will be to develop an approximate ultrarelativistic theory of QED which provides a simple description of extremely high energy phenomena. It is enlightening to contrast this approach to the more familiar nonrelativistic (low energy) limit of field theories. For example, the nonrelativistic limit of quantum electrodynamics affords tremendous computational simplifications and intuitive insights into low energy electrodynamic processes. We will argue here that our ultrarelativistic theory also clarifies the structure and behavior of quantum electrodynamics processes in the opposite limit of extreme energies.

In Chapter I we shall review the structure of the Poincare Group at infinite-momentum. We will interpret the infinite-momentum frame as a change of coordinate system to a new "time" axis $\tau = \frac{1}{\sqrt{2}}(t+z)$, and new "space" axis $\mathfrak{z} = \frac{1}{\sqrt{2}}(t-z)$. Thus our approach will avoid limiting procedures and approximations. We shall see that there exists a subgroup of the full Poincare Group which is isomorphic to the Galilean group in two dimensions. Working with this isomorphism and traditional nonrelativistic quantum mechanics in two dimensions, we then develop the fully relativistic theory of QED from the infinite-momentum point of view. Here $\tau = \frac{1}{\sqrt{2}}(t+z)$, a lightlike variable, is treated as the "time" axis, and $\mathfrak{z} = \frac{1}{\sqrt{2}}(t-z)$, another lightlike variable, is treated as a "space" axis. In order to study high energy scattering processes, we next include an external field in our formalism. We then obtain the high energy limit of the scattering operator describing the scattering of electrons and photons off the external field. A vivid physical picture emerges for these processes which allows us to view the scattering event in two stages. First, when two electromagnetic particles having large relative momenta exchange a fixed amount of momentum, the interaction can be viewed as occurring between the bare quanta which compose the incoming and outgoing scattering states. Furthermore, the interaction between these constituents is simply a relativistic generalization of the eikonal amplitude familiar from nonrelativistic scattering processes. This physical picture is, of course, a particular realization of Feynman's parton ideas.[5]

In Chapter III we turn to a program of computing the high energy limits of scattering amplitudes in QED. We will use perturbation theory to express the incoming and outgoing physical states in terms of bare constituents, but calculate the interaction of the constituents with the external field to all orders in the bare charge. We consider electron scattering, bremsstrahlung, pair production and Delbrück scattering. All of these amplitudes prove to be simple because of the two dimensional nonrelativistic structure of infinite-momentum dynamics.

Electroproduction of μ pairs and π pairs are also discussed as potential models for scaling in the deep inelastic region.

In Chapter IV we turn to a more speculative program of application. We interpret QED as a model of strong interactions, i.e. the bare charge is no longer considered small and it is no longer sensible to work to low order in perturbation theory. Our initial problem is to determine the most likely physical state of an energetic electron incident on an external field. We argue, in fact, that fixing the order of perturbation theory at 2n, the most probable physical state of the electron consists of a bare electron and n $e^+ - e^-$ pairs as presented in Figure 14. This physical state develops as the electron approaches the external field. The most likely interaction between the constituents and the external field is realized when the slowest pair on the chain picks up eikonal phases. A striking dilemna occurs now when one attempts to sum over n, the number of pairs in the physical state: the resulting scattering amplitude leads to a cross-section which violates the Froissart bound. It has been suggested that s-channel unitarity be restored by iterating the single chain graphs. In our physical picture this means we should include more than one chain of $e^+ - e^-$ constituents in the physical electron state. If we then treat these chains as independent entities and recalculate the scattering amplitude we no longer violate the Froissart bound, but obtain a cross-section which grows as the square of the logarithm of the incident energy. This is a popular result often referred to as the Regge-eikonal approximation.[6] However, we will argue on the basis of our physical picture that treating the chains as independent is a poor approximation. We substantiate this claim with a detailed calculation and are led to conclude that field theory's predictions for the high energy limits of scattering amplitude in strongly interacting theories is more elaborate than the

Regge-eikonal formalism. We conclude with a discussion of the physical content which should appear in a more realistic treatment of this ferocious problem.

II. Heuristic QED in the Infinite-Momentum Frame

A. A Short Review

The claim was presented in the Introduction that infinite-momentum dynamics possesses a deep nonrelativistic character. This point can be demonstrated by a simple group theoretic analysis of the Poincare algebra. The question we wish to address ourselves to in more detail, however, is whether this property of infinite-momentum is actually useful, i. e. can one, guided just by experience with non-relativistic quantum mechanics and the infinite-momentum presentation of the Poincare group, develop a fully relativistic field theory such as QED in a transparent fashion ?

To begin, we review our systematic treatment of relativistic kinematics in the infinite-momentum frame.[7] We regard the infinite momentum frame as the reference frame obtained by choosing new space-time coordinates $(\tau, x, y, \mathfrak{z})$ related to the usual coordinates (t, x, y, z) by

$$\begin{aligned} \tau &= \frac{1}{\sqrt{2}}(t+z) \\ \mathfrak{z} &= \frac{1}{\sqrt{2}}(t-z) \end{aligned} \qquad \text{(II.1)}$$

Thus the τ- and $\mathfrak{z}$-axes of the new frame lie on the light cone. The infinite-momentum frame is not a Lorentz frame, but is, in a certain sense, the limit of a Lorentz reference frame moving in the - z direction with nearly the speed of light. Let $\hat{x}^{\mu} = (\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3) = (\tau, x, y, \mathfrak{z})$ be the coordinates of a space-time point in the ordinary coordinate system, $x^\mu = (x^0, x^1, x^2, x^3) = (\tau, x, y, \mathfrak{z})$ be the new coordinates of the same point. Then

$$x^\mu = C^\mu{}_\nu \hat{x}^\nu \quad , \tag{II.2}$$

where

$$C^\mu{}_\nu = \begin{pmatrix} 2^{-\frac{1}{2}} & 0 & 0 & 2^{-\frac{1}{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2^{-\frac{1}{2}} & 0 & 0 & -2^{-\frac{1}{2}} \end{pmatrix} \tag{II.3}$$

For the purposes of this review, we use careted symbols for vectors and tensors in the ordinary coordinate system, uncareted symbols for vectors and tensors in the new coordinate system. In particular, we shall use $g_{\mu\nu}$ for the metric tensor in the new coordinate system:

$$g_{\mu\nu} = (C^{-1})^\alpha{}_\mu \, \hat{g}_{\alpha\beta} \, (C^{-1})^\beta{}_\nu \tag{II.4}$$

We take for the ordinary metric tensor $\hat{g}_{00} = 1$, $\hat{g}_{11} = \hat{g}_{22} = \hat{g}_{33} = -1$. Then

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{II.5}$$

We use $g_{\mu\nu}$ to lower indices, so that $a_0 = a^3$, $a_3 = a^0$; this may seem confusing, but it has important consequences. For instance, the wave operator $\partial_\mu \partial^\mu = 2\partial_0 \partial_3 - \partial_1 \partial_1 - \partial_2 \partial_2$ is only first order in $\partial_0 = \partial / \partial \tau$.

Let us consider the generators of the Poincaré group in the new notation. Our conventions for the Poincaré algebra in the ordinary notation are

$$\left[\hat{P}^\mu, \hat{P}^\nu\right] = 0 \qquad \left[\hat{M}_{\mu\nu}, \hat{P}_\rho\right] = i(\hat{g}_{\nu\rho}\hat{P}_\mu - \hat{g}_{\mu\rho}\hat{P}_\nu)$$

(II.6)

$$\left[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}\right] = i(\hat{g}_{\mu\sigma}\hat{M}_{\nu\rho} + \hat{g}_{\nu\rho}\hat{M}_{\mu\sigma} - \hat{g}_{\mu\rho} M_{\nu\sigma} - \hat{g}_{\nu\sigma}\hat{M}_{\mu\rho}) \ .$$

The generators of rotations and boosts are, respectively, $\hat{M}_{ij} = \epsilon_{ijk} J_k$ and $\hat{M}_{i0} = K_i$. Using the matrix $C^\mu{}_\nu$ to transform from the usual notation to the new notation, we obtain

$$P^\mu = (P^0, P^1, P^2, P^3) = (\eta, P^1, P^2, H) \qquad \text{(II.7)}$$

and

$$M_{\mu\nu} = \begin{pmatrix} 0 & -S_1 & -S_2 & K_3 \\ S_1 & 0 & J_3 & B_1 \\ S_2 & -J_3 & 0 & B_2 \\ -K_3 & -B_1 & -B_2 & 0 \end{pmatrix} , \qquad \text{(II.8)}$$

where

$$\eta = \frac{1}{\sqrt{2}} (\hat{P}^0 + \hat{P}^3)$$

$$H = \frac{1}{\sqrt{2}} (\hat{P}^0 - \hat{P}^3)$$

$$B_1 = \frac{1}{\sqrt{2}} (K_1 + J_2)$$

$$B_2 = \frac{1}{\sqrt{2}} (K_2 - J_1) \qquad \text{(II.9)}$$

$$S_1 = \frac{1}{\sqrt{2}} (K_1 - J_2)$$

$$S_2 = \frac{1}{\sqrt{2}} (K_2 + J_1) \ .$$

The commutation relations among these generators are, of course, given by (2.6) without the hats. The commutation relations among the operators $H, P^1, P^2, \eta, J_3, B_1, B_2$ are particularly interesting. They are the same as the commutation relations among the symmetry operators of non-relativistic quantum mechanics in two dimensions with

$H \longrightarrow$ hamiltonian ,

$\vec{P}_T$ — momentum ,

$\eta \longrightarrow$ mass ,

$J_3 \longrightarrow$ angular momentum ,

B_1 and $B_2 \longrightarrow$ generators of (Galilean) boosts in the x and y directions, respectively.

Indeed, we have

$$\begin{aligned} &[H,\vec{P}_T] = [H,\eta] = [\vec{P}_T,\eta] = [J_3,H] = [J_3,\eta] = [\vec{B},\eta] = 0 \\ &[J_3,P^k] = i\epsilon_{k\ell}P^\ell \qquad [J_3,B^k] = i\epsilon_{k\ell}B^\ell \\ &[B_k,H] = -i\,P^k \qquad [B_k,P^\ell] = -i\,\delta_{ij}\,\eta \ , \end{aligned} \tag{II . 10}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. The commutation relations (2.10) are the result of an isomorphism between the subgroup of the Poincaré group generated by P^μ, J_3, and $\vec{B}$ and the Galilean symmetry group of non-relativistic quantum mechanics in two dimensions. This isomorphism expresses the nonrelativistic structure for quantum mechanics in the infinite momentum frame.[3,4]

It is easy to verify that the subgroup of the Poincare group generated by P^1, P^2, η, J_3, B_1, and B_2 leaves the planes τ = const. invariant. We can, therefore, refer to these operators as "kinematic" symmetry operators.

The operator K_3 which generates z boosts will play an important role when we discuss infinite-momentum QED in the presence of an external field. It is easy to compute that our operators simple scale under Lorentz boosts in the z direction:

$$
\begin{aligned}
e^{i\omega K_3}\, \eta \, e^{-i\omega K_3} &= e^{\omega} \eta \\
e^{i\omega K_3}\, \vec{P}_T \, e^{-i\omega K_3} &= \vec{P}_T \\
e^{i\omega K_3}\, H \, e^{-i\omega K_3} &= e^{-\omega} H \\
e^{i\omega K_3}\, J_3 \, e^{-i\omega K_3} &= J_3 \\
e^{i\omega K_3}\, \vec{B} \, e^{-i\omega K_3} &= e^{\omega} \vec{B} \\
e^{i\omega K_3}\, \vec{S} \, e^{-i\omega K_3} &= e^{-\omega} \vec{S}
\end{aligned}
\qquad \text{(II.11)}
$$

These simple relations comprise one of the major practical advantages of the infinite-momentum frame variables.

Consider now the operators S_1 and S_2 in connection with our non-relativistic analogy. We find that S_1 and S_2 commute with each other and with the hamiltonian H. Thus they play the role of the "dynamical" symmetry operators sometimes encountered in non-relativistic quantum mechanics.[8] The operators S_1, S_2 form a vector $\vec{S}$ under rotations: $[J_3, S_k] = i\,\epsilon_{k\ell}\, S_\ell$. The commutation relations of $\vec{S}$ with η, $\vec{P}_T$, and $\vec{B}$ are

$$
\begin{aligned}
&[S_k, \eta] = -i\, P^k \qquad [S_k, P_\ell] = -i\delta_{k\ell}\, H \\
&[S_k, B_\ell] = -i\epsilon_{k\ell}\, J_3 + i\delta_{k\ell}\, K_3 \quad .
\end{aligned}
\qquad \text{(II.12)}
$$

B. Heuristic QED at Infinite Momentum

Armed with the nonrelativistic analogy expressed in (II.10) we turn now to an informal development of infinite-momentum QED. We begin with the mass shell condition for a free electron, $p_\mu p^\mu = m^2$, or $2\eta H - \underset{\sim}{p}^2 = m^2$. If we make the usual identification $p_\mu \to i\partial_\mu$ we arrive at the equation of motion for the free electron field (the Klein-Gordon equation):

$$i\partial_0 \Psi(x) = \frac{1}{2\eta} (\underset{\sim}{p}^2 + m^2) \Psi(x) , \tag{II.13}$$

where $1/\eta$ is the integral operator

$$\left[\frac{1}{\eta} \Psi\right](x) = \frac{1}{2i} \int d\xi\, \epsilon(\mathcal{y} - \xi)\, \Psi(\tau, \underset{\sim}{x}, \xi). \tag{II.14}$$

and plays the role of $(\text{mass})^{-1}$ in the nonrelativistic analogy. As we will see, it suffices to let $\Psi(x)$ have only two components. The two components are postulated to satisfy the equal-τ anticommutation relations

$$\left\{\Psi_\alpha(x), \Psi_\beta(x')\right\}_{\tau=\tau'} = \delta_{\alpha\beta}\, \delta(\mathcal{y} - \mathcal{y}')\, \delta^2(\underset{\sim}{x} - \underset{\sim}{x}') . \tag{II.15}$$

Free photons are described by the two transverse components $\underset{\sim}{A}(x)$ of the electromagnetic potential. As in the formal development[7], we use the infinite-momentum gauge, $A^0 = A_3 = 0$. The equal-τ commutation relations satisfied by $\underset{\sim}{A}(x)$ are

$$\begin{aligned} \left[A^j(x), A^k(x')\right]_{\tau=\tau'} &= \delta_{jk}\, i\Delta(x - x')_{\tau=\tau'} \\ &= \delta_{jk} \frac{1}{4i} \epsilon(\mathcal{y} - \mathcal{y}')\, \delta^2(\underset{\sim}{x} - \underset{\sim}{x}') \end{aligned} \tag{II.16}$$

The free photon Hamiltonian is

$$H_\gamma = \frac{1}{2} \sum_{k=1}^{2} \int d\underset{\sim}{x}\, dz\; A^k(x)\, \underset{\sim}{p}^2\, A^k(x) \;. \tag{II.17}$$

Using the commutation relations (II.16), this Hamiltonian leads to the expected equation of motion,

$$\left[A^k(x),\; H_\gamma\right] = i\partial_0\, A^k(x) = \frac{1}{2\eta}\, \underset{\sim}{p}^2\, A^k(x) \;. \tag{II.18}$$

The natural two component spinors w(s) and polarization vectors $\underset{\sim}{\epsilon}(\lambda)$ in this description are

$$\begin{aligned} w(+1/2) &= \begin{pmatrix}1\\0\end{pmatrix} & \qquad w(-1/2) &= \begin{pmatrix}0\\1\end{pmatrix} \\ \underset{\sim}{\epsilon}(+1) &= 2^{-1/2}(1, i) & \qquad \underset{\sim}{\epsilon}(-1) &= 2^{-1/2}(1,-i), \end{aligned} \tag{II.19}$$

where the arguments s, λ refer to the infinite-momentum helicity discussed in the formal development. Using these wave functions, the Fourier expansion of the fields Ψ, A take the form[9]

$$\Psi(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta} \sum_s \left\{\sqrt{2\eta}\; w(s)\, e^{-ip\cdot x}\, b(p,s) + \sqrt{2\eta}\; w(-s)\, e^{+ip\cdot x}\, d^\dagger(p,s)\right\} \;, \tag{II.20}$$

$$\underset{\sim}{A}(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta} \sum_\lambda \left\{\underset{\sim}{\epsilon}(\lambda)\, e^{-ip\cdot x}\, a(p,\lambda) + \underset{\sim}{\epsilon}(\lambda)^*\, e^{+ip\cdot x}\, a^\dagger(p,\lambda)\right\} \;. \tag{II.21}$$

The operator $b^\dagger(p,s)$, $d^\dagger(p,s)$, and $a^\dagger(p,s)$ are creation operators for electrons, positrons, and photons respectively. They satisfy the commutation relations

$$\begin{aligned} \left\{b(p,s),\; b^\dagger(p',s')\right\} &= \delta_{ss'}(2\pi)^3\; 2\eta\; \delta(\eta-\eta')\; \delta^2(\underset{\sim}{p}-\underset{\sim}{p}') \\ \left\{d(p,s),\; d^\dagger(p',s')\right\} &= \delta_{ss'}(2\pi)^3\; 2\eta\; \delta(\eta-\eta')\; \delta^2(\underset{\sim}{p}-\underset{\sim}{p}') \\ \left[a(p,\lambda),\; a^\dagger(p',\lambda')\right] &= \delta_{\lambda\lambda'}(2\pi)^3\; 2\eta\; \delta(\eta-\eta')\; \delta^2(\underset{\sim}{p}-\underset{\sim}{p}') \;. \end{aligned} \tag{II.22}$$

The electrodynamic interaction can be introduced into this formalism by writing the free electron wave equation in the form[10]

$$i\partial_0 \Psi = (m - i\underline{\sigma}\cdot \underline{p}) \frac{1}{2\eta} (m + i\underline{\sigma}\cdot \underline{p})\Psi \ , \tag{II.23}$$

then making the gauge-invariant substitution $p_\mu \longrightarrow p_\mu - eA_\mu$. Then, using the gauge choice $A_3 = 0$, the wave equation with interactions becomes

$$i\partial_0 \Psi = eA_0\Psi + (m - i\underline{\sigma}\cdot[\underline{p} - e\underline{A}]) \frac{1}{2\eta} (m + i\underline{\sigma}\cdot[\underline{p} - e\underline{A}])\Psi \tag{II.24}$$

The dependent variable A_0 is eliminated with the help of Maxwell's equations, $\partial^\mu F_{\nu\mu} = \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu = J_\nu$. Choosing $\nu = 3$ and recalling that $A_3 = 0$, we find that $-\partial_3(\partial_3 A_0 - \underline{\nabla}\cdot\underline{A}) = J_3$. From the formal development[7], we find that $J_3 = J^o = e\,\Psi^\dagger\Psi$. Therefore

$$A_0 = \frac{1}{\eta^2} e\Psi^\dagger\Psi + \frac{1}{\eta} \underline{p}\cdot\underline{A} \ , \tag{II.25}$$

where $1/\eta^2$ is the integral operator

$$\left[\frac{1}{\eta^2}\Psi\right](x) = -\frac{1}{2}\int d\xi \, |\zeta - \xi| \quad \Psi(\tau, \underline{x}, \xi) \, . \tag{II.26}$$

Now the equation of motion for Ψ reads

$$i\partial_0 \Psi = \Psi \frac{e^2}{\eta^2} \Psi^\dagger\Psi + \Psi \frac{e}{\eta} \underline{p}\cdot\underline{A} + (m - i\underline{\sigma}\cdot[\underline{p} - e\underline{A}]) \frac{1}{2\eta} (m + i\underline{\sigma}\cdot[\underline{p} - e\underline{A}])\Psi \tag{II.27}$$

Finally, from Eqs. (II.5), (II.17), and the Heisenberg relation $[iH, \Psi] = \partial_0 \Psi$, we can conjecture that the Hamiltonian for the theory is

$$H = \int d\underline{x}\, d\zeta \Bigg\{ \frac{e^2}{2} \Psi^\dagger\Psi \frac{1}{\eta^2} \Psi^\dagger\Psi + e\,\Psi^\dagger\Psi \frac{1}{\eta} \underline{p}\cdot\underline{A} \tag{II.28}$$

$$+ \Psi^\dagger(m - i\underline{\sigma}\cdot[\underline{p} - e\underline{A}]) \frac{1}{2\eta} (m + i\underline{\sigma}\cdot[\underline{p} - e\underline{A}])\Psi$$

$$+ \frac{1}{2} \sum_{k=1}^{2} A^k \underline{p}^2 A^k \Bigg\}$$

$$= h_0 + h_I \tag{II.29}$$

with $h_0 = H_{e=0}$.

The matrix elements of H are very simple when taken between the "infinite-momentum helicity" states created by the operators $b^\dagger(p, s), d^\dagger(p, s), a^\dagger(p, \lambda)$. The matrix elements are easily calculated using the expansions (II.20) and (II.21) of the fields:

1) Single photon emission (Fig. 1a):

$$\langle e^-(p', s')\, \gamma(q, \lambda)|H|\, e^-(p, s)\rangle$$
$$= (2\pi)^3 \delta(\eta_{out} - \eta_{in})\, \delta^2(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})[2\eta']^{1/2}[2\eta]^{-1/2}\; e\, w^\dagger(s')\, \underset{\sim}{j}(\underset{\sim}{p}', \underset{\sim}{p}) \cdot \underset{\sim}{\epsilon}^*(\lambda)\, w(s) \; , \tag{II.30}$$

where

$$w^\dagger(s')\underset{\sim}{j}(p', p) \cdot \underset{\sim}{\epsilon}^*(\lambda)\; w(s)$$
$$= w^\dagger(s')\Big\{\eta_q^{-1}\, \underset{\sim}{q} \cdot \underset{\sim}{\epsilon}^*(\lambda) - \underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon}^*(\lambda)\, [2\eta]^{-1}\, \underset{\sim}{\sigma} \cdot \underset{\sim}{p}$$
$$- \underset{\sim}{\sigma} \cdot \underset{\sim}{p}'\, [2\eta']^{-1}\, \underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon}^*(\lambda) - \frac{1}{2}\, im\; \underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon}^*(\lambda)\Big[\eta'^{-1} - \eta^{-1}\Big]\Big\} w(s) \; . \tag{II.31}$$

In Table I, we list all of the possible matrix elements $w^\dagger \underset{\sim}{j} \cdot \underset{\sim}{\epsilon}^* w$.[11]

The matrix elements for other processes involving two fermions and one photon can be obtained by the usual substitution rules. For instance, the matrix element for $\gamma \to e^- e^+$ is

$$\langle e^-(p', s')\; e^+(p, s)|H|\; \gamma(q, \lambda)\rangle$$
$$= (2\pi)^3\, \delta(\eta_{out} - \eta_{in})\, \delta^2(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})\; [2\eta]^{1/2}\; [2\eta']^{1/2} \tag{II.32}$$
$$\times \;\; e\, w^\dagger(s')\; \underset{\sim}{j}(\underset{\sim}{p}', -\underset{\sim}{p}) \cdot \underset{\sim}{\epsilon}^*(-\lambda)\; w(-s)$$

2) Instantaneous electron exchange (Fig. 1b):

$$\langle e^-(p_4, s_4)\; \gamma(p_3, \lambda_3)\,|H|\, e^-(p_1, s_1)\; \gamma(p_2, \lambda_2)\rangle$$
$$= (2\pi)^3\, \delta(\eta_{out} - \eta_{in})\, \delta^2(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})\, [2\eta_4]^{1/2}\; [2\eta_1]^{1/2} \tag{II.33}$$
$$\times\, e^2\; w^\dagger(s_4)\underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon}(\lambda_2)\; [2\eta_0]^{-1}\, \underset{\sim}{\sigma} \cdot \underset{\sim}{\epsilon}^*(\lambda_3)\; w(s_1) \;\; .$$

The spinor product is very simple:

$$w^{\dagger}(s_4)\, \underset{\sim}{\sigma}\cdot \underset{\sim}{\epsilon}(\lambda_2)\, [2\eta_0]^{-1}\, \underset{\sim}{\sigma}\cdot \underset{\sim}{\epsilon}^*(\lambda_3)\, w(s_1)$$

$$= \begin{cases} 1/\eta_0 & \text{if all the particles are right handed or if all the particles are left handed} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(II.34)}$$

3) Instantaneous scalar photon exchange (Fig. 1c):

$$\langle e^-(p_3, s_3)\, e^-(p_4, s_4) | H | e^-(p_1, s_1)\, e^-(p_2, s_2) \rangle$$
$$= (2\pi)^3\, \delta(\eta_{out} - \eta_{in})\, \delta^2(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})\, [2\eta_1\, 2\eta_2\, 2\eta_3\, 2\eta_4]^{1/2}\, e^2 (\eta_0)^{-2}\, \delta_{s_1 s_3} \delta_{s_2 s_4}$$
$$+ \text{ contribution from crossed diagram .} \qquad \text{(II.35)}$$

The veteran field theorist, armed with this information, will be able to construct the rules for old-fashioned perturbation diagrams by whatever formal methods suit his taste:

1) A factor $(H_f - H + i\epsilon)^{-1}$ for each intermediate state;

2) An overall factor $-2\pi i\, \delta(H_f - H_i)$;

3) For each internal line, a sum over spins and an integration $(2\pi)^{-3} \int d\underset{\sim}{p} \int_0^{\infty} d\eta/(2\eta)$;

4) For each vertex,

   a) a factor $(2\pi)^3\, \delta(\eta_{out} - \eta_{in})\, \delta(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})$,

   b) a factor $[2\eta]^{1/2}$ for each fermion line entering or leaving the vertex (the factors $[2\eta]^{1/2}$ associated with each internal fermion line have the effect of removing the factor $1/(2\eta)$ from the phase space integral),

   c) a simple matrix element (e.g., $e w^{\dagger} \underset{\sim}{j} \cdot \epsilon^* w$) .

5) These rules give the S-matrix element[12] $\langle f|S|i \rangle$. One obtains the differential cross section from the S-matrix in the conventional fashion.

This "derivation" of infinite-momentum perturbation theory rules is not meant to be a complete alternative to a more thorough and systematic discussion of field theory. In fact, this discussion relies on certain guesses which are, however, straightforwardly answered in the more conventional treatment of the problem. The real motivation for the present discussion comes from strong interactions. Here we do not know the equations of motion, so a conventional approach is of no help. However, the infinite-momentum discussion here is so transparent that it might be easier and more compelling to insert guesses concerning strong interaction dynamics into its structure than into the conventional formalism. This scheme has, in fact, been considered with some interesting results.[13]

C. High-Energy Scattering from an External Potential

The reformulation of QED described above was motivated in part by a desire to develop limiting theories to describe high-energy scattering. We next turn to such a theory to describe the scattering of high-energy electrons and photons in a prescribed external electromagnetic potential $a_\mu(x)$. The results of this section can be derived using the full arsenal of canonical formalism with an external potential included in the Lagrangian. However, the same results can be obtained by extending the heuristic discussion above. In the spirit of our general approach to these problems, we will consider the heuristic derivation here.

Begin by introducing the potential a_μ into the electron wave equation (II.27) according to the gauge-invariant substitution $p_\mu \rightarrow p_\mu - ea_\mu$. Then the equation of motion reads

$$(i\partial_0 - eA_0 - ea_0)\psi = [m - i\alpha\cdot(p - eA - ea)] \frac{1}{2(\eta - ea_3)} [m + i\alpha\cdot(p - eA - ea)]\psi . \tag{II.36}$$

Here $[\eta - ea_3]^{-1}$ is the integral operator

$$\left[\frac{1}{\eta - ea_3}\psi\right](x) = \int d\xi \frac{1}{2i}\epsilon(\mathfrak{z} - \xi) \exp\left(-i\int_\xi^{\mathfrak{z}} d\xi' \, a_3(\tau, x, \xi')\right) \psi(\tau, x, \xi) . \tag{II.37}$$

Now, just as in Section B, we can eliminate the dependent variable A_0 using Maxwell's equations and find the Hamiltonian H which gives $i[H, \psi] = \partial_0 \psi$. The result is

$$\begin{aligned} H(\tau) = \int dx\, d\mathfrak{z} \Big\{ & ea_0\psi^\dagger\psi + \frac{e^2}{2}\psi^\dagger\psi \frac{1}{\eta^2}\psi^\dagger\psi + e\psi^\dagger\psi \frac{1}{\eta} p\cdot A \\ & + \psi^\dagger[m - i\alpha\cdot(p - eA - ea)] \frac{1}{2(\eta - ea_3)} [m + i\alpha\cdot(p - eA - ea)]\psi \\ & + \frac{1}{2}\sum_{k=1}^{2} A^k p^2 A^k \Big\} . \end{aligned} \tag{II.38}$$

It will be convenient to imagine writing H in the form $H(\tau) = H_0(\tau) + V(\tau)$, where $H_0(\tau)$ is given by (II.38) with $a_\mu = 0$ and

$$V(\tau) = H(\tau) - H_0(\tau) . \tag{II.39}$$

Thus H_0 is the full Hamiltonian for quantum electrodynamics with no external potential, and V gives the additional effect of the potential.

Now let us look at the scattering matrix in the interaction picture with V as the interaction Hamiltonian. Define the interaction picture fields by

$$\begin{aligned} \psi_I(\tau, x, \mathfrak{z}) &= \exp(+iH_0(0)\tau)\, \psi(0, x_T, \mathfrak{z}) \exp(-iH_0(0)\tau) \\ A_I(\tau, x, \mathfrak{z}) &= \exp(+iH_0(0)\tau)\, A(0, x_T, \mathfrak{z}) \exp(-iH_0(0)\tau) , \end{aligned} \tag{II.40}$$

and let $V_I(\tau)$ be given by (II.38) and (II.39) with $\psi_I(x)$ and $A_I(x)$ substituted for $\psi(x)$ and $A(x)$. Then it is a familiar exercise to show that the scattering matrix can be written in the form

$$S_{fi} = \langle f | T \exp\left(-i \int d\tau\, V_I(\tau)\right) | i \rangle \ , \tag{II.41}$$

where T indicates τ-ordering and $|f\rangle$ and $|i\rangle$ are appropriate eigenstates of $H_0(0)$ (which may be evaluated in perturbation theory).

We are interested in the high energy limit of S_{fi} as $\eta_i, \eta_f \longrightarrow \infty$. To study this limit we let $|i_0\rangle$ and $|f_0\rangle$ be fixed states and calculate S_{fi} between the high energy states $|i\rangle = e^{-i\omega K_3}|i_0\rangle$ and $|f\rangle = e^{-i\omega K_3}|f_0\rangle$, where K_3 is the generator of Lorentz boosts in the z direction. Thus we want to calculate

$$\begin{aligned} S_{fi} &= \langle f_0 | e^{i\omega K_3}\, T \left\{ \exp\left(-i \int d\tau\, V_I(\tau)\right) \right\} e^{-i\omega K_3} | i_0 \rangle \\ &= \langle f_0 | T \left\{ \exp\left(-i \int d\tau\, e^{i\omega K_3}\, V_I(\tau)\, e^{-i\omega K_3}\right) \right\} | i_0 \rangle \end{aligned} \tag{II.42}$$

in the limit $\omega \longrightarrow \infty$.

We recall from the full canonical theory that the boost operator K_3 is given by

$$K_3 = \int d\underset{\sim}{x}\, d\mathscr{y} \left[\Psi^\dagger \frac{i}{2} \overleftrightarrow{\partial}_3 \Psi + (\partial_3 \underset{\sim}{A}) \cdot (\partial_3 \underset{\sim}{A}) \right]_{\tau=0} , \tag{II.43}$$

and that the fields transform very simply under boosts:[14]

$$\begin{aligned} e^{i\omega K_3}\, \Psi_I(\tau, \underset{\sim}{x}, \mathscr{y})\, e^{-i\omega K_3} &= e^{\omega/2}\, \Psi_I(e^{-\omega}\tau, \underset{\sim}{x}, e^{\omega}\mathscr{y}) \\ e^{i\omega K_3}\, \underset{\sim}{A}_I(\tau, \underset{\sim}{x}, \mathscr{y})\, e^{-i\omega K_3} &= \underset{\sim}{A}_I(e^{-\omega}\tau, \underset{\sim}{x}, e^{\omega}\mathscr{y}) \ . \end{aligned} \tag{II.44}$$

It is thus easy to calculate the effect of the boost operator on $V_I(\tau)$. The term $e a_0 \Psi^\dagger \Psi$ remains finite in the limit $\omega \longrightarrow \infty$ and the rest of the terms are of order $e^{-\omega}$; we indeed find that

$$\begin{aligned} e^{i\omega K_3} V_I(\tau)\, e^{-i\omega K_3} &= \int d\underset{\sim}{x} d\mathscr{y}\, e^{\omega} e a_0(\tau, \underset{\sim}{x}, \mathscr{y})\, \Psi_I^\dagger(e^{-\omega}\tau, \underset{\sim}{x}, e^{\omega}\mathscr{y})\, \Psi_I(e^{-\omega}\tau, \underset{\sim}{x}, e^{\omega}\mathscr{y}) + O(e^{-\omega}) \\ &= \int d\underset{\sim}{x} d\mathscr{y}\, e a_0(\tau, \underset{\sim}{x}, e^{-\omega}\mathscr{y})\, \Psi_I^\dagger(e^{-\omega}\tau, \underset{\sim}{x}, \mathscr{y})\, \Psi_I(e^{-\omega}\tau, \underset{\sim}{x}, \mathscr{y}) + O(e^{-\omega}) \end{aligned} \tag{II.45}$$

Upon going to the limit the operators are all evaluated at $\tau = 0$, so the τ-ordering can be ignored. (This may be checked by examining the power series expansion.) Thus we obtain as $\omega \to \infty$

$$\begin{aligned} S_{fi} &= \langle f_0 | \mathbb{F} | i_0 \rangle + O(e^{-\omega}) \\ &= \langle f | \mathbb{F} | i \rangle + O(e^{-\omega}) , \end{aligned} \tag{II.46}$$

where

$$\begin{aligned} \mathbb{F} &= \exp\left(-i \int d\tau \, d\underset{\sim}{x} d\mathscr{y} \; e a_0(\tau, \underset{\sim}{x}, 0) \, \Psi_I^\dagger(0, \underset{\sim}{x}, \mathscr{y}) \, \Psi_I(0, \underset{\sim}{x}, \mathscr{y})\right) \\ &= \exp\left\{-i \int d\underset{\sim}{x} \; X(\underset{\sim}{x}) \, \rho(\underset{\sim}{x})\right\} \end{aligned} \tag{II.47}$$

and

$$X(\underset{\sim}{x}) = e \int d\tau \;\; a_0(\tau, \underset{\sim}{x}, 0) \tag{II.48}$$

$$\rho(\underset{\sim}{x}) = \int d\mathscr{y} \, \Psi^\dagger(0, \underset{\sim}{x}, \mathscr{y}) \, \Psi(0, \underset{\sim}{x}, \mathscr{y}) \; . \tag{II.49}$$

This (formally) closed expression for the limiting form of the scattering operator is in fact the eikonal approximation,[15] and also establishes a connection with parton ideas. The initial state $|i\rangle$ is an eigenstate of H_0, the Hamiltonian for quantum electrodynamics with no external field. Thus it is a "dressed" electron, photon, or whatever. Imagine expanding $|i\rangle$ in terms of the "bare" quanta associated with the fields $\Psi(0, \underset{\sim}{x}, \mathscr{y})$, $\underset{\sim}{A}(0, \underset{\sim}{x}, \mathscr{y})$ at time $\tau = 0$:

$$\begin{aligned} |i\rangle = &+ \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_\lambda g(\underset{\sim}{p}, \eta, \lambda) \, a^\dagger(\underset{\sim}{p}, \eta, \lambda) |0\rangle + \ldots \\ &+ \int d\underset{\sim}{p}_1 \int_0^\infty \frac{d\eta_1}{2\eta_1} \int d\underset{\sim}{p}_2 \int_0^\infty \frac{d\eta_2}{2\eta_2} \sum_{s_1 s_2} h(\underset{\sim}{p}_1, \eta_1, s_1; \underset{\sim}{p}_2, \eta_2, s_2) \\ &\times \, b^\dagger(\underset{\sim}{p}_1, \eta_1, s_1) \, d^\dagger(\underset{\sim}{p}_2, \eta_2, s_2) |0\rangle + \ldots \; . \end{aligned} \tag{II.50}$$

Here, for example, $h(\underset{\sim}{p}_1, \eta_1, s_1; \underset{\sim}{p}_2, \eta_2, s_2)$ is the amplitude for the state $|i\rangle$ to contain a bare electron with momentum $\underset{\sim}{p}_1, \eta_1$ and spin s_1, and a bare positron with momentum $\underset{\sim}{p}_2, \eta_2$ and spin s_2.

