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FADDEEV'S EQUATIONS FOR LOCAL POTENTIALS

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FADDEEV'S EQUATIONS FOR LOCAL POTENTIALS

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This dissertation studies the non-relativistic quantum mechanical three-body problem. The three particle system is that described by three spinless particles interacting via a sum of two-body potentials. The dynamics of the system is determined by Schroedinger's equation. Recently L. D. Faddeev has shown how to re-express this equation in a form which is well defined and avoids the singularities which customarily accompany the three particle Schroedinger equation. Faddeev's work shows that solutions to the problem exist but does not obtain any of these solutions. Subsequent research has obtained specific numerical solutions under the assumption that the two-body potential is separable in momentum space. Mathematically this assumption means that the potential can be written $\langle \vec{p}|V|\vec{q}\rangle = f(\vec{p})h(\vec{q})$. Unfortunately, this assumption has little obvious justification for three-body systems of physical interest. The first aim of this work is to obtain solutions for an arbitrary potential. Secondly, we use these general solutions to test the validity of the assumption that two-body potentials may be represented as separable potentials.

Our method for dealing with the Faddeev equations that govern the three-body problem consists of two parts. These equations are first simplified. We are able to reduce the number of independent variables from six to two by expanding the system in terms of a complete set of angular momentum wave functions. This leaves us with a two-variable integral equation to solve. Next, special numerical techniques are developed so that we can efficiently represent this integral equation as a finite matrix problem. Solutions are obtained by solving this matrix problem with a computer.

This method is employed to obtain binding energies and wave functions for a system of three identical particles. Two often studied non-separable potentials (exponential and Yukawa) are used. The system manifests a number of three-body bound states depending on the strength of the two-body attraction. The bound state wave functions are described. Turning to the evaluation of the separable approximation in our example, we find this approximation to be useful only in the limit where the two-body forces are weak.

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CHAPTER I

FORMAL DEVELOPMENT OF FADDEEV'S EQUATIONS

This introduction outlines some of the background and motivation for this particular study of the quantum mechanical three-body problem. Here the three-body problem means the determination of both the bound state spectra and wave functions as well as elastic and inelastic scattering. Ideally we would like to have a single formalism which yields a simple, usable method for obtaining exact solutions for this problem. We work only with the non-relativistic treatment of the three-body problem, since it is clear that we must learn to solve this simpler problem before attacking the complications introduced by relativistic effects.

Early approaches to the three-body problem have been characterized by piecemeal attacks on various separate aspects of three-body systems. The variational method has been used to find approximate bound state energies and wave functions. Although this method gives a rigorous upper bound on the ground state energy it is difficult to prove convergence. Furthermore, the variational method does not yet lend itself to calculating solutions where inelastic processes occur.[1] The ground state for local potentials was solved by Baker, Gammel, Hill and Wills[2] by exploiting the different time dependences of the various eigenstates. If the time variable t is replaced by $i\tau$, where τ is real and positive, then the different eigenstate wave functions die out at rates that depend on their energies. This difference in decay rate may be used to isolate the ground state component present in an arbitrary trial wave function. While successful, this method cannot readily treat excited or scattering states. By using separable potentials Mitra[3] was able to solve the three particle Schroedinger equation for binding energies and scattering lengths. While the use of the separable potential greatly simplifies the problem, the underlying physics of this approach is not clear. Later Amado,[4] by

introducing quasiparticle states, was also able to arrive at a formulation for the three-body problem that has the same appearance as that of the separable method. Here again the physical content of this approach is unclear.

Faddeev's treatment[5-8] of the three-body problem avoids most of the difficulties described above. First of all his work treats the entire problem on the same footing. Bound states, elastic and inelastic scattering are all treated with one formalism. The method is able to avoid any ill-defined singular operators and is completely rigorous. Basically, Faddeev has shown us how to treat the three particle Schroedinger equation correctly. However, the form in which Faddeev leaves the three-body problem is not readily solvable. One can again employ separable potentials to simplify the situation but this just reduces the calculation to the model used by Mitra.

Omnes[9] has introduces a symmetric angular decomposition of the three-body system. This simplifies the problem but still leaves it in the form of an intractable integral equation in three variables. The present work[10] carries the decomposition of Omnes one step further. Here, without loss of generality, we are able to reduce the integral equations to equations in only two variables. A parallel result had been arrived at by Ahmadzadeh and Tjon[11] who treated the angular momentum decomposition by using a vector addition approach to construct the eigenstates of the total angular momentum operator. They too obtained a set of coupled integral equations in two variables.

We have studied various numerical methods for solving these equations. When our problem is restricted to the simple three identical particle system we can readily obtain accurate solutions for any potential. A similar effort by Wong and Zambotti[12] yielded results which differed from ours and are now known to be inaccurate. Recently Wong[13] has developed a new method using an expansion of the Schroedinger

equation in Strümian functions[14] and was able to reproduce our results for the Yukawa interaction.

We have used our numerical techniques to obtain the exact eigenvalue spectrum and the bound state wave functions. These new results are described. Finally, we are able to evaluate the validity of the separable potential approach by comparing various separable calculations with our exact solutions.

A. Mathematical Background

As an introduction to the three body problem, we present here a brief review of the derivation of Faddeev's equations. The present account is a synthesis of the work of L. D. Faddeev and C. Lovelace. Although Faddeev gave the first and only[5-8] completely rigorous derivation of the equations that bear his name we will adapt much from the mathematical notation and arguments used by Lovelace.[15,16] The primary reason for this is that the approach of Lovelace does not assume the extensive mathematical background needed for much of Faddeev's work. Most of Lovelace's proofs are simple and easily intelligible to physicists. Throughout this work it is assumed that there are no three-body forces.

In this chapter and the ones that follow we will study the two- and three-body problem only in its momentum space form. In the coordinate space versions of the two- and three-body problem the singular aspects of the problem reside in the asymptotic behavior of the functions. For example, wave functions of three particles may have a very complex asymptotic structure, oscillating in some directions of configuration space while falling off in others. However, after the Fourier transformation to momentum space has been made, the asymptotic singularities become poles or cuts in the complex energy plane. These types of singularities are much easier to treat than asymptotic ones. The price paid for this simplification

is that we must deal with singular kernels. With appropriate care, this can be done. The foregoing reason, we believe, explains why so little successful work has come from coordinate space approaches to the three-body problem while on the other hand the momentum space approach used by Faddeev provides an exact, albeit formal, solution to the problem.

Before proceeding to the various operator equations that describe the physics of two- and three-body scattering, we need to develop the mathematical machinery capable of dealing with these equations. The central aim of our discussion will be to arrive at a theorem which will tell us when the inverses of these operators exist. We first introduce the concept of a Schmidt norm. Let us consider the complete linear vector space formed of all square-integrable functions. This space is usually denoted as the Banach space of L_2 functions. Now suppose A is a linear operator that maps this space onto itself, and that may be represented as an integral transform. That is, given f_1, $f_2 \in L_2$ and $A f_1 = f_2$ then

$$f_2(x) = \int a(x,x')\ f_1(x')\, dx' \quad . \tag{I.1}$$

The Schmidt norm for the integral operator A is defined to be

$$\left\| A \right\|_s^2 \equiv \iint \left| a(x,\ x') \right|^2 dx\, dx' \quad . \tag{I.2}$$

An operator for which this norm is bounded is called a Schmidt operator. This norm should be contrasted with the usual operator norm which is defined as

$$\left\| A \right\|^2 = \sup_{f \in \ell_2} (Af,\ Af)/(f,f). \tag{I.3}$$

It is not difficult to show[17] that the Schmidt norm is weaker than the operator norm, so

$$||A||_s \geq ||A|| \quad . \tag{I.4}$$

The inequality is clearly satisfied by $a(x,x') = \delta(x-x')$. By definition this is the identity operator and $||A|| = 1$, but $||A||_s = \infty$, since the integrand is the square of a delta function. Some of the obvious properties of the Schmidt operators are that:

(a) the product of two Schmidt operators is also a Schmidt operator,

(b) the finite sum of Schmidt operators is a Schmidt operator.