We also imagine the final scattering state $|f\rangle$ to be expanded in terms of bare quanta ("partons") in the same way. If we know all the amplitudes g, h, etc., we can then evaluate S_{fi} by moving $\mathbb{F}$ to the right past all of the parton creation operators until $\mathbb{F}$ acts on the vacuum state $|0\rangle$. That is, we write

$$\mathbb{F}\, b^\dagger \circ\circ\circ\, a^\dagger|0\rangle = \mathbb{F} b^\dagger \mathbb{F}^{-1} \circ\circ\circ\, \mathbb{F} a^\dagger \mathbb{F}^{-1}\, \mathbb{F}|0\rangle\, . \tag{II.51}$$

We note that $\mathbb{F}$ is invariant under $\mathscr{y}$-translations, and thus commutes with the momentum operator η. Since $|0\rangle$ is the only state with $\eta = 0$, we conclude that $\mathbb{F}|0\rangle = |0\rangle$. (This result can be formally assured by considering the operators in $\rho(x)$ to be normal-ordered.) The effect of $\mathbb{F}$ on the creation operators $b^\dagger$, $d^\dagger$, $a^\dagger$ is easily calculated using the equal-τ commutation relations (II.15). We find first that

$$\mathbb{F}\, \psi^\dagger(0,\underset{\sim}{x},\mathscr{y})\, \mathbb{F}^{-1} = e^{-i\chi(\underset{\sim}{x})}\, \psi^\dagger(0,\underset{\sim}{x},\mathscr{y}) \quad . \tag{II.52}$$

Upon Fourier-transforming this relation we obtain the convolution integral

$$\mathbb{F} b^\dagger(\underset{\sim}{p},\eta;s)\, \mathbb{F}^{-1} = \int \frac{d\underset{\sim}{p}'}{(2\pi)^2}\, b^\dagger(\underset{\sim}{p}',\eta;s)\, F(\underset{\sim}{p}'-\underset{\sim}{p}) \quad , \tag{II.53}$$

where

$$F(\underset{\sim}{q}) = \int d\underset{\sim}{x}\; e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\, e^{-i\chi(\underset{\sim}{x})} \quad . \tag{II.54}$$

Thus when a high energy bare electron passes through the potential at position $\underset{\sim}{x}$, the only effect of the potential is to multiply the electron wave function by an eikonal phase factor $\exp(-i\chi(\underset{\sim}{x}))$. (Note that the phase $\chi(\underset{\sim}{x})$ is simply the integral of the potential along the trajectory of the electron.) The momentum component η of the bare electron and its infinite momentum helicity s are conserved in the process, and no pairs are created.

The effect of F on the positron creation operators is equally simple. In passing through the potential each bare positron receives the opposite phase:

$$\mathbb{F}\, d^\dagger(\underset{\sim}{p},\eta;s)\, \mathbb{F}^{-1} = \int \frac{d\underset{\sim}{p}'}{(2\pi)^2}\, d^\dagger(\underset{\sim}{p}',\eta;s)\, F_c(\underset{\sim}{p}'-\underset{\sim}{p}) \; , \tag{II.55}$$

where

$$F_c(\underset{\sim}{q}) = \int d\underset{\sim}{x}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\, e^{+i\chi(\underset{\sim}{x})} \quad . \tag{II.56}$$

Finally, we find that the bare photons are unaffected by the potential:

$$\mathbb{F}\, a^{\dagger}(\underset{\sim}{p},\eta;\lambda)\, \mathbb{F}^{-1} = a^{\dagger}(\underset{\sim}{p},\eta;\lambda) \quad . \tag{II.57}$$

After we have moved F to the right past all of the parton creation operators, we are left with an expansion of the state $\mathbb{F}|i\rangle$ in terms of parton states (similar to the expansion (II.50) of $|i\rangle$). Assuming that the expansion of the final state $|f\rangle$ is also known, it is then a simple matter to compute the overlap S_{fi} of $|f\rangle$ with $\mathbb{F}|i\rangle$.

Of course we do not in fact know the amplitudes involved in the expansions of the states $|i\rangle$ and $|f\rangle$ in terms of bare particle states. In the examples treated in the next section we are forced to use approximate amplitudes calculated from perturbation theory. What we wish to emphasize here is the physical picture that emerges from the present discussion:

1) The scattering of high energy physical particles from the external potential is <u>not</u> simple. For example, it is <u>not</u> described by a single eikonal phase.

2) The physical particles can be viewed as being composed of certain constituent particles (called partons in the language of Feynman). In the present case the partons are the "bare" quanta created by the fields Ψ and $\underset{\sim}{A}$ at $\tau = 0$.

3) The scattering of high energy partons from the potential <u>is</u> simple.

4) The interaction of the partons among themselves is complicated, but at high energies these interactions are slowed down by relativistic time dilation. Therefore no parton-parton interactions take place during the finite time interval during which the partons interact with the external field.

Thus the scattering of high energy particles from the external field occurs in three steps. First the partons in the initial state interact among themselves during the infinite time interval $-\infty<\tau<0$. Then each individual parton scatters in a simple way from the external potential. Finally, the partons again interact among themselves during the infinite time interval $0<\tau<\infty$.

III. Applications

In this chapter we calculate the high energy limits of the cross sections for several interesting scattering processes. As we have seen, the contribution to the high energy limit of the S-matrix from the scattering of the individual partons off the external field can be calculated exactly. However, the interactions among the partons in the initial and final states do not simplify in the high energy limit. Thus we include these interactions only to a finite order in perturbation theory. Nevertheless, the required calculations in perturbation theory are quite easy because of the simple form of the matrix elements of the Hamiltonian in the infinite momentum frame.

We begin with a short discussion of the methods involved in the calculations, then proceed to the calculation of cross sections for electron scattering with second order vertex corrections, bremsstrahlung, pair production, Delbruck scattering, and electroproduction of μ-pairs in an external field.

A. Calculational Methods

In all of our applications we must compute the amplitudes involved in the expansions (II.50) of the initial and final states in terms of bare particle states. To do this, we recall the definition of the unitary evolution operator $U(\tau',\tau) = \exp(ih_0\tau')\exp(-i[h_0+h_I][\tau'-\tau])\exp(-ih_0\tau)$, where h_0 is the free particle Hamiltonian and h_0+h_I is the full Hamiltonain for quantum electrodynamics with no external potential. The final physical scattering state $|f(b)\rangle$ consisting of outgoing particles with momenta and helicities labeled by 'b' is related to the corresponding bare particle state $|b\rangle$ by $f(b)| = \langle b|U(\infty, 0)$. Similarly, the physical initial state $|i(a)\rangle$ is related to the corresponding bare particle state $|a\rangle$ by $|i(a)\rangle = U(0, -\infty)|a\rangle$. Thus the high energy limit of the scattering matrix, Eq. (II.46), can be written as

$$\langle b|S|a\rangle = \langle f(b)|\mathbb{F}|i(a)\rangle = \langle b|U(\infty, 0)\, \mathbb{F}\, U(0, -\infty)|a\rangle . \tag{III.1}$$

We need the expansion (II.50) of $|f(b)\rangle$ in terms of bare particle states $|n\rangle$: $\langle f(b)| = \sum_n \langle b|U(\infty, 0)|n\rangle\langle n|$. The amplitudes $\langle b|U(\infty, 0)|n\rangle$ can be calculated to a finite order in perturbation theory using the familiar perturbation expansion of $U(\infty, 0)$:

$$\begin{aligned}\langle f(b)| = \langle b| + \sum_n \langle b|h_I|n\rangle \frac{1}{H_f - H_n + i\epsilon} \langle n| \\ + \sum_{m,n} \langle b|h_I|m\rangle \frac{1}{H_f - H_m + i\epsilon} \langle m|h_I{}'n\rangle \frac{1}{H_f - H_n + i\epsilon} \langle n| + \ldots ,\end{aligned} \tag{III. 2}$$

where H_f is the energy of the final state and $h_0|m\rangle = H_m|m\rangle$.

Similarly, the initial state can be written as

$$|i(a)\rangle = \sum_n |n\rangle\langle n|U(0, -\infty)|a\rangle = |a\rangle + \sum_n |n\rangle \frac{1}{H_i - H_n + i\epsilon} \langle n|h_I|a\rangle + \ldots$$

However, since the initial state in our examples is always a one particle state, it is convenient to factor the wave function renormalization constant $\sqrt{Z_a}$ out of this expansion[16]:

$$\begin{aligned}|i(a)\rangle = \sqrt{Z_a}\Big\{|a\rangle + \sum_n{}' |n\rangle \frac{1}{H_i - H_n} \langle n|h_I|a\rangle \\ + \sum_n{}' \sum_m{}' |n\rangle \frac{1}{H_i - H_n} \langle n|h_I|m\rangle \frac{1}{H_i - H_m} \langle m|h_I|a\rangle + \ldots\Big\} .\end{aligned} \tag{III.3}$$

If $|a\rangle$ is, say, a one electron state then the sums Σ' exclude one electron states; the $i\epsilon$ terms in the energy denominators are then irrelevant. Since $U(0, -\infty)$

is unitary, the renormalization constant $\sqrt{Z_a}$ can be determined from the requirement

$$\langle i(a) | i(a') \rangle = \langle a | a' \rangle \quad . \tag{III.4}$$

Let us return now to the formula (III.1) for $\langle b|S|a\rangle$. It will prove convenient to explicitly separate the uninteresting "no scattering" term $\langle b|a\rangle$ from $\langle b|S|a\rangle$ before doing any calculations. This can be accomplished by noting that $\langle b|U(\infty, 0)\, \mathbb{1}\, U(0,-\infty)|a\rangle = \langle b|U(\infty,-\infty)|a\rangle$ is the S-matrix for quantum electrodynamics with no external potential, which is simply $\langle b|U(\infty, -\infty)|a\rangle = \langle b|a\rangle$ if $|a\rangle$ is a (stable) one particle state. Thus

$$\langle b|S|a\rangle = \langle b|a\rangle + \langle b|U(\infty, 0)\,[\mathbb{F} - \mathbb{1}]\,U(0, -\infty)|a\rangle \quad . \tag{III.5}$$

It is, of course, only the second term in (III.5) which is related to cross sections. With the normalization conventions used in this paper, the exact relationship is

$$d\sigma = \frac{1}{2\eta_a} \frac{d\mathbf{p}_1\, d\eta_1}{(2\pi)^3\, 2\eta_1} \cdots \frac{d\mathbf{p}_N\, d\eta_N}{(2\pi)^3\, 2\eta_N} (2\pi)\, \delta\left(\eta_a - \sum_{j=1}^{N} \eta_j\right) |\langle b|\mathcal{T}|a\rangle|^2 \tag{III.6}$$

where the transition amplitude $\langle b|\mathcal{T}|a\rangle$ is defined by

$$\langle b|U(\infty, 0)\,[\mathbb{F} - \mathbb{1}]\,U(0,-\infty)|a\rangle = (2\pi)\,\delta(\eta_a - \eta_b)\,\langle b|\mathcal{T}|a\rangle \quad . \tag{III.7}$$

B. <u>Electron Scattering</u>

We wish to calculate the amplitude

$$S_{fi} - \delta_{fi} = \langle e^-(p', s')|U(\infty, 0)\,[\mathbb{F} - \mathbb{1}]\,U(0,-\infty)|e^-(p, s)\rangle \tag{III.8}$$

for high energy electron scattering off an external field. We will calculate the amplitude to second order in the structure of the physical electron. Using the expansion (III.3) for $\langle e^-|U(\infty, 0)$ and $U(0,-\infty)|e^-\rangle$ and keeping terms to order e^2 we find with the help of (II.53) that

$$S_{fi} - \delta_{fi} = (2\pi)\,\delta(\eta - \eta')\,2\eta\,Z_2\left[F(\underset{\sim}{p}' - \underset{\sim}{p}) - (2\pi)^2\,\delta^2(\underset{\sim}{p}' - \underset{\sim}{p})\right]$$
$$\times \left\{ \delta_{ss'} + (2\pi)^{-3}\int d\underset{\sim}{p}_2 \int_0^{\eta} \frac{d\eta_2}{2\eta_2} \sum_{s_1,\lambda_2} e^2 \right. \tag{III.9}$$
$$\left. \times \frac{w^{\dagger}(s')\,\underset{\sim}{j}(p', p' - p_2)\cdot\underset{\sim}{\epsilon}(\lambda_2)w(s_1)\quad w^{\dagger}(s_1)\,\underset{\sim}{j}(p - p_2, p)\cdot\underset{\sim}{\epsilon}^*(\lambda_2)\,w(s)}{[H(p') - H(p' - p_2) - \omega(p_2)]\,[H(p) - H(p - p_2) - \omega(p_2)]} \right\}$$

Here $H(p) = (\underset{\sim}{p}^2 + m^2)/2\eta$ is the free electron Hamiltonian, $\omega(p) = \underset{\sim}{p}^2/2\eta$ is the free photon Hamiltonian, and Z_2 is the electron wave function renormalization constant (to be calculated to order e^2). The two terms in Eq. (III.9) are represented by τ-ordered diagrams in Fig. 2a and 2b. The figures also clarify the kinematic notation chosen here. The black dots in the diagrams refer to the eikonal factor $[F(\underset{\sim}{p}' - \underset{\sim}{p}) - (2\pi)^2\,\delta^2(\underset{\sim}{p}' - \underset{\sim}{p})]$.

In order to discuss the general form of the scattering amplitude, let us write (III.9) in the abbreviated form

$$S_{fi} - \delta_{fi} = (2\pi)\,\delta(\eta - \eta')\,2\eta[F(\underset{\sim}{q}) - (2\pi)^2\,\delta^2(\underset{\sim}{q})]\quad w^{\dagger}(s')\,M(\underset{\sim}{p}',\eta; \underset{\sim}{p},\eta)\,w(s) \tag{III.10}$$

where $q^{\mu} = p'^{\mu} - p^{\mu}$. One important result which we notice immediately is that the second order vertex correction does not destroy the proportionality between the scattering amplitude and the eikonal factor that one finds if the electron structure is neglected altogether.[1] However, it should be pointed out that if the scattering amplitude were calculated to fourth order in the structure of the electron, a diagram like Fig. 3 would appear and this proportionality would be lost.[12]

The effects of the electron structure are contained in the factor $w^{\dagger}Mw$. It will come as no surprise that the four matrix elements of M are simply related to two invariant form factors $F_{1,2}(q^2)$. It is instructive to derive this relation using the invariance principles which appear naturally in the infinite momentum frame. Using Eq. (III.10) and the table of matrix elements, Table I, then we can easily verify that $w^\dagger M w$ is invariant under the following symmetry operations:

1) Lorentz z-boosts: momenta transform according to $(\eta, \underset{\sim}{p}) \longrightarrow (e^{\omega}\eta, \underset{\sim}{p})$, helicities remain unchanged.

2) "Galilean boosts": momenta transform according to $(\eta, \underset{\sim}{p}) \longrightarrow (\eta, \underset{\sim}{p} + \eta u)$, helicities remain unchanged.

3) Rotations in the (x^1, x^2)-plane.

4) "Parity": momenta transform according to $(\eta, p^1, p^2) \longrightarrow (\eta, p^1, -p^2)$, helicities are reversed.

For $q \neq 0$, the four matrices $\mathbb{1}$, $\underset{\sim}{q} \cdot \underset{\sim}{\sigma}$, $\underset{\sim}{q} \times \underset{\sim}{\sigma} = q^1\sigma^2 - q^2\sigma^1$, σ_z are linearly independent. Thus M can be written in the form

$$M(p', p) = a\,\mathbb{1} + b\,\underset{\sim}{q} \cdot \underset{\sim}{\sigma} + c\,\underset{\sim}{q} \times \underset{\sim}{\sigma} + d\sigma_z \ . \tag{III.11}$$

The coefficients a, b, c, d will then be functions of p' and p, or, equivalently, of $\eta(=\eta')$, $\underset{\sim}{p}' + \underset{\sim}{p}$, $\Theta_q = \tan^{-1}(q^2/q^1)$, and $\underset{\sim}{q}^2$. But the invariance of $w^\dagger M w$ under Lorentz z-boosts implies that the coefficients are independent of η; invariance under "Galilean boosts" implies that they are independent of $\underset{\sim}{p}' + \underset{\sim}{p}$; and rotational invariance implies that they are independent of Θ_q. Thus each coefficient is a function of $\underset{\sim}{q}^2$ only. Finally, invariance of $w^\dagger M w$ under the "parity" operation implies that $c(\underset{\sim}{q}^2) = -c(\underset{\sim}{q}^2)$ and $d(\underset{\sim}{q}^2) = -d(\underset{\sim}{q}^2)$; hence $c = d = 0$. The remaining form factors a and b are functions of $\underset{\sim}{q}^2$; but since $\eta_q = 0$,

$$q^2 \equiv q^\mu q_\mu = 2\eta_q H_q - \underset{\sim}{q}^2 = -\underset{\sim}{q}^2 \tag{III.12}$$

Therefore the expansion of M takes the form

$$M(p', p) = a(q^2)\,\mathbb{1} + b(q^2)\,\underset{\sim}{q} \cdot \underset{\sim}{\sigma} \tag{III.13}$$

This analysis can be compared to the general analysis of electron scattering from a weak external field which concludes that the S-matrix, calculated to first order in the external potential and all orders in the structure of the electron, takes the form

$$S_{fi} - \delta_{fi} = -i\int d^4x \, ea_\mu(x) \, e^{iq\cdot x} \times \overline{U}(p', s') \left\{\gamma^\mu F_1(q^2) + \frac{i}{2m}\sigma^{\mu\nu} q_\nu F_2(q^2)\right\} U(p, s) . \tag{III.14}$$

In the high energy limit, Eq. (III.14) becomes

$$S_{fi} - \delta_{fi} = -2\pi i \, \delta(\eta' - \eta) \left[\int d\mathbf{x} \, e^{-i\mathbf{q}\cdot\mathbf{x}} \int d\tau \, ea_0(\tau, \mathbf{x}, 0)\right] \overline{U}(p', s') \left\{\gamma^0 F_1(q^2) + \frac{i}{2m} \sigma^{0\nu} q_\nu F_2(q^2)\right\} U(p, s).$$

When this result is converted to the notation used in this paper, it reads

$$S_{fi} - \delta_{fi} = (2\pi) \, \delta(\eta' - \eta) \, 2\eta \left[-i\int d\mathbf{x} \, \chi(\mathbf{x}) \exp(-i\mathbf{q}\cdot\mathbf{x})\right] \times w^\dagger(s') \left\{F_1(q^2) \, \mathbb{1} + \frac{i}{2m} F_2(q^2) \, \mathbf{q}\cdot\boldsymbol{\sigma}\right\} w(s). \tag{III.15}$$

Comparison of this result with (III.10) and (III.13) shows that the form factor $a(q^2)$ can be identified with $F_1(q^2)$ and $b(q^2)$ can be identified with $[i/(2m)]F_2(q^2)$. Thus our result is

$$S_{fi} - \delta_{fi} = (2\pi) \, \delta(\eta' - \eta) \, 2\eta \left[F(\mathbf{q}) - (2\pi)^2 \delta^2(\mathbf{q})\right] \times w^\dagger(s') \left\{F_1(q^2) \, \mathbb{1} + \frac{i}{2m} F_2(q^2) \, \mathbf{q}\cdot\boldsymbol{\sigma}\right\} w(s). \tag{III.16}$$

Apparently the amplitude for scattering with no change in helicity is proportional to $F_1(q^2)$, whereas the helicity flip amplitudes are proportional to $F_2(q^2)$. For instance

$$S_{fi}\left(s'=\tfrac{1}{2}, s=\tfrac{1}{2}\right) = \delta_{fi} + (2\pi)\,\delta(\eta' - \eta) \, 2\eta\left[F(\mathbf{q}) - (2\pi)^2 \delta^2(\mathbf{q})\right] F_1(q^2) \tag{III.17}$$

$$S_{fi}\left(s'=-\tfrac{1}{2}, s=\tfrac{1}{2}\right) = (2\pi)\,\delta(\eta' - \eta) \, 2\eta\left[F(\mathbf{q}) - (2\pi)^2 \delta^2(\mathbf{q})\right] \frac{iq_+}{\sqrt{2}\,m} F_2(q^2), \tag{III.18}$$

where $q_\pm = 2^{-1/2}(q^1 \pm iq^2)$.

We are now in a position to return to Eq. (III.9) in order to calculate the electron form factors. We begine with the helicity flip amplitude and the form factor F_2. It is convenient to choose a coordinate system (by transforming the coordinates with a "Galilean boost" if necessary) so that

$$p^\mu = (\eta, -\underset{\sim}{p}', H), \qquad p'^\mu = (\eta, \underset{\sim}{p}', H), \qquad q^\mu = (0, 2\underset{\sim}{p}', 0) .$$

Then the energy denominators in (III.9) become

$$\begin{aligned} H(p') - H(p' - p_2) - \omega(p_2) &= -\frac{1}{2\eta}\left[\frac{(\underset{\sim}{p}_2 - \beta\underset{\sim}{p}')^2 + \beta^2 m^2}{\beta(1-\beta)}\right] \\ H(p) - H(p - p_2) - \omega(p_2) &= -\frac{1}{2\eta}\left[\frac{(\underset{\sim}{p}_2 + \beta\underset{\sim}{p}')^2 + \beta^2 m^2}{\beta(1-\beta)}\right], \end{aligned} \qquad \text{(III.19)}$$

where

$$\beta = \eta_2/\eta .$$

The numerator factor in the helicity flip amplitude is trivially calculated with the aid of Table I:

$$\sum_{s_1, \lambda_2} w^\dagger(-1/2)\, \underset{\sim}{j} \cdot \underset{\sim}{\epsilon}\, w\, w^\dagger \underset{\sim}{j} \cdot \underset{\sim}{\epsilon}^* w(+1/2)$$

$$= \frac{im\ \eta_2}{\sqrt{2}\,\eta(\eta - \eta_2)}\left(\frac{p_{2+}}{\eta_2} - \frac{p'_+}{\eta}\right) + \left(\frac{p_{2+}}{\eta_2} - \frac{p'_+}{\eta}\right)\frac{-im\ \eta_2}{\sqrt{2}\,\eta(\eta - \eta_2)}$$

$$= \frac{1}{\eta^0}\sqrt{2}\, im \frac{\beta}{1-\beta}\, p'_+ \ . \qquad \text{(III.20)}$$

If we insert these results (III.19) and (III.20) back into (III.9) and use (III.18) to identify $F_2(q^2)$ we find[19]

$$F_2(\underset{\sim}{q}^2) = \frac{4\alpha m^2}{(2\pi)^2}\int_0^1 d\beta\, \beta^2(1-\beta)\int d\underset{\sim}{p}_2 \times \left[\left(\underset{\sim}{p}_2^2 + \beta^2\left[\tfrac{1}{4}\underset{\sim}{q}^2 + m^2\right]\right)^2 - \beta^2(\underset{\sim}{p}_2 \cdot \underset{\sim}{q})^2\right]^{-1}.$$

(III.21)

The integrals are elementary and we find without difficulty

$$F_2(\underset{\sim}{q}^2) = \frac{\alpha}{2\pi}\left[\frac{2m^2}{|\underset{\sim}{q}|\,(\underset{\sim}{q}^2+4m^2)^{1/2}}\;\log\left(\frac{(\underset{\sim}{q}^2+4m^2)^{1/2}+|\underset{\sim}{q}|}{(\underset{\sim}{q}^2+4m^2)^{1/2}-|\underset{\sim}{q}|}\right)\right]. \tag{III.22}$$

We recognize this equation as a familiar expression for the second order contribution ot $F_2(q^2)$.[20] Letting $\underset{\sim}{q}^2 \to 0$, we obtain

$$F_2(0) = \frac{\alpha}{2\pi} \tag{III.23}$$

which is the famous anomalous magnetic moment of the electron.

Before turning to consider the form factor $F_1(q^2)$, we shall point out the calculational advantages that the formulation of infinite momentum perturbation theory used here has over others that have appeared in the literature.[21] First, no high energy approximation has to be used to extract the important pieces of the energy denominators and vertices. This occurs because of the simple scaling behavior our kinematic variables have under boosts in the z-direction. Secondly, the electrodynamic vertices between infinite momentum helicity states are so simple that traces can be altogether avoided.

We now turn our attention to the helicity nonflip amplitude and the form factor F_1. Using Table I, we calculate the numerator factor in the amplitude (III.9):

$$\sum_{s_1,\lambda_2} w^\dagger(+1/2)\,\underset{\sim}{j}\cdot\underset{\sim}{\epsilon}\,w\,w^\dagger\underset{\sim}{j}\cdot\underset{\sim}{\epsilon}^*\,w(+1/2)$$

$$= \left(\frac{p_{2+}}{\eta_2} - \frac{p'_+ - p_{2+}}{\eta-\eta_2}\right)\left(\frac{p_{2-}}{\eta_2} + \frac{p'_- + p_{2-}}{\eta-\eta_2}\right)$$

$$+\left(\frac{p_{2-}}{\eta_2} - \frac{p'_-}{\eta}\right)\left(\frac{p_{2+}}{\eta_2} + \frac{p'_+}{\eta}\right) + \left(\frac{m}{\sqrt{2}}\,\frac{\eta_2}{\eta(\eta-\eta_2)}\right)^2 \tag{III.24}$$

$$= \left[2\eta^2\,\beta^2(1-\beta)^2\right]^{-1}\left\{\left(\underset{\sim}{p}_2^2 - \beta^2\underset{\sim}{p}'^2\right)(1+(1-\beta)^2) + m^2\beta^4\right.$$

$$\left. - 2i(\underset{\sim}{p}'\times\underset{\sim}{p}_2)\;\beta^2(\beta-2)\right\},$$

where we have used the fact that $2k_+p_- = \underset{\sim}{k}\cdot\underset{\sim}{p} - i\underset{\sim}{k}\times\underset{\sim}{p}$.

If we substitute the expressions (III.24) and (III.19) for the numerator and energy denominators in (III.9) and use (III.17) to identify $F_1(q^2)$, we find

$$F_1(\underset{\sim}{q}^2) = Z_2(1 + I(\underset{\sim}{q}^2)), \qquad \text{(III.25)}$$

where

$$I(\underset{\sim}{q}^2) = \frac{2\alpha}{(2\pi)^2}\int_0^1 d\beta \int d\underset{\sim}{p}_2\, \beta^{-1}\left[\left(\underset{\sim}{p}_2^2 - \tfrac{1}{4}\beta^2\underset{\sim}{q}^2\right)(1+(1-\beta)^2) + m^2\beta^4\right]$$
$$\times\left[\left(\underset{\sim}{p}_2^2 + \beta^2\left[m^2 + \tfrac{1}{4}\underset{\sim}{q}^2\right]\right)^2 - \beta^2(\underset{\sim}{p}_2\cdot\underset{\sim}{q})^2\right]^{-1} \qquad \text{(III.26)}$$

In (III.26) we have used the fact that the term in the numerator proportional to $\underset{\sim}{p}_2\times\underset{\sim}{q}$ will not contribute to the integral.

The integral defining $I(q^2)$ diverges as $\beta\longrightarrow 0$ and as $\underset{\sim}{p}_2^2\longrightarrow\infty$. However these divergences are cancelled by corresponding divergences in Z_2, just as in conventional treatments of the second order vertex. If we calculate Z_2 to order α using

$$Z_2\left\langle e^-\left(p', \tfrac{1}{2}\right)\middle| U(\infty, 0)\, U(0, -\infty)\middle| e^-\left(p, -\tfrac{1}{2}\right)\right\rangle = (2\pi)^3\, 2\eta\,\delta(\eta' - \eta)\,\delta^2(\underset{\sim}{p}' - \underset{\sim}{p}), \qquad \text{(III.27)}$$

we find easily that

$$Z_2 = (1 + I(0))^{-1}\ . \qquad \text{(III.28)}$$

Thus $F_1(\underset{\sim}{q}^2)$, calculated to order α, is

$$F_1(\underset{\sim}{q}^2) = (1 + I(0))^{-1}\,(1 + I(\underset{\sim}{q}^2))$$
$$= 1 + (I(\underset{\sim}{q}^2) - I(0)) \qquad \text{(III.29)}$$

The integral defining $I(\underset{\sim}{q}^2)_{\text{renormalized}} = I(\underset{\sim}{q}^2) - I(0)$ is now better defined: the β-integral converges for fixed $\underset{\sim}{p}_2$ and the $\underset{\sim}{p}_2$-integral converges for fixed β. However, the integral still has the familiar infrared divergence coming from the region near $\beta = 0$, $\underset{\sim}{p}_2 = 0$. In an explicit evaluation of $F_1(q^2)$, this infrared divergence could be eliminated by inserting a small photon mass in the energy denominators.

Before proceeding to the next example, we should point out that the use of the eikonal approximation in (III.8) is self-consistent, even though Fig. 2b includes a loop. This is true because the loop integrals are well behaved in the region $\beta \approx 1$, where the electron in the intermediate state is no longer a "right mover." If the integrals had diverged at the endpoint $\beta = 1$, the claim that Eq. (III.8) closely approximates the effect of external field on the physical particle would have been unjustified.

C. Bremsstrahlung

In this section we shall calculate the helicity amplitudes for the experimentally interesting process of bremsstrahlung off an external field. The matrix element of interest is then,

$$S_{fi} = \langle\, e(p', s')\gamma(k,\lambda) \mid U(\infty, 0)(\mathbb{F}-1)\, U(0, -\infty) \mid e(p, s) \,\rangle \ . \tag{III.30}$$

If we insert our expression for the physical states from Section III.A accurate to terms of order e, we readily find

$$\begin{aligned} S_{fi} = {} & (2\pi)\, \delta(\eta - \eta' - \eta_k)\, 2 \left[\eta \eta'\right]^{\frac{1}{2}} \left[F(\underset{\sim}{p}' + \underset{\sim}{k} - \underset{\sim}{p}) - (2\pi)^2 \delta^2(\underset{\sim}{p}' + \underset{\sim}{k} - \underset{\sim}{p})\right] \\ & \times e \left[\frac{w^\dagger(s')\, \underset{\sim}{j}(p', p' + k) \cdot \underset{\sim}{\epsilon}^*(\lambda) w(s)}{H(p') + \omega(k) - H(p' + k)} + \frac{w^\dagger(s')\, \underset{\sim}{j}(p - k, p) \cdot \underset{\sim}{\epsilon}^*(\lambda)\, w(s)}{H(p) - \omega(k) - H(p - k)} \right] . \end{aligned} \tag{III.31}$$

The terms in this expression can be visualized with the aid of Fig. 4a and b respectively.

In order to discuss bremsstrahlung conveniently we choose a coordinate system with its z-axis along the direction of the outgoing photon. The energy denominators in Eq. (III.31) become,

$$H(p') + \omega(k) - H(p'+k) = \frac{\eta_k}{2\eta\eta'}(\underset{\sim}{p}'^2 + m^2)$$

$$H(p) - \omega(k) - H(p-k) = -\frac{\eta_k}{2\eta\eta'}(\underset{\sim}{p}^2 + m^2) \; .$$

Finally, if we choose definite helicities for the incoming and outgoing particles we obtain, with the aid of Table 1, the infinite momentum helicity amplitudes for bremsstrahlung,

$$S_{fi} = (2\pi)\delta(\eta - \eta' - \eta_k)\, 2[\eta\eta']^{\frac{1}{2}} \left[F(\underset{\sim}{p}'-\underset{\sim}{p}) - (2\pi)^2\delta^2(\underset{\sim}{p}'-\underset{\sim}{p})\right] e\, M(s \to s', \lambda)$$

$$M(\tfrac{1}{2} \to \tfrac{1}{2}, 1) = \frac{2\eta}{\eta_k}\left\{-\frac{p'_-}{\underset{\sim}{p}'^2 + m^2} + \frac{p_-}{\underset{\sim}{p}^2 + m^2}\right\} \qquad \text{(III.32)}$$

$$M(\tfrac{1}{2} \to \tfrac{1}{2}, -1) = \frac{2\eta'}{\eta_k}\left\{-\frac{p'_+}{\underset{\sim}{p}'^2 + m^2} + \frac{p_+}{\underset{\sim}{p}^2 + m^2}\right\}$$

$$M(\tfrac{1}{2} \to -\tfrac{1}{2}, 1) = \sqrt{2}\; i m\left\{-\frac{1}{\underset{\sim}{p}'^2 + m^2} + \frac{1}{\underset{\sim}{p}^2 + m^2}\right\}$$

$$M(\tfrac{1}{2} \to -\tfrac{1}{2}, -1) = 0 \; .$$

These results should prove useful in detailed calculations with specified external fields. For cases in which the external field can be treated perturbatively, one can easily show that Eqs. (III.32)lead to the high energy limit of the Bethe-Heitler formula.

D. Pair Production

We wish to calculate the scattering amplitude

$$S_{fi} = \langle e^-(p_1, s_1) e^+(p_2, s_2) \mid U(\infty, 0) [\mathbb{F}-\mathbb{1}] \; U(0, -\infty) \mid \gamma(k, \lambda) \rangle . \qquad \text{(III.33)}$$

Proceeding along familiar lines, we insert perturbation expansions of the physical states accurate to first order in e and find

$$S_{fi} = (2\pi)\,\delta(\eta_k - \eta_1 - \eta_2)\; 2\left[\eta_1\eta_2\right]^{\frac{1}{2}}\; e \int \frac{d\underset{\sim}{p}}{(2\pi)^2}\; \frac{w^\dagger(s_1)\,\underset{\sim}{j}\,(p, p-k)\cdot \underset{\sim}{\epsilon}(\lambda)\, w(-s_2)}{\omega(k) - H(p) - H(k-p)}$$

$$\times \left[F(\underset{\sim}{p}_1 - \underset{\sim}{p})\, F_c(\underset{\sim}{p}_2 + \underset{\sim}{p} - \underset{\sim}{k}) - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \underset{\sim}{p})\,\delta^2(\underset{\sim}{p}_2 + \underset{\sim}{p} - \underset{\sim}{k})\right] , \qquad \text{(III.34)}$$

which can be visualized with the aid of Fig. 5.

If we now choose the z-axis along the direction of the photon and calculate helicity amplitudes, we find

$$S_{fi} = (2\pi)\delta\,(\eta_k - \eta_1 - \eta_2)\; 2\left[\eta_1\eta_2\right]^{\frac{1}{2}}\; e\,(2\pi)^{-2} \int d\underset{\sim}{p}\; M(\lambda \to s_1, s_2)$$

$$\times \left[F(\underset{\sim}{p}_1 - \underset{\sim}{p})\, F_c(\underset{\sim}{p}_2 + \underset{\sim}{p}) - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \underset{\sim}{p})\,\delta^2(\underset{\sim}{p}_2 + \underset{\sim}{p})\right] , \qquad \text{(III.35)}$$

where

$$M(1 \to \tfrac{1}{2}, -\tfrac{1}{2}) = \left(\frac{-2\eta_1}{\eta_k}\right) \frac{p_+}{\underset{\sim}{p}^2 + m^2}$$

$$M(1 \to -\tfrac{1}{2}, \tfrac{1}{2}) = \left(\frac{2\eta_2}{\eta_k}\right) \frac{p_+}{\underset{\sim}{p}^2 + m^2}$$

$$M(1 \to \tfrac{1}{2}, \tfrac{1}{2}) = \left(\sqrt{2}\, i m\right) \frac{1}{\underset{\sim}{p}^2 + m^2}$$

$$M(1 \to -\tfrac{1}{2}, -\tfrac{1}{2}) = 0 \quad .$$

It is interesting to convert the momentum integration in (III.35) to an integration in coordinate space in order to appreciate the two-dimensional Galilean invariance group which manifests itself in the infinite momentum frame. To begin, we drop the special requirement that the transverse momentum $\underset{\sim}{k}$ of the photon be zero and return to the energy denominator in (III.34):

$$\omega(k, \eta_k) - H(\underset{\sim}{p}, \eta_1) - H(\underset{\sim}{k} - \underset{\sim}{p}, \eta_2) = (2\eta_k)^{-1}\left[\underset{\sim}{p} + (\underset{\sim}{k} - \underset{\sim}{p})\right]^2 - (2\eta_1)^{-1}\left[\underset{\sim}{p}^2 + m^2\right] - (2\eta_2)^{-1}\left[(\underset{\sim}{k}-\underset{\sim}{p})^2 + m^2\right]$$

This is a rather messy function of the momentum $\underset{\sim}{p}$ of the electron and the momentum $(\underset{\sim}{k} - \underset{\sim}{p})$ of the positron in the intermediate state. As is usual with two body problems in "non-relativistic" quantum mechanics, it pays to change variables to the total momentum, $\underset{\sim}{k}$, of the two particles and their relative momentum. Since η plays the role of particle mass in the nonrelativistic analogy, the relative momentum is

$$\underset{\sim}{q} = \bar{\eta} \left\{ \frac{\underset{\sim}{p}}{\eta_1} - \frac{\underset{\sim}{k} - \underset{\sim}{p}}{\eta_2} \right\} , \qquad \text{(III.36)}$$

where

$$\bar{\eta} = \eta_1 \eta_2 / (\eta_1 + \eta_2)$$

is the "reduced mass" of the pair. When written as a function of $\underset{\sim}{k}$ and $\underset{\sim}{q}$, the energy denominator is independent of $\underset{\sim}{k}$:

$$\omega(k, \eta_k) - H(\underset{\sim}{p}, \eta_1) - H(\underset{\sim}{k} - \underset{\sim}{p}, \eta_2) = -(2\bar{\eta})^{-1}\left[\underset{\sim}{q}^2 + m^2\right] \qquad \text{(III.37)}$$

(In non-relativistic terms, this is minus the "internal energy" of the pair.) Similarly, the vertex matrix element $w^{\dagger} \underset{\sim}{j} \epsilon \cdot w$ in (III.34) is a function of the relative momentum q only. After a little algebra we obtain the explicit form,

$$\frac{e\, w^{\dagger}(s_1)\, \underset{\sim}{j}\,(\underset{\sim}{p},\eta_1; p-k, -\eta_2)\cdot \underset{\sim}{\epsilon}(\lambda)\, w(-s_2)}{\omega(k,\eta_k) - H(p,\eta_1) - H(\underset{\sim}{k}-\underset{\sim}{p},\eta_2)} \equiv w^{\dagger}(s_1)\, \widetilde{\underset{\sim}{G}}(\underset{\sim}{q};\eta_1,\eta_2)\, w(-s_2)\cdot \underset{\sim}{\epsilon}(\lambda) \;,$$

$$\widetilde{\underset{\sim}{G}}(\underset{\sim}{q};\eta_1,\eta_2) = e\left\{\left(\frac{\eta_2-\eta_1}{\eta_k}\right) q - i(\underset{\sim}{q}\times\hat{z})\,\sigma_z + im\,\underset{\sim}{\sigma}\right\}(\underset{\sim}{q}^2+m^2)^{-1} \;, \qquad \text{(III.38)}$$

where $\underset{\sim}{q}\times\hat{z} = (q^2, -q^1)$.