Less obviously but equally important is the property that a product of a bounded operator, namely an operator with $||\cdot|| < \infty$, with a Schmidt operator is a Schmidt operator. Riesz and Sz. Nagy give a simple proof of this.[18] An additional useful property of these integral type operators is that the kernels may be approximated to an arbitrary accuracy by kernels of finite rank (e.g. by finite sums of separable terms). This approximation property is sufficient to show that operators are compact.[19] Here a compact operator is an operator which maps a bounded sequence into a sequence containing a limit point. This property of compactness means that most of the results of integral equation theory will be valid for our operator equations.

The next concept about operators needed is that of operator analyticity. Let μ be a complex number in some domain D. An operator valued function $T(\mu)$ is one for which the results of analytic function theory remain valid provided we use the norm $||\cdot||$ in place of the absolute value of a complex function.

The sufficient conditions for a Schmidt operator, represented by a kernel $K(x,y;\mu)$, to be operator analytic are

(i) For each x,y, $K(x,y;\mu)$ must be analytic for $\mu \in D$.

(ii) There exists a uniform bound $B < \infty$ such that

$$\left|\left| K(\mu) \right|\right|_S < B \qquad \text{all } \mu \in D .$$

We are now in a position to state a theorem which gives us sufficient conditions for insuring the existence of the inverse of the operators we will be using.

Theorem I. Let $T(\mu)$ be a compact operator in some Banach space (here the Banach space of bounded linear operators mapping L_2 onto itself). Suppose $T(\mu)$ is operator analytic for all μ in the connected domain D; then the following alternative holds:

(a) $1 - T(\mu)$ has no bounded inverse for any point in D, or

(b) the inverse of $1 - T(\mu)$ exists except for a finite number of isolated points.

In the neighborhood of a singular point, μ_0, the Laurent expansion is valid. The inverse can be written as a pole term with a residue of finite rank and a bounded term. For $\mu \simeq \mu_0$ the kernel of the resolvent (inverse) operator $R(x, y; \mu)$ has the representation

$$R(x,y;\mu) = \sum_{i=1}^{N} \frac{\phi_{\mu_0}^{(i)}(x)\ \psi_{\mu_0}^{(i)}(y)}{\mu - \mu_0} + B(x,y;\mu) \qquad (I.5)$$

where $B(x,y;\mu)$ is bounded, e.g. $||B(\mu)|| < C < \infty$, and $\phi_{\mu_0}^{(i)}$ and $\psi_{\mu_0}^{(i)}$ are homogeneous solutions to $1 - T(\mu_0)$ and its adjoint. It is this theorem that is

the key to our method. Although Lovelace[15] states a slightly altered version of this theorem he offers no proof. A proof has been given by J. Nuttal[20] who relates the theorem to more general results described by Dunford and Schwartz.[21] Yet another proof has been given by George Tiktopoulos.[22]

The simplest way to rule out alternative a is to find a point λ in D such that $||T(\lambda)|| < 1$. For such a λ the formal series expansion of $\left(1 - T(\lambda)\right)^{-1}$ is absolutely convergent and thus $\left(1 - T(\lambda)\right)^{-1}$ exists and is bounded by $\left(1 - ||T(\lambda)||\right)^{-1}$.

It is instructive to compare this theorem with the well known expansion of $(E - H)^{-1}$, where H is the usual self-adjoint Hamiltonian in quantum mechanics. The result is

$$(E - H)^{-1} = E^{-1}(1 - E^{-1}H)^{-1}$$

$$= \sum_b \frac{|\psi_b><\psi_b|}{E - E_b} + \int_0^\infty \frac{|\psi_\mu><\psi_\mu|}{E - \mu} \, d\mu \qquad \text{(I. 6)}$$

where $|\psi_b>$ is a ket describing the bound state of energy E_b, while $|\psi_\mu>$ is a non-renormalizable state of the continuum. We see that Eq. (I. 6) is a particular form of the resolvent representation (I. 5). However, (I. 6) is not general enough for our purposes since the operator analyticity in E obtains trivially for the operator $E^{-1}H$ since the dependence on E is just a scalar multiple of H.

B. Two Body Formulation

We now turn our attention to the formulation of two-body scattering. In the center-of-mass system there is one free momentum vector $\vec{p}$, which is the momentum of either of the two particles. The Hilbert space is that of $L_2(\vec{p})$, the space of square integrable functions of $\vec{p}$. In order to distinguish between

similar operators in the two- and three-particle Hilbert space we will adopt the following notation. Capital letters will refer to operators in the three-particle space while small letters will be reserved for operators in the two-particle Hilbert space.

The unperturbed two-particle Hamiltonian, h_0, is represented in momentum space by

$$\langle \vec{p} | h_0 | \vec{p}' \rangle = \frac{p^2}{2\mu} \delta_3(\vec{p} - \vec{p}') \tag{I.7}$$

where μ is the reduced mass $m_1 m_2/(m_1 + m_2)$. The resolvent operator for this Hamiltonian is defined to be

$$g_0(z) = (h_0 - z)^{-1} \tag{I.8}$$

and whose momentum space kernel is

$$\langle \vec{p} | g_0(z) | \vec{p}' \rangle = \frac{1}{\left(\frac{p^2}{2\mu} - z\right)} \delta(\vec{p} - \vec{p}') \tag{I.9}$$

Scattering is caused by a potential v_α. Locality requires that it depend only on $\vec{p} - \vec{p}'$. Specifically

$$\langle \vec{p} | v_\alpha | \vec{p}' \rangle = v_\alpha (\vec{p} - \vec{p}') \ . \tag{I.10}$$

Our theory will assume that the potential satisfies the following conditions. First we assume $v_\alpha(\vec{k})$ is square integrable. Secondly $v_\alpha(\vec{k})$ is analytic in k for a domain containing the real axis. This analyticity is assured if v_α can be written as a superposition of Yukawa potentials in the following form.

$$v_\alpha(\vec{p}) = \int_{\mu_0}^{\infty} \frac{\sigma(\mu)}{p^2 + \mu^2} \, d\mu \qquad \text{(I.11)}$$

where $\int_{\mu_0}^{\infty} \sigma(\mu)\, d\mu$ is required to be of bounded variation. Finally, time reversal invariance requires that v_α have the real-valued property

$$v_\alpha(-\vec{p}) = \overline{v_\alpha(\vec{p})} \quad . \qquad \text{(I.12)}$$

The full two-particle Hamiltonian is given by

$$h_\alpha = h_0 + v_\alpha \quad . \qquad \text{(I.13)}$$

Its companion resolvent $g_\alpha(z)$ is

$$g_\alpha(z) = (h_\alpha - z)^{-1} \quad . \qquad \text{(I.14)}$$

In the above equation existence of the operator g_α results from the self-adjoint properties of h_α which lead to the expansion given by (I.6). We note that $g_\alpha(z)$ of course will not exist for all values of z. For z equal to the energy of a bound state or for z lying on the real positive continuum, then g_α ceases to exist.