Using these results, we can write (III.34) as a coordinate-space integral. Let $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ be the coordinates of the electron and positron respectively in the Fourier expansions (II.54) and (II.55) of the eikonal factors, and define

$$\underset{\sim}{R} = \eta_k^{-1}(\eta_1 \underset{\sim}{x}_1 + \eta_2 \underset{\sim}{x}_2) = \text{coordinate of the center of "mass" of the pair}$$

$$\underset{\sim}{r} = \underset{\sim}{x}_1 - \underset{\sim}{x}_2 = \text{relative coordinate.} \qquad \text{(III.39)}$$

Then we find

$$S_{fi} = (2\pi)\,\delta(\eta_k-\eta_1-\eta_2)\, 2\left[\eta_1\eta_2\right]^{\frac{1}{2}} \int d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\, e^{-i\underset{\sim}{p}_1\cdot \underset{\sim}{x}_1}\, e^{-i\underset{\sim}{p}_2\cdot\underset{\sim}{x}_2}\left[e^{-i\chi(\underset{\sim}{x}_1)}\, e^{+i\chi(\underset{\sim}{x}_2)} - 1\right]$$

$$\times\; w^{\dagger}(s_1)\, \underset{\sim}{G}(\underset{\sim}{r};\eta_1,\eta_2)\, w(-s_2)\cdot \underset{\sim}{\epsilon}(\lambda)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}} \;, \qquad \text{(III.40)}$$

where

$$\underset{\sim}{G}(\underset{\sim}{r};\eta_1,\eta_2) = (2\pi)^{-2}\int d\underset{\sim}{q}\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\, \widetilde{\underset{\sim}{G}}(\underset{\sim}{q};\eta_1,\eta_2) \;.$$

It is interesting to interpret the various factors in (III.40). First, $\underset{\sim}{\epsilon}(\lambda)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{R})$ is the wave function of the initial bare photon. Multiplying this by $\underset{\sim}{G}(\underset{\sim}{r})$ tells us the composition of the physical photon in terms of its constituents, which, to first order, are an electron and a positron.[22] Hence we might refer to $\underset{\sim}{G}(\underset{\sim}{r}) \cdot \underset{\sim}{\epsilon}(\lambda) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{R})$ as the first order approximation to the wave function of the physical photon. The "internal" wave function $\underset{\sim}{G}(\underset{\sim}{r})$ satisfies a two-dimensional Schroedinger equation with a point source,

$$\left(-\frac{1}{2\eta}\underset{\sim}{\nabla}^2 + \frac{m^2}{2\eta}\right)\underset{\sim}{G}(\underset{\sim}{r}) = \frac{e}{2\eta}\left\{-i\left(\frac{\eta_2-\eta_1}{\eta_k}\right)\underset{\sim}{\nabla} - (\underset{\sim}{\nabla}\times\hat{z})\,\sigma_z + im\underset{\sim}{\sigma}\right\}\delta^2(\underset{\sim}{r}) .$$

The solution of this equation which vanishes as $|r| \to \infty$ is simply related to the modified Bessel function K_0 :

$$\underset{\sim}{G}(\underset{\sim}{r}) = \frac{e}{2\pi}\left\{-i\left(\frac{\eta_2-\eta_1}{\eta_k}\right)\underset{\sim}{\nabla} - (\underset{\sim}{\nabla}\times\hat{z})\,\sigma_z + im\underset{\sim}{\sigma}\right\} K_0(m\,|\,\underset{\sim}{r}\,|) .$$

The next factor in Eq. (III.40), the eikonal phase factor, tells us how the constituents of the physical photon interact with the external field. Finally, the factors $w^{\dagger}(s_1)\exp(-i\underset{\sim}{p}_1\cdot\underset{\sim}{x}_1)$ and $w(-s_2)\exp(-i\underset{\sim}{p}_2\cdot\underset{\sim}{x}_2)$ are the wave functions of the final electron and positron (calculated to zeroth order). Evaluation of the S-matrix is completed by integrating over the coordinates $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ of the electron and positron and multiplying by (2π) times an η-conserving delta function and by a fermion normalization factor $(2\eta_1)^{\frac{1}{2}}\,(2\eta_2)^{\frac{1}{2}}$.

E. Delbruck Scattering

Let us turn our attention now to the problem of photon scattering off an external field. We shall see that our scattering theory gives a clear and concise derivation of the amplitude for this process.

The matrix element we wish to calculate is

$$S_{fi} - \delta_{fi} = \langle\, \gamma(p',\lambda')\,|\, U(\infty,0)\,[\mathbb{F}-1]\, U(0,-\infty)\,|\,\gamma(p,\lambda)\,\rangle \,. \qquad \text{(III.41)}$$

If we insert the expansion of the physical photon state into (III.41) and calculate to order e^2, we find

$$S_{fi} - \delta_{fi} = e^2 (2\pi)^{-4} \, \delta(\eta' - \eta) \int_0^{\eta} d\eta_1 \int d\underset{\sim}{p}_1 \, d\underset{\sim}{p}_1' \sum_{s_1, s_2}$$

$$\times \left[F(\underset{\sim}{p}_1' - \underset{\sim}{p}_1) \, F_c(\underset{\sim}{p}_2' - \underset{\sim}{p}_2) - (2\pi)^4 \delta^2 (\underset{\sim}{p}_1 - \underset{\sim}{p}_1') \delta^2 (\underset{\sim}{p}_2' - \underset{\sim}{p}_2) \right]$$

$$\times \, w^{\dagger}(s_1) \, \underset{\sim}{j}\, (p_1, - p_2) \cdot \underset{\sim}{\epsilon}(\lambda) \, w(-s_2) \, w^{\dagger}(-s_2) \, j\, (-p_2', p_1') \cdot \epsilon^{*}(\lambda') w(s_1) \qquad \text{(III.42)}$$

$$\times \left[\omega(p) - H(p_1) - H(p_2) \right]^{-1} \left[\omega(p') - H(p_1') - H(p_2') \right]^{-1}$$

where

$$p_1 = (\underset{\sim}{p}_1, \eta_1) \qquad p_1' = (\underset{\sim}{p}_1', \eta_1)$$

$$p_2 = (\underset{\sim}{p} - \underset{\sim}{p}_1, \eta - \eta_1) \qquad p_2' = (\underset{\sim}{p}' - \underset{\sim}{p}_1', \eta - \eta_1)$$

This formula is visualized, and its kinematics are defined, in the τ-ordered diagram Fig. 6.

We are now faced with two related problems. First, the integrand in (III.42) is a very messy function of the independent momenta $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_1'$. Second, the momentum integration is divergent: if the integrals are cut off in an arbitrary non-covariant fashion, the result will depend on the cutoff parameter. The remedy is simple. Since S_{fi} is invariant under the Galilean symmetry group discussed in Section II.A and Section III.B, it will be to our advantage to use integration variables which are invariant under this group.

We choose to make use of four Galilean-invariant momenta $\underset{\sim}{r}$, $\underset{\sim}{q}$, $\underset{\sim}{\ell}$ and $\underset{\sim}{Q}$. The momenta $\underset{\sim}{r}$ and $\underset{\sim}{q}$ are defined so that the momentum transfer from the external potential to the electron in the intermediate state is $\underset{\sim}{r} + \underset{\sim}{q}$ and the momentum transfer to the positron is $\underset{\sim}{r} - \underset{\sim}{q}$:

$$\begin{aligned} \underset{\sim}{p}_1' - \underset{\sim}{p}_1 &= \underset{\sim}{r} + \underset{\sim}{q} \\ \underset{\sim}{p}_2' - \underset{\sim}{p}_2 &= \underset{\sim}{r} - \underset{\sim}{q} \; . \end{aligned} \tag{III.43}$$

The momenta $\underset{\sim}{\ell}$ and $\underset{\sim}{Q}$ are defined so that the "relative momentum" of the electron-positron pair is $\underset{\sim}{\ell} - \underset{\sim}{Q}$ before the interaction with the external field, and $\underset{\sim}{\ell} + \underset{\sim}{Q}$ after the interaction:

$$\begin{aligned} \underset{\sim}{\ell} - \underset{\sim}{Q} &= \bar{\eta} \left[\frac{\underset{\sim}{p}_1}{\eta_1} - \frac{\underset{\sim}{p}_2}{\eta_2} \right] \\ \underset{\sim}{\ell} + \underset{\sim}{Q} &= \bar{\eta} \left[\frac{\underset{\sim}{p}_1'}{\eta_1} - \frac{\underset{\sim}{p}_2'}{\eta_2} \right] \; , \end{aligned} \tag{III.44}$$

where $\bar{\eta} = \eta_1 \eta_2 / \eta$ is the "reduced mass" of the pair. We will use $\underset{\sim}{q}$ and $\underset{\sim}{\ell}$ as integration variables instead of $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_1'$. The momentum $\underset{\sim}{r}$ is, of course, fixed by the external momenta: $2\underset{\sim}{r} = \underset{\sim}{p}' - \underset{\sim}{p}$. We find with a little algebra that $\underset{\sim}{Q}$ is given in terms of $\underset{\sim}{r}$ and $\underset{\sim}{q}$ by

$$\underset{\sim}{Q} = \tfrac{1}{2} (\underset{\sim}{r} + \underset{\sim}{q}) - \alpha \underset{\sim}{r} \; , \tag{III.45}$$

where we have defined

$$\alpha = \eta_1 / \eta \; . \tag{III.46}$$

When this change of variables has been made, the scattering matrix takes the form

$$S_{fi} - \delta_{fi} = e^2 (2\pi)^{-4} \eta\, \delta(\eta' - \eta) \int d\underset{\sim}{q} \left[F(\underset{\sim}{r} + \underset{\sim}{q})\, F_c(\underset{\sim}{r} - \underset{\sim}{q}) - (2\pi)^4 \delta(\underset{\sim}{r} + \underset{\sim}{q}) \delta(\underset{\sim}{r} - \underset{\sim}{q}) \right] M_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') \tag{III.47}$$

where

$$M_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') = \int_0^1 d\alpha \int^\Lambda d\underset{\sim}{\ell} \sum_{s_1 s_2}$$

$$w^\dagger(s_1)\, \underset{\sim}{j}(p_1, -p_2) \cdot \underset{\sim}{\epsilon}(\lambda)\, w(-s_2)\; w^\dagger(-s_2)\, \underset{\sim}{j}(-p_2', p_1') \cdot \underset{\sim}{\epsilon}^*(\lambda') w(s_1) \tag{III.48}$$

$$\left[\omega(p) - H(p_1) - H(p_2)\right]^{-1} \left[\omega(p') - H(p_1') - H(p_2')\right] \;.$$

Eq. (III.47) has the attractive property that the integrand of the $\underset{\sim}{q}$-integration decomposes into two factors: one describing the interaction with the external field, and a second, called the photon impact factor by Cheng and Wu[23], describing the composition of the physical photon as a bare pair.

A technical complication arises because the impact factor M depends on a cutoff Λ in the $\underset{\sim}{\ell}$- integration. However, we will see that the cutoff does not affect the scattering amplitude, and therefore has no physical significance.

It is quite easy to write down the explicit form of M_Λ using the variables $\underset{\sim}{\ell}$ and $\underset{\sim}{Q} = \frac{1}{2}(\underset{\sim}{r} + \underset{\sim}{q}) - \alpha \underset{\sim}{r}$. The energy denominators are

$$\omega(p) - H(p_1) - H(p_2) = -(2\bar{\eta})^{-1} \left[(\underset{\sim}{\ell} - \underset{\sim}{Q})^2 + m^2\right] = -\left[2\eta\,\alpha(1-\alpha)\right]^{-1} \left[(\underset{\sim}{\ell} - \underset{\sim}{Q})^2 + m^2\right]$$

$$\omega(p') - H(p_1') - H(p_2') = -(2\bar{\eta})^{-1} \left[(\underset{\sim}{\ell} + \underset{\sim}{Q})^2 + m^2\right] = -\left[2\eta\,\alpha(1-\alpha)\right]^{-1} \left[(\underset{\sim}{\ell} + \underset{\sim}{Q})^2 + m^2\right] .$$

By making use of the Galilean invariance of the numerator factors $w^\dagger j w$, we can write them in terms of ℓ and Q immediately:

$$w^\dagger(s_1) j(p_1, -p_2) w(-s_2) = w^\dagger(s_1) j(\ell - Q, \eta_1; \ell - Q, -\eta_2) w(-s_2)$$

$$w^\dagger(-s_2) j(-p_2', p_1') w(s_1) = w^\dagger(-s_2) j(\ell + Q, -\eta_2; \ell + Q, \eta_1) w(s_1) .$$

Thus M_Λ takes the form

$$M_\Lambda(q, r; \lambda, \lambda') = \int_0^1 d\alpha \int^\Lambda d\ell \left[2\eta\,\alpha(1-\alpha)\right]^2 n(\ell, Q; \lambda, \lambda') \left[(\ell - Q)^2 + m^2\right]^{-1} \left[(\ell + Q)^2 + m^2\right]^{-1} , \tag{III.49}$$

where

$$n(\ell, Q; \lambda, \lambda') = \sum_{s_1 s_2} w^\dagger(s_1) j(\ell - Q, \eta_1; \ell - Q, -\eta_2) \cdot \epsilon(\lambda) w(-s_2) \times w^\dagger(-s_2) j(\ell + Q, -\eta_2; \ell + Q, \eta_1) \cdot \epsilon^*(\lambda') w(s_1) . \tag{III.50}$$

Let us consider the helicity flip case first. Reading from Table I, we find

$$n(\ell, Q; 1, -1) = -2(\eta_1\eta_2)^{-1}(\ell_+ - Q_+)(\ell_+ + Q_+) = -2\eta^{-2}\left[\alpha(1-\alpha)\right]^{-1}\left[\ell_+\ell_+ - Q_+Q_+\right] . \tag{III.51}$$

Thus

$$M_\Lambda(q, r, +1, -1) = -8 \int_0^1 d\alpha\, \alpha(1-\alpha) \int^\Lambda d\ell \left[\ell_+\ell_+ - Q_+Q_+\right]\left[(\ell - Q)^2 + m^2\right]^{-1} \left[(\ell + Q)^2 + m^2\right]^{-1} . \tag{III.52}$$

The helicity non-flip amplitude is also quite simple. Reading from Table I, we find

$$n(\underset{\sim}{\ell}, \underset{\sim}{Q}; +1, +1) = \left[\eta_1^{-2} + \eta_2^{-2}\right](\ell_+ - Q_+)(\ell_- + Q_-) + \tfrac{1}{2} m^2 \bar{\eta}^{-2}$$

$$= \tfrac{1}{2}\left[\eta\,\alpha(1-\alpha)\right]^{-2}\left\{\left[\alpha^2 + (1-\alpha)^2\right](\underset{\sim}{\ell}^2 - \underset{\sim}{Q}^2 - 2i\,\underset{\sim}{\ell} \times \underset{\sim}{Q}) + m^2\right\} . \tag{III.53}$$

The term proportional to $\underset{\sim}{\ell} \times \underset{\sim}{Q}$ can be dropped since it will not contribute to M_Λ. Thus we obtain

$$M(\underset{\sim}{q}, \underset{\sim}{r}; +1, +1) = 2\int_0^1 d\alpha \int^{\Lambda} d\underset{\sim}{\ell}\left[(\alpha^2 + [1-\alpha]^2)(\underset{\sim}{\ell}^2 - \underset{\sim}{Q}^2) + m^2\right]\left[(\underset{\sim}{\ell} - \underset{\sim}{Q})^2 + m^2\right]^{-1}\left[(\underset{\sim}{\ell} + \underset{\sim}{Q})^2 + m^2\right]^{-1} . \tag{III.54}$$

As mentioned earlier, the impact factors M_Λ given in (III.52) and (III.54) depend on the cutoff parameter Λ used to avoid the logarithmic divergence in the $\underset{\sim}{\ell}$- integration. However, we can verify that the cutoff does not affect the scattering amplitude in the limit $\Lambda \rightarrow \infty$ by writing M_Λ in the form

$$M_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') = \widetilde{M}_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') + M_\Lambda(\underset{\sim}{r}, \underset{\sim}{r}; \lambda, \lambda') . \tag{III.55}$$

The term $\widetilde{M}_\Lambda$ defined by (III.55) is evidently finite in the limit $\Lambda \rightarrow \infty$. If we use the simple observation that

$$\int d\underset{\sim}{q}\left[F(\underset{\sim}{r} + \underset{\sim}{q})\, F_c(\underset{\sim}{r} - \underset{\sim}{q}) - (2\pi)^4 \delta^2(\underset{\sim}{r} + \underset{\sim}{q})\delta^2(\underset{\sim}{r} - \underset{\sim}{q})\right] = 0 ,$$

we see that the cutoff dependent part of $M_\Lambda(\underset{\sim}{q}, r; \lambda, \lambda')$, namely $M_\Lambda(\underset{\sim}{r}, \underset{\sim}{r}; \lambda, \lambda')$, does not contribute to the scattering amplitude (III.47) and therefore has no special significance.

In addition, we may note that because of its definition $\tilde{M}_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda')$ is zero at $\underset{\sim}{q} = \underset{\sim}{r}$. It is also zero at $\underset{\sim}{q} = -\underset{\sim}{r}$. (Indeed, it is an even function of $\underset{\sim}{q}$, as can be verified by making the change of variables $\alpha \longrightarrow (1-\alpha)$ in (III.52) and (III.54).) Thus the scattering amplitude (III.47) remains finite even if the eikonal factors are singular at $\underset{\sim}{q} = \pm r$, as they are in the case that $A_\mu(x)$ is a static Coulomb potential. The renormalized impact factors $\tilde{M}_\infty(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda')$ are identical (aside from a factor $-e^4(2\pi)^{-3}$) to the impact factors for the photon found by other techniques by Cheng and Wu.[23]

F. Electroproduction of μ Pairs; Scaling

We wish to discuss here a "model" calculation which, hopefully, has important features in common with electron-nucleon inelastic scattering. We imagine the process pictured in Figs. 7a and 7b: a virtual photon, produced by the scattered electron, creates a pair of muons which diffract through an external field (e. g. a nucleus). In the spirit of inelastic electron-nucleon scattering we put eikonal phases only on the members of the pair and treat all particles as distinguishable.

One purpose of the model is to investigate the scaling property recently discovered in electron-nucleon scattering.[24] To do this, we assume that only the final electron is observed and construct the cross section $d\sigma/dQ^2 d\nu$, where Q^2 is the four-momentum transfer from the electron line and ν is the energy transfer. We then ask whether the diffractive mechanism envisioned here leads to scale invariant expressions for the form factors σ_T and σ_S in the limit $Q^2 \longrightarrow \infty$.[25]

To begin, we construct the scattering amplitude corresponding to Figs. 7a and 7b:

$$S_{fi} = e^2 (2\pi)\, \delta(\eta - \eta' - \eta_1 - \eta_2) \left[2\eta 2\eta' 2\eta_1 2\eta_2\right]^{\frac{1}{2}} \int \frac{d\underset{\sim}{p}_1'}{(2\pi)^2}$$

$$\times \left\{ \frac{\Sigma_\lambda w^\dagger(s')\, \underset{\sim}{j}\,(p', p)\cdot \epsilon^*(\lambda)\, w(s) \quad w^\dagger(s_1)\, \underset{\sim}{j}\,(p_1', -p_2')\cdot \underset{\sim}{\epsilon}(\lambda)\, w(-s_2)}{(2\eta_q)\,(H(p) - H(p') - \omega(q))} + (\eta_q)^{-2} \delta_{s', s}\, \delta_{s_1, -s_2} \right\}$$

$$\times \left[H(p) - H(p') - H_\mu(p_1') - H_\mu(p_2')\right]^{-1} \left[F(\underset{\sim}{p}_1 - \underset{\sim}{p}_1')\, F_c(\underset{\sim}{p}_2 - \underset{\sim}{p}_2') - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \underset{\sim}{p}_1')\, \delta^2(\underset{\sim}{p}_2 - \underset{\sim}{p}_2')\right] , \tag{III.56}$$

where

$$\underset{\sim}{q} = \underset{\sim}{p} - \underset{\sim}{p}' , \qquad \eta_q = \eta - \eta'$$

$$\underset{\sim}{p}_2' = -\underset{\sim}{p}_1' + \underset{\sim}{q} , \qquad \eta_1' = \eta_1 , \; \eta_2' = \eta_2 .$$

The first term in braces in (III.56) corresponds to exchange of transverse photons (Fig. 7a); the second term corresponds to the exchange of a "scalar photon" (Fig. 7b). The function $H_\mu(p)$ refers to the free muon Hamiltonian $(\underset{\sim}{p}^2 + \mu^2)/2\eta$, where μ is the muon mass.

Before proceeding further, it is convenient (as usual) to change variables in the momentum integration from $\underset{\sim}{p}_1'$ to $\underset{\sim}{k}$, where $\underset{\sim}{k}$ is the "relative momentum" of the virtual μ-pair:

$$\underset{\sim}{k} = \frac{\eta_1 \eta_2}{\eta_q} \left(\frac{\underset{\sim}{p}_1'}{\eta_1} - \frac{\underset{\sim}{p}_2'}{\eta_2} \right) = \underset{\sim}{p}_1' - \alpha \underset{\sim}{q} , \tag{III.57}$$

where

$$\alpha = \eta_1 / \eta_q . \tag{III.58}$$

It is also convenient to let $-Q^2$ stand for the square of the 4-momentum transferred from the electron line:

$$-Q^2 = (p - p')^{\mu} (p - p')_{\mu} \quad . \tag{III.59}$$

In terms of these variables, the energy denominators in (III.56) have the simple forms

$$H(p) - H(p') - \omega(q) = -Q^2/(2\eta_q) \quad ,$$

(III.60)

$$H(p) - H(p') - H_{\mu}(p_1') - H_{\mu}(p_2') = -\frac{Q^2}{2\eta_q} - \frac{\eta_q}{2\eta_1\eta_2}(\mathbf{k}^2 + \mu^2) \quad .$$

The numerator functions $w^{\dagger}\mathbf{j}\cdot\boldsymbol{\epsilon}^{*}w \; w^{\dagger}\mathbf{j}\cdot\boldsymbol{\epsilon}w$ can be read from Table I, and are also simple functions of $\mathbf{k}$.

We are now prepared to write out S_{fi} in a form suitable for calculating the cross section. Let us choose the z-axis in the direction of the beam, so $\mathbf{p} = 0$, and consider S_{fi} for the choice of spins $s = s' = s_1 = \frac{1}{2}$, $s_2 = -\frac{1}{2}$. Then when we substitute the expressions from Table I and Eq. (III.60) into (III.56) we obtain

$$S_{fi} = (2\pi)\,\delta(\eta - \eta' - \eta_1 - \eta_2)\left[2\eta\, 2\eta' 2\eta_1 2\eta_2\right]^{\frac{1}{2}} \left(\frac{-2e^2}{Q^2\eta_q}\right) \widetilde{M}(\mathbf{p}_1, \mathbf{p}_2) \quad , \tag{III.61}$$

where

$$\widetilde{M}(\mathbf{p}_1, \mathbf{p}_2) = (2\pi)^{-2}\int d\mathbf{k}\, \widetilde{f}(\mathbf{k})$$

$$\left[F(\mathbf{p}_1 - \alpha\mathbf{q} - \mathbf{k})\, F_c(\mathbf{p}_2 - (1-\alpha)\mathbf{q} + \mathbf{k}) - (2\pi)^4 \delta^2(\mathbf{p}_1 - \alpha\mathbf{q} - \mathbf{k})\,\delta^2(\mathbf{p}_2 - (1-\alpha)\mathbf{q} + \mathbf{k})\right] \quad ,$$

(III.62)

and

$$\tilde{f}(\underset{\sim}{k}) = \tilde{f}_R(\underset{\sim}{k}) + \tilde{f}_L(\underset{\sim}{k}) + \tilde{f}_S(\underset{\sim}{k})$$
$$= \left\{\left(\frac{\eta}{\eta'} p'_-\right) \alpha k_+ - p'_+ \left[1-\alpha\right] k_- + \alpha(1-\alpha) Q^2\right\} \left[\underset{\sim}{k}^2 + \alpha(1-\alpha)Q^2 + \mu^2\right]^{-1} . \tag{III.63}$$

The three terms in $f(\underset{\sim}{k})$ arise from exchange of a right handed photon, a left handed photon, and a "scalar photon" respectively.

The physics of the amplitude $\tilde{M}(p_1, p_2)$ is more apparent if we write it as a Fourier transform by inserting the expansions of the eikonal factors into (III.62). The resulting structure of $\tilde{M}(\underset{\sim}{p}_1, \underset{\sim}{p}_2)$, and its physical interpretation will be familiar from the discussion of pair production by real photons in Section III.D. We find

$$\tilde{M}(\underset{\sim}{p}_1, \underset{\sim}{p}_2) = \int d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\, e^{-i\underset{\sim}{p}_1\cdot\underset{\sim}{x}_1}\, e^{-i\underset{\sim}{p}_2\cdot\underset{\sim}{x}_2}\, M(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$$
$$= \int d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\, e^{-i\underset{\sim}{p}_1\cdot\underset{\sim}{x}_1}\, e^{-i\underset{\sim}{p}_2\cdot\underset{\sim}{x}_2} \left[\exp\left(-i\chi(\underset{\sim}{x}_1)\right) \exp\left(+i\chi(\underset{\sim}{x}_2)\right) - 1\right] f(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{R}} \tag{III.64}$$

where $\underset{\sim}{R} = \eta_q^{-1}(\eta_1 \underset{\sim}{x}_1 + \eta_2 \underset{\sim}{x}_2)$ and $f(\underset{\sim}{r})$ is the Fourier transform of $\tilde{f}(\underset{\sim}{k})$. Explicit evaluation gives the wave function of the virtual muon pair, $f(\underset{\sim}{r})$, in terms of modified Bessel functions K_0 and K_1:

$$f(\underset{\sim}{r}) = (2\pi)^{-2} \int d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, \tilde{f}_R(\underset{\sim}{k}) + \tilde{f}_L(\underset{\sim}{k}) + \tilde{f}_S(\underset{\sim}{k}) = f_R(\underset{\sim}{r}) + f_L(\underset{\sim}{r}) + f_S(\underset{\sim}{r}) \ , \tag{III.65}$$

$$f_R(\underset{\sim}{r}) = \frac{i}{2\pi}\left(\frac{\eta}{\eta'}p'_-\right)\alpha\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}\frac{r_+}{r}K_1\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}r\right)$$

$$f_L(\underset{\sim}{r}) = -\frac{i}{2\pi}p'_+(1-\alpha)\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}\frac{r_-}{r}K_1\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}r\right) \tag{III.66}$$

$$f_S(\underset{\sim}{r}) = \frac{1}{2\pi}\alpha(1-\alpha)Q^2 K_0\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}r\right).$$

We will see in the sequel that, for our purposes, this expression for $f(\underset{\sim}{r})$ is not as formidable as it seems.

With a useable expression for S_{fi} now at hand, we are ready to construct the cross section $d\sigma$ integrated over the unobserved momenta of the muon pair. Using (III.61) in Eq. (III.6) we obtain

$$d\sigma = d\underset{\sim}{p}'\,d\eta'\left(\frac{4e^4}{(2\pi)^4Q^4\eta_q}\right)\int_0^1 d\alpha\,(2\pi)^{-4}\int d\underset{\sim}{p}_1\,d\underset{\sim}{p}_2\,|\,\widetilde{M}(p_1,p_2)\,|^2 . \tag{III.67}$$

Since $M(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ is simpler than $\widetilde{M}(\underset{\sim}{p}_1,\underset{\sim}{p}_2)$, we write the $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$ - integral as

$$\begin{aligned}(2\pi)^{-4}\int d\underset{\sim}{p}_1\,d\underset{\sim}{p}_2\,|\,\widetilde{M}(\underset{\sim}{p}_1,\underset{\sim}{p}_2)\,|^2 &= \int d\underset{\sim}{x}_1\,d\underset{\sim}{x}_2\,|\,M(\underset{\sim}{x}_1,\underset{\sim}{x}_2)\,|^2\\ &= \int d\underset{\sim}{x}_1\,d\underset{\sim}{x}_2\,|\,f(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\,|^2\left[2-2\cos\left(\chi(\underset{\sim}{x}_1)-\chi(\underset{\sim}{x}_2)\right)\right]\\ &= \int d\underset{\sim}{r}\,|\,f(\underset{\sim}{r})\,|^2\int d\underset{\sim}{b}\left[2-2\cos\left(\chi(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{r})-\chi(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{r})\right)\right]\end{aligned} \tag{III.68}$$

Assuming that the potential has cylindrical symmetry about the z-axis, we can replace $|\,f(r)\,|^2$ by $|\,f_R(\underset{\sim}{r})\,|^2 + |\,f_L(\underset{\sim}{r})\,|^2 + |\,f_S(\underset{\sim}{r})\,|^2$ in (III.68), since the various cross terms will vanish when the integration over the angle of $\underset{\sim}{r}$ is performed. Thus the cross section separates into a part due to the exchange of a "transverse photon", $d\sigma_T = d\sigma_R + d\sigma_L$, and a part due to the exchange of a "scalar photon", $d\sigma_S$. If we substitute the expressions for $|f_R|^2$, $|f_L|^2$, and $|f_S|^2$ obtained from (III.66) into (III.68) and (III.67) and interchange the roles of α and $(1-\alpha)$ in $d\sigma_L$, we obtain

$$\begin{aligned} d\sigma &= d\sigma_T + d\sigma_S \\ &= d\underset{\sim}{p}'\, d\eta' \left(4e^4 (2\pi)^{-6} Q^{-4} \eta_q^{-1}\right) \int_0^1 d\alpha \int d\underset{\sim}{r} \\ &\Big\{ \tfrac{1}{4} \Big[\Big(\frac{\eta}{\eta'}\Big)^2 + 1\Big] \underset{\sim}{p}'^2\, \alpha^2 \left[\alpha(1-\alpha)Q^2 + \mu^2\right] \Big[K_1\Big(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}} r\Big)\Big]^2 \\ &+ \alpha^2 (1-\alpha)^2 Q^4 \Big[K_0\Big(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}} r\Big)\Big]^2 \Big\} \\ &\int d\underset{\sim}{b} \left[2 - 2\cos\left(\chi(\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{r}) - \chi(\underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{r})\right)\right] . \end{aligned} \qquad \text{(III.69)}$$

This expression gives the cross section in the high energy limit discussed in Section II i.e. in the limit $\eta, \eta' \to \infty$ with η/η' and Q^2 fixed. It remains now to evaluate $d\sigma$ in the limit $Q^2 \to \infty$. To take this limit we have only to note that the modified Bessel functions appearing in (III.69) are large only for small values of their arguments, so that the main contribution to the $\underset{\sim}{r}$-integral comes from the region $\underset{\sim}{r}^2 < \left[\alpha(1-\alpha)Q^2 + \mu^2\right]^{-1}$.

Physically, this means that for large Q^2 the transverse separation r between the muons as they pass through the external potential is small. If the separation were zero the two muons would receive exactly opposite eikonal phases; thus for small r the net phase received by the muon pair is proportional not to χ but to $\nabla\chi$.

Mathematically, this means that the $Q^2 \to \infty$ limit of $d\sigma$ can be obtained by substituting for the $\underset{\sim}{b}$-integral in (III.69) its limiting form as $r \to 0$.[26] This limiting form is easily evaluated:

$$\int d\underset{\sim}{b}\left[2-2\cos\left(\chi(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{r})-\chi(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{r})\right)\right]\sim\int d\underset{\sim}{b}\left[2-2\cos\left(\underset{\sim}{r}\cdot\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right)\right]$$
$$\sim\int d\underset{\sim}{b}\left[\underset{\sim}{r}\cdot\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2 \qquad \text{(III.70)}$$
$$=\tfrac{1}{2}r^2\int d\underset{\sim}{b}\left[\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2 .$$

(In the last step we have used the assumed cylindrical symmetry of $\chi(\underset{\sim}{b})$.)

Once the limiting form (III.70) of the $\underset{\sim}{b}$-integral has been substituted into (III.69), the $\underset{\sim}{r}$-integral can be evaluated using the formula[27]

$$\int_0^\infty dx\left[K_J(x)\right]^2 x^{s-1} = 2^{s-3}\left[\Gamma(\tfrac{1}{2}s)\right]^2\frac{\Gamma(\tfrac{1}{2}s+J)\,\Gamma(\tfrac{1}{2}s-J)}{\Gamma(s)} .$$

This leads to

$$d\sigma = d\underset{\sim}{p}'d\eta'\left(\tfrac{2}{3}e^4(2\pi)^{-5}Q^{-4}\eta_q^{-1}\right)\int d\underset{\sim}{b}\left[\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2$$
$$\times\left\{\tfrac{1}{2}\left[\left(\frac{\eta}{\eta'}\right)^2+1\right]\underset{\sim}{p}'^2\int_0^1 d\alpha\,\frac{\alpha^2}{\alpha(1-\alpha)Q^2+\mu^2}+\int_0^1 d\alpha\left[\frac{\alpha(1-\alpha)Q^2}{\alpha(1-\alpha)Q^2+\mu^2}\right]^2\right\} . \qquad \text{(III.71)}$$

Evaluating the α-integrals in the limit $Q^2\longrightarrow\infty$, we find

$$d\sigma = d\sigma_T + d\sigma_S$$
$$\sim d\underset{\sim}{p}'\,d\eta'\,\frac{2e^4}{3(2\pi)^5Q^4\eta_q}\int d\underset{\sim}{b}\left[\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2 \qquad \text{(III.72)}$$
$$\times\left\{\tfrac{1}{2}\left[\left(\frac{\eta}{\eta'}\right)^2+1\right]\left(\frac{\underset{\sim}{p}'^2}{Q^2}\right)\left[\log\frac{Q^2}{\mu^2}+0(1)\right]+1\right\} .$$

We recall that this is the cross section for the choice of spins $s = s' = \frac{1}{2}$, $s_1 = \frac{1}{2}$, $s_2 = -\frac{1}{2}$. It is not difficult to see that the choice $s = s' = \frac{1}{2}$, $s_1 = -\frac{1}{2}$, $s_2 = +\frac{1}{2}$ leads to the same cross section. Each of the other six possible choices for the spins of the final particles gives a cross section $d\sigma_S = 0$ and a cross section $d\sigma_T$ which is small compared to the cross section in (III.72) as $Q^2 \to \infty$.[28] Thus the limiting cross section for $s = \frac{1}{2}$ (or $s = -\frac{1}{2}$), summed over final spins, is two times the cross section in (III.72).

In order to make contact with standard notation and identify the form factors $\sigma_T(Q^2, \nu)$, $\sigma_S(Q^2, \nu)$, let us define

$$\begin{aligned} E &= \text{lab energy of the incident electron} = 2^{-\frac{1}{2}}[\eta + H(p)] \\ E' &= \text{lab energy of the scattered electron} = 2^{-\frac{1}{2}}\left[\eta' + H(p')\right] \\ \nu &= E - E' \quad . \end{aligned} \tag{III.73}$$

Apparently in the high energy limit

$$\eta = 2^{\frac{1}{2}} E, \quad \eta' = 2^{\frac{1}{2}} E', \quad \eta_q = 2^{\frac{1}{2}} \nu \quad . \tag{III.74}$$

We recall also the definition of Q^2:

$$Q^2 = -(p-p')^{\mu}(p-p')_{\mu} = -2\eta_q\left(H(p) - H(p')\right) + \underset{\sim}{p}'^2 = \frac{\eta}{\eta'}\underset{\sim}{p}'^2 + \frac{\eta_q^2}{\eta\eta'} m^2 \; . \tag{III.75}$$

Thus in the high energy limit, and neglecting m^2 compared to Q^2, we can replace $\underset{\sim}{p}'^2$ by

$$\underset{\sim}{p}'^2 = \frac{E'}{E} Q^2 \, . \tag{III.76}$$

When we make these replacements we find for the cross section summed over final spins,

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{d\sigma_T}{dQ^2 d\nu} + \frac{d\sigma_S}{dQ^2 d\nu}$$
$$\sim \frac{2\alpha^2}{3\pi^2} \frac{1}{\nu Q^4} \frac{E'}{E} \left\{ \frac{E^2 + E'^2}{2EE'} \log\left(\frac{Q^2}{\mu^2}\right) + 1 \right\} \int d\underset{\sim}{b} \left[\underset{\sim}{\nabla} \chi(\underset{\sim}{b}) \right]^2 . \qquad \text{(III.77)}$$

Using (III.77) we can extract the form factors σ_S and σ_T[29]:

$$\sigma_S(Q^2, \nu/Q^2) \sim \frac{2\alpha}{3\pi} \frac{1}{Q^2} \int d\underset{\sim}{b} \left[\underset{\sim}{\nabla} \chi(\underset{\sim}{b}) \right]^2$$
$$\sigma_T(Q^2, \nu/Q^2) \sim \frac{2\alpha}{3\pi} \frac{1}{Q^2} \log\left(\frac{Q^2}{\mu^2}\right) \int d\underset{\sim}{b} \left[\underset{\sim}{\nabla} \chi(\underset{\sim}{b}) \right]^2 \qquad \text{(III.78)}$$

It is interesting to compare the behavior of σ_S and σ_T in the present model with the famous scaling behavior of the same form factors for deep inelastic electron-nucleon scattering.[30] In this model, $\sigma_S(Q^2, \nu/Q^2)$ is scale invariant: for large ν and Q^2, $Q^2 \sigma_S$ is a function of (ν/Q^2) only. However, the factor $\log(Q^2/\mu^2)$ spoils the scaling behavior of σ_T.[31]

In the somewhat hypothetical limit of an external field which varies in space slowly compared the lepton Compton-wavelength $(1/\chi|\nabla\chi| \ll \mu^{-1})$, the formula (III.71) for σ_s/σ_T is valid for all Q^2. The direct evaluation is shown in Fig. 8; we see that σ_s/σ_T is never larger than 0.26.