The behavior of the resolvents g_α and g_0 are governed by two identities known as the first and second resolvent equations. The first resolvent equation for g_α may be obtained directly from (I.14)

$$\begin{aligned} g_\alpha(z_1) - g_\alpha(z_2) &= g_\alpha(z_1)\,(h_\alpha - z_1)\left[g_\alpha(z_1) - g_\alpha(z_2)\right]\,(h_\alpha - z_2)\,g_\alpha(z_2) \\ &= g_\alpha(z_1)\left[1\cdot(h_\alpha - z_2) - (h_\alpha - z_1)\cdot 1\right] g_\alpha(z_2) \\ &= (z_1 - z_2)\, g_\alpha(z_1)\, g_\alpha(z_2) \end{aligned} \qquad \text{(I.15)}$$

Similarly the first resolvent equation for g_0 is

$$g_0(z_1) - g_0(z_2) = (z_1 - z_2)\, g_0(z_1)\, g_0(z_2) \tag{I.16}$$

The second resolvent equation relates $g_0(z)$ to $g_\alpha(z)$. This result follows by using both (I.14) and (I.8).

$$\begin{aligned} g_\alpha(z) &= g_0(z)\,(h_0 - z)\, g_\alpha(z) \\ &= g_0(z)\,(h_0 + v_\alpha - z)\, g_\alpha(z) - g_0(z)\, v_\alpha g_\alpha(z) \qquad \text{(I.17)} \\ &= g_0(z) - g_0(z)\, v_\alpha g_\alpha(z) \end{aligned}$$

Since $g_0(z)\,(h_0 - z) = 1$ and $g_\alpha(z)$ commute, the second resolvent equation has another form given by

$$g_\alpha(z) = g_0(z) - g_\alpha(z)\; v_\alpha\, g_0(z) \tag{I.18}$$

The formal solution for g_α may be obtained from (I.17). If $\left[1 + g_0(z)v_\alpha\right]^{-1}$ exists then we may write g_α as

$$g_\alpha(z) = \left[1 + g_0(z)\, v_\alpha\right]^{-1}\; g_0(z) \tag{I.19}$$

$$= g_0(z)\; \left[1 + v_\alpha g_0(z)\right]^{-1} \tag{I.20}$$

We can use Theorem I to show that these inverses exist. The conditions of Theorem I are statisfied if $g_0(z)\, v_\alpha$ is both a Schmidt operator and analytic. The kernel representation of $g_0(z)\, v_\alpha$ is

$$\langle \vec{p} \mid g_0(z)\, v_\alpha \mid \vec{p}' \rangle = \frac{v(\vec{p} - \vec{p}')}{\frac{p^2}{2\mu} - z} \tag{I.21}$$

and its Schmidt norm is

$$\|g_0(z)\, v_\alpha\|_S^2 = \int d\vec{p} \int d\vec{p}' \, \frac{\left| v_\alpha(\vec{p}' - \vec{p}) \right|^2}{\left| \frac{p^2}{2\mu} - z \right|^2}$$

$$= \frac{(2\mu\pi)^2}{\left| \mathrm{Im}\sqrt{2\mu z} \right|} \int d\vec{p}' \left| v_\alpha(\vec{p}') \right|^2 \qquad \text{(I.22)}$$

Thus for $\mathrm{Im}\sqrt{z} \neq 0$, $g_0(z)v_\alpha$ is a Schmidt operator. The analyticity of the kernel is evident from (I.21), and if we choose the domain of z to be $D_\epsilon = \left\{ z: \left| \mathrm{Im}\sqrt{z} \right| > \sqrt{\epsilon}\,, \epsilon > 0 \right\}$, then the kernel is uniformly bounded by $\epsilon^{-1} \left| v(\vec{p}-\vec{p}') \right|$. Thus all the conditions for Theorem I are satisfied. Furthermore, alternative a is ruled out by the following argument. By (I.22) we can find $z \in D_\epsilon$ such that $\| g_0(z)v_\alpha \| < 1$, thus we have proved both $\left[1 + v_\alpha g_0(z) \right]^{-1}$ and $\left[1 + g_0(z)\, v_\alpha \right]^{-1}$ are operator meromorphic in D_ϵ.

We now want to demonstrate the simple connection between our resolvents $g_\alpha(z)$ and the eigenfunctions of h_α. Let s be a singular point of $g_\alpha(z)$; then

$$\lim_{z \to s} (s - z)\, g_\alpha(z) \equiv P_s \qquad \text{(I.23)}$$

is a projection operator for an eigenstate of h_α of eigenvalue s. We prove this by the following simple manipulations. Suppose ϕ is any function in $L_2(\vec{p})$. We have

$$\begin{aligned} h_\alpha \lim_{z \to s} (s - z)\, g_\alpha(z)\, \phi &= \lim_{z \to s} (s - z)\, (h_\alpha - z + z)\, g_\alpha(z)\, \phi \\ &= \lim_{z \to s} (s - z)\phi + \lim_{z \to s} z(s - z)\, g_\alpha(z)\, \phi \\ &= s \lim_{z \to s} (s - z)\, g_\alpha(z)\, \phi \end{aligned} \qquad \text{(I.24)}$$

Thus we have shown

$$h_{\alpha} P_s \phi = s P_s \phi \equiv s \psi(s) . \tag{I.25}$$

Combining this last result with our forms for the second resolvent equations (I.11) and (I.18) we get the Lippman-Schwinger equation for wavefunctions. To do this, define $\phi(s)$ to be an eigenfunction of h_0,

$$h_0 \phi(s) = s \phi(s),$$

which represents an incoming wave. Obviously

$$\lim_{z \to s} (s - z) \; g_0(z) \; \phi(s) = \phi(s) . \tag{I.26}$$

Applying (I.17) to $\phi(s)$, multiplying by $(s - z)$ and taking the limit gives us via (I.26) and (I.25):

$$\psi(s) = \phi(s) - \lim_{z \to s} g_0(z) \; v_{\alpha} \psi(s) \tag{I.27}$$

If we use (I.18) in place of (I.17) then the result is

$$\psi(s) = \phi(s) - \lim_{z \to s} g_{\alpha}(z) \; v_{\alpha} \; \phi_{\alpha}(s) \tag{I.28}$$

In order to complete our discussion of the two-body problem it is convenient to introduce the t-matrix or transition amplitude operator. It is well known[23] that the transition amplitude probability between an incoming state $\phi_b(s)$ and an outgoing state $\phi_a(s)$ produced by scattering from a potential v_{α} is

$(\phi_a, v_\alpha \psi_b)$. The transition operator is defined to be the operator t such that

$$\left(\phi_a, t_\alpha(s)\ \phi_b\right) = (\phi_a, v_\alpha \psi_b) \tag{I.29}$$

for all ϕ_a and ϕ_b. From (I.28) we see that

$$\left(\phi_a, t_\alpha(s)\ \phi_b\right) = \left(\phi_a, \left[v_\alpha - v_\alpha\ g_\alpha(s + i\epsilon)\ v_\alpha\right]\phi_b\right) \tag{I.30}$$

Because ϕ_a and ϕ_b are arbitrary it follows that

$$t_\alpha(s + i\epsilon) = v_\alpha - v_\alpha\ g_\alpha(s + i\epsilon)\ v_\alpha \tag{I.31}$$

The most general version of t_α is obtained by replacing $s + i\epsilon$ with z, where z is any point in the complex plane.

From the operator definition of $t_\alpha(z)$, Eq. (I.31), we can obtain two very useful identities by multiplying this equation from the left and from the right by $g_0(z)$. We have

$$\begin{aligned} t_\alpha(z)\ g_0(z) &= \left[v_\alpha - v_\alpha\ g_\alpha(z)\ v_\alpha\right] g_0(z) \\ &= v_\alpha\left[g_0(z) - g_\alpha(z)\ v_\alpha\ g_0(z)\right] \\ &= v_\alpha\ g_\alpha(z). \end{aligned} \tag{I.32}$$

The last equality follows from the second resolvent Eq. (I.18). Alternately,

$$\begin{aligned} g_0(z)\ t_\alpha(z) &= g_0(z)\ \left[v_\alpha - v_\alpha\, g_\alpha(z)\ v_\alpha\right] \\ &= \left[g_0(z) - g_0(z)\, v_\alpha\ g_\alpha(z)\right]\ v_\alpha \\ &= g_\alpha(z)\ v_\alpha \end{aligned} \tag{I.33}$$

where (I.17) has been used.

Identities (I. 32) and (I. 33) may now be used to re-cast our t-matrix equation into forms where the exact resolvent (known only if the h_α problem has been solved) is removed. Using (I. 32) we get

$$t_\alpha(z) = v_\alpha - t_\alpha(z)\, g_0(z)\, v_\alpha . \tag{I. 34}$$

With (I. 33) the result is

$$t_\alpha(z) = v_\alpha - v_\alpha\, g_0(z)\, t_\alpha(z) . \tag{I. 35}$$

Both (I. 34) and (I. 35) are operator versions of the Lippman-Schwinger equation for the t-matrix. The operator equations may be transformed into integral equations just by using the completeness relation for the Hamiltonian, h_0, and taking expectation values between eigenstates $|\vec{q}>$ and $|\vec{q}'>$. Thus (I. 35) becomes

$$<\vec{q}|t_\alpha(z)|\vec{q}'> = <\vec{q}|v_\alpha|\vec{q}'> - \int d\vec{q}'' \frac{<\vec{q}|v_\alpha|\vec{q}''><\vec{q}''|t_\alpha(z)|\vec{q}'>}{\frac{q''^2}{2\mu} - z} . \tag{I. 36}$$

We note that for $<\vec{q}|t_\alpha(z)|\vec{q}'>$, $\vec{q}$ is the variable of the integral equations and $(z, \vec{q}')$ are parametric labels.

Furthermore since we proved earlier that $g_0(z)v_\alpha$ is operator analytic and compact the solution of (I. 34) may be written

$$\begin{aligned} t_\alpha(z) &= [1 + v_\alpha g_0(z)]^{-1} v_\alpha \\ &= v_\alpha [1 + g_0(z) v_\alpha]^{-1} \end{aligned} \tag{I. 37}$$

One immediate consequence of (I. 37) is that the solution of the integral equation (I. 36) is unique except at poles $[1 + g_0(z) v_\alpha]^{-1}$.

To summarize our two-body results, all the important equations governing two-body scattering have been expressed in operator form. Furthermore, we have shown that inverses of these operators exist. The key equations in obtaining our results were the first and second resolvent equations. Simple algebraic manipulations then lead to the Lippman-Schwinger equations for the t matrix. Aside from its simplicity the merit of the above approach to two-body scattering is that the same techniques lead to the solution of the three-body scattering problem. Our program for analyzing the three-body problem will consist of a study of the resolvent operators on the three-body Hilbert space. By using the first and second resolvent identities together with our two-body results we will obtain Faddeev's equations.

C. Derivation of Faddeev's Equations

Turning to three-body scattering we must first describe the coordinate system of momentum variables we are using. Let $\vec{p}_i$ denote the momenta of the i^{th} particle, i = 1, 2, 3. We shall always assume that the total center-of-mass momentum $\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$ is zero. This condition means that only two of the three momentum vectors are independent. The center-of-mass transformation is effected by first transforming one pair to its relative center-of-mass system and then applying the center-of-mass transformation to that pair and the remaining particle. Defining $\vec{q}_{ij}$ to be the reduced center-of-mass momentum of particles i and j, then

$$\vec{q}_{ij} = (m_j \vec{p}_i - m_i \vec{p}_j) / (m_i + m_j) \quad . \tag{I.38}$$

The momentum configuration of a system can be described by the pair of vectors $(\vec{q}_{ij}, \vec{p}_k)$, where i, j and k are cyclic. However, other choices for the two

independent vectors are possible, such as $(\vec{p}_i, \vec{p}_j)$ or even $(\vec{q}_{ij}, \vec{q}_{ik})$. Occasionally we will drop the subscripts on $(\vec{q}, \vec{p})$ when the choice is obvious or irrelevant. Expressed in these momentum vectors the unperturbed three-particle Hamiltonian is

$$H_0 = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + \frac{\vec{p}_3^{\,2}}{2m_3} \tag{I.39}$$

$$H_0 = \frac{1}{2\mu_{ij}} \vec{q}_{ij}^{\,2} + \frac{1}{2n_k} \vec{p}_k^{\,2} \tag{I.40}$$

where

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j}; \qquad n_k = \frac{m_k (m_i + m_j)}{m_i + m_j + m_k}; \tag{I.41}$$

The appropriate three-particle Hilbert space for these variables is that of square integrable functions of six real variables--or $L_2(\vec{p}, \vec{q})$. Our first task is to define the form of a two-body interaction in the three-particle Hilbert space. Suppose v_{ij} is the potential between particles i and j. Such a two-body interaction will not involve particle k and thus the three-body potential V_{ij} may be written.

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_2, \vec{p}_3 \,|\, V_{ij} \,|\, \vec{p}'_1, \vec{p}'_2, \vec{p}'_3 \rangle &= \delta(\vec{p}_k - \vec{p}'_k) \langle \vec{p}_i \vec{p}_j \,|\, v_{ij} \,|\, \vec{p}'_i \vec{p}'_j \rangle \\ &= \delta(\vec{p}_k - \vec{p}'_k) \langle \vec{q}_{ij} \,|\, v_{ij} \,|\, \vec{q}'_{ij} \rangle \end{aligned} \tag{I.42}$$

Adding any one of these interactions V_{ij} to H_0 gives us a new three-particle Hamiltonian and a corresponding resolvent. Specifically we have

$$H_i = H_0 + V_{jk} \tag{I.43}$$

with

$$G_i(z) = (H_i - z)^{-1} \tag{I.44}$$

$$G_0(z) = (H_0 - z)^{-1} \tag{I.45}$$

As a first exercise in dealing with the Hilbert space $L_2(\vec{p}, \vec{q})$ let us calculate the scattering equations when just one interaction, V_{jk}, is present. That is, we want to solve the scattering problem for H_i. As before we start with the second resolvent equation relating $G_0(z)$ and $G_i(z)$. Algebraic manipulations identical to those that gave us (I.17) and (I.18) show that

$$G_i(z) = G_0(z) - G_0(z)\, V_{jk}\, G_i(z) \tag{I.46}$$

$$G_i(z) = G_0(z) - G_i(z)\, V_{jk}\, G_0(z) \tag{I.47}$$

We define the three-particle transition amplitude operator to be

$$T_i(z) = V_i - V_i\, G_i(z)\, V_i \tag{I.48}$$

where we have used the notation $V_i \equiv V_{jk}$. Combining (I.48) with (I.46) and (I.47) gives us

$$T_i(z) = V_i - V_i\, G_0(z)\, T_i(z) \tag{I.49}$$

$$= V_i - T_i\, G_0(z)\, V_i \tag{I.50}$$

Since one of the particles, i, is a spectator and never participates in the scattering we can construct the solutions of (I.49) from those of $t_i(z)$--the two

particle transition amplitude for $v_\alpha = v_{jk} \equiv v_i$. Taking the expectation values of (I. 48) tells us that $T_i(z)$ must have the form

$$\langle \vec{p}_i, \vec{q}_{ik} \mid T_i(z) \mid \vec{p}_i', \vec{q}_{jk}' \rangle = \delta(\vec{p}_i - \vec{p}_i')\, f(\vec{p}_i, z, \vec{q}_{jk}, \vec{q}_{jk}') \tag{I. 51}$$

If we write out (I. 50) in detail using (I. 51) we find

$$\delta(\vec{p}_i - \vec{p}_i')\, f(\vec{p}_i, z, \vec{q}_{jk}, \vec{q}_{jk}') = \delta(\vec{p}_i - \vec{p}_i') \langle \vec{q}_{jk} \mid v_{jk} \mid \vec{q}_{jk}' \rangle$$

$$- \delta(\vec{p}_i - \vec{p}_i') \int d\vec{q}_{jk}'' \frac{\langle \vec{q}_{jk} \mid v_{jk} \mid \vec{q}_{jk}'' \rangle f(\vec{p}_i, z, \vec{q}_{jk}'', \vec{q}_{jk}')}{\frac{p_i^2}{2n_i} + \frac{q_{jk}''^2}{2\mu_{jk}} - z} \tag{I. 52}$$