It is not clear what direct connection these calculations have with respect to hadron electroproduction. While there appears to be a diffractive mechanism[32] operating in both cases, the details (e.g., the scaling behavior of σ_T) are different. However it may be that some features of the process, such as the importance of small transverse distances $(\Delta x)^2 \lesssim Q^{-2}$ at large Q^2 are common to both.

G. Electroproduction of π Pairs

While the calculation discussed in the previous section did not yield scaling, it has the attractive property of predicting a small number (< 0.26) for the ratio σ_S/σ_T. This result can be traced to the fact that the constituents of the photon (muons) has spin $\frac{1}{2}$. To display this fact explicitly we consider the process of electroproduction of π's (spin zero) off an external field.

Our first task is to develop the infinite-momentum version of QED for spin zero bosons. By arguments analogous to those of Chapter II, it is easy to obtain the Hamiltonian which governs such a physical system:

$$H = \phi^\dagger(\underset{\sim}{p}^2 + m^2)\phi + \tfrac{1}{2}\underset{\sim}{A}\cdot\underset{\sim}{p}^2\underset{\sim}{A} - e\phi^\dagger(\overrightarrow{\underset{\sim}{p}} - \overleftarrow{\underset{\sim}{p}})\phi\cdot\underset{\sim}{A} + e^2\underset{\sim}{A}^2\phi^\dagger\phi + e\phi^\dagger(\overrightarrow{\eta} - \overleftarrow{\eta})\phi\frac{e}{\eta}\underset{\sim}{p}\cdot\underset{\sim}{A} + + \tfrac{1}{2}e^2\left[\phi^\dagger(\overrightarrow{\eta} - \overleftarrow{\eta})\phi\right]\frac{1}{\eta^2}\left[\phi^\dagger(\overrightarrow{\eta} - \overleftarrow{\eta})\phi\right] \qquad \text{(III.79)}$$

We are interested, in particular, in the two graphs shown in Figure (7). The upper line in the graphs is familiar since we know how the electron couples to both the transverse and scalar photons. We can read off from (III. 79) the matrix elements for the pair production of spin zero bosons from a transverse photon:

a) $$(2\pi)^3\delta(\eta_q - \eta_1' - \eta_2')\delta^2(\underset{\sim}{q} - \underset{\sim}{p}_1' - \underset{\sim}{p}_2')\left(\underset{\sim}{p}_2' - \underset{\sim}{p}_1' - \frac{\eta_2' - \eta_1'}{\eta_q}\underset{\sim}{q}\right)\cdot\underset{\sim}{\epsilon}(\lambda) \qquad \text{(III.80)}$$

and the matrix element for an electron to create a pair of spin zero bosons through instantaneous photon exchange:

(b) $$(2\pi)^3\delta(\eta-\eta'-\eta_1'-\eta_2')\delta^2(\underset{\sim}{p}-\underset{\sim}{p}'-\underset{\sim}{p}_1'-\underset{\sim}{p}_2')\, e^2\delta_{s's}\sqrt{2\eta 2\eta'}\;\frac{(\eta_2'-\eta_1')}{(\eta_2'+\eta_1')^2} \qquad \text{(III. 81)}$$

The final ingredient in the calculation is the limiting form of the S-matrix describing the scattering of the energetic constituents ($\pi^+\pi^-$) off the external field. The details of the argument follow those of Chapter II, and one can easily find that the same answer emerges: each π receives an eikonal phase $\exp(-ie\int a_0(\tau, x, \underset{\sim}{y})d\tau)$ as it passes through the external field. This result is very plausible since the eikonal formula depends only on the charge density of the constituents and not on their more detailed properties.

With these preliminaries done we can record the S-matrix element for Figure (7):

$$S_{fi} = e^2(2\pi)\delta(\eta-\eta'-\eta_2-\eta_3)\sqrt{2\eta 2\eta'}$$

$$\int\frac{d\underset{\sim}{p}_1'}{(2\pi)^2}\left\{\frac{\sum_\lambda \omega^\dagger(s')\underset{\sim}{j}(p',p)\cdot\underset{\sim}{\epsilon}^*(\lambda)\omega(s)\left(\underset{\sim}{p}_2'-\underset{\sim}{p}_1'-\frac{\eta_2'-\eta_1'}{\eta_q}\underset{\sim}{q}\right)\cdot\underset{\sim}{\epsilon}(\lambda)}{2\eta_q(H(p)-H(p')-\omega(q))}+\frac{(\eta_2-\eta_1)}{\eta_q^2}\delta_{s's}\right\}$$

$$\left[H(p)-H(p')-H_\pi(p_1')-H_\pi(p_2')\right]^{-1}\left[F(\underset{\sim}{p}_1-\underset{\sim}{p}_1')F_c(\underset{\sim}{p}_2-\underset{\sim}{p}_2')-(2\pi)^4\delta(\underset{\sim}{p}_1-\underset{\sim}{p}_1')\delta(\underset{\sim}{p}_2-\underset{\sim}{p}_2')\right] \qquad \text{(III.82)}$$

where the kinematics are defined as in Section (F). The pion current term can be written much more simply in terms of $\underset{\sim}{k}$, the "relative momentum" of the pair,

$$\underset{\sim}{p}_2'-\underset{\sim}{p}_1'-\frac{\eta_2-\eta_1}{\eta_1+\eta_2}(\underset{\sim}{p}_2'+\underset{\sim}{p}_1')=\frac{2}{(\eta_1+\eta_2)}\left[\eta_1\underset{\sim}{p}_2'-\eta_2\underset{\sim}{p}_1'\right]=2\underset{\sim}{k} \qquad \text{(III. 83)}$$

If we choose $s = s' = \frac{1}{2}$ the numerator in the term corresponding to Figure (7a) can be worked out with the help of Table I,

$$2 \sum_{\lambda} \omega^{\dagger}(\tfrac{1}{2}) \underset{\sim}{j}(p', p) \cdot \underset{\sim}{\epsilon}^*(\lambda) \omega(\tfrac{1}{2}) \underset{\sim}{k} \cdot \epsilon(\lambda) = - \frac{2\eta}{\eta'(\eta - \eta')} p'_- k_+ - \frac{2}{\eta - \eta'} p'_+ k_- \qquad \text{(III.84)}$$

Furthermore, if we define the variable

$$\alpha = \eta_1 / \eta_q \qquad \text{(III.85)}$$

and do the manipulations which led from (III.58) to (III.62) in Section (F) we obtain,

$$S_{fi} = e^2 (2\pi) \delta(\eta - \eta' - \eta_1 - \eta_2) \sqrt{2\eta 2\eta'} \left(\frac{4\eta_1 \eta_2}{Q^2 \eta_q^2} \right) \int \frac{d\underset{\sim}{p}'_1}{(2\pi)^2} \left[\frac{\frac{\eta}{\eta'} p'_- k_+ + p'_+ k_- + (\frac{1}{2} - \alpha) Q^2}{\underset{\sim}{k}^2 + \alpha(1-\alpha) Q^2 + m_\pi^2} \right]$$

$$\times \left[F(\underset{\sim}{p}_1 - \alpha \underset{\sim}{q} - \underset{\sim}{k}) F_c(\underset{\sim}{p}_2 - (1-\alpha) \underset{\sim}{q} + \underset{\sim}{k}) - (2\pi)^4 \delta(\underset{\sim}{p}_1 - \alpha \underset{\sim}{q} - \underset{\sim}{k}) \delta(\underset{\sim}{p}_2 - (1-\alpha) \underset{\sim}{q} + \underset{\sim}{k}) \right] \qquad \text{(III.86)}$$

It will prove convenient to write this expression in the form appearing in Section (F). We have

$$S_{fi} = (2\pi) \delta(\eta - \eta' - \eta_1 - \eta_2) \sqrt{2\eta 2\eta' 2\eta_1 2\eta_2} \left(\frac{2e^2}{Q^2 \eta_q} \right) \tilde{m}_\pi (\underset{\sim}{p}_1, \underset{\sim}{p}_2) \qquad \text{(III. 87)}$$

where

$$\tilde{m}_\pi (\underset{\sim}{p}_1, \underset{\sim}{p}_2) = \int \frac{d\underset{\sim}{k}}{(2\pi)^2} \tilde{f}_\pi (\underset{\sim}{k}) \left[F(\underset{\sim}{p}_1 - \alpha \underset{\sim}{q} - \underset{\sim}{k}) F_c(\underset{\sim}{p}_2 - (1-\alpha) \underset{\sim}{q} + \underset{\sim}{k}) - (2\pi)^4 \delta(\underset{\sim}{p}_1 - \alpha \underset{\sim}{q} - \underset{\sim}{k}) \delta(\underset{\sim}{p}_2 - (1-\alpha) \underset{\sim}{q} + \underset{\sim}{k}) \right] \qquad \text{(III.88)}$$

and

$$\tilde{f}_\pi(\underset{\sim}{k}) = \tilde{f}_\pi^{(R)}(\underset{\sim}{k}) + \tilde{f}_\pi^{(L)}(\underset{\sim}{k}) + \tilde{f}_\pi^{(S)}(\underset{\sim}{k}) = \sqrt{\alpha(1-\alpha)}\,\frac{\left(\frac{\eta}{\eta'}p'_-k_+ + p'_+k_- + (\frac{1}{2}-\alpha)Q^2\right)}{\left[\underset{\sim}{k}^2 + \alpha(1-\alpha)Q^2 + m_\pi^2\right]} \qquad \text{(III. 89)}$$

The interpretation of the various terms is the same as in Section (F). We observe that the f_π's differ from the f_μ's of Section (F) only in their α dependence. In fact,

$$\tilde{f}_\pi^{(R)}(\underset{\sim}{r}) = \sqrt{\frac{1-\alpha}{\alpha}}\,\tilde{f}_\mu^{(R)}(\underset{\sim}{r}),\ \tilde{f}_\pi^{(L)}(\underset{\sim}{r}) = \sqrt{\frac{\alpha}{1-\alpha}}\,\tilde{f}_\mu^{(L)}(\underset{\sim}{r}),\ \tilde{f}_\pi^{(S)}(\underset{\sim}{r}) = \frac{(\frac{1}{2}-\alpha)}{\sqrt{\alpha(1-\alpha)}}\,\tilde{f}_\mu^{(S)}(\underset{\sim}{r}) \qquad \text{(III.90)}$$

Analogous steps which led from (III.67) to (III.72) in Section (F) give the transverse and scalar differential cross-section,

$$d\sigma_T = d\underset{\sim}{p}_1'd\eta' \left(\frac{\frac{2}{3}e^4}{(2\pi)^5Q^4\eta_q}\right)\int d\underset{\sim}{b}\left[\nabla X(\underset{\sim}{b})\right]^2 \tfrac{1}{2}\left[(\tfrac{\eta}{\eta'})^2+1\right]\int_0^1 d\alpha\,\frac{\alpha(1-\alpha)}{\left[\alpha(1-\alpha)Q^2+m_\pi^2\right]}$$

$$d\sigma_S = d\underset{\sim}{p}_1'd\eta' \left(\frac{\frac{2}{3}e^4}{(2\pi)^5Q^4\eta_q}\right)\int d\underset{\sim}{b}\left[\nabla X(\underset{\sim}{b})\right]^2 \int_0^1 d\alpha\,\frac{\alpha(1-\alpha)(\frac{1}{2}-\alpha)^2Q^4}{\left[\alpha(1-\alpha)Q^2+m_\pi^2\right]^2} \qquad \text{(III.91)}$$

We can identify the form factors $\sigma_S(Q^2, \frac{\nu}{Q^2})$ and $\sigma_T(Q^2, \frac{\nu}{Q^2})$ from these expressions in the usual way

$$\sigma_T(Q^2, \frac{\nu}{Q^2}) = \frac{2\alpha}{3\pi}\int d\underset{\sim}{b}\left[\nabla X(\underset{\sim}{b})\right]^2 \int_0^1 \frac{\alpha(1-\alpha)}{\left[\alpha(1-\alpha)Q^2+m_\pi^2\right]}\,d\alpha$$

$$\sigma_S(Q^2, \frac{\nu}{Q^2}) = \frac{2\alpha}{3\pi}\int d\underset{\sim}{b}\left[\nabla X(\underset{\sim}{b})\right]^2 \frac{1}{Q^2}\int_0^1 \frac{\alpha(1-\alpha)(\frac{1}{2}-\alpha)^2}{\left[\alpha(1-\alpha)Q^2+m_\pi^2\right]^2}\,d\alpha \qquad \text{(III.92)}$$

Taking the ratio of these quantities and letting $Q^2 >> m_\pi^2$ we have

$$\frac{\sigma_S}{\sigma_T} \rightarrow \frac{\log(Q^2/m_\pi^2)}{2} \qquad \text{(III.93)}$$

In this model σ_T scales, but σ_S grows logarithmically with Q^2. This is just the opposite behavior of the μ electroproduction model. It is clear from (III.92) that this difference stems from the different longitudinal momentum distributions of the constituents (μ's or π's) in the physical electron. Although neither model scales, the first (spin $\frac{1}{2}$ constituents) is more appealing since the ratio σ_S/σ_T was, at least, a small number.

IV. Applications in Strong Interactions

A. Description of the Approach

We now turn to a more speculative program of applications. We are interested, in particular, in attempting to obtain predictions relevant to strong interaction phenomena from our field theory model. The motivation for using field theory as a framework for such a discussion is the fact that, unlike other available theories, it respects the basic principles of unitarity, analyticity and crossing which one would like a "solution" of strong interactions to possess. However, since perturbation theory is the only calculational technique at ones disposal, it is difficult to see how one can obtain predictions from field theory which do not rely on the unrealistic assumption of weak coupling. One particular way out has been, of course, to sum infinite sets of Feynman diagrams. This procedure is, in general, difficult to defend for two reasons. First, one can question whether one possesses the physically most important set of diagrams; and second, a mathematical justification for the summing procedures (which are usually approximate in nature) is lacking.

Regardless of the difficulties inherent in this program, it is certainly a worthwhile task, in light of the absence of a competing fundamental theory of strong interactions, to attempt to obtain some insights from field theory. Whether one can obtain detailed and reliable predictions is another much more difficult question we will be addressing in the subsequent chapters. Many of the processes we will consider have been discussed in a different light by H. Cheng and T. T. Wu.[32] We will obtain several of their results in a more understandable and physical context; but, more importantly, we will also argue that some of their recent predictions concerning total cross-sections at asymptotic energies are based on an unrealistic set of approximations which omit many crucial characteristics of strong interactions.

Since the calculations to be presented are rather technical and tedious, it behooves us to dwell at some length on their physical features before proceeding further. We consider an energetic electron scattering off an external field. We relax our calculational procedure of working only to finite order in the number of bare constituents that make up the physical electron and ask for that set of graphs which give the dominant contribution to the scattering amplitude. It should come as no surprise that the graphs are similar in general structure to those considered in multiperipheral models. In fact, we argue that, fixing the order of perturbation theory at 2n, the most probable physical state of the electron consists of a bare electronand n e^+-e^- pairs as presented in Figure (14). This diagram is a slight variation on the more familiar t-channel ladder graphs in ϕ^3 models of Regge poles. A distinctive kinematic feature of these graphs is the longitudinal momentum distribution of the pairs down the chain: each pair on the chain has (on the average) a fixed fraction r of the longirudinal momentum of the pair just above it. Thus, as is common in multiperipheral models, the longitudinal momentum distribution of the $e^+ - e^-$ pairs is a constant function of $\log \eta$ ($\approx$ rapidity).

This physical state develops as the electron approaches the external field. The most likely interaction between the constituents and the external field is realized when the slowest pair on the chain picks up eikonal phases. The last step in the elastic scattering process is the reconstruction of the outgoing electron. This, naturally, occurs through the same machinery that developed the initial physical state. Thus, the entire scattering process can be diagrammed as in Figure (14).

A striking dilemna occurs now when one attempts to sum over n, the number of pairs in the physical state: the resulting scattering amplitude leads to a cross-section which violates the Froissart bound. The physics responsible for such an effect is actually very simple. Consider Figure (14) again and view it from the t-channel. If the $e^+ - e^-$ pairs were absent and the two photons coming off the electron belonged to the external field, the scattering amplitude would be negative and proportional to the c.m. energy of the collision. In the usual way, this leads to a constant cross-section. New imagine putting the $e^+ - e^-$ pairs back into the diagram. They provide a mechanism whereby the 2 photons can attract one another and tend to bind. Invoking the familiar Regge argument, this binding in the t-channel implies a stronger energy dependence of the scattering amplitude in the s-channel. In more detail, we will find that the scattering amplitude, as represented in the complex angular momentum plane, possesses a cut which extends to $J = 1 + \frac{11\pi}{32} \alpha^2$.

It has been suggested that s-channel unitarity be restored by iterating the single chain graphs. The motivation for such a suggestion comes from the eikonal formula (II.47). This simplest eikonal formula occurs from the iteration of single photons in the s-channel. One might guess (and we will prove this in detail) that if a more complicated t-channel structure is iterated in the s-channel, a similar eikonal scattering amplitudes emerges. However, now the eikonal phase X is associated with the single chain diagrams instead of single photon exchange. Since the scattering amplitude for single chain exchange is purely absorptive, the associated phase X is negative and could serve to damp out the absurdly large single chain amplitude. This is indeed what happens: the result of the s-channel iteration procedure is a scattering amplitude which saturates but does not violate the Froissart bound.

Our physical picture sheds considerable light on the implicit assumptions contained in this procedure. First, we will see that these unitarity corrections occur when more than one chain of $e^+ - e^-$ pairs occur in the physical electron state. So, a typical contribution to the scattering might be a diagram as shown in Figure (18). A suspicious assumption of the iteration scheme is that the various multiperipheral chains do not communicate with one another during the scattering process. However, according to our physical picture, the chains scatter simultaneously off the external field, and hence are nearby in real space-time. There is also considerable overlap between the various chains in momentum space, i. e. electrons and positrons from different chains often occur with small relative subenergies. Hence, it would be very odd to suppose that they would not interact and link up their respective chains. Naive considerations such as these lead us to suspect that it is unrealistic to treat the chains as independent.

A detailed calculation to confirm this belief is carried out in Section E. There we consider, in a somewhat simpler model than QED, a set of graphs which lie outside the proposed iteration scheme. We find that these graphs as depicted in Figure (20) grow even faster with energy than the single chain amplitude. On the basis of this work we are led to conclude that the popular Regge-eikonal formalisms are too naive to be considered indicative of field theory's predictions for diffraction scattering. We conjecture that more effort is needed to sift out the general features a more reasonable and physically appealing picture of diffraction scattering should possess before attempting more detailed perturbation theory calculations.

B. Single Chain

In this section we wish to emphasize some of the physical features and important formulas for the multiperipheral process shown in Figures (9-14). These processes have been considered in the literature, so we relegate our explicit and lengthy calculations of the associated scattering amplitudes to the Appendices. The reader is, however, advised to familiarize himself with some of the main features of these calculations before venturing on.

We will also find it slightly more convenient for these applications to modify our perturbation theory rules somewhat. These rules are, aside from certain normalization changes, simply those which follow from the formal field theoretic development. We shall associate the following factors with the parts of a certain τ-ordered diagram:

(i) wave functions $u(p,s)$, $\bar{u}(p,s)$, $\bar{u}_c(p,s)$, $u_c(p,s)$, and $e_\lambda(p)$ for the external lines;

(ii) $(\not{p} + m) = \sum_s u(p,s)\,\bar{u}(p,s)$ for electron propagators;

$(-\not{p} + m) = -\sum_s u_c(p,s)\,\bar{u}_c(p,s)$ for positron propagators;

$\sum_\lambda e_\lambda(p)^\mu\, e_\lambda(p)^\nu$ for photon propagators;

(iii) $e\gamma_\mu$ for each vertex in Fig. 1a.

$e^2 \delta_3^\mu \delta_3^\nu \frac{1}{\eta_0^2} \ldots \gamma_\mu \ldots \gamma_\nu \ldots$ for each vertex in Fig. 1b, where η_0 is the total η transferred across the vertex

$e^2 \gamma_\nu \gamma^o \gamma_\mu \frac{1}{\eta_0}$ for each vertex in Fig. 1c;

(iv) a factor $(2\pi)^3 \delta(\eta_{out} - \eta_{in}) \delta(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})$ for each vertex;

(v) a factor $(H_f - H + i\epsilon)^{-1}$ for each intermediate state;

(vi) an integration $(2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta}$ and a sum over spins for each internal line;

(vii) an eikonal phase factor for a chosen intermediate state.

Let us now return to the multiperipheral diagrams of Figure 14. Our physical picture sllows us to look at these scattering processes in three parts. First, the incoming physical electron dissociates into a state of bare constituents which in this case are $e^+ - e^-$ pairs. The scattering ampilitude receives its dominant (leading logarithm of energy) contribution from multiperipheral chains which are stronglv ordered. i.e. the ratios of the longitudinal momenta of successive virtual photons down the chain are small. So, to good approximation the physical electron consists of a chain of $e^+ - e^-$ pairs whose longitudinal momentum decreases the further down the chain we move. The chain of constituents next scatters off the external field when the slowest $e^+ - e^-$ pair picks up eikonal phases (Fig. 14). Finally, the scattered state of constituents recombines into the outgoing physical electron.

According to (C.10) of Appendix C, the scattering amplitude (forward direction) for this process reads,

$$S^{(1\ chain)}(\eta) = -(2\pi)(2\eta)\ \delta(\eta - \eta')\ \left[M(\eta/\eta_{min}) - 1\right] \qquad \text{(IV.1)}$$

where

$$M(\eta/\eta_{min}) = \frac{1}{\lambda^2}\int \frac{d\beta}{(2\pi)^2}\ f^*(\beta,1)\ f(\beta,1)\left(\frac{\eta}{\eta_{min}}\right)^{\left(\frac{2\alpha}{\pi}\right)^2 \tilde{C}(\beta)} \theta\left(\frac{\eta}{\eta_{min}} - 1\right) \quad (IV.2)$$

and η refers to the incident electron and the functions f and C are derived in Appendix D. The minus sign in (IV.1) indicates that the scattering is pure absorption. The scattering amplitude can be understood more clearly by transforming it to the complex angular momentum plane. As discussed in Appendix C, the Mellin transform of (IV.2) reads,

$$M(J) = \frac{1}{\lambda^2}\int \frac{d\beta}{(2\pi)^2}\ f^*(\beta,1)\ f(\beta,1)\left[\frac{1}{J - \left(\frac{2\alpha}{\pi}\right)^2 \tilde{C}(\beta)}\right] \quad (IV.3)$$

which possesses a cut over that range of J where the denominator, $J - \left(\frac{2\alpha}{\pi}\right)^2 C(\beta)$, can vanish. That range is, in fact, from $J = 0$ to $J = 11\pi\alpha^2/32$. As derived in Appendix C this implies that the energy dependence of the S matrix element reads,

$$S^{(1\text{-chain})}(\eta) \sim - \frac{\eta^{1 + \frac{11\pi\alpha^2}{32}}}{\sqrt{\log \eta}} \quad (IV.4)$$

This result has been obtained by Frolov et al.,[5] and disagrees slightly with the cut claimed by Cheng and Wu.[4] We see that (IV.4) violates the Froissart bound no matter how small α is. In effect, the multiperipheral chain has provided a mechanism whereby the two photons coming off the through-going electron line tend to attract one another. This effect elevates the energy dependence of the S matrix from η, characteristic of spin one photons, to $\eta^{1+(11\pi\alpha^2/32)}$.

It is this violation of the Froissart bound which has caused several authors to consider diagrams of iterations of the multiperipheral chains as a possible mechanism for softening the energy dependence of (IV.4). The success of this scheme relies upon the observation that such diagrams have alternating signs and hence tend to cancel when summed. This behavior is similar in character to the simpler and more familiar s-channel iteration procedures used in eikonal approximations. We will see in the next section that such s-channel iterations can be easily computed and understood from our infinite momentum point of view.

C. TWO-CHAIN DIAGRAMS

In this section we will look in detail at the diagram in Fig. 15. The incident electron emits two photons which break up into two pairs which scatter simultaneously in the external field before they each coalesce back into photons which subsequently land on the outgoing electron. In addition to the particular τ-ordered graph drawn in Fig.15 there is a graph for each other allowable τ-ordering of the vertices. The crucial point, however, is that although a particular τ-ordered graph is complicated, the sum of all the graphs is simple.

To see this consider Fig.16 which shows a particular τ-ordered diagram which contributes to the incident physical electron state. Each vertical line in the figure denotes a certain intermediate state and energy denominator. In addition to just this τ-ordered diagram, there are diagrams for each of the 5 other permutations of the vertices (1234). There are two points concerning these diagrams we must make. First, to leading order in η_P, the vertices on the through-going electron line do not distinguish between the order of emission of the various photons. This is so because the η's of all the photons are predominantly small compared to η_P, and the photons couple to the electron line through γ^0 which behaves like

$$\bar{u}(p,s)\ \gamma^0\ u(P,S) = 2\sqrt{\eta_P \eta}\ \delta_{sS} \approx 2\eta_P\ \delta_{sS} \qquad \text{(IV.5)}$$

Secondly, the infinite momentum energy of the through-going electron can be ignored in the energy denominators for each diagram. Using this simplification we will write down the energy denominator factors for each diagram.

$$
\begin{aligned}
&(1234) \quad \left\{\omega(k)\,\left[H(p_3)+H(p_4)\right]\left[\omega(q)+H(p_3)+H(p_4)\right]\left[H(p_1)+H(p_2)+H(p_3)+H(p_4)\right]\right\}^{-1} \\
&(1324) \quad \left\{\omega(k)\left[\omega(k)+\omega(q)\right]\left[\omega(q)+H(p_3)+H(p_4)\right]\left[H(p_1)+H(p_2)+H(p_3)+H(p_4)\right]\right\}^{-1} \\
&(3124) \quad \left\{\omega(q)\left[\omega(q)+\omega(k)\right]\left[\omega(q)+H(p_3)+H(p_4)\right]\left[H(p_1)+H(p_2)+H(p_3)+H(p_4)\right]\right\}^{-1} \\
&(1342) \quad \left\{\omega(k)\left[\omega(k)+\omega(q)\right]\left[\omega(k)+H(p_1)+H(p_2)\right]\left[H(p_1)+H(p_2)+H(p_3)+H(p_4)\right]\right\}^{-1} \\
&(3142) \quad \left\{\omega(q)\left[\omega(q)+\omega(k)\right]\left[\omega(k)+H(p_1)+H(p_2)\right]\left[H(p_1)+H(p_2)+H(p_3)+H(p_4)\right]\right\}^{-1} \\
&(3412) \quad \left\{\omega(q)\left[H(p_1)+H(p_2)\right]\left[\omega(k)+H(p_1)+(p_2)\right]\left[H(p_1)+H(p_2)+H(p_3)+H(p_4)\right]\right\}^{-1}
\end{aligned}
\qquad \text{(IV.6)}
$$

The sum is computed efficiently if we combine the first three terms, then the second three terms and sum the results to obtain,

$$
\left\{\omega(k)\,\left[H(p_3)+H(p_4)\right]\right\}^{-1}\left\{\omega(q)\left[H(p_1)+H(p_2)\right]\right\}^{-1} \qquad \text{(IV.7)}
$$

This important factorization property means that the two bare pairs in the physical state are independent of one another (in the region of phase space which gives the dominant contribution to the scattering amplitude). Using the ideas in this example, it is not difficult to construct an inductive proof of the factorization property for any strongly-ordered multi-chain graph.[24] In fact, QED experts will recognize this factorization property as simply a slight variation on an argument familiar from bremsstrahlung and infrared problems.

We now return to Fig. 15, and write the scattering amplitude,

$$S = \frac{1}{2!} \frac{e^4}{(2\pi)^9} \delta(\eta_p - \eta_{p'}) \int M^{(1)}(\underset{\sim}{p}' - \underset{\sim}{p}; \eta)\; 2\eta\delta(\eta' - \eta)$$

$$d\underset{\sim}{p}' d\underset{\sim}{k}_1 d\underset{\sim}{p}_1 d\underset{\sim}{p}_4 d\eta_{k_1} d\eta_1 d\eta' \;(2\eta)^{-2} (2\eta_{k_1})^{-2} (2\eta_1)^{-2} (2\eta_4)^{-2}$$

$$\left[\omega(k_1)\right]^{-1}\left[\omega(k_2)\right]^{-1}\left[H(p_1)+(p_2)\right]^{-1}\left[H(p_3)+H(p_4)\right]^{-1}\sum_{S,S'} \bar{u}(P'S')\gamma^o u(p's')\bar{u}(p's')\gamma^o u(P,S)$$

$$\mathrm{tr}\left[(\not{p}_1+m)\gamma^o(\not{p}_4+m)\gamma^3(-\not{p}_3+m)\gamma^o(-\not{p}_2+m)\gamma^3\right]\left[F(\underset{\sim}{p}_4-\underset{\sim}{p}_1)F_c(\underset{\sim}{p}_3-\underset{\sim}{p}_2)-(2\pi)^2\delta(\underset{\sim}{p}_4-\underset{\sim}{p}_1)\delta(\underset{\sim}{p}_3-\underset{\sim}{p}_2)\right]$$

(IV.8)

where

$$\underset{\sim}{k}_2 - \underset{\sim}{k}_1 = \underset{\sim}{P}' - \underset{\sim}{P} - \underset{\sim}{p}' + \underset{\sim}{p}$$

$$\underset{\sim}{p}_2 = \underset{\sim}{k}_1 - \underset{\sim}{p}_1 = \underset{\sim}{P} - \underset{\sim}{p} - \underset{\sim}{p}_1, \quad \underset{\sim}{p}_3 = \underset{\sim}{k}_2 - \underset{\sim}{p}_4 = \underset{\sim}{P}' - \underset{\sim}{p}' - \underset{\sim}{p}_4$$

and where we have identified the scattering amplitude for the "inner" loop and have used (A.13). The factor of $\frac{1}{2!}$ occurs because when we sum over the permutations of the vertices we effectively double count individual diagrams. We can further simplify (IV.8) by noting that the η' integration is done by the δ function coming from the inner loop. Finally, identifying the scattering amplitude for the "outer" loop,

$$S = (2\pi)\; 2\eta_p \delta(\eta_p - \eta_{p'}) \;\frac{1}{2!} \int \frac{d\underset{\sim}{p}'}{(2\pi)^2}\; M^{(1)}(\underset{\sim}{p}' - \underset{\sim}{p}; \eta_p)\; M^{(1)}(\underset{\sim}{P}' - \underset{\sim}{P} - \underset{\sim}{p}' + \underset{\sim}{p}; \eta_p) \qquad \text{(IV.9)}$$

This convolution integral can be factored by transforming to $\underset{\sim}{x}$-space. Define,

$$M^{(1)}(\underset{\sim}{q}; \eta_p) = \int d\underset{\sim}{x}\; e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}}\; M^{(1)}(\underset{\sim}{x}; \eta_p) \qquad \text{(IV.10)}$$

Then,

$$S = 2\eta_p(2\pi)\,\delta(\eta_p - \eta_{p'}) \int d\underset{\sim}{x}\; e^{i(\underset{\sim}{P}' - \underset{\sim}{P})\cdot \underset{\sim}{x}} \frac{1}{2!} \left[M^{(1)}(\underset{\sim}{x}, \eta_p) \right]^2 \qquad \text{(IV.11)}$$

which shows the beginnings of the expected "eikonalization."

Using the same techniques we can obtain the scattering amplitude for the diagrams indicated in Fig. 17. The result is, using the notation of Appendix A and B,

$$S = 2\eta_p(2\pi)\,\delta(\eta_p - \eta_{p'}) \int \frac{d\underset{\sim}{p}'}{(2\pi)^2}\; M^{(1)}(\underset{\sim}{p}' - \underset{\sim}{p}; \eta_p)\; M^{(2)}(\underset{\sim}{P}' - \underset{\sim}{P} - \underset{\sim}{p}' + \underset{\sim}{p}; \eta_p)$$

$$\text{(IV.12)}$$

$$S = 2\eta_p(2\pi)\,\delta(\eta_p - \eta_{p'}) \int d\underset{\sim}{x}\; e^{i(\underset{\sim}{P}' - \underset{\sim}{P})\cdot \underset{\sim}{x}}\, M^{(1)}(\underset{\sim}{x}, \eta_p)\; M^{(2)}(\underset{\sim}{x}, \eta_p)$$

In this case, when summing over the permutations of the vertices, there is no overcounting of diagrams.

Continuing the argument to contain all two chain diagrams, as indicated in Fig. 18, we conclude that the scattering amplitude for this class of diagrams reads,

$$S^{(2\,\text{chain})} = 2\eta_p(2\pi)\,\delta(\eta_p - \eta_{p'}) \int d\underset{\sim}{x}\; e^{i(\underset{\sim}{P}' - \underset{\sim}{P})\cdot \underset{\sim}{x}}$$

$$\frac{1}{2!} \left[M^{(1)}(\underset{\sim}{x}, \eta_p) + M^{(2)}(\underset{\sim}{x}, \eta_p) + \cdots \right] \left[M^{(1)}(\underset{\sim}{x}, \eta_p) + \cdots \right] \qquad \text{(IV.13)}$$

Identifying $M^{(1\,\text{chain})}(x, \eta_p)$ as in Appendix C, we have

$$S^{(2\,\text{chain})} = 2\eta_p(2\pi)\,\delta(\eta_p - \eta_{p'}) \int d\underset{\sim}{x}\; e^{+i(\underset{\sim}{P}' - \underset{\sim}{P})\cdot \underset{\sim}{x}} \frac{1}{2!} \left[M^{(1\,\text{chain})}(\underset{\sim}{x}, \eta_p) \right]^2$$

$$\text{(IV.14)}$$

D. THE REGGE-EIKONAL FORMULA - A CRITICISM

The arguments presented in the previous section generalize straightforwardly to diagrams which contain N chains scattering off an external field. Again, after summing over all τ-ordered diagrams, the chains become independent (in that region of phase space which contributes the leading log to the scattering amplitude), and the amplitude reads,

$$S^{(N)}(\underset{\sim}{P}'-\underset{\sim}{P}) = 2\,\eta_P(2\pi)\delta(\eta_P-\eta_{P'})\int d\underset{\sim}{x}\; e^{i(\underset{\sim}{P}'-\underset{\sim}{P})\cdot\underset{\sim}{x}}\;\frac{1}{N!}\left[M^{(1\,\text{chain})}(\underset{\sim}{x},\eta_P)\right]^N \tag{IV.15}$$

Summing over N, we find

$$S(\underset{\sim}{P}'-\underset{\sim}{P}) = -2\,\eta_P(2\pi)\,\delta(\eta_P-\eta_{P'})\int d\underset{\sim}{x}\; e^{i(\underset{\sim}{P}'-\underset{\sim}{P})\cdot\underset{\sim}{x}}\left[1-e^{M^{(1\,\text{chain})}(\underset{\sim}{x},\,\eta_P)}\right] \tag{IV.16}$$

which is the Regge-eikonal form for the scattering amplitude.

(IV.16) has been investigated in detail for QED and $\lambda\phi^3$ field theories.[35] The calculation for $\lambda\phi^3$ is particularly simple, and using the techniques of this paper or otherwise, it is easy to find that,

$$M^{(1\,\text{chain})}(\underset{\sim}{x},\eta_P) \sim -\frac{\eta_P^{\alpha(0)-1}}{\log\;\eta_P}\; e^{-\frac{\underset{\sim}{x}^2}{2|\alpha'(0)|\log\;\eta_P}} \tag{IV.17}$$

where $\alpha(0)$ and $\alpha'(0)$ are, respectively, the intercept and slope of the leading Regge trajectory. In the forward direction then, (IV.17) receives significant contributions only from $|\underset{\sim}{x}| \leq 0\,(\log\;\eta_P)$, and leads to an elastic cross section which saturates, but does not violate, the Froissart bound. In QED one finds that $M^{(1\,\text{chain})}(\underset{\sim}{x},\eta_P)$ possesses a fixed cut at $J = 1 + \frac{11\pi}{32}\,\alpha^2$ which is modulated by a complicated function of $\underset{\sim}{x}$ which behaves like $e^{-\lambda|\underset{\sim}{x}|}$ for $|\underset{\sim}{x}| \gg \frac{1}{\lambda}$. So, as in the case of $\lambda\phi^3$, although $M^{(1\,\text{chain})}$ taken alone violates the Froissart bound, the Regge-eikonal formula leads to a cross section which increases only as $\log^2 \eta_P$. It is this phenomena which has led several authors to take the Regge-eikonal formula very seriously.

However, from the point of view of the physical picture developed in this paper it is not even clear that the Regge-eikonal scheme is at all reasonable. In particular, (IV.17) assumes that the chains never interact among themselves. However, we have seen that the chains all scatter simultaneously off the external field, so they are really overlapping and crowded together in real space-time. So, even if the photons linking the pairs together were given a large mass (short range), such photons could easily propagate between two chains and link them up. The simplest example of such a process is shown in Fig. 19 . This is an interference effect between a two-chain and a one-chain diagram. These will be studied in considerable detail in the next section.

E. INTERFERENCE EFFECTS

We wish to consider the simplest type of interference graphs in some detail. These are the 2 chain-1 chain graphs, an example of which is drawn in Fig. 20. As in previous sections we are content to calculate only the leading logarithms of each diagram. In this approximation the photons forming the right hand chain in Fig. 20 are strongly ordered in the usual sense. This fact then allows us to formally sum over the subsections indicated by letters A, B, and C in Fig. 20 and replace them by their Regge form. This fact is stated pictorially in Fig. 21 where we have a Reggeon, defined diagrammatically in Fig. 22, interacting with an elementary particle (massive photon) through the exchange of another elementary particle. Just as in the calculation of the single chain, this exchange gives rise to an attractive potential between the particle and the Reggeon. Figure 21 has the advantage of showing that this class of interference terms reduces to a quasi-two body calculation, and should, in principle, be solvable.

Although the single chain gives a fixed Regge cut in QED, there is no reason to restrict our considerations to just this case. We will, in fact, develop a formalism in which we can insert an input Reggeon (pole or cut) into Fig. 21, and then deduce some characteristics (energy dependence, at least) of the output Reggeon. In particular we will see cases in which Fig. 21 generates a scattering amplitude which grows faster with energy than the scattering amplitude corresponding to a single Regge exchange. Our method of analysis consists of several steps: first, write an integral equation which sums up diagrams of Fig. 21; second, specialize to the forward direction and obtain a simpler (Fredholm) integral equation; third, use variational principles to obtain lower bounds on the highest eigenvalue of the kernel; finally, relate the bound on the eigenvalue to a lower bound on the energy dependence of the scattering amplitude.

We begin the analysis by writing the S-matrix in a more convenient form. We define a function W which is related to the S-matrix by the removal of the photon legs 1, 2, 3, and 4 shown in Fig. 21. So,

$$S = (2\pi)(2\eta)\,\delta(\eta-\eta')\int d\underset{\sim}{p}_1\, d\underset{\sim}{\ell}_1\, d\underset{\sim}{q}_1\, \delta(\underset{\sim}{\Delta}-\underset{\sim}{p}_1-\underset{\sim}{\ell}_1-\underset{\sim}{q}_1)\, \frac{d\alpha}{2\alpha}\, d\underset{\sim}{p}\, d\underset{\sim}{\ell}\, d\underset{\sim}{q}\, \delta(\underset{\sim}{\Delta}-\underset{\sim}{p}-\underset{\sim}{\ell}-\underset{\sim}{q})$$

$$\left(\frac{1}{(\underset{\sim}{\ell}_1^2+\lambda^2)(\underset{\sim}{q}_1^2+\lambda^2)}\right)\left(\frac{1}{(\underset{\sim}{\ell}^2+\lambda^2)(\underset{\sim}{q}^2+\lambda^2)}\right) W\left(\underset{\sim}{\ell}_1+\underset{\sim}{q}_1, \underset{\sim}{p}_1; \underset{\sim}{\ell}+\underset{\sim}{q}, \underset{\sim}{p};\ \frac{\alpha}{\eta_{min}}\right)$$

$$S = (2\pi)(2\eta)\,\delta(\eta-\eta')\int d\underset{\sim}{p}_1\, d\underset{\sim}{\ell}_1\, \frac{d\alpha}{2\alpha}\, d\underset{\sim}{p}\, d\underset{\sim}{\ell}\left(\frac{1}{[(\underset{\sim}{\Delta}-\underset{\sim}{p}_1-\underset{\sim}{\ell}_1)^2+\lambda^2][\underset{\sim}{\ell}_1^2+\lambda^2]}\right) \tag{IV.18}$$

$$\left(\frac{1}{[(\underset{\sim}{\Delta}-\underset{\sim}{p}-\underset{\sim}{\ell})^2+\lambda^2][\underset{\sim}{\ell}^2+\lambda^2]}\right)\quad W\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \underset{\sim}{p}_1;\ \underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p};\ \frac{\alpha}{\eta_{min}}\right)$$

The $\underset{\sim}{\ell}$ and $\underset{\sim}{\ell}_1$ integrals can be done, giving

$$S = (2\pi)(2\eta)\,\delta(\eta-\eta')\int d\underset{\sim}{p}_1\, d\underset{\sim}{p}\int_{\eta_{min}}^{\eta}\frac{d\alpha}{\alpha}\, k\,(|\underset{\sim}{\Delta}-\underset{\sim}{p}_1|)\, k(|\underset{\sim}{\Delta}-\underset{\sim}{p}|)\, W\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \underset{\sim}{p}_1; \underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p}; \frac{\alpha}{\eta_{min}}\right) \tag{IV.19}$$

where

$$k(|\underset{\sim}{\Delta}-\underset{\sim}{p}|) \equiv \int d\underset{\sim}{\ell}\,\frac{1}{[(\underset{\sim}{\Delta}-\underset{\sim}{p}_1-\underset{\sim}{\ell}_1)^2+\lambda^2][\underset{\sim}{\ell}_1^2+\lambda^2]} = \int d\underset{\sim}{x}\; e^{-i(\underset{\sim}{\Delta}-\underset{\sim}{p})\cdot\underset{\sim}{x}}\, K_0^2\,(\mu|\underset{\sim}{x}|) \tag{IV.20}$$

The function W now represents the propagation and interaction of the Reggeon and elementary particle in the t-channel. We can write an integral equation for W in terms of the Reggeon, the photon propagator and the interaction between them. The integral equation is represented pictorially in Fig. 23. If we represent the Reggeon by the function $R(\underset{\sim}{k},\ \eta/\eta_{min})$, the integral equation becomes,

$$W\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1\underset{\sim}{p}_1;\ \underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p};\ \frac{\eta}{\eta_{min}}\right) = \delta(\underset{\sim}{p}-\underset{\sim}{p}_1)\, R\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1;\ \frac{\eta}{\eta_{min}}\right)\frac{1}{(\underset{\sim}{p}_1^2+\lambda^2)} +$$

$$\frac{e^4}{2(2\pi)^3}\int_{\eta_{min}}^{\eta}\frac{d\alpha}{\alpha}\int\frac{d\underset{\sim}{h}}{[(\underset{\sim}{p}_1-\underset{\sim}{h})^2+\lambda^2]}\cdot\frac{1}{(\underset{\sim}{p}_1^2+\lambda^2)}\; F(\underset{\sim}{h}-\underset{\sim}{p}_1, \underset{\sim}{\Delta}-\underset{\sim}{h})\, R(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \eta/\alpha)\, F(\underset{\sim}{p}_1, \underset{\sim}{h}-\underset{\sim}{p}_1)\times$$

$$W\left(\underset{\sim}{h}, \underset{\sim}{\Delta}-\underset{\sim}{h}; \underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p};\ \frac{\alpha}{\eta_{min}}\right) \tag{IV.21}$$

where the function F describes the possible momentum dependence in the coupling between the Reggeon and photon. By iterating (IV.21) one can verify that it indeed sums the diagrams of Fig. 20. Instead of discussing this integral equation in its full generality we will specialize to a rather naive model in which F is momentum independent and R is a simple pole. The real situation in QED will be discussed at the end of this section.

With these simplifications the integral equation now reads,

$$W\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \underset{\sim}{p}_1; \underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p}; \frac{\eta}{\eta_{min}}\right) = \delta(\underset{\sim}{p}-\underset{\sim}{p}_1)\, \frac{1}{(\underset{\sim}{p}_1^2+\lambda^2)} \left(\frac{\eta}{\eta_{min}}\right)^{\beta(\underset{\sim}{\Delta}-\underset{\sim}{p}_1)} \theta\left(\frac{\eta}{\eta_{min}} - 1\right)$$
$$+ \frac{e^4}{2(2\pi)^3} \int_{\eta_{min}}^{\eta} \frac{d\alpha}{\alpha} \int \frac{d\underset{\sim}{h}}{[(\underset{\sim}{p}_1-\underset{\sim}{h})^2+\lambda^2]}\, \frac{\lambda^2}{[\underset{\sim}{p}_1^2+\lambda^2]} \left(\frac{\eta}{\alpha}\right)^{\beta(\underset{\sim}{\Delta}-\underset{\sim}{p}_1)} W\left(\underset{\sim}{h}, \underset{\sim}{\Delta}-\underset{\sim}{h}; \underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p}; \frac{\alpha}{\eta_{min}}\right) \tag{IV.22}$$

where β is the trajectory function of the input Reggeon. W is, of course, different from zero only for $\eta/\eta_{min} > 1$. It will suffice for the purposes at hand to consider the somewhat simpler integral equation for the function,

$$T\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \underset{\sim}{p}_1; \frac{\eta}{\eta_{min}}\right) = \int d\underset{\sim}{p}\, k(|\underset{\sim}{\Delta}-\underset{\sim}{p}|)\, W\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \underset{\sim}{p}_1; \underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p}; \frac{\alpha}{\eta_{min}}\right) \tag{IV.23}$$

T satisfies the integral equation,

$$T\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \underset{\sim}{p}_1; \frac{\eta}{\eta_{min}}\right) = k(|\underset{\sim}{\Delta}-\underset{\sim}{p}_1|)\, \frac{1}{(\underset{\sim}{p}_1^2+\lambda^2)} \left(\frac{\eta}{\eta_{min}}\right)^{\beta(\underset{\sim}{\Delta}-\underset{\sim}{p}_1)} \theta\left(\frac{\eta}{\eta_{min}} - 1\right)$$
$$+ \frac{e^4}{2(2\pi)^3} \int_{\eta_{min}}^{\eta} \frac{d\alpha}{\alpha} \int \frac{d\underset{\sim}{h}}{[(\underset{\sim}{p}_1-\underset{\sim}{h})^2+\lambda^2]}\, \frac{\lambda^2}{(\underset{\sim}{p}_1^2+\lambda^2)} \left(\frac{\eta}{\alpha}\right)^{\beta(\underset{\sim}{\Delta}-\underset{\sim}{p}_1)} T\left(\underset{\sim}{h}, \underset{\sim}{\Delta}-\underset{\sim}{h}; \frac{\alpha}{\eta_{min}}\right) \tag{IV.24}$$

Care must be taken in writing the order of the first two arguments in T, because this ordering reflects the exchange character of the interaction between the Reggeon and elementary particle. This integral equation becomes much simpler if we introduce Mellin transforms. Recall the definition of the transform,

$$T(\underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p}; J) = \int_0^{\infty} T(\underset{\sim}{\Delta}-\underset{\sim}{p}, \underset{\sim}{p}; y)\, y^{-J-1}\, dy$$

and its inverse,

$$T(\underset{\sim}{\Delta}-\underset{\sim}{p},\underset{\sim}{p};y) = \frac{1}{2\pi i}\int_C T(\underset{\sim}{\Delta}-\underset{\sim}{p},\underset{\sim}{p};J)\, y^J\, dJ$$

where the contour is chosen to the right of all the singularities of $T(\underset{\sim}{\Delta}-\underset{\sim}{p},\underset{\sim}{p};J)$. It is an easy exercise to obtain the integral equation for the transforms,

$$T(\underset{\sim}{\Delta}-\underset{\sim}{p}_1,\underset{\sim}{p}_1;J) = k(|\underset{\sim}{\Delta}-\underset{\sim}{p}_1|)\, \frac{1}{[J-\beta(\underset{\sim}{\Delta}-\underset{\sim}{p}_1)]\,[p_1^2+\lambda^2]} + \frac{e^4}{2(2\pi)^3}\int \frac{d\underset{\sim}{h}}{[(\underset{\sim}{p}_1-\underset{\sim}{h})^2+\lambda^2]}\, \frac{\lambda^2}{[\underset{\sim}{p}_1^2+\lambda^2]\,[J-\beta(\underset{\sim}{\Delta}-\underset{\sim}{p}_1)]}\, T(\underset{\sim}{h},\underset{\sim}{\Delta}-\underset{\sim}{h};J) \qquad \text{(IV.25)}$$

Consider this equation in the forward ($\underset{\sim}{\Delta}=0$) direction. The driving term then depends only upon $\underset{\sim}{p}_1^2$. T will inherit this symmetry, so the angular integral in the homogeneous term can be done. If we carry out this integral and define,

$$t = \underset{\sim}{p}_1^2 \qquad t' = \underset{\sim}{h}^2$$

the integral equation becomes,

$$T(t;J) = \frac{k(t)}{[J-\beta(t)]\,[t+\lambda^2]} + \frac{e^4}{4(2\pi)^2}\int_0^\infty \frac{dt'}{\sqrt{(t+t'+\lambda^2)^2-4tt'}}\, \frac{\lambda^2}{[t+\lambda^2]\,[J-\beta(t)]}\, T(t';J) \qquad \text{(IV.26)}$$

This one dimensional integral equation can be written with a symmetric kernel if we simply define,

$$W(t;J) = \sqrt{J-\beta(t)}\,\sqrt{t+\lambda^2}\; T(t;J) \qquad \text{(IV.27)}$$

and note that

$$W(t;J) = \frac{k(t)}{\sqrt{J-\beta(t)}\,\sqrt{t+\lambda^2}} + \frac{e^4}{4(2\pi)^2}\int_0^\infty K_S(t,t')\, W(t';J)\, d\left(\frac{t'}{\lambda^2}\right) \qquad \text{(IV.28)}$$

where

$$K_S(t,t') = \left(\frac{1}{\sqrt{J-\beta(t)}\,\sqrt{t+\lambda^2}}\right)\left(\frac{\lambda^4}{\sqrt{(t+t'+\lambda^2)^2-4tt'}}\right)\left(\frac{1}{\sqrt{J-\beta(t')}\,\sqrt{t'+\lambda^2}}\right)$$

Since $K_S(t,t')$ is a symmetric, real, square-integrable kernel, it must have a discrete spectrum of real eigenvalues. It will become evident shortly that if we can obtain the highest eigenvalue of K_S, then we will have found the leading energy dependence of the set of graphs of interest. Actually, we will be content to obtain a rather weak lower bound on the highest eigenvalue of K_S and thereby obtain a lower bound on the energy dependence of the amplitude.

In order to see explicitly the connection between the eigenvalue problem and the energy dependence of the scattering amplitude we go back to (IV.19) and write it in terms of W(t;J). From (IV.19) and (IV.20)we have

$$S = (2\pi)(2\eta)\,\delta(\eta-\eta')\int d\underset{\sim}{p}_1 \int_{\eta_{min}}^{\eta} \frac{d\alpha}{\alpha}\; k(|\underset{\sim}{\Delta}-\underset{\sim}{p}_1|)\; T\left(\underset{\sim}{\Delta}-\underset{\sim}{p}_1, \underset{\sim}{p}_1; \frac{\alpha}{\eta_{min}}\right) \tag{IV.30}$$

Then, introducing Mellin transforms and specializing to the forward direction, we have using (IV.27)

$$S = (2\pi)(2\eta)\,\delta(\eta-\eta')\; M\left(\frac{\eta}{\eta_{min}}\right) \tag{IV.31}$$

where, for large η/η_{min},

$$M\left(\frac{\eta}{\eta_{min}}\right) = \pi\int_0^{\infty} dt\; k(t) \int_C \frac{W(t;J)}{J\sqrt{J-\beta(t)}\sqrt{t+\lambda^2}} \left(\frac{\eta}{\eta_{min}}\right)^J dJ \tag{IV.32}$$

where the contour C lies to the right of all the singularities of the integrand. Consider the integral equation for W and write it schematically,

$$W = B + g\,K_S W \tag{IV.33}$$

where $g = \frac{e^4}{4(2\pi)^2}$. (VI. 16) has the formal solution

$$W = (1-g K_S)^{-1} B \tag{IV.34}$$

which may be written under appropriate circumstances as a series,

$$W(t;J) = \sum_{n=0}^{\infty} W^{(n)}(t;J) = \sum_{n=0}^{\infty} (g K_S)^{(n)} B \tag{IV.35}$$

The nth approximate of W(t:J) is given by

$$W^{(n)}(t;J) = g^n \int \cdots \int dt_1 \ldots dt_n \; K_S(t,t_1) \, K_S(t_1,t_2) \ldots K_S(t_{n-1},t_n) B(t_n) \tag{IV.36}$$

Since K_S is a Fredholm kernel it has a discrete spectral decomposition,[36]

$$K_S(t,t') = \sum_{n=1}^{\infty} \mu_n f_n(t) f_n(t') \tag{IV.37}$$

where μ_n and $f_n(t)$ are respectively the nth eigenvalue and eigenfunction of K_S. If we then approximate $W^{(n)}(t;J)$ by withholding only the highest eigenvalue (μ_1) of K_S, (IV.36) becomes,

$$W^{(n)}(t;J) \sim g^n \mu_1^n (J) \; f_1(t;J) \int dt' \; f_1(t';J) \; B(t') \tag{IV.38}$$

where we have indicated explicitly that the eigenvalues and eigenfunctions can depend upon J. If we now introduce the Mellin transform of $M\left(\frac{\eta}{\eta_{min}}\right)$,

$$M(J) = \pi \int_0^{\infty} dt \; k(t) \; \frac{W(t:J)}{J\sqrt{J-\beta(t)}\sqrt{t+\lambda^2}} \tag{IV.39}$$

we have from (IV.38) that its nth approximate is,

$$M^{(n)}(J) \sim g^n \mu_1^n (J) \left[\int_0^{\infty} dt \; \frac{f_1(t;J)}{J\sqrt{J-\beta(t)}\sqrt{t+\lambda^2}}\right] \left[\int_0^{\infty} dt' \; f_1(t';J) B(t')\right] \tag{IV.40}$$

Therefore, summing over n,

$$M(J) = \sum_{n=0}^{\infty} M^{(n)}(J) = \left(\frac{1}{1-g\mu_1(J)}\right)\left[\int_0^{\infty} dt \frac{f_1(t;J)}{J\sqrt{J-\beta(t)}\sqrt{t+\lambda^2}}\right]\left[\int_0^{\infty} dt'\, f_1(t';J)B(t')\right] \tag{IV.41}$$

So, it is clear that M(J) will develop a pole at $J=J_0$ where

$$1 - g\mu_1(J_0) = 0 \tag{IV.42}$$

By doing a simple variational calculation to obtain a lower bound on $\mu_1(J_0)$, we can obtain estimates of the location of the solution J_0 of (IV.42).

Recall the Rayleigh-Reitz variational principle which states that the largest eigenvalue of K_S is given by

$$\mu_1 = \sup_{f\in L_2} \frac{\left(f(t), K_S(t,t')f(t')\right)}{\left(f(t), f(t)\right)} \tag{IV.43}$$

where f(t) is any square integrable function. So, if we choose f(t) at random, we can be sure that (IV.43) will give a lower bound on μ_1. We choose,

$$f(t) = \frac{1}{\lambda}\sqrt{J-\beta(t)}\sqrt{t+\lambda^2}\; e^{-at} \tag{IV.44}$$

where "a" is a parameter which must be chosen such that f(t) is normalized to unity. From (IV.29) and (IV.44), we compute

$$(f, K_S f) = \frac{1}{\lambda^2}\int dt\, dt' \frac{e^{-a(t+t')}}{\sqrt{(t+t'+\lambda^2)^2 - 4tt'}} \tag{IV.45}$$

By changing the integration variables to $t_+=t+t'$ and $t_-=t-t'$, it is not difficult to reduce (IV.45) to a one dimensional integral,

$$(f, K_S f) = \frac{1}{2a\lambda^2}\, e^{\frac{1}{2}a\lambda^2}\int_{(a\lambda^2/2)}^{\infty} e^{-y}\, y^{-1}\, dy \tag{IV.46}$$

The relation between "a" and J is given in this approach by the normalization condition,

$$(f, f) = 1 = \frac{1}{\lambda^4} \int_0^{\infty} [J-\beta(t)] \; [t+\lambda^2] \; e^{-2at} \, dt \tag{IV.47}$$

We suppose for illustration that the trajectory is linear, $\beta(t) = \beta_0 - \beta_1 t$. Then,

$$(f, f) = (J-\beta_0) \frac{1}{(2a\lambda^2)} + (J-\beta_0 + \beta_1\lambda^2) \frac{2}{(2a\lambda^2)^2} + \beta_1\lambda^2 \frac{6}{(2a\lambda^2)^3} \tag{IV.48}$$

Two extreme cases of (IV.46) and (IV.48) are quite simple and illustrate clearly the mechanism at work. First consider $(a\lambda^2) \gg 1$. Then

$$(f, K_S f) \sim \frac{1}{(a \; \lambda^2)^2} \qquad (f, f) \sim \frac{J-\beta_0}{2(a \; \lambda^2)} = 1 \tag{IV.49}$$

Thus,

$$\mu_1(J) > \sqrt{\frac{2}{J-\beta_0}}$$

Inserting this inequality into (IV.42), we find that M(J) develops a pole at

$$J_0 > \beta_0 + 2g^2 \tag{IV.50}$$

which is further to the right on the complex J-plane than the input Reggeon pole which is located at $J = \beta_0$. We cannot take this example too seriously, however, because it corresponds to strong coupling. However, we can consider another case which is closer to QED. Imagine that $(a\lambda^2)$ is small and $\beta_1 = 0$. Then,

$$(f, K_S f) = \frac{1}{2a\lambda^2} \log\left(\frac{2}{a\lambda^2}\right) > \frac{1}{2a\lambda^2} \; , \quad (f, f) = \frac{2(J-\beta_0)}{(2a\lambda^2)^2} \tag{IV.51}$$

So,

$$\mu_1(J) > \frac{1}{\sqrt{2(J-\beta_0)}}$$

which means, according to (IV.42) , that

$$J_0 > \beta_0 + \frac{g^2}{2} \tag{IV.53}$$

in this weak coupling case.

These two very crude examples serve to illustrate the point that it is not difficult to find t channel exchanges which lead to stronger energy dependences than simple chains. The relation, however, of these model calculations to QED is more delicate. As we saw in Section B, the chain diagrams in QED generate a Regge cut, which we have written as a linear superposition of poles. In principle, we can treat this case because our integral equation is also linear. Another difference between QED and the model calculation is that the coupling between the Regge cut and the photon is actually momentum dependent. This fact could change the character of the integral equation, but it does not change the fact that the exchanged photons in Fig. 21 tend to bind the t-channel system. So, although the spectrum of $K_S(t, t')$ may no longer be discrete, diagrams like Fig. 21 could still possess a stonger energy dependence than the simple chain.

Potentially more interesting than the 2 chain-1 chain diagrams considered here are the 2 chain-2 chain diagrams. We can give these a pictorial representation shown in Fig. 24, and recognize that they are iterated Mandelstam cut diagrams. Such diagrams have been considered in the literature and it has been conjectured that they generate Regge poles, although this point has not been verified.[37]

F. CONCLUSIONS AND DISCUSSION

Guided by a clear physical picture, we have accumulated evidence that graphs more complicated than simple multiperipheral chains might play a substantial role in diffraction scattering. One might now take the diagrams of Fig. 20 and use them as the input of the s-channel iteration procedure described in Section D. Since these diagrams have a stronger energy dependence than the single multiperipheral chains, they certainly give a significant contribution to the S-matrix. We cannot, however, claim that they contribute significantly to the total cross section since this quantity depends upon the range of the graphs in the transverse $|\underset{\sim}{x}|$ plane (which we have not determined) as well as their energy dependence. More importantly, however, we have argued that the s-channel iteration scheme is not a physically convincing procedure. So, we do not take this proposal seriously. It appears that present field theoretic approaches to diffraction scattering lack a compelling mechanism to enforce s-channel unitarity. Until this deep problem is understood more clearly, detailed perturbation theory calculations will not resolve additional really interesting and important questions in this field.

The results of this investigation suffer from the technical limitation in any leading logarithm calculation. It has been argued[38] that leading logarithm calculations are only accurate when the couplings of the particles are small enough. However, the spirit of this investigation is to obtain results which do not rely upon the size of coupling constants. It is the unspoken hope of this investigation that although the leading logarithm approach is not perfectly accurate, it remains indicative of the truth if the coupling constants become fairly large. One might argue, for example, that it would certainly be bizarre if the energy dependence of the interference terms decreased relative to the single multiperipheral chain as the coupling constant increased! One might also question the usefulness of perturbation theory in this entire program. We saw in Section B that the single multiperipheral chain violates the Froissart bound by a power of the energy. The s-channel iteration procedure then reduced the energy dependence of the scattering amplitude until it just saturated the bound. However, the success of this procedure relied upon the detailed cancellation among graphs, each of which was absurdly large. One might now ask whether this feat was profound or accidental. For example, are there other graphs which further reduce the energy dependence of the final result?

We have emphasized our space-time picture of high energy scattering throughout this paper. In particular we have argued in Section E that it is not unlikely for two particles in the physical state of the projectile to possess small relative subenergy and to propagate near one another. These conditions are, however, the ideal ones in which the particles are likely to interact significantly (e.g., resonant). This is a problem which exists (and is often ignored) even in multiperipheral models in which only one chain of constituents is allowed. However, this effect can reach extreme proportions for physical particle states consisting of more than one chain of constituents. For example, e^+e^- pairs on different chains are likely to overlap in both momentum and configuration space. These pairs will certainly interact and their respective chains might often be linked up in the process. A simple example of such a possibility is shown in Fig. 25. Unfortunately, perturbation theory is not an efficient tool for computing these effects. Perhaps the effective field technique from statistical physics provides a better calculational and conceptual framework for this problem. Anyway, in light of the complexity of Section E, more theoretical effort is needed in deciding questions of this general nature than in the calculations of minute details of diagrams which just happen to be exactly computable.

APPENDIX A. SINGLE LOOP

In this Appendix we will illustrate our calculational methods by extracting the leading energy dependence of the simplest graph in the class to be considered. According to our perturbation theory there are four diagrams (Fig. 9). However, if we recall that the infinite momentum polarization vectors satisfy

$$-g^{\mu\nu} = \sum_{\lambda=1}^{2} \epsilon_{\lambda}^{\mu}(p)\ \epsilon_{\lambda}^{\nu}(p) + \frac{1}{\eta^2}(2\eta H - \underset{\sim}{p}^2)\,\delta_3^{\mu}\ \delta_3^{\nu} \tag{A.1}$$

it is easy to see that the four diagrams can be combined into one (Fig. 10). Now, however, instead of associating a factor $\sum_{\lambda} \epsilon_{\lambda}(p)^{\mu}\ \epsilon_{\lambda}(p)^{\nu}$ with each internal photon we make the association with $-g^{\mu\nu}$. Throughout this paper this simplification of our perturbation theory rules will be tacitly understood.

Now it is straightforward to write down the amplitude for this diagram. Since this diagram has been studied previously in the literature by Cheng and Wu[32], and others[33], we will try to use notation as similar to theirs as possible. Using the kinematics indicated in Fig. 12,

$$S^{(1)} = \frac{e^4}{(2\pi)^9}\,\delta(\eta_P - \eta_{P'}) \int d\underset{\sim}{p}\,d\underset{\sim}{k}_1\,d\underset{\sim}{p}_1\,d\underset{\sim}{p}_4\,d\eta_{k_1}\,d\eta_1 (2\eta)^{-2}(2\eta_{k_1})^{-2}(2\eta_1)^{-2}(2\eta_2)^{-2}$$

$$\left[H(P)-H(p)-\omega(k_1)\right]^{-1}\left[H(P)-H(p)-H(p_1)-H(p_2)\right]^{-1}\left[H(P')-H(p')-\omega(k_2)\right]^{-1}$$

$$\left[H(P')-H(p')-H(p_3)-H(p_4)\right]^{-1} \sum_{s,s'} \bar{u}(P'S')\ \gamma^{\sigma} u(p',s')\,\bar{u}(p',s')\,\gamma^0 u(p,s)$$

$$\bar{u}(p,s)\gamma^{\mu} u(P,S)\ \mathrm{tr}\left[(\not{p}_1+m)\gamma^0(\not{p}_4+m)\gamma_{\sigma}(-\not{p}_3+m)\gamma^0(-\not{p}_2+m)\gamma_{\mu}\ \right]$$

$$\left[F(\underset{\sim}{p}'-\underset{\sim}{p})F(\underset{\sim}{p}_4-\underset{\sim}{p}_1)F_c(\underset{\sim}{p}_3-\underset{\sim}{p}_2)-(2\pi)^6\,\delta(\underset{\sim}{p}'-\underset{\sim}{p})\,\delta(\underset{\sim}{p}_4-\underset{\sim}{p}_1)\,\delta(\underset{\sim}{p}_3-\underset{\sim}{p}_2)\right] \tag{A.2}$$

where

$$(\underset{\sim}{p}, \eta) = (\underset{\sim}{P}-\underset{\sim}{k}, \eta_P - \eta_{k_1}) \quad , \quad (\underset{\sim}{k}_2, \eta_{k_2}) = (\underset{\sim}{P}'-\underset{\sim}{p}', \eta_{P'} - \eta')$$

$$(\underset{\sim}{p}_2, \eta_2) = (\underset{\sim}{k}_1-\underset{\sim}{p}_1, \eta_{k_1} - \eta_1) \quad , \quad (\underset{\sim}{p}_3, \eta_3) = (\underset{\sim}{P}'-\underset{\sim}{p}'-\underset{\sim}{p}_4, \eta_{P'} - \eta' - \eta_4)$$

Since we are involved in a leading log calculation, we should treat the external field perturbatively. Through fourth order the possible diagrams are listed in Fig. 11a and b. However, the diagrams in Fig. 11a prove to be larger than those in Fig. 11b by a factor of log η_p, so we will limit our attention to them. Now expanding the eikonal factor in (A. 2) and withholding only the appropriate terms,

$$[F(\underset{\sim}{p}'-\underset{\sim}{p}) F(\underset{\sim}{p}_4-\underset{\sim}{p}_1) F_c(\underset{\sim}{p}_3-\underset{\sim}{p}_2) - (2\pi)^6 \delta(\underset{\sim}{p}'-\underset{\sim}{p}) \delta(\underset{\sim}{p}_4-\underset{\sim}{p}_1) \delta(\underset{\sim}{p}_3-\underset{\sim}{p}_2)] =$$

$$(2\pi)^2 \delta(\underset{\sim}{p}'-\underset{\sim}{p}) e^4 \left\{ \frac{1}{[(\underset{\sim}{p}_4-\underset{\sim}{p}_1)^2 + \lambda^2][(\underset{\sim}{p}_3-\underset{\sim}{p}_2)^2 + \lambda^2]} - \frac{1}{2} \delta(\underset{\sim}{p}_4-\underset{\sim}{p}_1) \int d\underset{\sim}{q} \frac{1}{[\underset{\sim}{q}^2 + \lambda^2][(\underset{\sim}{p}_3-\underset{\sim}{p}_2-\underset{\sim}{q})^2 + \lambda^2]} \right.$$

$$\left. - \frac{1}{2} \delta(\underset{\sim}{p}_3-\underset{\sim}{p}_2) \int d\underset{\sim}{q} \frac{1}{[\underset{\sim}{q}^2 + \lambda^2][(\underset{\sim}{p}_4-\underset{\sim}{p}_1-\underset{\sim}{q})^2 + \lambda^2]} \right\} \qquad \text{(A. 3)}$$

Substituting this into (A. 2) we have,

$$S^{(1)} = \frac{e^8}{(2\pi)^7} \delta(\eta_P - \eta_{P'}) \int d\underset{\sim}{k}_1 d\underset{\sim}{p}_1 d\underset{\sim}{p}_4 d\eta_{k_1} d\eta_1 (2\eta)^{-2} (2\eta_{k_1})^{-2} (2\eta_1)^{-2} (2\eta_2)^{-2}$$

$$[H(P) - H(p) - \omega(k_1)]^{-1} [H(P) - H(p) - H(p_1) - H(p_2)]^{-1} [H(P') - H(p') - \omega(k_2)]^{-1}$$

$$[H(P') - H(p') - H(p_3) - H(p_4)]^{-1} \sum_{s,s'} \bar{u}(P'S')\gamma^\sigma u(p', s') \bar{u}(p's')\gamma^0 u(p, s) \bar{u}(p, s)\gamma^\mu u(P, S)$$

$$\mathrm{tr}[(\not{p}_1 + m)\gamma^0 (\not{p}_4 + m)\gamma_\sigma (-\not{p}_3 + m)\gamma^0 (-\not{p}_2 + m)\gamma_\mu] \left\{ \frac{2}{[(\underset{\sim}{p}_4-\underset{\sim}{p}_1)^2 + \lambda^2][(\underset{\sim}{p}_3-\underset{\sim}{p}_2)^2 + \lambda^2]} - \right.$$

$$\delta(\underset{\sim}{p}_4-\underset{\sim}{p}_1)\int d\underset{\sim}{q}\,\frac{1}{[\underset{\sim}{q}^2+\lambda^2]\,[(\underset{\sim}{p}_3-\underset{\sim}{p}_2-\underset{\sim}{q})^2+\lambda^2]} - \delta(\underset{\sim}{p}_4-\underset{\sim}{p}_1)\int d\underset{\sim}{q}\,\frac{1}{[\underset{\sim}{q}^2+\lambda^2]\,[(\underset{\sim}{p}_4-\underset{\sim}{p}_1-\underset{\sim}{q})^2+\lambda^2]}\Bigg\} \quad \text{(A. 4)}$$

It is not difficult to infer from (A. 4) that the region $\eta_{k_1} << \eta_P$ of phase space gives the dominant contribution to the scattering amplitude. Furthermore, in this region of phase space the sum over the indices σ and μ receive their dominant contributions from the values $\sigma = \mu = 0$ (upper indices!). This fact can be checked in detail, and can be understood fron the observation that γ^0 scales like η under z boosts (favoring large η in the through-going electron line), and γ^3 scales like H under z boosts (favoring small η for the virtual photons). And lastly we can neglect H(P) and H(p') in all the energy denominators since they are $0(\frac{1}{\eta_P})$. These observations lead to simplifications of many factors in S_{fi}:

$$\bar{u}(p,s)\,\gamma^0 u(P,S) = 2\sqrt{\eta_P\eta}\quad\delta_{Ss} \approx 2\eta_P\,\delta_{Ss}$$

$$H(P)-H(p)-\omega(k_1) \approx -\omega(k_1) = -\frac{\underset{\sim}{k}_1^2+\lambda^2}{2\eta_{k_1}} \qquad \text{(A. 5)}$$

$$H(P)-H(p)-H(p_1)-H(p_2) \approx -H(p_1)-H(p_2) = -\frac{\underset{\sim}{p}_1^2+m^2}{2\eta_1} - \frac{\underset{\sim}{p}_2^2+m^2}{2\eta_2}\,.$$

It will prove convenient to scale the η dependence out of the integrand, so introduce the dimensionless variables α and β,

$$\eta_{k_1} = \alpha\,\eta_P \quad , \qquad \eta_1 = \beta\eta_{k_1} = \beta(\alpha\eta_P)$$

where

$$0 \le \alpha \le 1 \quad , \quad 0 \le \beta \le 1$$

Now,

$$S^{(1)} \cong \frac{e^8}{2(2\pi)^7} \eta_P \delta(\eta_P - \eta_{P'}) \delta_{SS'} \int d\underset{\sim}{k}_1 d\underset{\sim}{p}_1 d\underset{\sim}{p}_4 \quad \frac{d\alpha}{\alpha} d\beta \, [\underset{\sim}{k}_1{}^2 + \lambda^2]^{-1} [\underset{\sim}{k}_2{}^2 + \lambda^2]^{-1}$$

$$\frac{\mathrm{tr}\,[(\not p_1 + m)\gamma^0 (\not p_4 + m)\gamma^3 (-\not p_3 + m)\gamma^0 (-\not p_2 + m)\gamma^3]}{[(1-\beta)(\underset{\sim}{p}_1^2 + m^2) + \beta(\underset{\sim}{p}_2^2 + m^2)]\,[(1-\beta)(\underset{\sim}{p}_4^2 + m^2) + \beta(\underset{\sim}{p}_3^2 + m^2)]}$$

$$\Bigg\{ \frac{1}{[(\underset{\sim}{p}_4 - \underset{\sim}{p}_1)^2 + \lambda^2]\,[(\underset{\sim}{p}_3 - \underset{\sim}{p}_2)^2 + \lambda^2]} - \frac{1}{2}\delta(\underset{\sim}{p}_4 - \underset{\sim}{p}_1) \int d\underset{\sim}{q} \frac{1}{[\underset{\sim}{q}^2 + \lambda^2][(\underset{\sim}{p}_3 - \underset{\sim}{p}_2 - \underset{\sim}{q})^2 + \lambda^2]} -$$

$$\frac{1}{2}\,\delta(\underset{\sim}{p}_3 - \underset{\sim}{p}_2) \int d\underset{\sim}{q} \frac{1}{[\underset{\sim}{q}^2 + \lambda^2]\,[(\underset{\sim}{p}_4 - \underset{\sim}{p}_1 - \underset{\sim}{q})^2 + \lambda^2]} \Bigg\} \qquad \text{(A. 6)}$$

This expression becomes considerably more transparent if we change integration variables. First choose a frame such that

$$\underset{\sim}{P}' = -\underset{\sim}{P}$$

and define new integration variables $\underset{\sim}{q}$ and $\underset{\sim}{q}'$,

$$\underset{\sim}{k}_1 = \underset{\sim}{P} + \underset{\sim}{q} \quad , \quad \underset{\sim}{p}_4 = \underset{\sim}{P}' + \underset{\sim}{p}_1 - \underset{\sim}{q}'$$

Furthermore, if we introduce a function

$$K(\underset{\sim}{P}'; \underset{\sim}{q}, \underset{\sim}{q}') = \frac{e^4}{2(2\pi)^4} \int d\underset{\sim}{p}_1 d\beta \Bigg\{ - \frac{\mathrm{tr}\,[(\not p_1 + m)\gamma^0 (\not P' - \not q' + \not p_1 + m)\gamma^3 (\not p_1 + \not q - \not q' + m)\gamma^0 (\not p_1 + \not P' - \not q + m)\gamma^3]}{[(1-\beta)\underset{\sim}{p}_1^2 + \beta(\underset{\sim}{P}' - \underset{\sim}{q} + \underset{\sim}{p}_1)^2 + m^2]\,[(1-\beta)(\underset{\sim}{P}' - \underset{\sim}{q}' + \underset{\sim}{p}_1)^2 + \beta(\underset{\sim}{q}' - \underset{\sim}{q} - \underset{\sim}{p}_1)^2 + m^2]}$$

$$+ (\underset{\sim}{q}' = \underset{\sim}{P}') + (\underset{\sim}{q}' = -\underset{\sim}{P}') \Bigg\} \qquad \text{(A. 7)}$$

the scattering amplitude can be written in the form,

$$S^{(1)} = (2\pi e^4) 2\eta_P \, \delta(\eta_P - \eta_{P'}) \, \delta_{SS'} \int \frac{d\underset{\sim}{q}}{(2\pi)^2} \frac{d\underset{\sim}{q}'}{(2\pi)^2} \frac{d\alpha}{\alpha} \left[(\underset{\sim}{P}'-\underset{\sim}{q})^2+\lambda^2\right]^{-1} \left[(\underset{\sim}{P}'+\underset{\sim}{q})^2+\lambda^2\right]^{-1}$$

$$\left[(\underset{\sim}{P}'-\underset{\sim}{q}')^2+\lambda^2\right]^{-1} \left[(\underset{\sim}{P}'+\underset{\sim}{q}')^2+\lambda^2\right]^{-1} K(\underset{\sim}{P}';\underset{\sim}{q},\underset{\sim}{q}') \qquad (A.8)$$

We see from this expression that the function K describes the composition of the virtual photon as a bare pair, and predicts how effectively such a system scatters off an external field. This function has been obtained and simplified previously by Frolov et. al.[33], and Cheng and Wu.[32] Our analysis agrees with theirs, and after a lengthy Feynman parameter calculation we find,

$$K(0;\underset{\sim}{q},\underset{\sim}{q}') = \left(\frac{8\alpha^2}{\pi}\right) \int_0^1 dx \int_0^1 dy \, \frac{\left[x(1-x)+y(1-y)\right] \underset{\sim}{q}^2 \underset{\sim}{q}'^2 - 2x(1-x)y(1-y)\left[2\underset{\sim}{q}^2\underset{\sim}{q}'^2 + (\underset{\sim}{q}\cdot\underset{\sim}{q}')^2\right]}{x(1-x)\underset{\sim}{q}^2 + y(1-y)\underset{\sim}{q}'^2 + m^2} \qquad (A.9)$$

in the forward direction.