Dividing out the delta function in (I. 52) we see that f is the solution of the two-body Lippman-Schwinger equation if we make the identification that in (I. 36) $z \longrightarrow z - \frac{p_i^2}{2n_i}$. Thus

$$f(\vec{p}_i, z, \vec{q}_{jk}, \vec{q}_{jk}') = \langle \vec{q}_{jk} \mid t_i \left(z - \frac{p_i^2}{2n_i} \right) \mid \vec{q}_{jk}' \rangle \tag{I. 53}$$

Note that this identification implicitly uses the fact that the two-body Lippman-Schwinger equation has a unique solution. We now have the sought-after relationship between two- and three-body transition amplitudes,

$$\langle \vec{p}_i, \vec{q}_{jk} \mid T_i(z) \mid \vec{p}_i', \vec{q}_{jk}' \rangle = \delta(\vec{p}_i - \vec{p}_i') \langle \vec{q}_{jk} \mid t_i \left(z - \frac{p_i^2}{2n_i} \right) \mid \vec{q}_{jk}' \rangle . \tag{I. 54}$$

It is instructive to attempt to solve the Lippman-Schwinger equation when all three two-body potentials are present. Let us define the total potential U to be

$$U = V_{12} + V_{13} + V_{23} \tag{I.55}$$

The Hamiltonian and exact Green's function are then

$$H = H_0 + U \tag{I.56}$$

and

$$G(z) = (H - z)^{-1} \tag{I.57}$$

The second resolvent equation relating $G(z)$ and $G_0(z)$ is

$$G(z) = G_0(z) - G_0(z)\, U\, G(z) \tag{I.58}$$

$$G(z) = G_0(z) - G(z)\, U\, G_0(z) \tag{I.59}$$

Let Φ_n be a three-body plane wave state satisfying

$$H_0\, \Phi_n = E_n\, \Phi_n \tag{I.60}$$

Then as in (I.24) we have

$$\lim_{i\epsilon \longrightarrow 0} \; -i\epsilon\; G(E_n + i\epsilon)\; \Phi_n = \psi_n \tag{I.61}$$

where

$$H\, \psi_n = E\, \psi_n \; .$$

Applying (I.58) to Φ_n results in the wave function version of the Lippman-Schwinger equation

$$\psi_n = \Phi_n - G_0(E_n + i\epsilon)\; U\, \psi_n \tag{I.62}$$

where we have used

$$\lim_{i\epsilon \longrightarrow 0} \; - i\epsilon \; G_0(E_n + i\epsilon) \; \phi_n = \phi_n \tag{I.63}$$

Now suppose ϕ_n^i is an eigenfunction of H_i

$$H_i \; \phi_n^i = E_n \; \phi_n^i$$

We know such a state exists because to a bound state between particles j and k of energy $\gamma_i < E_n$ we may add the kinetic energy of the free particle so that

$$E_n = \gamma_i + \frac{p_i^2}{2n_i} \tag{I.64}$$

It is not difficult to show that[5]

$$\lim_{i\epsilon \longrightarrow 0} \; - i\epsilon \; G_0 \; (E_n + i\epsilon) \; \phi_n^i = 0 \tag{I.65}$$

Thus applying (I.58) to ϕ_n^i gives us

$$\psi_n^i = - \; G_0 \; (E_n + i\epsilon) \; U \psi_n^i \tag{I.66}$$

where

$$\lim_{i\epsilon \longrightarrow 0} \; - i\epsilon \; G \; (E_n + i\epsilon) \; \phi_n^i = \psi_n^i \tag{I.67}$$

Equation (I.66) shows that we have constructed a homogeneous solution for the Lippman-Schwinger equation given in (I.62). The presence of such homogeneous solutions means we cannot use (I.62) to determine ψ_n uniquely. Such homogeneous solutions must be ruled out by asymptotic boundary conditions. Such subsidiary conditions destroy the usefulness of (I.62) as an integral equation.

In analogy with the definition of the two-body transition amplitude (I.31) we define the three-body transition amplitude by

$$T(z) = U - U\,G(z)\,U \tag{I.68}$$

This equation together with resolvent identities gives us

$$\begin{aligned} T(z) &= U - U\,G_0(z)\,T(z) \\ &= U - T(z)\,G_0(z)\,U \end{aligned} \tag{I.69}$$

Although these equations are valid they are not useful for obtaining $T(z)$. The difficulty, which is related to the non-uniqueness discussed above, is that $T(z)$ is not compact, because it contains delta functions. The delta function contained in the operator $V_{12}G_0(z)$ persists in terms arising from higher iterates--e.g., $V_{12}G_0(z)\,V_{12}G_0(z)$ has the same delta function.

We shall now derive Faddeev's equations and show that the operator equations we obtain are compact. We begin by considering a decomposition of $T(z)$ that is suggested by (I.68). Define $\hat{T}_{ij}(z)$ as

$$\hat{T}_{ij}(z) = \delta_{ij}\,V_i - V_i\,G(z)\,V_j \ . \tag{I.71}$$

Clearly from (I.68) we have

$$T(z) = \sum_{ij} \hat{T}_{ij}(z) \ . \tag{I.72}$$

In order to transform (I.69) into an integral equation we need to express $G(z)V_j$ or $V_iG(z)$ in terms of the $\hat{T}_{ij}$. Let us start with $G(z)V_j$. Using the second

resolvent identity relating $G_0(z)$ and $G(z)$ equation (I. 58) gives us

$$\begin{aligned} G(z)V_j &= \left[G_0(z) - G_0(z)\ U\ G(z) \right] V_j \\ &= G_0(z)V_j - G_0(z) \sum_{i=1}^{3} V_i\ G(z)V_j \\ &= G_0(z) \sum_{i=1}^{3} \left[\delta_{ij}\ V_i - V_i\ G(z)\ V_j \right] \qquad \text{(I. 73)} \\ G(z)V_j &= G_0(z) \sum_{i=1}^{3} \hat{T}_{ij}(z) \ . \end{aligned}$$

For $V_i G(z)$ we use (I. 59) to find

$$\begin{aligned} V_i\ G(z) &= V_i \left[G_0(z) - G(z)\ U\ G_0(z) \right] \\ &= \sum_{j=1}^{3} \left[\delta_{ij}\ V_i - V_i\ G(z)\ V_j \right] G_0(z) \qquad \text{(I. 74)} \\ V_i\ G(z) &= \sum_{j=1}^{3} \hat{T}_{ij}(z)\ G_0(z) \ . \end{aligned}$$

Combining (I. 73) and (I. 71) we get an integral equation for $\hat{T}_{ij}$:

$$\hat{T}_{ij}(z) = \delta_{ij}\ V_i - V_i\ G_0(z) \sum_{k=1}^{3} \hat{T}_{ki}(z) . \qquad \text{(I. 75)}$$

Or alternately using (I. 72) leads to

$$\hat{T}_{ij}(z) = \delta_{ij}\ V_i - \sum_{k=1}^{3} \hat{T}_{ik}(z)\ G_0(z)\ V_j \qquad \text{(I. 76)}$$

If we iterate these equations once more we will find non-compact operators of the type $V_i G_0(z) V_i G_0(z)$, etc. We will now use the trick of Faddeev to eliminate such terms. The term $\hat{T}_{ij}(z)$ appears on both sides of (I. 75). Collecting both the $\hat{T}_{ij}(z)$ on the left we have

$$\left[1 + V_i G_0(z)\right] \hat{T}_{ij}(z) = \delta_{ij} V_i - V_i G_0(z) \sum_{k\neq i}^{3} \hat{T}_{kj}(z) \tag{I.77}$$