We want only to observe at this time that K does not depend upon α. Hence, the scattering amplitude apparently diverges logarithmically. However, an improved analysis of this process (our method, for instance, interchanges limits and integrations freely) shows that the α integral should be cutoff at the point where the virtual photon is becoming "wee." Such a procedure is physically sensible since the pair intermediate state is no longer long-lived once the photon's longitudinal momentum falls to order unity. Thus,

$$\int \frac{d\alpha}{\alpha} \rightarrow \int_{\frac{\eta_{min}}{\eta_P}}^{1} \frac{d\alpha}{\alpha} = \log\left(\frac{\eta_P}{\eta_{min}}\right) \qquad (A.10)$$

The scattering amplitude depends logarithmically on the energy of the incident electron,

$$S^{(1)} = -(8\pi e^4)\ \eta_P \log \eta_P\, \delta(\eta_P - \eta_{P'})\, \delta_{SS'} \int \frac{d\underset{\sim}{q}}{(2\pi)^2}\ \frac{d\underset{\sim}{q}'}{(2\pi)^2}\ \left[(\underset{\sim}{P}'-\underset{\sim}{q})^2+\lambda^2\right]^{-1} \left[(\underset{\sim}{P}'+\underset{\sim}{q})^2+\lambda^2\right]^{-1}$$

$$\left[(\underset{\sim}{P}'-\underset{\sim}{q}')^2+\lambda^2\right]^{-1} \left[(\underset{\sim}{P}'+\underset{\sim}{q}')^2+\lambda^2\right]^{-1} \quad K(\underset{\sim}{P}';\underset{\sim}{q},\underset{\sim}{q}') \qquad \text{(A. 11)}$$

For later analysis it will prove useful to define

$$M^{(1)} = -e^4 \log \eta_P\, \delta_{S'S} \int \frac{d\underset{\sim}{q}}{(2\pi)^2}\ \frac{d\underset{\sim}{q}'}{(2\pi)^2}\ \left[(\underset{\sim}{P}'-\underset{\sim}{q})^2+\lambda^2\right]^{-1} \left[(\underset{\sim}{P}'+\underset{\sim}{q})^2+\lambda^2\right]^{-1} \left[(\underset{\sim}{P}'-\underset{\sim}{q}')^2+\lambda^2\right]^{-1}$$

$$\left[(\underset{\sim}{P}'+\underset{\sim}{q}')^2+\lambda^2\right]^{-1} \quad K(\underset{\sim}{P}';\underset{\sim}{q},\underset{\sim}{q}') \qquad \text{(A. 12)}$$

and write,

$$S^{(1)} = 2\eta_P (2\pi)\ \delta(\eta_P - \eta_{P'})\, M^{(1)} \qquad \text{(A. 13)}$$

APPENDIX B. SINGLE CHAIN

We wish to study the multiperipheral diagrams in Fig. 13 and indicate the arguments necessary to obtain the amplitude for a chain of N e^+e^- pairs in Fig. 14. In placing the eikonal vertices on just the second pair in Fig. 13 we have anticipated the fact that the external field will be treated perturbatively as in the previous section, and only the leading behavior of the diagram will be found. According to our perturbation theory rules the amplitude for Fig. 13a reads,

$$S_a^{(2)} = -\frac{e^8}{(2\pi)^{13}}\,\delta(\eta_P-\eta_{P'})\int d\underset{\sim}{k}_1 d\underset{\sim}{p}_1 d\eta_{k_1} d\eta_1 (2\eta_{k_1})^{-2}(2\eta_1)^{-2}(2\eta_2)^{-2}(2\eta)^{-2}$$

$$\left[H(P)-H(p)-\omega(k_1)\right]^{-1}\left[H(P)-H(p)-H(p_1)-H(p_2)\right]^{-1}\left[H(P')-H(p)-\omega(k_2)\right]^{-1}\left[H(P')-H(p)-H(p_3)-H(p_4)\right]^{-1}$$

$$\sum_{s,s'}\bar{u}(P',S')\gamma^{\nu}u(p,s)\,\bar{u}(p,s)\gamma^{\mu}u(P,S)\,\mathrm{tr}\left[(\not{p}_1+m)\gamma^{\rho}(\not{p}_4+m)\gamma_{\nu}(-\not{p}_3+m)\gamma^{\sigma}(-\not{p}_2+m)\gamma_{\mu}\right]$$

$$d\underset{\sim}{k}_3 d\underset{\sim}{p}_5 d\underset{\sim}{p}_8 d\eta_{k_3} d\eta_5 (2\eta_{k_3})^{-2}(2\eta_5)^{-2}(2\eta_6)^{-2}\left[H(P)-H(p)-H(p_2)-H(p_4)-\omega(k_3)\right]^{-1}$$

$$\left[H(P)-H(p)-H(p_4)-H(p_2)-H(p_5)-H(p_6)\right]^{-1}\left[H(P')-H(p)-H(p_2)-H(p_4)-\omega(k_4)\right]^{-1}$$

$$\left[H(P')-H(p)-H(p_4)-H(p_2)-H(p_8)-H(p_7)\right]^{-1}\mathrm{tr}\left[(\not{p}_5+m)\gamma^{0}(\not{p}_8+m)\gamma_{\sigma}(-\not{p}_7+m)\gamma^{0}(-\not{p}_6+m)\gamma_{\rho}\right]$$

$$\left[F(\underset{\sim}{p}_8-\underset{\sim}{p}_5)F_c(\underset{\sim}{p}_7-\underset{\sim}{p}_6)-(2\pi)^2\delta(\underset{\sim}{p}_8-\underset{\sim}{p}_5)\,\delta(\underset{\sim}{p}_7-\underset{\sim}{p}_6)\right] \qquad \text{(B.1)}$$

The overall minus sign occurs because photon 3 attaches to an electron line while photon 4 attaches to a positron. Clearly $S^{(2)}$ is in general untractable. However, as we expect on the basis of the analysis of the previous Appendix,

$S^{(2)}$ receives its dominant η dependence from that region of phase space where

$$\eta_{k_1} << \eta_P \qquad , \qquad \eta_{k_3} << \eta_2$$

This means, in the language of multiperipheral models, that only strongly-ordered diagrams contribute to the calculation. Furthermore, in this region of phase space we can set

$$\mu = \nu = \rho = \sigma = 0 \qquad \text{(upper indices!)}$$

and approximate the energy denominators,

$$H(P) - H(p) - \omega(k_1) \approx -\omega(k_1)$$

$$H(P) - H(p) - H(p_4) - H(p_2) - H(p_5) - H(p_6) \approx -H(p_5) - H(p_6) \quad \text{, etc.}$$

The amplitude simplifies to read,

$$S^{(2)} = -\frac{e^8}{(2\pi)^{13}} \delta(\eta_P - \eta_{P'}) \int d\underset{\sim}{k}_1 d\underset{\sim}{p}_1 d\eta_{k_1} d\eta_1 (2\eta_{k_1})^{-2} (2\eta_1)^{-2} (2\eta_2)^{-2} (2\eta)^{-2}$$

$$\left[\omega(k_1)\right]^{-1} \left[\omega(k_2)\right]^{-1} \left[H(p_1) + H(p_2)\right]^{-1} \left[H(p_3) + H(p_4)\right]^{-1}$$

$$\sum_{s,s'} \bar{u}(P', S')\gamma^0 u(p,s)\, \bar{u}(p,s)\gamma^0 u(P,S)\, \mathrm{tr}\left[(\not{p}_1 + m)\gamma^0 (\not{p}_4 + m)\gamma^3 (-\not{p}_3 + m)\gamma^0 (-\not{p}_2 + m)\gamma^3\right]$$

$$d\underset{\sim}{k}_3 d\underset{\sim}{p}_5 d\underset{\sim}{p}_8 d\eta_{k_3} d\eta_5 (2\eta_{k_3})^{-2} (2\eta_5)^{-2} (2\eta_6)^{-2} \left[\omega(k_3)\right]^{-1} \left[\omega(k_4)\right]^{-1} \left[H(p_5) + H(p_6)\right]^{-1}$$

$$\left[H(p_7) + H(p_8)\right]^{-1} \mathrm{tr}\left[(\not{p}_5 + m)\gamma^0 (\not{p}_8 + m)\gamma^3 (-\not{p}_7 + m)\gamma^0 (-\not{p}_6 + m)\gamma^3\right]$$

$$\left[F(\underset{\sim}{p}_8 - \underset{\sim}{p}_5) F_c(\underset{\sim}{p}_7 - \underset{\sim}{p}_6) - (2\pi)^2 \delta(\underset{\sim}{p}_8 - \underset{\sim}{p}_5)\, \delta(\underset{\sim}{p}_7 - \underset{\sim}{p}_6)\right] \qquad \text{(B. 2)}$$

Treating the eikonal perturbatively and introducing integration variables for the "lower" loop as we did for Fig. 5,

$$\underset{\sim}{k}_3 = \underset{\sim}{P}+\underset{\sim}{q}' \qquad \underset{\sim}{p}_8 = \underset{\sim}{P}'+\underset{\sim}{p}_5-\underset{\sim}{q}''$$

$$\eta_{k_3} = \gamma\eta_P \qquad \eta_5 = \delta\eta_{k_3} = \delta(\gamma\eta_P)$$

we can identify a factor of -K coming from the lower loop

$$S^{(2)} = \frac{e^8}{(2\pi)^5}\, 2\delta(\eta_P-\eta_{P'})\int d\underset{\sim}{k}_1 d\underset{\sim}{p}_1 d\eta_{k_1} d\eta_1\, \theta(\eta-\eta_{k_1})\theta(\eta_1-\eta_{k_3})(2\eta_{k_1})^{-2}(2\eta_1)^{-2}(2\eta_2)^{-2}$$

$$(2\eta)^{-1}[\omega(k_1)]^{-1}[\omega(k_2)]^{-1}[H(p_1)+H(p_2)]^{-1}[H(p_3)+H(p_4)]^{-1}\sum_{s,s'}\bar{u}(P',S')\gamma^0 u(p,s)\bar{u}(p,s)\gamma^0 u(P,S)$$

$$\mathrm{tr}[(\not{p}_1+m)\gamma^0(\not{p}_4+m)\gamma^3(-\not{p}_3+m)\gamma^0(-\not{p}_2+m)\gamma^3]\frac{d\gamma}{\gamma}\, d\underset{\sim}{q}' d\underset{\sim}{q}''\, [(\underset{\sim}{P}'-\underset{\sim}{q}')^2+\lambda^2]^{-1}$$

$$[(P'+q)^2+\lambda^2]^{-1}[(P'-q'')^2+\lambda^2]^{-1}[(P'+q'')^2+\lambda^2]^{-1}\, K(\underset{\sim}{P}';\underset{\sim}{q}',\underset{\sim}{q}'') \qquad \text{(B. 3)}$$

where the θ functions enforce the fact that all the η's in the diagram must be positive. If we now change variables in the "upper" loop in the usual way,

$$\underset{\sim}{k}_1 = \underset{\sim}{P}+\underset{\sim}{q} \qquad \underset{\sim}{p}_1 = \underset{\sim}{P}'+\underset{\sim}{p}_4-\underset{\sim}{q}'$$

$$\eta_{k_1} = \alpha\eta_P \qquad \eta_1 = \beta\eta_{k_1} = \beta(\alpha\eta_P)$$

and add in Figs. 13b, c, d and compare to Fig. 11a, we can recognize a factor $K(\underset{\sim}{P}';\underset{\sim}{q},\underset{\sim}{q}')$ emerging for the "upper" loop. Finally,

$$S^{(2)} = -(4\pi e^4)\eta_P\, \delta(\eta_P - \eta_{P'})\, \delta_{S'S} \int \frac{d\underset{\sim}{q}}{(2\pi)^2} \frac{d\underset{\sim}{q}'}{(2\pi)^2} \frac{d\underset{\sim}{q}''}{(2\pi)^2} \int \frac{d\alpha}{\alpha} \frac{d\gamma}{\gamma}\, \theta(1-\alpha)\, \theta(\alpha-\gamma)$$

$$\left[(\underset{\sim}{P}'-\underset{\sim}{q})^2+\lambda^2\right]^{-1}\left[(\underset{\sim}{P}'+\underset{\sim}{q})^2+\lambda^2\right]^{-1}\left[(\underset{\sim}{P}'-\underset{\sim}{q}')^2+\lambda^2\right]^{-1}\left[(\underset{\sim}{P}'+\underset{\sim}{q}')^2+\lambda^2\right]^{-1}\left[(\underset{\sim}{P}'-\underset{\sim}{q}'')^2+\lambda^2\right]^{-1}\left[(\underset{\sim}{P}'+\underset{\sim}{q}'')^2+\lambda^2\right]^{-1}$$

$$K(\underset{\sim}{P}';\underset{\sim}{q},\underset{\sim}{q}')\; K(\underset{\sim}{P}';\underset{\sim}{q}',\underset{\sim}{q}'') \qquad \text{(B.4)}$$

The integrations over the η-fractions of the photons must be cutoff from below in the same way as done in the single loop diagram,

$$\int_{\frac{\eta_{min}}{\eta_P}}^{1} \frac{d\alpha}{\alpha} \int_{\frac{\eta_{min}}{\eta_P}}^{\alpha} \frac{d\lambda}{\lambda} \cong \frac{1}{2!} \log^2\left(\frac{\eta_P}{\eta_{min}}\right) \qquad \text{(B.5)}$$

We have done enough analysis now to see that the scattering amplitude with N e^+e^- loops must be given by,

$$S^{(N)} = -(4\pi e^4)\eta_P\, \delta(\eta_P - \eta_{P'})\, \delta_{S'S} \frac{1}{N!} \log^N\left(\frac{\eta_P}{\eta_{min}}\right) I^{(N+1)}(\underset{\sim}{P}') \qquad \text{(B.6)}$$

where

$$I^{(N)}(\underset{\sim}{P}') = \int \frac{d\underset{\sim}{k}_1}{(2\pi)^2} \cdots \frac{d\underset{\sim}{k}_N}{(2\pi)^2} \frac{1}{\left[(\underset{\sim}{P}'-\underset{\sim}{k}_1)^2+\lambda^2\right]\left[(\underset{\sim}{P}'+\underset{\sim}{k}_1)^2+\lambda^2\right] \cdots \left[(\underset{\sim}{P}'-\underset{\sim}{k}_N)^2+\lambda^2\right]\left[(\underset{\sim}{P}'+\underset{\sim}{k}_N)^2+\lambda^2\right]}$$

$$K(\underset{\sim}{P}';\underset{\sim}{k}_1,\underset{\sim}{k}_2)\; K(\underset{\sim}{P}';\underset{\sim}{k}_2,k_3) \cdots K(\underset{\sim}{P}';\underset{\sim}{k}_{N-1},\underset{\sim}{k}_N) \qquad \text{(B.7)}$$

The crucial factors of $\log^N \eta_P$ and N! arise as they did for the 2-loop diagram.

In particular, the amplitude receives a factor from the N strongly-ordered photons of,

$$\int_{\frac{\eta_{min}}{\eta_P}}^{1} \frac{dx_1}{x_1} \int^{x_1} \frac{dx_2}{x_2} \cdots \int^{x_{N-1}} \frac{dx_N}{x_N} \cong \frac{1}{N!} \log^N \left(\frac{\eta_P}{\eta_{min}} \right) \qquad (B.8)$$

In the next Appendix we will see that we can sum all the $S^{(N)}$ in the forward direction.

APPENDIX C. BRANCH CUT (m=λ=0)

We wish to study $S^{(N)}$ in the forward direction. For $\underset{\sim}{P}'=\underset{\sim}{P}=0$, we have

$$S^{(N)}(\underset{\sim}{P}=0) = -(4\pi e^4)\,\eta_P\,\delta(\eta_P-\eta_{P'})\,\frac{1}{N!}\,\log^N(\eta_P)\,I^{(N+1)}(\underset{\sim}{P}=0) \tag{C.1}$$

where

$$I^{(N)}(0) = \int \frac{d\underset{\sim}{k}_1}{(2\pi)^2}\cdots\frac{d\underset{\sim}{k}_N}{(2\pi)^2}\,\frac{1}{(\underset{\sim}{k}_1^2+\lambda^2)^2\cdots(\underset{\sim}{k}_N^2+\lambda^2)^2}\,K(\underset{\sim}{k}_1,\underset{\sim}{k}_2)K(\underset{\sim}{k}_2,\underset{\sim}{k}_3)\cdots K(\underset{\sim}{k}_{N-1},\underset{\sim}{k}_N)$$

and we have defined,

$$K(\underset{\sim}{k}_i,\underset{\sim}{k}_{i+1}) \equiv K(\underset{\sim}{P}'=0;\underset{\sim}{k}_i,\underset{\sim}{k}_{i+1})$$

We can solve for $I^{(N)}$ explicitly, and sum the amplitudes $S^{(N)}$ in the case $m=\lambda=0$. (However, λ must be held non-zero in the first and last propagators on the chain in order to avoid a spurious infra-red divergence.) We might argue, instead of setting $m=\lambda=0$, that we are integrating only over that part of phase space for which $\underset{\sim}{k}^2 \gg m^2, \lambda^2$. In that case we will obtain here at least a lower bound on the "real" scattering amplitude.

Recall from Appendix B that when m=0,

$$K(\underset{\sim}{k}_1,\underset{\sim}{k}_2) = \left(\frac{8\alpha^2}{\pi}\right)\underset{\sim}{k}_1^2\underset{\sim}{k}_2^2\;B_0(\underset{\sim}{k}_1,\underset{\sim}{k}_2) \tag{C.2}$$

where

$$B(\underset{\sim}{k}_1,\underset{\sim}{k}_2) = \int_0^1 dx\int_0^1 dy\,\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)\underset{\sim}{k}_1^2+y(1-y)\underset{\sim}{k}_2^2}$$

and we have averaged over the free angle $\underset{\sim}{k}_1\cdot\underset{\sim}{k}_2$ already. Substituting this into (C.1), we have,

$$I^{(N)} = \left(\frac{8\alpha^2}{\pi}\right)^{N-1} \frac{1}{(2\pi)^{2N}} \int d\underset{\sim}{k}_1 d\underset{\sim}{k}_N \frac{\underset{\sim}{k}_1^2}{(\underset{\sim}{k}_1^2+\lambda^2)^2} \frac{\underset{\sim}{k}_N^2}{(\underset{\sim}{k}_N^2+\lambda^2)^2} B(\underset{\sim}{k}_1,\underset{\sim}{k}_2) B(\underset{\sim}{k}_2,\underset{\sim}{k}_3) \dots B(\underset{\sim}{k}_{N-1},\underset{\sim}{k}_N) \cdot$$

$$d\underset{\sim}{k}_2 \dots d\underset{\sim}{k}_{N-1} \tag{C.3}$$

It will prove convenient to scale the (momentum)$^{-2}$ dimension out of B. To do this we change variables,

$$\left|\underset{\sim}{k}_1\right| = \kappa e^{\xi_1}, \dots, \left|\underset{\sim}{k}_N\right| = \kappa e^{\xi_N}$$

and note that,

$$B(\kappa e^{\xi_1}, \kappa e^{\xi_2}) = \kappa^{-2} B(e^{\xi_1}, e^{\xi_2})$$

then,

$$I^{(N)} = \left(\frac{8\alpha^2}{\pi}\right)^{N-1} \frac{1}{(2\pi)^N} \frac{1}{\kappa^2} \int d\xi_1 d\xi_N \frac{e^{4(\xi_1+\xi_N)}}{\left(e^{2\xi_1}+\frac{\lambda^2}{\kappa^2}\right)^2 \left(e^{2\xi_N}+\frac{\lambda^2}{\kappa^2}\right)^2}$$

$$e^{2(\xi_2+\xi_3+\dots+\xi_{N-1})} B\left(e^{\xi_1}, e^{\xi_2}\right) B\left(e^{\xi_2}, e^{\xi_3}\right) \dots B\left(e^{\xi_{N-1}}, e^{\xi_N}\right) \tag{C.4}$$

Notice that,

$$e^{(\xi_1+\xi_2)} B\left(e^{\xi_1}, e^{\xi_2}\right) = \int dxdy \frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)\exp\left[+(\xi_1-\xi_2)\right] + y(1-y)\exp\left[-(\xi_1-\xi_2)\right]} \equiv C(\xi_1-\xi_2)$$

is a function only of the difference $(\xi_1-\xi_2)$. We should, therefore, change integration variables, So,

$$I^{(N)} = \left(\frac{8\alpha^2}{\pi}\right)^{N-1} \frac{1}{(2\pi)^N} \frac{1'}{\kappa^2} \int d\eta_1 \ldots d\eta_N \frac{e^{3(\eta_1+\eta_N)}}{\left(e^{2\eta_1}+\frac{\lambda^2}{\kappa^2}\right)^2 \left(e^{2\eta_N}+\frac{\lambda^2}{\kappa^2}\right)^2} \cdot \tag{C.5}$$

$$C(\eta_2)C(\eta_3)\ldots C(\eta_{N-1})C(\eta_1-\eta_2-\eta_3-\cdots-\eta_N)$$

The resulting convolution integral is factored upon introducing Fourier transforms,

$$C(\eta) \equiv \int \frac{d\beta}{(2\pi)} e^{i\eta\beta} \tilde{C}_0(\beta)$$

Then,

$$I^{(N)} = \left(\frac{8\alpha^2}{\pi}\right)^{N-1} \frac{1}{(2\pi)^N \kappa^2} \int \frac{d\beta}{(2\pi)} e^{i(\eta_1-\eta_N)\beta} \frac{e^{3(\eta_1+\eta_N)}}{\left(e^{2\eta_1}+\frac{\lambda^2}{\kappa^2}\right)^2 \left(e^{2\eta_N}+\frac{\lambda^2}{\kappa^2}\right)^2}$$

$$\left[\tilde{C}(\beta)\right]^{N-1} d\eta_1 d\eta_N \tag{C.6}$$

Introducing the function

$$f(\beta,\gamma) = \int d\eta e^{-i\eta\beta} \frac{e^{3\eta}}{(e^{2\eta}+\gamma^2)^2} \tag{C.7}$$

We can write finally,

$$I^{(N)} = \left(\frac{8\alpha^2}{\pi}\right)^{N-1} \frac{1}{(2\pi)^N \kappa^2} \int \frac{d\beta}{(2\pi)} f^*(\beta, \tfrac{\lambda}{\kappa}) f(\beta, \tfrac{\lambda}{\kappa}) \left[\tilde{C}(\beta)\right]^{N-1} \tag{C.8}$$

The properties of the functions f and C are derived in Appendix D. From (D.4, 6, 10), we have

$$f(\beta,\gamma) = \gamma^{-(1+i\beta)} f(\beta, 1)$$

$$f(\beta, 1) = \frac{1}{2}\Gamma\left(\frac{1}{2}+\frac{i\beta}{2}\right)\Gamma\left(\frac{3}{2}-\frac{i\beta}{2}\right) = \frac{1}{2}B\left(\frac{1}{2}+\frac{i\alpha}{2}, \frac{3}{2}-\frac{i\alpha}{2}\right)$$

$$\tilde{C}(0) = \frac{11\pi^3}{2^7}, \quad \tilde{C}'(0) = 0, \quad \tilde{C}''(0) = -\frac{11\pi^3}{32}\left(\frac{\pi^2}{12} - \frac{1}{44}\right) \qquad \text{(C.9)}$$

Having calculated $I^{(N)}$ we can return to the scattering amplitude and note that $\sum_{N=1}^{\infty} S^{(N)}$ is just an exponential series,

$$S^{(1\text{ chain})} \equiv \sum_{N=1}^{\infty} S^{(N)} = -(4\pi e^4)\,\eta_P\,\delta(\eta_P - \eta_{P'})\left[\frac{1}{\lambda^2}\int\frac{d\beta}{(2\pi)^2} f^*(\beta, 1) f(\beta, 1) \cdot e^{\left(\frac{2\alpha}{\pi}\right)^2 \tilde{C}(\beta)\log\eta_P} - 1\right] \qquad \text{(C.10)}$$

If we imagine letting $\eta \to \infty$, we can evaluate the leading part of the integral straightforwardly,

$$S^{(1\text{ chain})} \cong -(4\pi e^4)\,\eta_P\,\delta(\eta_P - \eta_{P'})\, e^{\left(\frac{2\alpha}{\pi}\right)^2 \tilde{C}(0)\log\eta_P} \int\frac{d\beta}{(2\pi)^2} f^*(\beta, 1) f(\beta, 1) \cdot e^{\left(\frac{2\alpha}{\pi}\right)^2 \left[\tilde{C}(\beta) - C(0)\right]\log\eta_P} \qquad \text{(C.11)}$$

$$S^{(1\ \text{chain})} \approx -(4\pi e^4)\,\eta_P\,\delta(\eta_P-\eta_{P'})\,e^{\left(\frac{2\alpha}{\pi}\right)^2 \tilde{C}(0)\log\eta_P} \quad .$$

$$\frac{f^*(0,1)f(0,1)}{(2\pi)^2\lambda^2}\int d\beta\; e^{-\frac{1}{2}\left(\frac{2\alpha}{\pi}\right)^2\left|\tilde{C}''(0)\right|\log\eta_P\cdot\beta^2}$$

$$S^{(1\ \text{chain})} \approx -(4\pi e^4)\,\frac{|f(0,1)|^2}{(2\pi)^2\lambda^2}\left(\frac{2\pi}{\left|\left(\frac{2\alpha}{\pi}\right)^2\tilde{C}''(0)\right|}\right)^{\frac{1}{2}}\delta(\eta_P-\eta_{P'})\,\frac{(\eta_P)^{1+\left(\frac{2\alpha}{\pi}\right)^2\tilde{C}(0)}}{\sqrt{\log\eta_P}} \tag{C.12}$$

The factor $\log^{-\frac{1}{2}}\eta_P$ is indicative of the square root character of the branch cut responsible for $S^{(1\ \text{chain})}$. We see also that the branch cut extends to $J=1+\left(\frac{2\alpha}{\pi}\right)^2\tilde{C}(0)=1+\frac{11\pi}{32}\alpha^2$, which shows that the single chain multiperipheral diagrams summed alone violate the Froissart bound.

In preparation for a subsequent discussion, consider in more detail the structure of this cut singularity in the J-plane. Rewrite (C.10),

$$S^{(1\ \text{chain})} = -(2\pi)(2\eta_P)\,\delta(\eta_P-\eta_P')\left[M\left(\frac{\eta_P}{\eta_{min}}\right)-1\right] \tag{C.13}$$

where

$$M\left(\frac{\eta_P}{\eta_{min}}\right) = \frac{1}{\lambda^2}\int\frac{d\beta}{(2\pi)^2}\,f^*(\beta,1)f(\beta,1)\left(\frac{\eta_P}{\eta_{min}}\right)^{\left(\frac{2\alpha}{\pi}\right)^2\tilde{C}(\beta)}\theta\left(\frac{\eta_P}{\eta_{min}}-1\right) \tag{C.14}$$

The θ-function simply states the trivial fact that the S-matrix is different from unity only when $\eta_P > \eta_{min}$. To discuss the behavior of the scattering amplitude from the complex angular momentum point of view, we turn to the Mellin transform of M(J). We easily compute from (C.13) that

$$M(J) = \frac{1}{\lambda^2}\int \frac{d\beta}{(2\pi)^2} f^*(\beta,1) f(\beta,1)\left[\frac{1}{J-(\frac{2\alpha}{\pi})^2\tilde{C}(\beta)}\right] \tag{C.15}$$

Therefore, M(J) possesses a cut over that range of J for which there is a solution to the equation

$$J = (\frac{2\alpha}{\pi})^2 \tilde{C}(\beta) \tag{C.16}$$

We recall from Appendix D that $\tilde{C}(\beta)$ is an even, positive function with a maximum at $\beta=0$, and decreases monotonically to zero as β increases. Therefore, the cut extends from $J_{min}=0$ to $J_{max}=(\frac{2\alpha}{\pi})^2\tilde{C}(0)=\frac{11\pi}{32}\alpha^2$. The discontinuity of M(J) across the cut is,

$$\text{Disc } M(J) = -\frac{\pi}{\lambda^2}\int\frac{d\beta}{(2\pi)^2} f^*(\beta,1) f(\beta,1)\,\delta\left[J-(\frac{2\alpha}{\pi})^2\tilde{C}(\beta)\right] \tag{C.17}$$

Since the high energy behavior of the scattering amplitude is controlled by the behavior of Disc M(J) near J_{max}, we shall obtain the right hand side of (C.17) explicitly in this region. To do this it suffices to solve (C.16) for β in terms of J near the endpoint $\beta=0$. (C.16) reads, to second order in β,

$$J = \left(\frac{2\alpha}{\pi}\right)^2 \left[\widetilde{C}(0) + \frac{1}{2}\widetilde{C}''(0)\beta^2\right]$$

So,

$$\beta = \sqrt{\frac{J_{max} - J}{D}}$$

where

$$D = -\frac{1}{2}\left(\frac{2\alpha}{\pi}\right)^2 \widetilde{C}''(0) = \frac{11\pi}{16}\left(\frac{\pi^2}{12} - \frac{1}{44}\right)\alpha^2$$

Finally, for J less than but near J_{max},

$$\text{Disc } M(J) \cong -\frac{\pi}{\lambda^2}\int \frac{d\beta}{(2\pi)^2} f^*(\beta, 1) f(\beta, 1) \frac{\delta\left(\beta - \sqrt{\frac{J_m - J}{D}}\right)}{2\sqrt{D(J_m - J)}}$$

$$\text{Disc } M(J) \cong \left[\frac{\pi}{2^7 \lambda^2 \sqrt{D}}\right] \frac{1}{\sqrt{J_m - J}} \qquad \text{(C.18)}$$

where we have substituted the numerical value $f(0,1) = \frac{\pi}{4}$. (C.18) shows the square root character of the cut. Furthermore, it is easy to take (C.18) and invert the Mellin transform

$$M(y) = \frac{1}{\pi}\int_0^{J_{max}} y^J \text{Disc } M(J)\, dJ$$

and rederive (C.12).

APPENDIX D PROPERTIES OF f AND C

In order to complete the discussion of the branch cut we should simplify the functions

$$f(\alpha,\beta) = \int_{-\infty}^{\infty} d\eta \; e^{-i\eta\alpha} \; \frac{e^{3\eta}}{(e^{2\eta}+\beta^2)^2} \qquad \text{(D. 1)}$$

$$C(\xi) = \int dxdy \; \frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)e^{+\xi}+y(1-y)e^{-\xi}} \qquad \text{(D. 2)}$$

which were introduced in Appendix C.

Consider the function $f(\alpha,\beta)$ first. It is easy to see that the function's dependence on β can be scaled out. If we define a new integration variable y,

$$\frac{1}{\beta}e^{\eta} = y$$

we can rewrite (D. 1) as,

$$f(\alpha,\beta) = \beta^{-(1+i\alpha)} f(\alpha, 1)$$

$$f(\alpha, 1) = \int_0^{\infty} dy \; \frac{y^{2-i\alpha}}{(y^2+1)^2}$$

Furthermore, $f(\alpha, 1)$ can be identified as Beta function if we change variables in the integrand to,

$$u = \frac{1}{y^2+1}$$

then,

$$f(\alpha,1) = \frac{1}{2}\int_0^1 du \;\; u^{-\frac{1}{2}+\frac{i\alpha}{2}} \, (1-u)^{\frac{1}{2}-\frac{i\alpha}{2}}$$

which is just,

$$f(\alpha, 1) = \frac{1}{2}B\left(\frac{1}{2}+\frac{i\alpha}{2}, \frac{3}{2}-\frac{i\alpha}{2}\right) = \Gamma\left(\frac{1}{2}+\frac{i\alpha}{2}\right)\Gamma\left(\frac{3}{2}-\frac{i\alpha}{2}\right) \qquad (D.3)$$

Our determination of the character of the branch cut relied upon the nonvanishing of f(0, 1). In fact,

$$f(0, 1) = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{\pi}{4} \qquad (D.5)$$

Now we turn to the function $C(\xi)$. We wish to compute its Fourier transform,

$$\widetilde{C}(\beta) = \int d\xi e^{-i\beta\xi} C(\xi) \qquad (D.6)$$

Using the transform,

$$\int d\xi\, e^{-i\beta\xi}\left(\frac{1}{x(1-x)e^{\xi}+y(1-y)e^{-\xi}}\right) = \frac{\pi}{2\cosh\left(\frac{\pi}{2}\beta\right)}\left[x(1-x)\right]^{-\frac{1}{2}+i\frac{\beta}{2}}\left[y(1-y)\right]^{-\frac{1}{2}-i\frac{\beta}{2}} \qquad (D.7)$$

we have,

$$\widetilde{C}(\beta) = \frac{\pi}{2\cosh(\frac{\pi}{2}\beta)}\int dxdy\left\{\left[x(1-x)\right]^{\frac{1}{2}+i\frac{\beta}{2}}\left[y(1-y)\right]^{-\frac{1}{2}-i\frac{\beta}{2}} + \right.$$

$$\left.\left[x(1-x)\right]^{-\frac{1}{2}+i\frac{\beta}{2}}\left[y(1-y)\right]^{-\frac{1}{2}-i\frac{\beta}{2}} - 5\left[x(1-x)\right]^{\frac{1}{2}+i\frac{\beta}{2}}\left[y(1-y)\right]^{\frac{1}{2}-i\frac{\beta}{2}}\right\} \qquad (D.8)$$

which we recognize as the sum of products of Beta functions. So,

$$\widetilde{C}(\beta) = \frac{\pi}{2\cosh(\frac{\pi}{2}\beta)} \left\{ \frac{\left[\Gamma(\frac{3}{2}+i\frac{\beta}{2})\right]^2 \left[\Gamma(\frac{1}{2}-i\frac{\beta}{2})\right]^2}{\Gamma(3+i\beta)\Gamma(1-i\beta)} + \frac{\left[\Gamma(\frac{3}{2}-i\frac{\beta}{2})\right]^2 \left[\Gamma(\frac{1}{2}+i\frac{\beta}{2})\right]^2}{\Gamma(3-i\beta)\Gamma(1+i\beta)} \right.$$

$$\left. -5\frac{\left[\Gamma(\frac{3}{2}+i\frac{\beta}{2})\right]^2 \left[\Gamma(\frac{3}{2}-i\frac{\beta}{2})\right]^2}{\Gamma(3+i\beta)\Gamma(3-i\beta)} \right\} \qquad \text{(D. 9)}$$

Furthermore, if we use various gamma function identities such as the "Reflection formula", (D. 9) can be written in the final form,

$$\widetilde{C}(\beta) = \frac{\pi^2}{16\beta} \left(\frac{11+3\beta^2}{4+\beta^2}\right) \frac{\sinh(\frac{\pi}{2}\beta)}{\cosh^2(\frac{\pi}{2}\beta)} \qquad \text{(D. 10)}$$

APPENDIX E: EIKONAL APPROXIMATION

It is the purpose of this short appendix to shed more light on the limiting form of the S-matrix derived in the text. We consider the simpler problem of the high energy scattering of a bare Dirac particle off an external field. We will see that the eikonal approximation emerges again, but we will also withhold the correction terms to this leading approximation. These corrections, which are of kinematic origin, are at most proportional to the reciprocal of the incident particle's energy.

Consider a bare Dirac particle in an external field $a_\mu(x)$:

$$(i\partial_0 - ea_0)\psi = [m - i\underset{\sim}{\sigma}\cdot(\underset{\sim}{p} - e\underset{\sim}{a})]\frac{1}{2(\eta - ea_3)}[m + i\underset{\sim}{\sigma}\cdot(\underset{\sim}{p} - e\underset{\sim}{a})]\psi \tag{E.1}$$

where

$$\left[\frac{1}{\eta - ea_3}\psi\right](x) = \int d\xi\, \frac{1}{2i}\,\epsilon(\mathscr{z} - \xi)\, e^{-i\int_\xi^{\mathscr{z}} d\xi' a_3(\tau,\underset{\sim}{x},\xi')}\,\psi(\tau,\underset{\sim}{x},\xi) \tag{E.2}$$

Let us assume that the incoming electron is very energetic, so that we can write,

$$\psi(\tau,\underset{\sim}{x},\mathscr{z}) = e^{-i\eta\mathscr{z}}\,\psi'(\tau,\underset{\sim}{x},\mathscr{z}) \tag{E.3}$$

where η is very large and ψ' varies slowly with $\mathscr{z}$. Then the integral in (E.2) receives its leading contribution from the neighborhood $\mathscr{z} \approx \xi$. To see this write (E.2) in the form

$$\left[\frac{1}{\eta - ea_3}\psi\right](x) = \int d\xi\, e^{-i\eta\xi}\,\epsilon(\mathscr{z} - \xi)\; F(\tau,\underset{\sim}{x},\mathscr{z};\xi) \tag{E.4}$$

where

$$F(\tau,\underset{\sim}{x},\mathscr{Y};\xi) = \frac{1}{2i}\, e^{-i\int_{\xi}^{\mathscr{Y}} a_3(\tau,\underset{\sim}{x},\xi')d\xi'}\, \psi'(\tau,\underset{\sim}{x},\xi) \tag{E.5}$$

Expansing $F(\tau,\underset{\sim}{x},\mathscr{Y};\xi)$ in a Taylor series around $\mathscr{Y}=\xi$,

$$F(\tau,\underset{\sim}{x},\mathscr{Y};\xi) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\partial^n F}{\partial\xi^n}(\tau,\underset{\sim}{x},\mathscr{Y};\xi)\right](\mathscr{Y}-\xi)^n \tag{E.6}$$

and substituting this expansion into (E.4), we find that $\left[\frac{1}{\eta-ea_3}\psi\right](x)$ can be written as a power series in the factor η^{-1},

$$\left[\frac{1}{\eta-ea_3}\psi\right](x) = \sum_{n=0}^{\infty} \frac{1}{i^n\eta^{n+1}}\, e^{i\eta\mathscr{Y}}\left[\frac{\partial^n F}{\partial\xi^n}(\tau,\underset{\sim}{x},\mathscr{Y};\mathscr{Y})\right] \tag{E.7}$$

where we have used the identity

$$2in!\int_{-\infty}^{\infty}\epsilon(\mathscr{Y})\mathscr{Y}^n e^{-i\eta\mathscr{Y}}d\mathscr{Y} = \frac{1}{i^n\eta^{n+1}} \tag{E.8}$$

to extract powers of η. For the purposes of illustration let us restrict our considerations to the first term in (E.7). Then (E.1) becomes,

$$i\partial_0\psi = ea_0\psi + \frac{\left[m - i\underset{\sim}{\sigma}\cdot(\underset{\sim}{p}-e\underset{\sim}{a})\right]\left[m + i\underset{\sim}{\sigma}\cdot(\underset{\sim}{p}-e\underset{\sim}{a})\right]}{2\eta}\psi + \ldots \tag{E.9}$$

where η is now just a kinematic (c number) variable. Some additional spinology allows us to write,

$$i\partial_0\psi = \left\{ea_0 + \frac{1}{2\eta}\left[(\underset{\sim}{p}-e\underset{\sim}{a})^2 - e\sigma_z B_z + m^2\right] + \ldots\right\}\psi \tag{E.10}$$

where

$$B_z = \partial_1 a^2 - \partial_2 a^1$$

We recognize (E.10) as the Pauli-Schrödinger equation in 2 dimensions for a spin $\frac{1}{2}$ particle of mass η and gyromagnetic ratio 2.

This final equation of motion also motivates the appearance of the eikonal phase found more formally in the text. Simply letting $\eta \rightarrow \infty$, (E.9) becomes,

$$i\partial_0 \psi = e\, a_0 \psi \tag{E.11}$$

which admits the eikonal solution

$$\psi(\tau, \underset{\sim}{x}, \gamma) = e^{-ie \int_{-\infty}^{\tau} a_0(\tau', \underset{\sim}{x}, \gamma) d\tau'} \phi(\underset{\sim}{x}, \gamma) \tag{E.12}$$

One might also withhold the leading corrections to the eikonal formula in order to study the range of validity of this popular approximation.

References and Footnotes

1. S. Fubini and G. Furlan, Physics 1, 229 (1965).
2. S. Weinberg, Phys. Rev. 150, 1313 (1966).
3. L. Susskind, Phys. Rev. 165, 1535 (1968). L. Susskind, "Lectures given at the 1968 Summer Institute for Theoretical Physics at the University of Colorado", unpublished.
4. K. Bardakci and M. B. Halpern, Phys. Rev. 176, 1686 (1968).
5. R. P. Feynman, invited paper at the Third Topical Conference on High Energy Collisions of Hadrons, Stony Brook, New York, September 1969; Phys. Rev. Letters 23, 1415 (1969).
6. H. Cheng and T. T. Wu, Phys. Rev. Letters 24, 1456 (1970).
7. J. B. Kogut and D. E. Soper, Phys. Rev. D1, 2901 (1970).
8. Cf. L. I. Schiff, Quantum Mechanics (McGraw Hill, Inc., New York, 1968); p. 234 ff, Third Edition.
9. Note that $d\underset{\sim}{p}\, d\eta/2\eta$ is the Lorentz-invariant surface element $dp_x dp_y dp_z/2E$ on the mass shell. The η-integration runs from 0 to ∞, thus covering the forward mass shell.
10. Recall that the nonrelativistic equation of motion is written $i\partial_0 \Psi = \frac{1}{2m} \underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, \underset{\sim}{\sigma} \cdot \underset{\sim}{p}\, \Psi$ before introducing the minimal substitution in order to obtain the correct $\underset{\sim}{\sigma} \cdot \underset{\sim}{B}$ term.
11. The reader will note that such combinations in Table I as $(\underset{\sim}{q}/\eta_q) - (\underset{\sim}{p}/\eta)$ transform under this subgroup like (momentum/mass)-(momentum/mass) and are therefore invariant under "Galilean boosts". This invariance can often be used to practical advantage in calculations.

12. With the present normalization conventions, $< f \mid S \mid i > = (2\pi)^4 \delta^4 (p_f - p_i) M$, where M is the invariant amplitude calculated with the conventions of Bjorken and Dress using Dirac spinors normalized to $\bar{u}u = 2m$. See J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, (McGraw-Hill, New York, 1965); Appendix B.

13. J. D. Bjorken, SLAC-PUB-905 ("Partons").

14. These relations are physically rather obvious. Since $\psi^\dagger \psi$ represents a current density it must scale like $(\text{vol.})^{-1}$ under boosts. Hence the multiplicative factor $\exp(\omega/2)$. But the combination $\int \underset{\sim}{A}\, \psi^\dagger \psi \, d\underset{\sim}{x}\, dz$ enters the Hamiltonian which must be invariant under boosts. So, $\underset{\sim}{A}$ should not change scale under boosts.

15. A short discussion of the eikonal approximation in contained in Appendix E.

16. We also use this formula for a one particle final state.

17. This relationship can be obtained by using a wave packet for the initial state (cf., M. L. Goldberger and K. M. Watson, Collision Theory (John Wiley, New York, 1964); Section 3.3)). In the high energy limit in which $\eta \sim \sqrt{2}\, E$ this reduces to the more familiar result with η replaced everywhere by E in Eqs. (3.6) and (3.7).

18. Cf., H. Cheng and T. T. Wu, Phys. Rev. 184, 1868 (1969).

19. To calculate F_2 to order e^2, we can use the value $Z_2 = 1$, which is correct to order e^0.

20. S. J. Chang and S. K. Ma have used different infinite momentum techniques to obtain this result, Phys. Rev. 180, 1506 (1969).

21. S. D. Drell, D. J. Levy, T. M. Yan, Phys. Rev. Letters 22, 744 (1969); H. Cheng and T. T. Wu, Phys. Rev. D1, 1069 (1970); S. J. Chang and S. K. Ma, op. cit.

22. The amplitude $\sqrt{Z_3}$ for a physical photon to be a bare photon is 1 to lowest order, but does not, of course, contribute to pair production.

23. H. Cheng and T. T. Wy, Phys. Rev. 182, 1852 (1969).

24. E. Bloom et al., SLAC-PUB-642.

25. Note that the limit $\nu \to \infty$ is already implicit in our formalism.

26. More precisely: let $d\sigma'$ be the limiting form of $d\sigma$ so obtained. Then it is not difficult to prove that $d\sigma_S = d\sigma_S'[1 + 0(1/Q^2)]$, $d\sigma_T = d\sigma_T'[1 + 0(1/\log Q^2)]$ as $Q^2 \to \infty$, assuming that the potential is sufficiently well behaved.

27. A. Erdelyi, ed., Tables of Integral Transforms, (McGraw-Hill, New York, 1954); Vol. 1, pg. 334.

28. If the helicity is flipped on the electron line $d\sigma_T$ is suppressed by a factor (m^2/Q^2) as $Q^2 \to \infty$. If the helicity is flipped on the muon line, $d\sigma_T$ is suppressed by a factor $[1/\log(Q^2/\mu^2)]$.

29. J. D. Bjorken, Phys. Rev. D1, 1376 (1970). The relevant formula reads,

$$\frac{d\sigma}{dQ^2 d\nu}(s=\tfrac{1}{2}) = \frac{\alpha}{\pi}\frac{1}{\nu Q^2}\frac{E'}{E}\left[\frac{E^2+E'^2}{2EE'}\sigma_T + \sigma_S\right]$$

30. J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

31. Strictly speaking, scale invariance for σ_T means that $Q^2\sigma_T$ approaches a finite limit as $Q^2 \to \infty$ with (ν/Q^2) held constant. However, we have evaluated σ_T in this model in the limit $(\nu/Q^2) \to \infty$ with Q^2 held constant, and then we have let $Q^2 \to \infty$. It is not impossible for σ_T to exhibit scale invariance in the limit $Q^2 \to \infty$, $(\nu/Q^2) = $ const., but not in the reversed limit used here.

32. H. Cheng and T. T. Wu, Phys. Rev. 186, 1611 (1969), Phys. Rev. D1, 2775 (1970).

33. G. V. Frolov, V. N. Gribov, L. N. Lipatov, Phys. Letters 31B, 34 (1970).

34. A somewhat simpler proof can be given by appealing to Feynman diagrams instead of the old-fashioned diagrams considered here (cf. S. J. Chang and P. M. Fishbane, Phys. Rev. D2, 1104 (1970)). However, in so doing one loses the space-time picture which will prove important later in the paper.

35. S. J. Chang and T. M. Yan, Phys. Rev. Letters 25, 1586 (1970). Unfortunately the model considered in this paper is not indicative of the true behavior of $\lambda\phi^3$ field theory (cf. references in footnote 10). In fact, the single ladder graph of structureless, spin zero mesons violates the Froissart bound only when the coupling λ is large. Then the leading logarithm technique becomes unreliable. Nevertheless, some features of these author's calculation (such as the strong absorption which saturates the Froissart bound) should survive a better treatment of the problem, and so we discuss their result briefly.

36. F. Smithies, Integral Equations (Cambridge University Press, Cambridge, 1962).

37. P. V. Landshoff and J. C. Polkinghorne, Phys. Rev. 181, 1989 (1969).

38. I. J. Musinich, G. Tiktopoulos and S. B. Treiman, Phys. Rev. D3, 1041 (1971).
H. Cheng and T. T. Wu, DESY Preprint, December and March, 1971.

TABLE I

MATRIX ELEMENTS FOR PHOTON EMISSION

$$p_{\pm} = 2^{-1/2}(p^1 \pm ip^2) \ , \quad q = p - p'$$

s	s'	λ	$w^{\dagger}(s')\, \underline{j}(p', p) \cdot \epsilon^*(\lambda)\, w(s)$
1/2	1/2	1	$(q_-/\eta_q) - (p'_-/\eta')$
1/2	1/2	-1	$(q_+/\eta_q) - (p_+/\eta)$
1/2	-1/2	1	$-2^{-1/2}\, \text{im}\ \eta_q/(\eta\eta')$
1/2	-1/2	-1	0
-1/2	1/2	1	0
-1/2	1/2	-1	$-2^{-1/2}\, \text{im}\ \eta_q/(\eta\eta')$
-1/2	-1/2	1	$(q_-/\eta_q) - (p_-/\eta)$
-1/2	-1/2	-1	$(q_+/\eta_q) - (p'_+/\eta')$

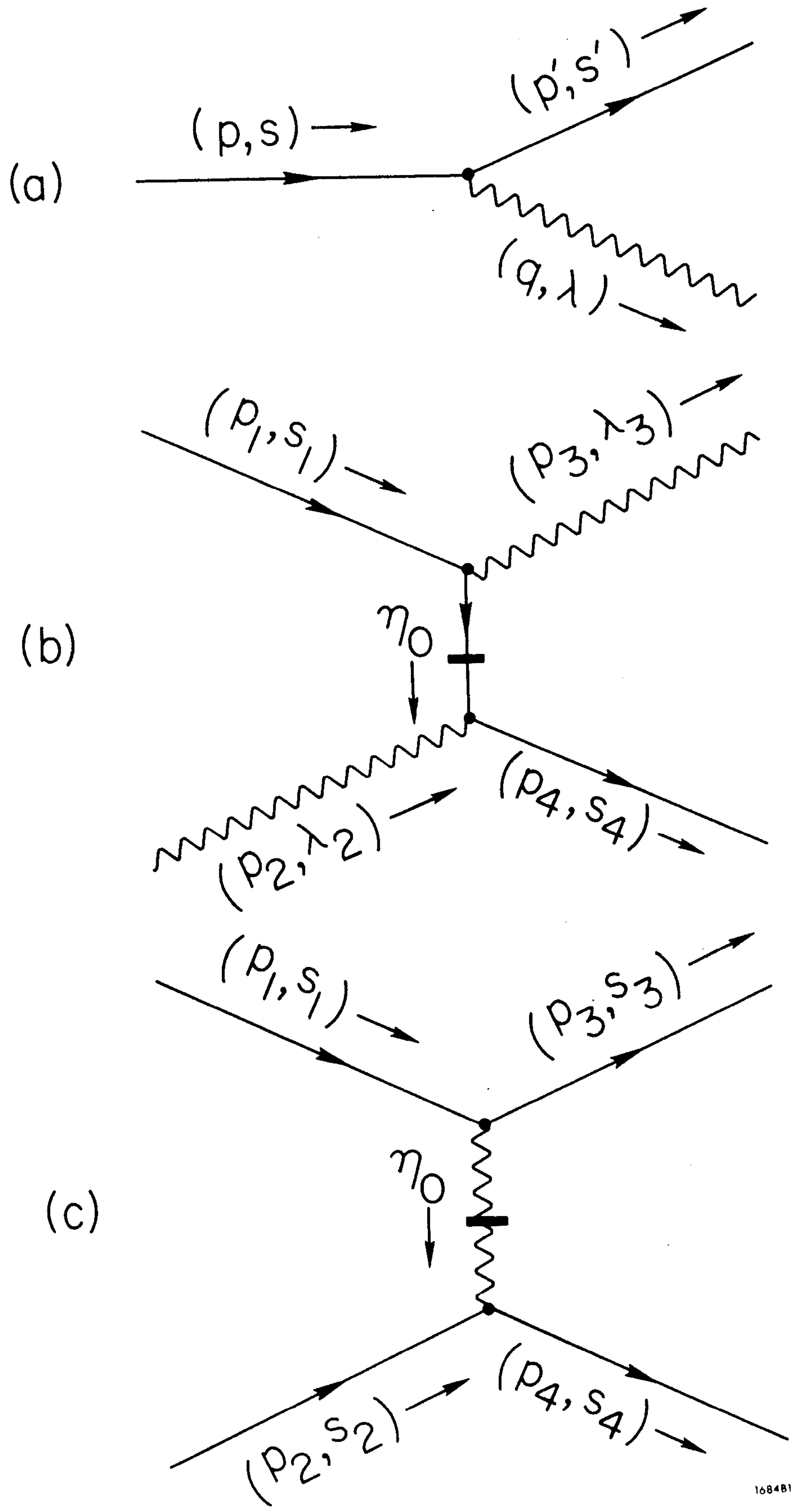

(a)
(p, s)
(p′, s′)
(q, λ)
(b)
(p_1, s_1)
(p_3, λ_3)
η_0
(p_2, λ_2)
(p_4, s_4)
(c)
(p_1, s_1)
(p_3, s_3)
η_0
(p_2, s_2)
(p_4, s_4)
1684B1

Fig. 1

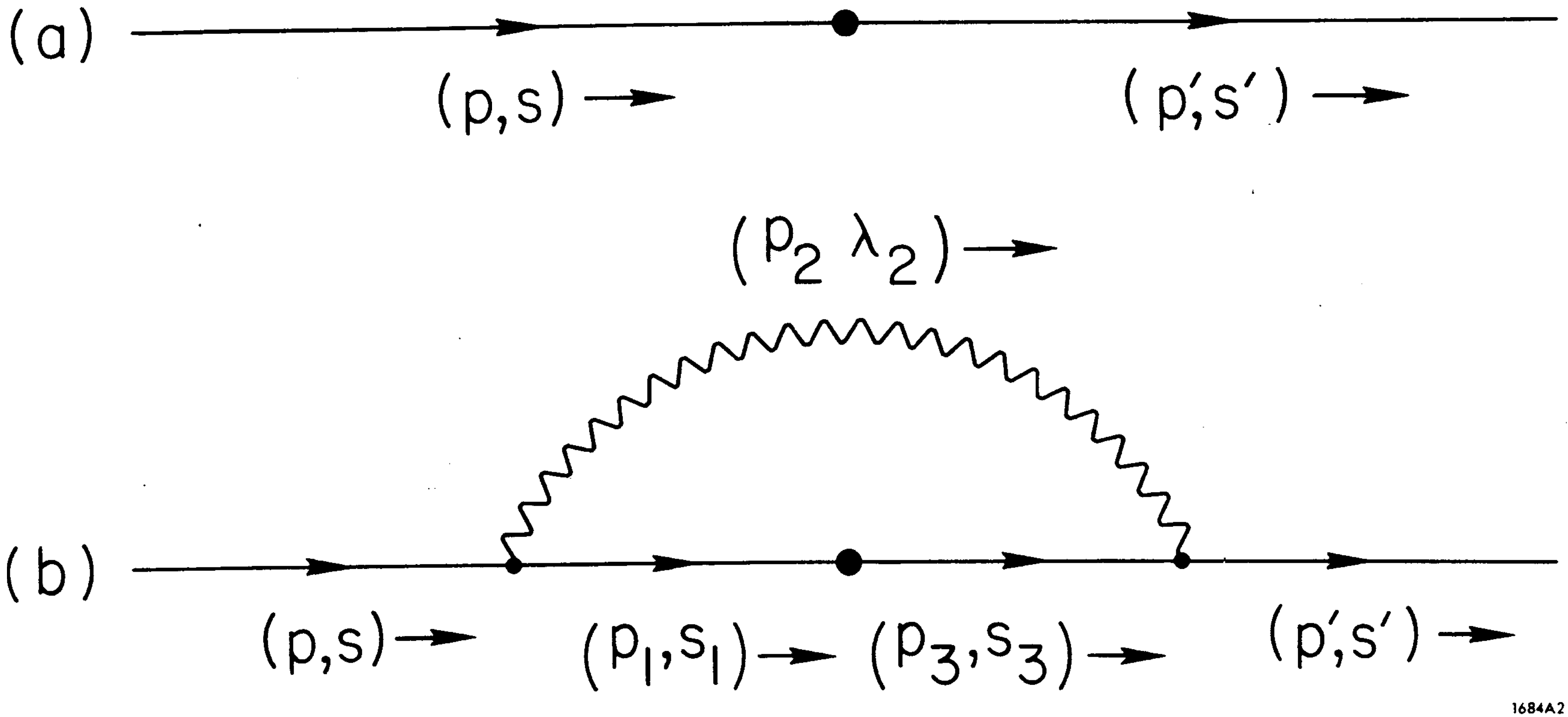


Fig. 2

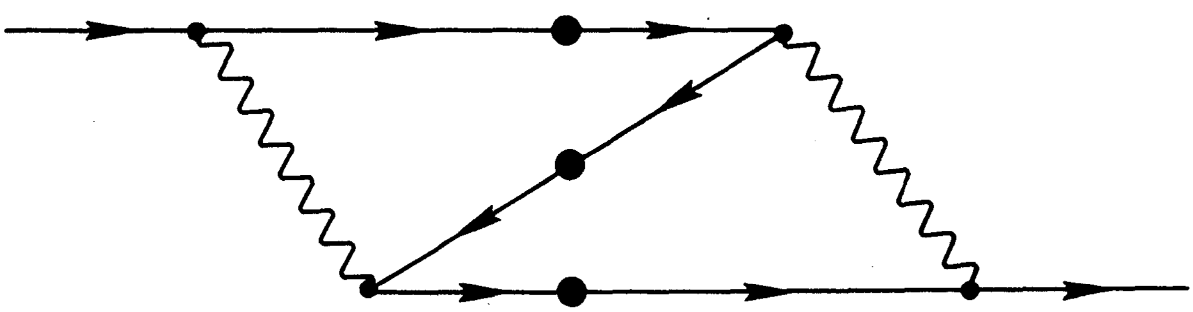


Fig. 3

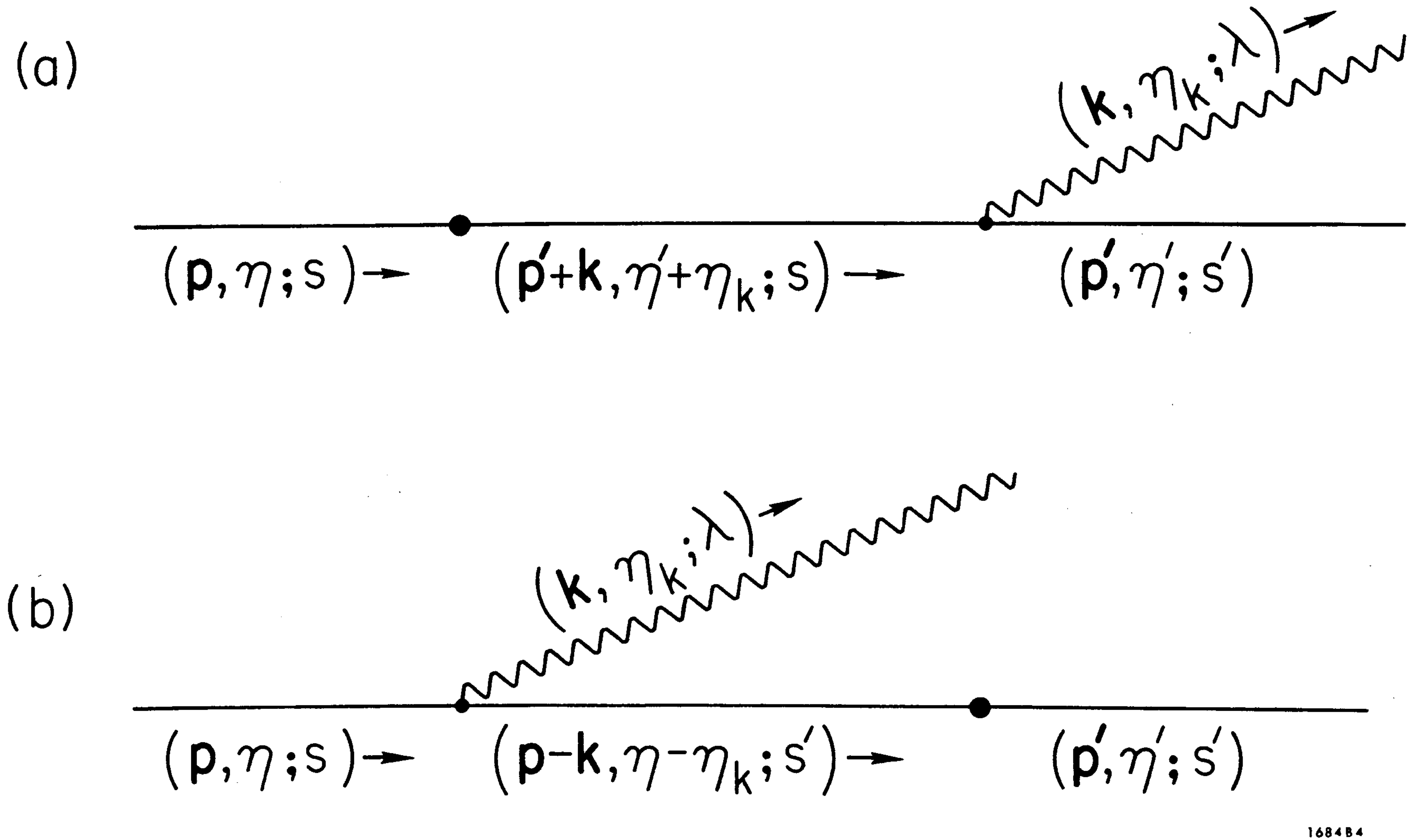

(a)
$(\mathbf{k}, \eta_k; \lambda)$
$(\mathbf{p}, \eta; s)$
$(\mathbf{p}'+\mathbf{k}, \eta'+\eta_k; s)$
$(\mathbf{p}', \eta'; s')$
(b)
$(\mathbf{k}, \eta_k; \lambda)$
$(\mathbf{p}, \eta; s)$
$(\mathbf{p}-\mathbf{k}, \eta-\eta_k; s')$
$(\mathbf{p}', \eta'; s')$
1684B4

Fig. 4

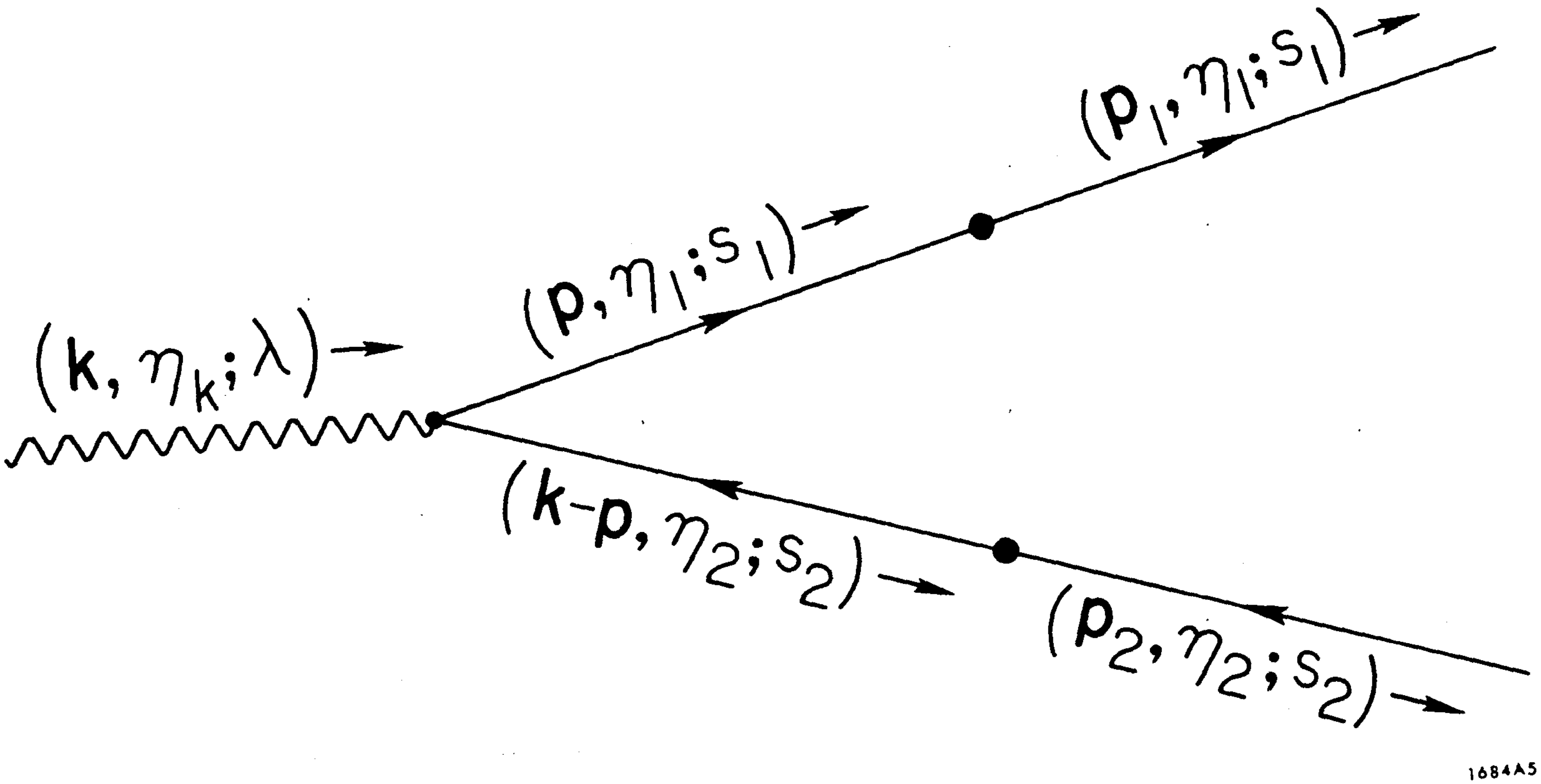


Fig. 5

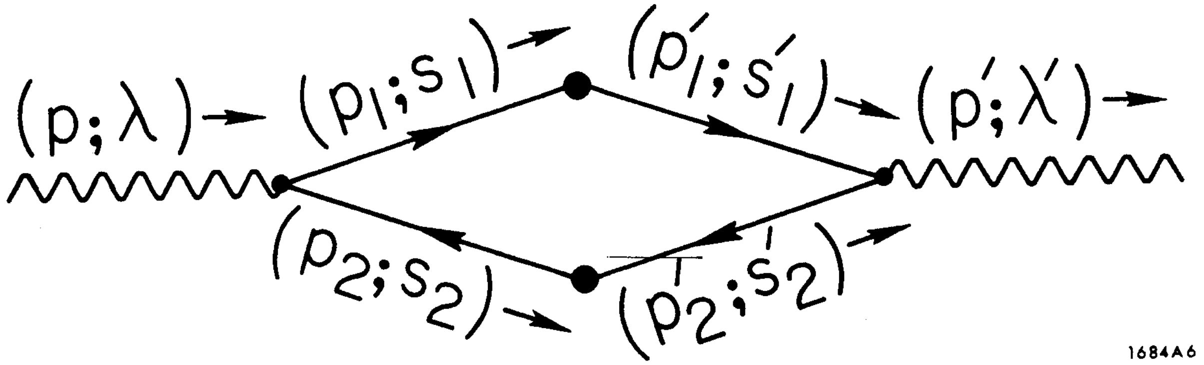


Fig. 6

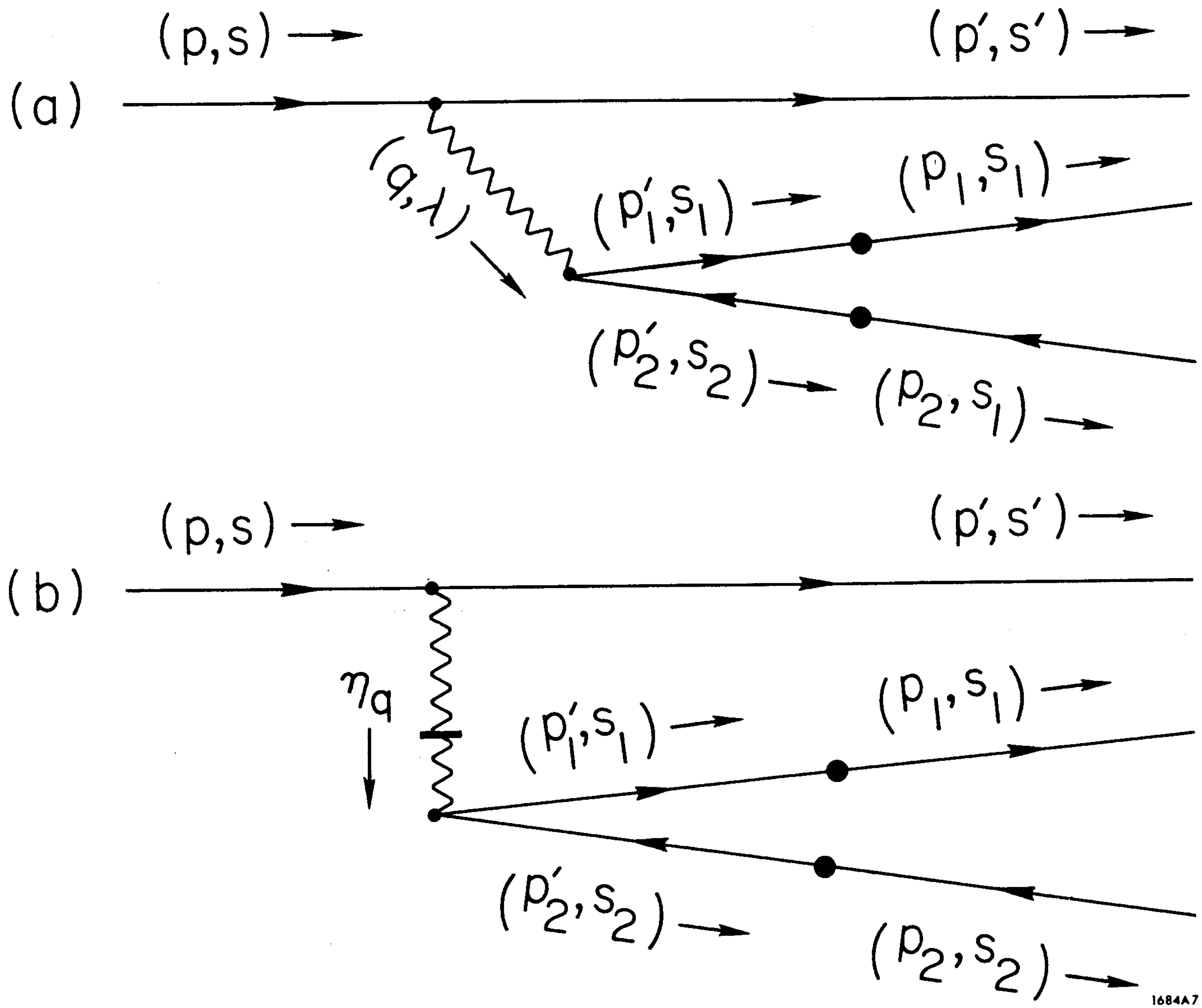

(a)
(p, s)
(p′, s′)
(q, λ)
(p′1, s1)
(p1, s1)
(p′2, s2)
(p2, s1)
(b)
(p, s)
(p′, s′)
ηq
(p′1, s1)
(p1, s1)
(p′2, s2)
(p2, s2)
1684A7