Multiplying by $\left(1 + V_i G_0(z)\right)^{-1}$ gives us

$$\hat{T}_{ij}(z) = \delta_{ij} \left[1 + V_i G_0(z)\right]^{-1} V_i - \left[1 + V_i G_0(z)\right]^{-1} V_i G_0(z) \sum_{k\neq i}^{3} \hat{T}_{kj}(z) \quad . \tag{I.78}$$

But $\left(1 - V_i G_0(z)\right)^{-1} V_i$ is just the three-body amplitude for the one two-body interaction described in Eq. (I. 49) through (I. 54). Thus (I. 75) simplifies to

$$\hat{T}_{ij}(z) = \delta_{ij} T_i(z) - T_i(z) G_0(z) \sum_{k\neq i}^{3} \hat{T}_{kj}(z) \tag{I.79}$$

Had we used (I. 76) instead of (I. 75) we would have been led to

$$\hat{T}_{ij}(z) = \delta_{ij} T_i(z) - \sum_{k\neq j}^{3} \hat{T}_{ik}(z) G_0(z) T_j \tag{I.80}$$

Now define T^i and $\bar{T}^i$ by

$$T^i(z) \equiv \sum_{j=1}^{3} \hat{T}_{ij}(z) \tag{I.81}$$

$$\bar{T}^i(z) \equiv \sum_{j=1}^{3} \hat{T}_{ji}(z) \tag{I.82}$$

By virtue of (I. 72) we have

$$T(z) = \sum_{i=1}^{3} T^{i}(z) = \sum_{i=1}^{3} \bar{T}^{i}(z) \tag{I. 83}$$

Summing (I. 79) over j gives us

$$T^{i}(z) = T_{i}(z) - T_{i}(z)\, G_{0}(z) \left[T^{j}(z) + T^{k}(z) \right] \tag{I. 84}$$

Summing (I. 80) over i and replacing the remaining j with i gives us

$$\bar{T}^{i}(z) = T_{i}(z) - \left[\bar{T}^{j}(z) + \bar{T}^{k}(z) \right] G_{0}(z)\, T_{i}(z) . \tag{I. 85}$$

Equation (I. 84) and Eq. (I. 85) are Faddeev's equations. Either of the two forms above are complete and differ only in the manner which the total transition amplitude is decomposed into three parts. The physical interpretation of these equations is simple. In (I. 84) the appearance of $T_{i}(z)$, the two-body scattering operator between particles j and k, as a common factor to the left of all other operators means that $T^{i}(z)$ is the portion of $T(z)$ in which the pair (jk) scatter first and then all possible scatterings happen. Analogously $\bar{T}^{i}(z)$ is that portion of the three-body scattering in which the pair (jk) scatter last after all other possible scatterings have occurred.

In order for our Faddeev equations to be integral equations they must have the property of closure. Consequently we must consider the unknown function to be the vector $\left\{ T^{1}(z),\ T^{2}(z),\ T^{3}(z) \right\}$, or $\left\{ \bar{T}^{1}(z),\ \bar{T}^{2}(z),\ \bar{T}^{3}(z) \right\}$. Thus Faddeev's equations are basically matrix operator equations. It is interesting to write out these equations in their matrix form.

$$\begin{pmatrix} T^1(z) \\ T^2(z) \\ T^3(z) \end{pmatrix} = \begin{pmatrix} T_1(z) \\ T_2(z) \\ T_3(z) \end{pmatrix} - \begin{pmatrix} 0 & T_1(z)G_0(z) & T_1(z)G_0(z) \\ T_2(z)G_0(z) & 0 & T_2(z)G_0(z) \\ T_3(z)G_0(z) & T_3(z)G_0(z) & 0 \end{pmatrix} \begin{pmatrix} T^1(z) \\ T^2(z) \\ T^3(z) \end{pmatrix}$$

(I. 86)

Now we want to establish that the square of Faddeev's matrix operator is a Schmidt operator. The square of the matrix in (I. 86) will be composed of finite sums of terms of the kind

$$T_i(z)\, G_0(z)\; T_j(z)\; G_0(z) \quad .$$

where $i \neq j$. This term may be rewritten as

$$T_i(z)G_0(z)\, T_j(z)G_0(z) = \left[1 + G_0(z)\, V_i\right]^{-1} V_i G_0(z)\, V_j\, G(z)$$

$$= \left[1 + G_0(z)\, V_i\right]^{-1} V_i\, G_0(z)\, V_j\, G_0(z)\, \left[1 + V_j G_0(z)\right]^{-1} \quad . \qquad \text{(I. 87)}$$

For the domain $\mathrm{Im}\sqrt{z} \geq \epsilon > 0$, the factors, $\left[1 + G_0(z)\, V_i\right]^{-1}$ and $\left[1 + V_j G_0(z)\right]^{-1}$, are bounded and analytic except for the two-body bound state poles. Thus it is sufficient to show that $V_i G_0(z) V_j G_0(z)$ is square integrable. By taking the expectation value between momentum states, using equations like (I. 42) and the expectation value of the resolvent $G_0(z)$,

$$\langle \vec{p}_i\, ,\, \vec{q}_{jk} \,\big|\, G_0(z) \,\big|\, \vec{p}_i', \vec{q}_{jk}' \rangle = \frac{\delta(\vec{p}_i - \vec{p}_i')\;\; \delta(\vec{q}_{jk} - \vec{q}_{jk}')}{\frac{\vec{p}_i^{\,2}}{2n_i} + \frac{\vec{q}_{jk}^{\,2}}{2\mu_{jk}} - z} \qquad \text{(I. 88)}$$

we may work out explicitly that $||V_i G_0(z) V_j G_0(z)||^2$ is finite for z not on the real axis. Thus the inverse of the identity plus Faddeev's matrix operator exists according to Theorem I. In fact, much stronger results hold than these. Rubin, Sugar and Tiktopoulos[24] have shown using Fredholm theory that the square of Faddeev's matrix kernel is Fredholm even on the real axis. This strong result should be contrasted with that of Faddeev who was only able to show that the fifth power of the kernel was compact. The reason Faddeev could not obtain the stronger result was that he assumed only that the potentials involved satisfied smoothness conditions whereas here and in the work of Lovelace it is assumed that the potential is analytic in its momentum variables.

D. Green's Functions and Wave Functions

We now want to turn to formulating Faddeev's equations for the exact resolvent or Green's function. From the Green's function formulation it is just a few steps to obtain the integral equations for three body wave functions. We start with the identity,

$$T(z)\ G_0(z) = U\ G(z) \tag{I.89}$$

which is proved just like (I.32). Thus the second resolvent equation for $G_0(z)$ and $G(z)$ may be recast as follows

$$\begin{aligned} G(z) &= G_0(z) - G_0(z)\ U\ G(z) \\ &= G_0(z) - G_0(z)\ T(z)\ G_0(z) \\ &= G_0(z) - \sum_{i=1}^{3} G_0(z)\ T^i(z)\ G_0(z) \end{aligned} \tag{I.90}$$

If we define an operator $G^i(z)$ to be

$$G^i(z) = G_0(z)\, T^i(z)\, G_0(z) \tag{I.91}$$

then we have the following decomposition of $G(z)$

$$G(z) = G_0(z) - \sum_{i=1}^{3} G^i(z) \tag{I.92}$$

We obtain Faddeev's equations for the Green's function if we pre and post multiply Eq. (I.84) by $G_0(z)$ and use Eq. (I.91). Doing this we have

$$\begin{aligned} G^i(z) &= G_0(z)\, T_i(z)\, G_0(z) - G_0(z)\, T_i(z)\, G_0(z) \left[T^j(z) + T^k(z)\right] G_0(z) \\ &= G_0(z)\, T_i(z)\, G_0(z) - G_0(z)\, T_i(z) \left[G^j(z) + G^k(z)\right] \end{aligned} \tag{I.93}$$

This may be further simplified by using the identity

$$\begin{aligned} G_i(z) &= G_0(z) - G_0(z)\, V_i\, G(z) \\ &= G_0(z) - G_0(z)\, T_i(z)\, G_0(z) \end{aligned} \tag{I.94}$$

Faddeev's-Green's function equations thus may be written

$$G^i(z) = G_0(z) - G_i(z) - G_0(z)\, T_i(z) \left[G^j(z) + G^k(z)\right] \tag{I.95}$$

We note that if we put Eq. (I.95) into matrix form that we have the same matrix operator written out explicitly in Eq. (I.86), with the one exception that $G_0(z)$ and $T_i(z)$ have been permuted.

Let us now transform our Green's function equation into one describing wave functions. Suppose our incoming state ϕ_n^3 is the n^{th} bound state between particles (12) and a plane wave state for the remaining particle. Thus we have

$$H_3 \, \phi_n^3 = (H_0 + V_i) \, \phi_n^3 = E_n \, \phi_n^3 \qquad \text{(I.96)}$$

and

$$\lim_{i\epsilon \to 0} \; -i\epsilon \; G_i \, (E_n + i\epsilon) \; \phi_n^3 = \delta_{i3} \; \phi_n^3 \qquad \text{(I.97)}$$

Applying (I.95) to ϕ_n^3, and taking the limiting process $\epsilon \to +0$ gives us the desired wave function. Recalling (I.65) the driving term becomes

$$-i\epsilon \left[G_0 \, (E_n + i\epsilon) - G_i \, (E_n + i\epsilon) \right] \; \Phi_{3n} = - \; \delta_{i3} \, \phi_n^3 \qquad \text{(I.98)}$$

By definition of the exact Green's function $G(E_n + i\epsilon)$, the outgoing three-body scattering state, Ψ_{3n}, is given by

$$-i\epsilon \; G(E_n + i\epsilon) \; \phi_n^3 = \Psi_{3n} \qquad \text{(I.99)}$$

Now the decomposition of the Green's function given in Eq. (I.90) leads to

$$\Psi_{3n} = \sum_{i=1}^{3} \Psi_{3n}^i, \quad \Psi_{3n}^i = +i\epsilon \; G^i(E_n + i\epsilon) \; \Phi_{3n} \qquad \text{(I.100)}$$

Putting all this together gives Faddeev's equation for Ψ_{3n}^i in the form

$$+ \Psi_{3n}^i = - \phi_n^3 \; \delta_{3i} - G_0(z) \; T_i(z) \left[+ \Psi_{3n}^j + \Psi_{3n}^k \right] \qquad \text{(I.101)}$$

We will complete our discussion of Faddeev's method by giving an example of a transition amplitude for a specific physical process. Suppose we want to calculate the amplitude for the rearrangement process

$$(1,\ 2) + 3 \rightarrow 1 + (2,3)$$

The incoming state is assured to be a bound state between particles 1 and 2, and the third particle has momentum $\vec{p}_3^{\,0}$. The final state is one in which particles 2 and 3 are bound and particle 1 has momentum $\vec{p}_1^{\,f}$. The three-body energy before and after scattering is defined as E. If $-\gamma_3^2/2\mu_{12}$ is the bound state energy of the incoming (1,2) system and $-\gamma_1^2/2\mu_{23}$ is the bound state energy of the outgoing (2,3) system then conservation of energy tells us

$$E = \frac{\vec{p}_3^{\,02}}{2n_3} - \frac{\gamma_3^2}{2\mu_{12}} = \frac{\vec{p}_1^{\,f2}}{2n_1} - \frac{\gamma_1^2}{2\mu_{23}} \qquad \text{(I.102)}$$

The incoming system asymptotically satisfies the Hamiltonian H_3. Let $|\psi_3(E, \vec{p}_3^{\,0})>$ be the ket describing the incoming state; then

$$H_3 \left| \psi_3(E, \vec{p}_3^{\,0})> = E \right| \psi_3(E, \vec{p}_3^{\,0})> \qquad \text{(I.103)}$$

Representing $|\psi_3(E, \vec{p}_3^{\,0})>$ in the three-body plane wave space, the wave-function is written

$$<\vec{q}_{12},\ \vec{p}_3 \left| \psi_3 (E, \vec{p}_3^{\,0})> = \delta(\vec{p}_3^{\,0} - \vec{p}_3)\ \ \phi_3(\vec{q}_{12}) \qquad \text{(I.104)}$$

where ϕ_3 is the two-body bound state wave function describing the γ_3 bound state. Likewise for the outgoing state we have

$$H_1 \left| \psi_1 (E, \vec{p}_1^{\,f})> = E \right| \psi_1 (E, \vec{p}_1^{\,f})> \qquad \text{(I.105)}$$

with

$$\langle \vec{q}_{23},\ \vec{p}_1 \mid \psi_1(E, \vec{p}_1^{\,f}) \rangle = \delta(\vec{p}_1 - \vec{p}_1^{\,f})\ \phi_1(\vec{q}_{23}) \qquad \text{(I.106)}$$

The transition amplitude may now be written

$$\langle \psi_3(E, \vec{p}_3^{\,0}) \mid T(E + i\epsilon) \mid \psi_1(E, \vec{p}_1^{\,f}) \rangle =$$

$$= \int d\vec{p}_3\, d\vec{q}_{12} \int d\vec{p}_1'\, d\vec{q}_{23}' \langle \psi_3(E, \vec{p}_3^{\,0}) \mid \vec{p}_3, \vec{q}_{12} \rangle \langle \vec{p}_3, q_{12} \mid T(E + i\epsilon) \mid \vec{p}_1'\, \vec{q}_{23}' \rangle \qquad \text{(I.107)}$$

$$\times \langle \vec{p}_1', \vec{q}_{23}' \mid \psi_1(E, \vec{p}_1^{\,f}) \rangle$$

$$= \int d\vec{q}_{12}\, d\vec{q}_{23}'\, \phi_3^*(\vec{q}_{12}) \langle \vec{p}_3^{\,0}, \vec{q}_{12} \mid T(E + i\epsilon) \mid \vec{p}_1^{\,f}, \vec{q}_{23}' \rangle \phi_1(\vec{q}_{23})$$

Thus in order to obtain this rearrangement scattering amplitude we need the two-body bound-state wave functions ϕ_3 and ϕ_1 as well as the three-body plane wave transition amplitude $\langle \vec{p}, \vec{q} \mid T(E + i\epsilon) \mid \vec{p}', \vec{q}' \rangle$.

The integral equation for $\langle \vec{p}, \vec{q} \mid T(E + i\epsilon) \mid \vec{p}', \vec{q}' \rangle$ is simply derived from our operator Eq. (I.84) by taking the plane wave expectation values and using the completeness relation for the plane wave states, Thus we have

$$\langle \vec{p}, \vec{q} \mid T^i(z) \mid \vec{p}', \vec{q}' \rangle = \langle \vec{p}, \vec{q} \mid T_i(z) \mid \vec{p}', \vec{q}' \rangle$$

$$- \int d\vec{p}''\, d\vec{q}'' \frac{\langle \vec{p}, \vec{q} \mid T_i(z) \mid \vec{p}''\, \vec{q}'' \rangle}{\frac{p''^2}{2n} + \frac{q''^2}{2\mu} - z} \langle \vec{p}'', \vec{q}'' \mid \left[T^j(z) + T^k(z) \right] \mid \vec{p}', \vec{q}' \rangle \qquad \text{(I.108)}$$