Fig. 7

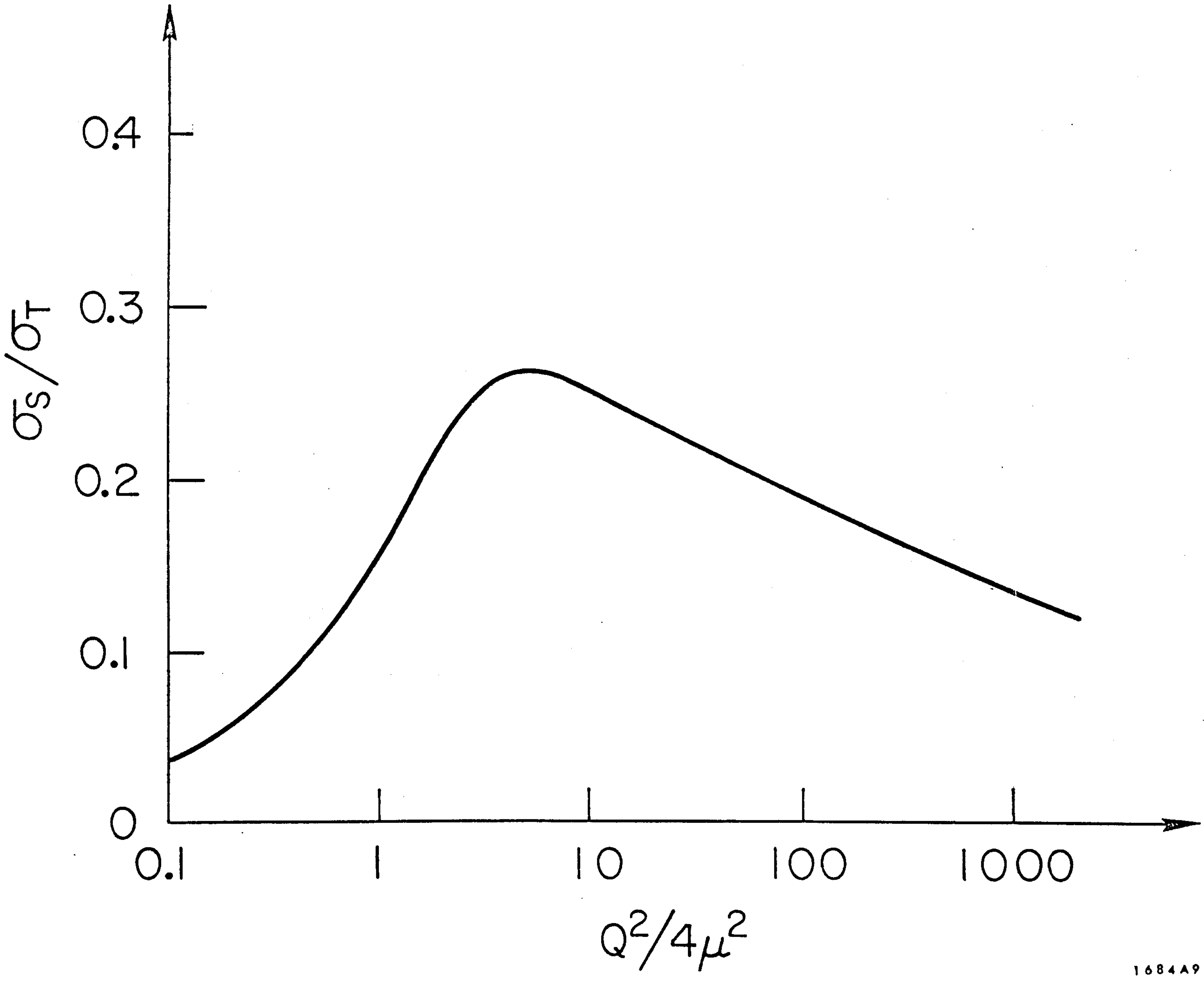


Fig. 8

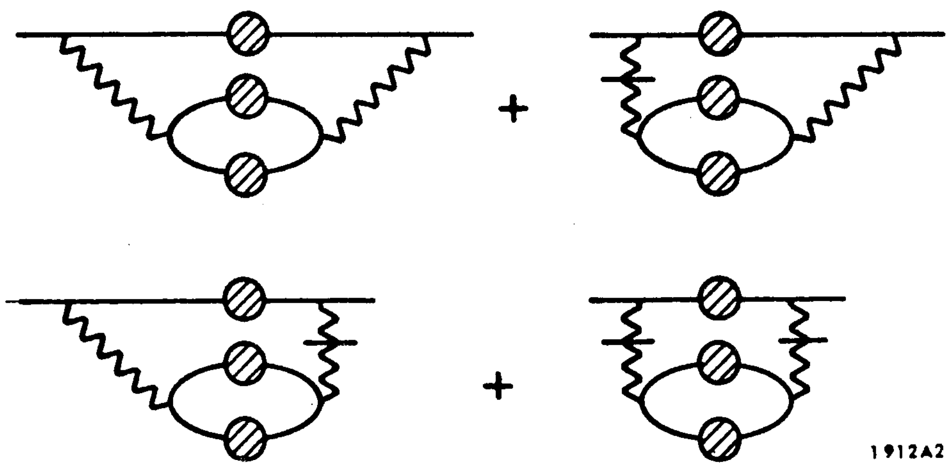


Fig. 9

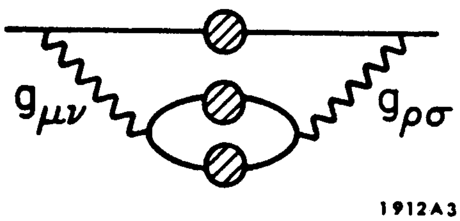


Fig. 10

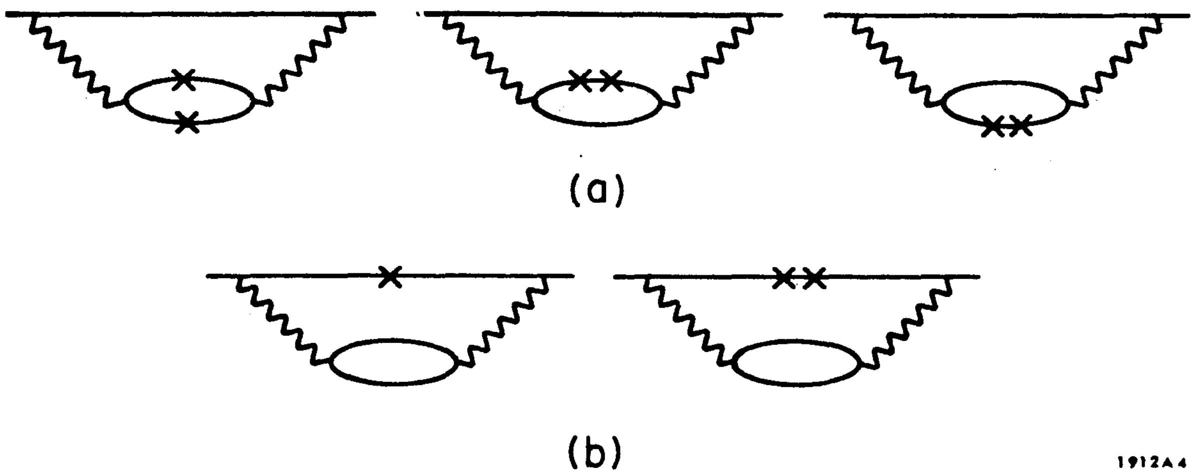


Fig. 11

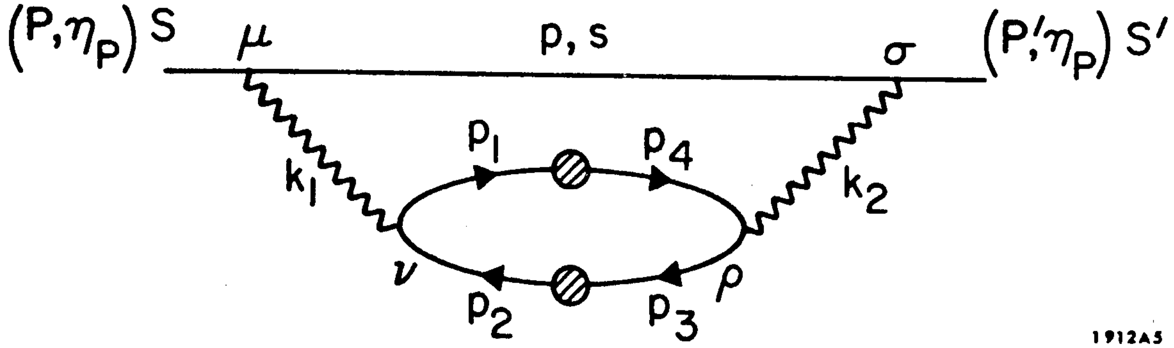


Fig. 12

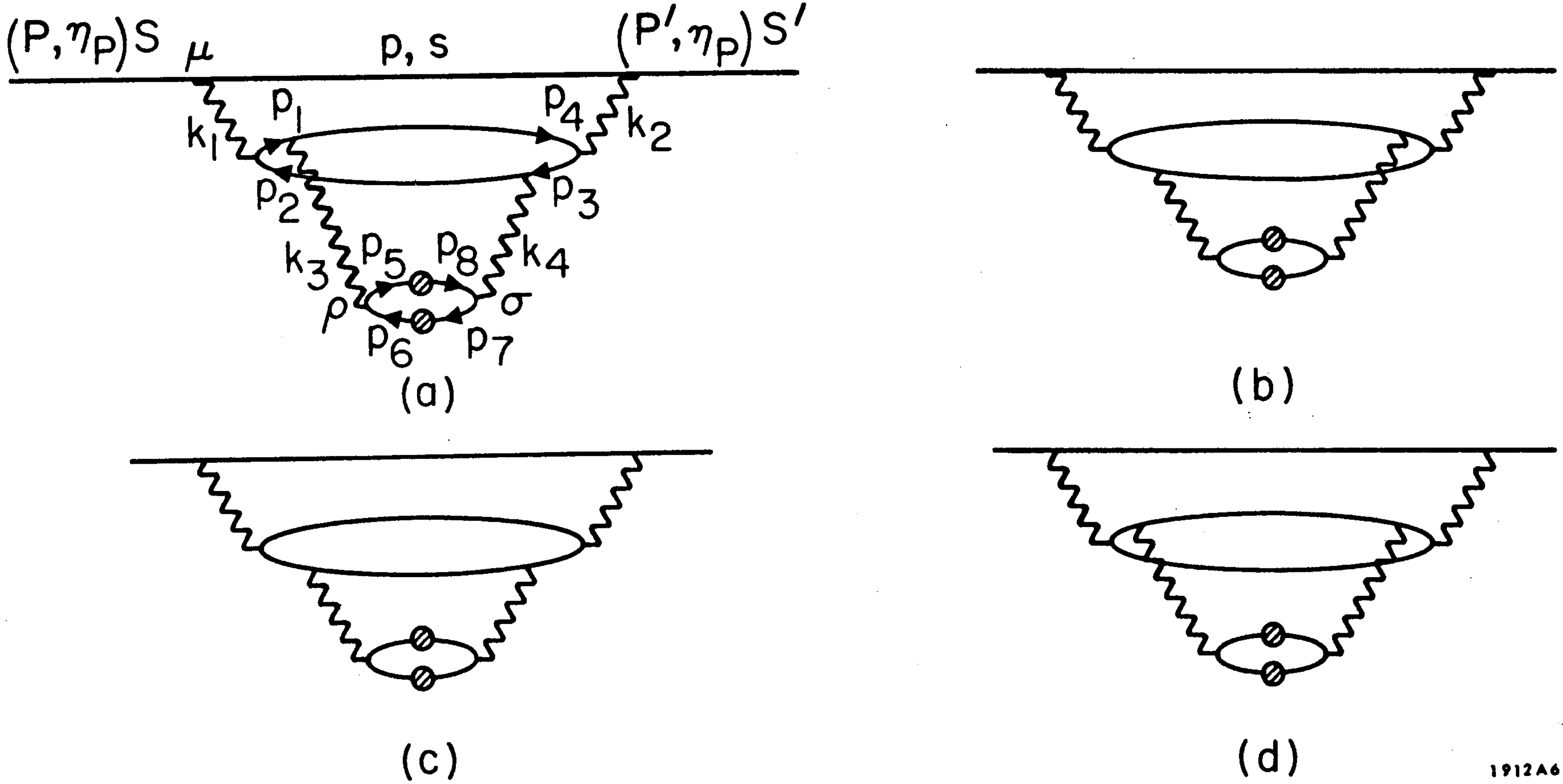


Fig. 13

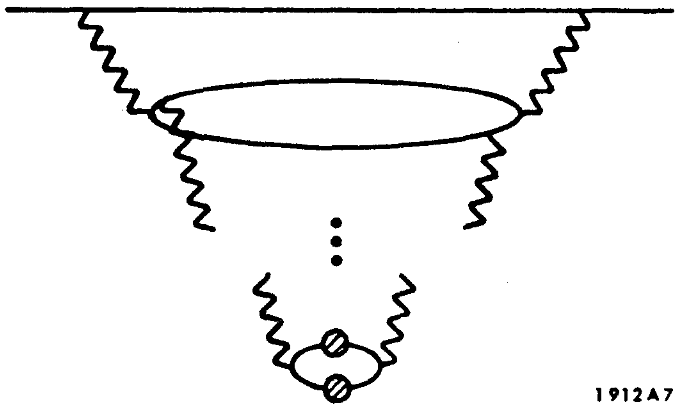


Fig. 14

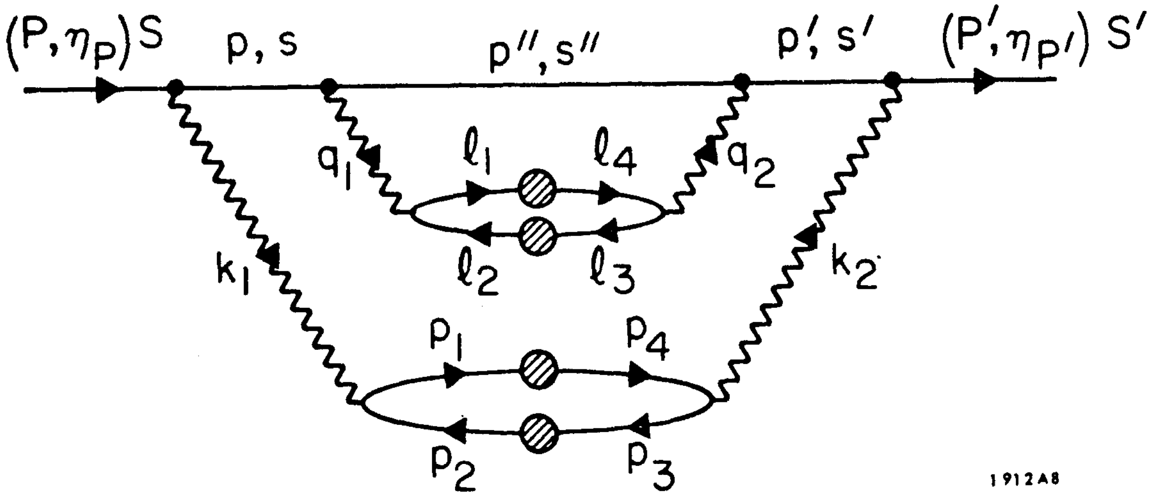


Fig. 15

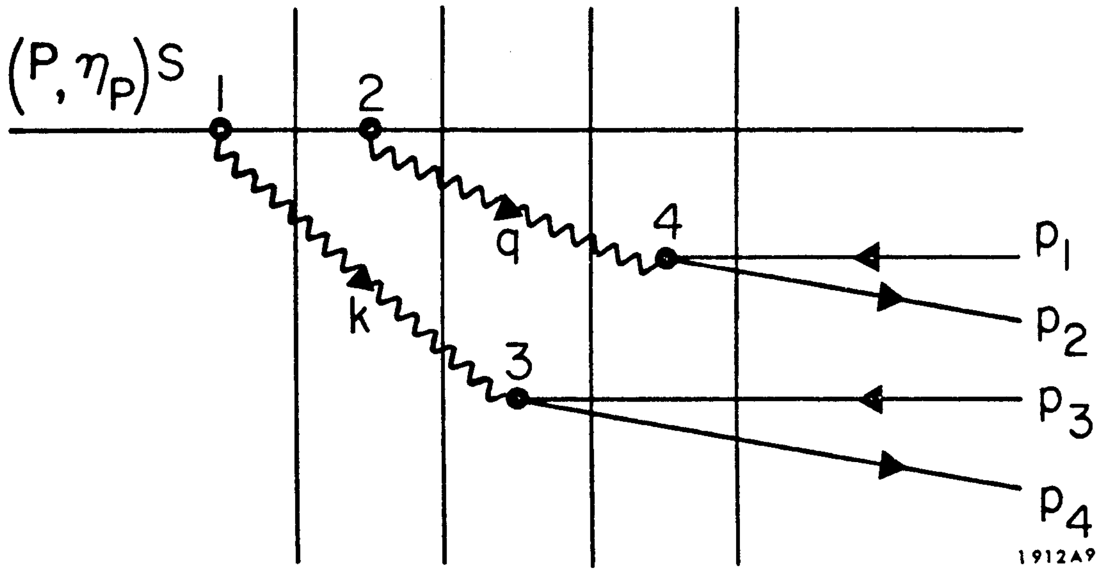


Fig. 16

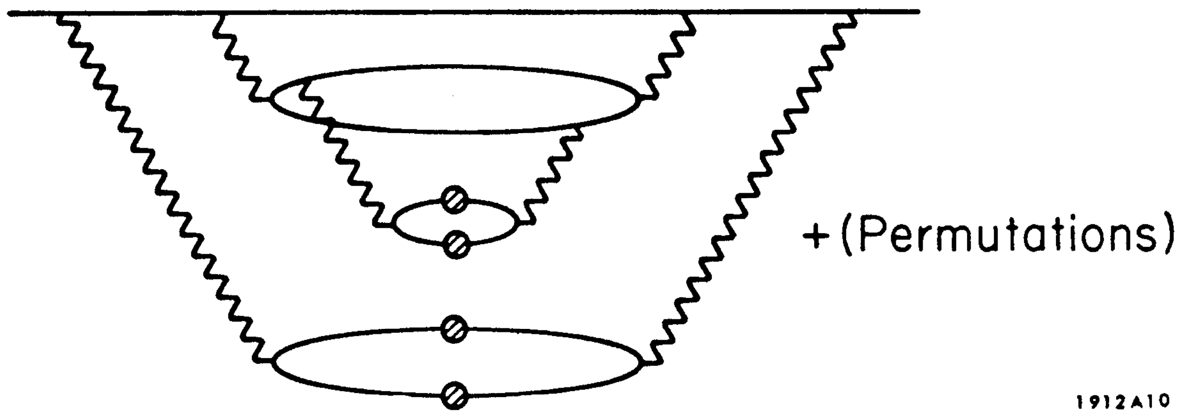


Fig. 17

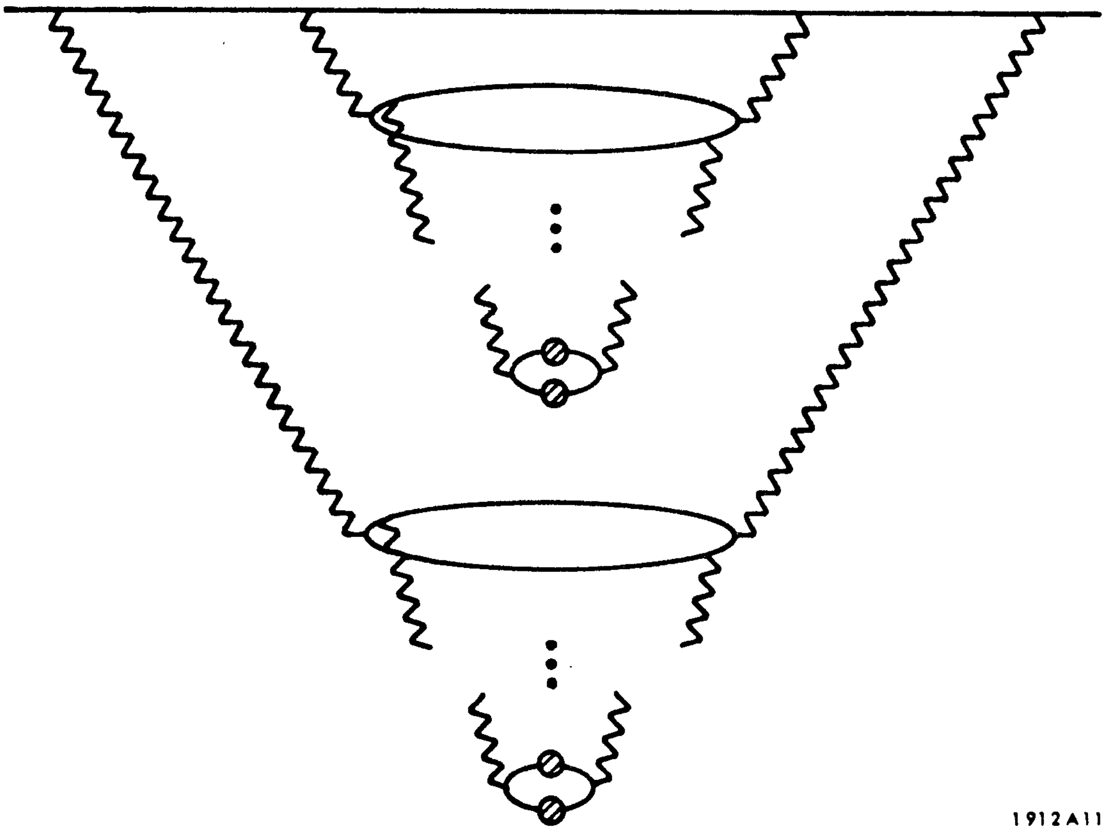


Fig. 18

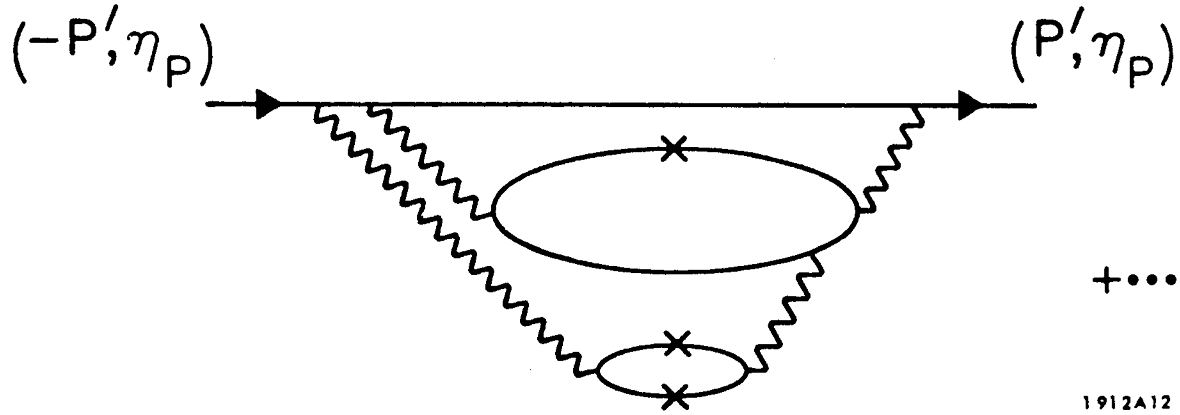


Fig. 19

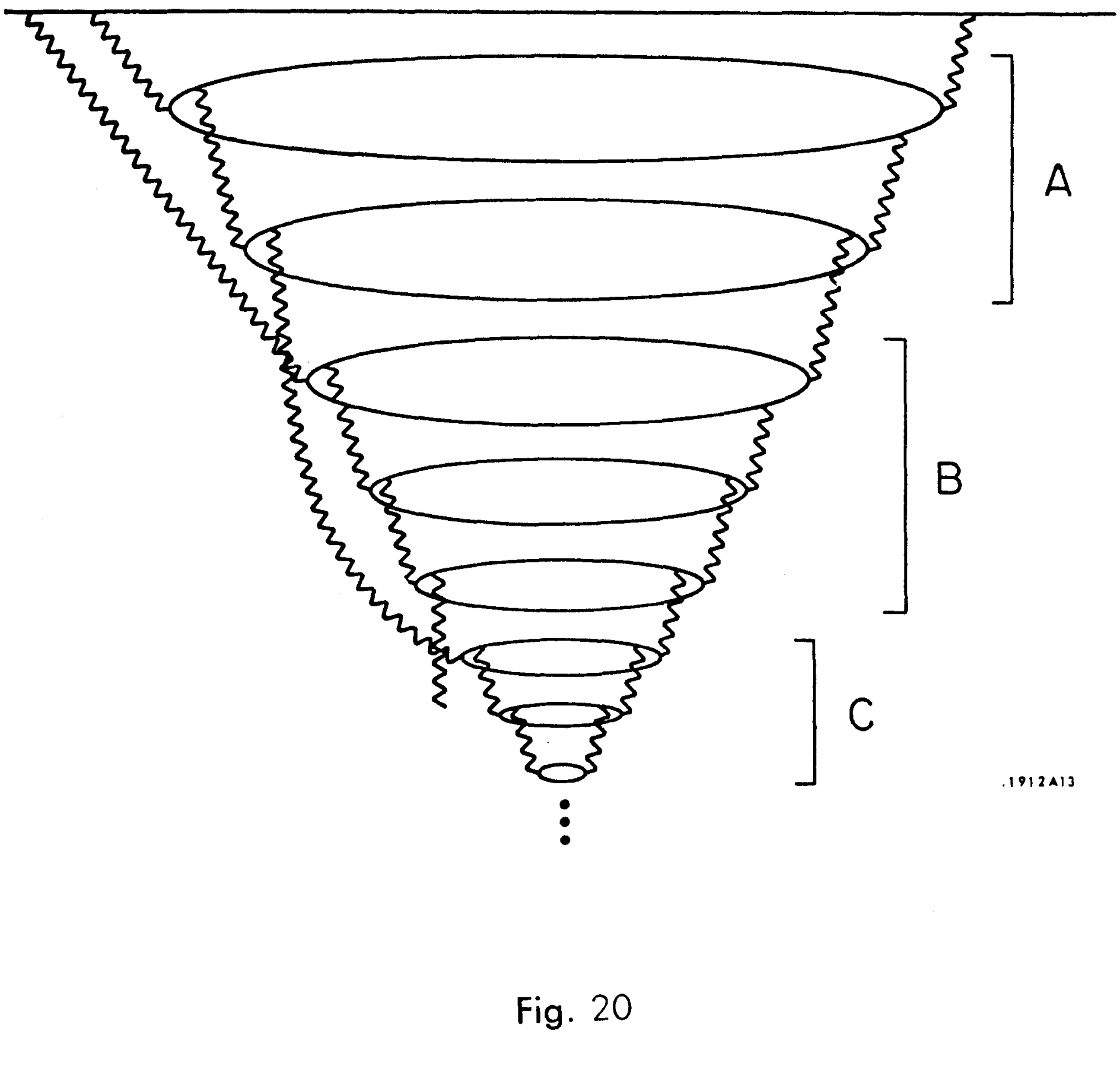


Fig. 20

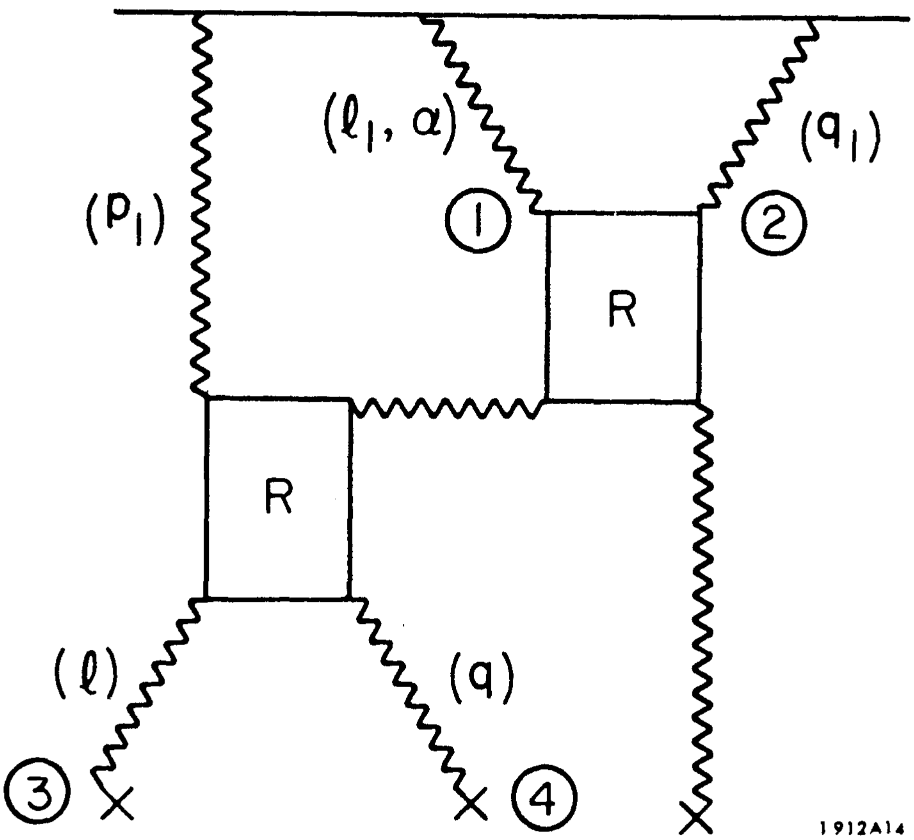


Fig. 21

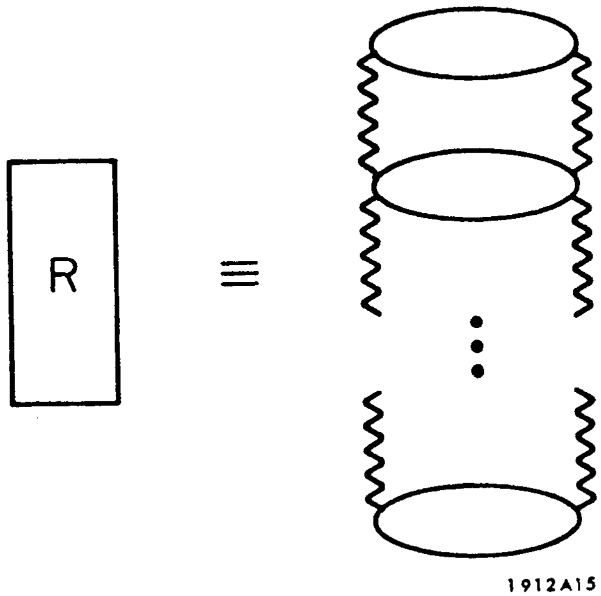
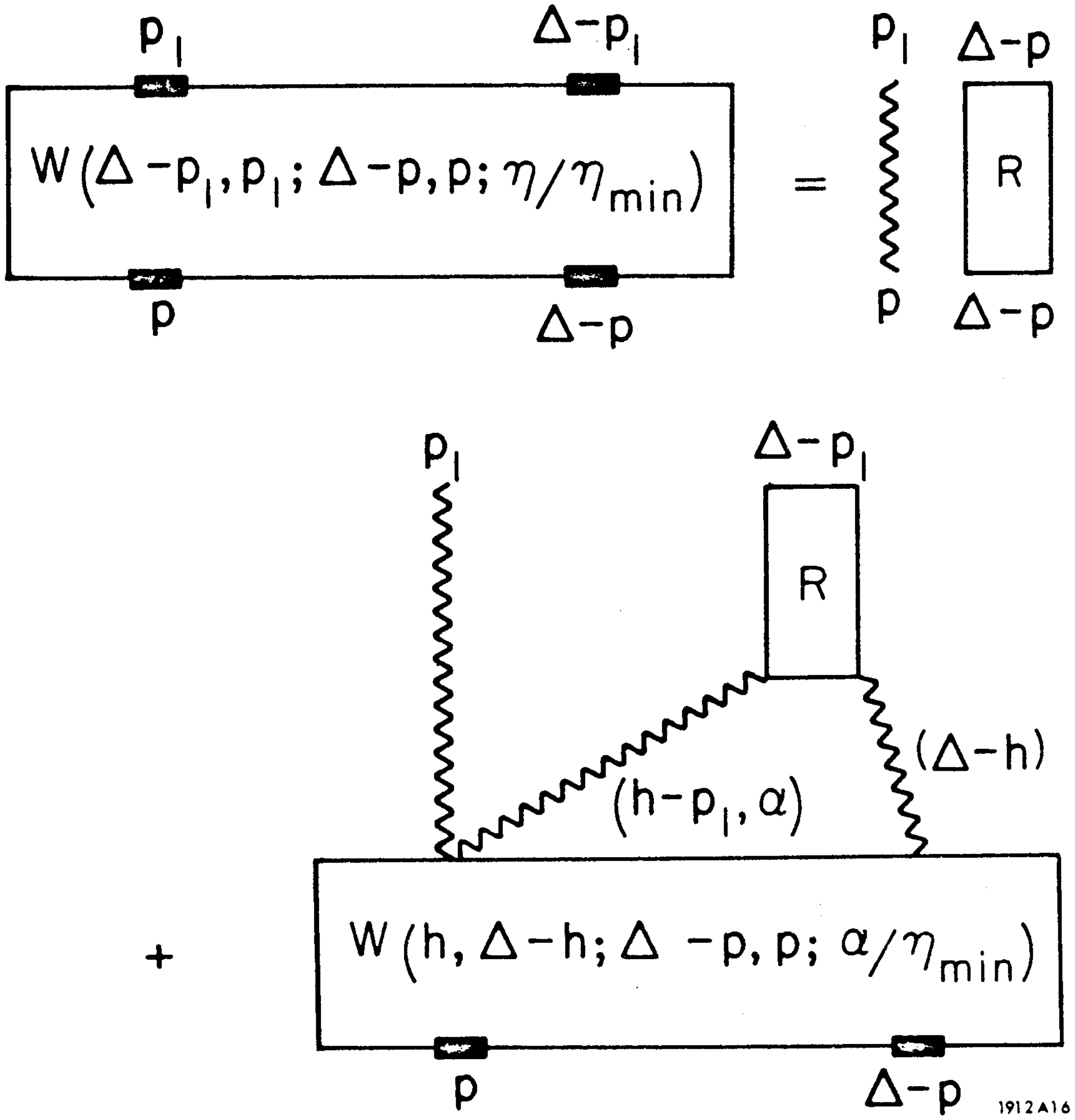


Fig. 22



p1
Δ-p1
W(Δ-p1, p1; Δ-p, p; η/ηmin)
p
Δ-p
=
p1
Δ-p
R
p
Δ-p
+
p1
Δ-p1
R
(Δ-h)
(h-p1, α)
W(h, Δ-h; Δ -p, p; α/ηmin)
p
Δ-p
1912A16

Fig. 23

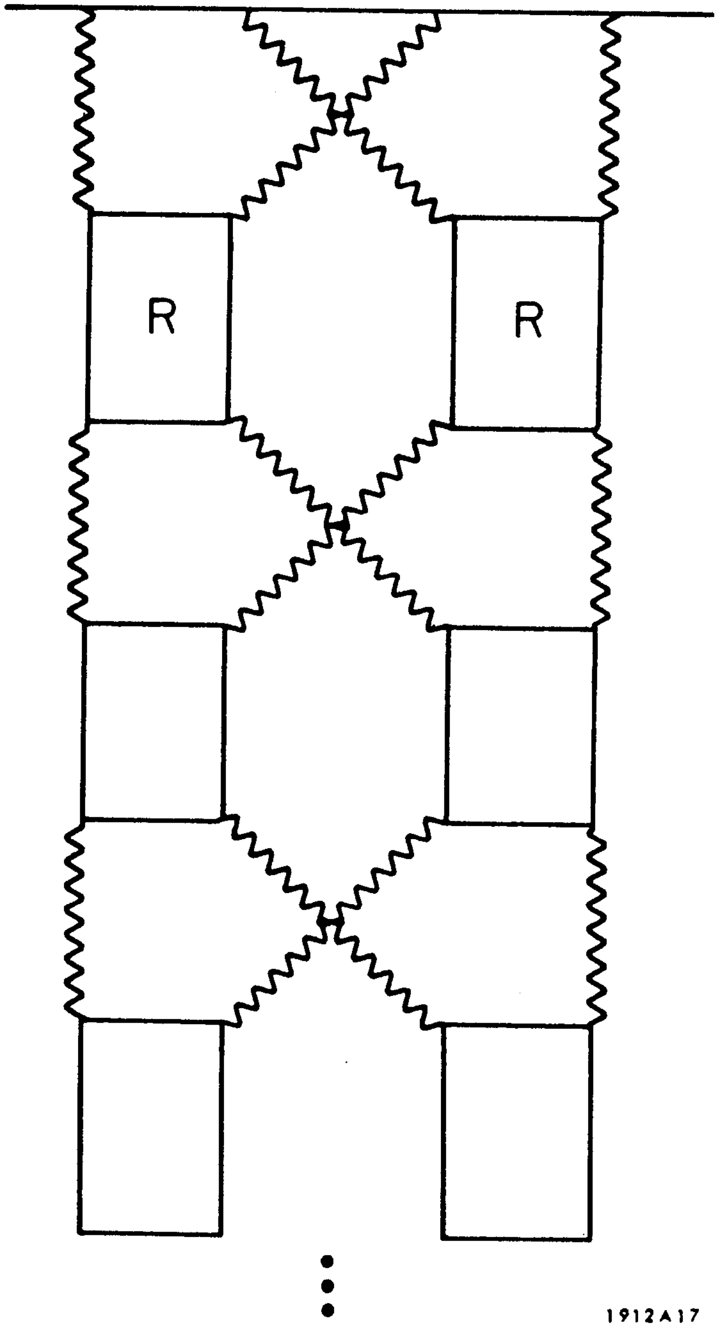


Fig. 24

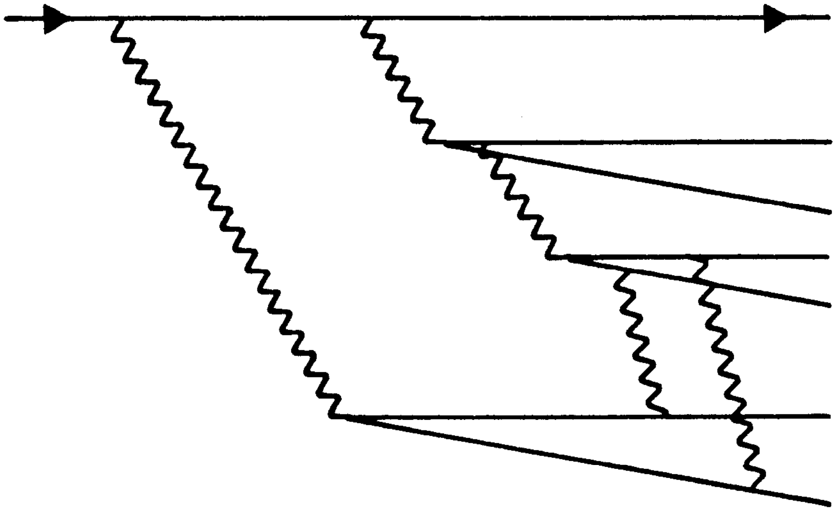


Fig. 25