We can make this plane wave version more explicit by using the relationship, (I.54), between two and three-body amplitudes. Our plane wave equation

becomes

$$\langle \vec{p}_i, \vec{q}_{jk} \mid T^i(z) \mid \vec{p}_i', \vec{q}_{jk}' \rangle = \delta(\vec{p}_i - \vec{p}_i') \langle \vec{q}_{jk} \mid t_i \left(z - \frac{p_i^2}{2n_i} \right) \mid \vec{q}_{jk}' \rangle$$

$$- \int d\vec{p}_i'' \, d\vec{q}_{jk}'' \; \frac{\delta(\vec{p}_i - \vec{p}_i'') \langle \vec{q}_{jk} \mid t_i \left(z - \frac{p_i^2}{2n_i} \right) \mid \vec{q}_{jk}'' \rangle}{\frac{p''^2}{2n} + \frac{q''^2}{2\mu} - z} \langle \vec{p}_i'' \vec{q}_{jk}'' \mid T^j(z) + T^k(z) \mid \vec{p}_i' \, \vec{q}_{jk}' \rangle \quad - \quad (I.109)$$

This last form makes explicit the fact that the principal ingredient contained in the kernel for Faddeev's integral equation is the off-shell two-body transition amplitude $\langle \vec{q}_{jk} \mid t_i \, (z - p_i^2/2n_i) \mid \vec{q}_{jk}'' \rangle$. Once we have obtained this kernel and have solved (I.109) then we have solved all three-body physics for the three particles interacting through potentials v_i. It should be emphasized that as Eq. (I.109) stands it is rather difficult to obtain even numerical solutions. The equation is an integral equation in six real variables $(\vec{p}_i, \vec{q}_{jk})$ and has seven variables $(z, \vec{p}_i', \vec{q}_{jk}')$ which enter parametrically.

CHAPTER II

ANGULAR MOMENTUM REDUCTION OF OMNES

In this section we turn our attention to the angular momentum reduction of Faddeev's equations. The benefits which accrue from this type of analysis for the three-body problem are similar to those which result from the partial wave reduction of the two-body problem. The plane wave representation of the two-body Lippman-Schwinger equations is given in (I. 36). The scattering amplitude $\langle \vec{q} \,|t(z)|\, \vec{q}' \rangle$ is a function of seven continuous variables, four of which are parameters and three of which are variables of integration. When the angular momentum reduction is completed the four variables describing the directions of $\vec{q}$ and $\vec{q}'$ have been replaced by four discrete indices ℓ, ℓ', m, m'. Furthermore because the angular momentum operator $\vec{L}$ commutes with h, the Hamiltonian of the two-body system, the amplitude is diagonal with respect to the indices ℓ and ℓ', and m and m'. Hence we are left with an integral equation in one radial variable, and a parametric dependence on ℓ, z and q'. In the two-body problem we are even able to sum out the m dependence. Thus our motive in doing the angular momentum reduction of the three-body problem is to eliminate and diagonalize as much of the thirteen variables in $\langle \vec{p}, \vec{q} \,|T(t)|\, \vec{p}', \vec{q}' \rangle$ as is possible. Our final result given in Chapter IV will be to reduce Faddeev's equations to a sum integral equations in just two variables. The first stage in our reduction follows closely the angular momentum analysis given by Omnes.[9] We will pursue this analysis in some detail, because of its importance to our later results and because it is more subtle than angular momentum reductions in the two-body problem.

Looking at our three-body transition amplitude, $\langle \vec{p}, \vec{q} \,|\, T(z) \,|\, \vec{p}', \vec{q}' \rangle$, the angular decomposition immediately suggested is to expand $|\vec{p}', \vec{q}' \rangle$ in terms of the two radial variables p, q and four angular variables described by $\theta_p, \phi_p, \theta_q, \phi_q$.

The angular space would then be spanned by sums of $Y_{\ell m}\left(\theta_p, \phi_p\right) Y_{\ell' m'}\left(\theta_q, \phi_q\right)$. This approach has been carried out.[11] One should note that this procedure is not symmetric in the following sense. There are three equivalent choices of $(\vec{p}, \vec{q})$. Each choice leads to a different, but complete, set of four angular variables.

A. Eigenstate Basis and Completeness

In contradistinction to the above approach let us write the transition amplitude in the symmetric form $<\vec{p}_1, \vec{p}_2, \vec{p}_3 \left| T(z) \right| \vec{p}'_1, \vec{p}'_2, \vec{p}'_3>$, where we recall that only two of the three particle momenta are independent because of total momentum conservation.

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{p}'_1 + \vec{p}'_2 + \vec{p}'_3 = 0 \qquad \text{(II.1)}$$

Geometrically speaking the state $\left| \vec{p}'_1, \vec{p}'_2, \vec{p}'_3 \right>$ may be thought of as a triangle, with a $\vec{p}_i$ for each side. Thus $<\vec{p}_1, \vec{p}_2, \vec{p}_3 \left| T(z) \right| \vec{p}'_1, \vec{p}'_2, \vec{p}'_3>$ may be interpreted as giving the probability that one triangle $\left(\vec{p}_1, \vec{p}_2, \vec{p}_3\right)$ will scatter into another $\left(\vec{p}'_1, \vec{p}'_2, \vec{p}'_3\right)$. Now let us describe these triangles by their shape and by their spacial orientation. The shape is completely specified by lengths of the three sides p_1, p_2, p_3. Once the shape of the triangle is fixed it is a rigid body whose spacial orientation can be described by three Euler angles. Here and in what follows we will use the definitions given by Edmonds[25] for the Euler angles. Let (ψ, θ, ϕ) be the three Euler angles describing some body-fixed axis in the triangle. Defining the independent particle energy, ω_i, to be

$$\omega_i = \frac{p_i^2}{2m_i} \qquad i = 1, 2, 3$$

we can choose our six independent variables to be $\left(\omega_1, \omega_2, \omega_3, \psi, \theta, \phi\right)$. It is this symmetric coordinate system in which we shall perform the angular

momentum reduction. With this change of variables our plane wave state may be written as

$$|\vec{p}, \vec{q}> = |\vec{p_1}, \vec{p_2}, \vec{p_3}> = |\omega_1, \omega_2, \omega_3, \psi, \theta, \phi> . \qquad \text{(II. 2)}$$

An alternate complete set of six commuting operators is the three particle energies $\omega_1, \omega_2, \omega_3$ the square of the total angular momentum $J^2 = J(J+1)$, the projection J_z of $\vec{J}$ on the spacially fixed z axis and the projection J_λ of $\vec{J}$ on to the body-fixed axis. In terms of the quantum numbers of these operators our wave function may be written

$$|\omega_1, \omega_2, \omega_3, \ J, \ M, \ \lambda >$$

In order to evaluate operator products we need to know how to transform in the three-body Hilbert space from the basis $|\vec{p_1}, \vec{p_2}, \vec{p_3}>$ to the basis $|\omega_1, \omega_2, \omega_3, J, M, \lambda >$ Since the rotation matrix element $\mathcal{D}^J_{M\lambda}(\psi, \theta, \phi)$ is just the eigenfunction of J^2, J_z and J_λ on the angular space (ψ, θ, ϕ) we have

$$<\vec{p_1}, \vec{p_2}, \vec{p_3}|\omega_1, \omega_2, \omega_3, \ J, \ M, \lambda> = A \prod_{i=1}^{3} \delta\left(\frac{p_i^2}{2m_i} - \omega_i\right) \mathcal{D}^{*J}_{\lambda M}(\psi, \theta, \phi) \qquad \text{(II. 3)}$$

where A is some scalar to be determined. Once we fix our normalizations for the states $|\vec{p_1}, \vec{p_2}, \vec{p_3}>$ and $|\omega_1, \omega_2, \omega_3, \ J, \ M, \lambda>$ we can determine A. We choose

$$<\vec{p_1}, \vec{p_2}, \vec{p_3} | \vec{p_1'}, \vec{p_2'}, \vec{p_3'}> = \delta(\vec{p_i} - \vec{p_i'}) \ \delta(\vec{p_j} - \vec{p_j'}) \qquad \text{(II. 4)}$$

where $\vec{p}_i$ and $\vec{p}_j$ are an arbitrary choice of the two independent momentum vectors. Normalization for $|\omega_1, \omega_2, \omega_3, J, M, \lambda>$ is taken to be

$$<\omega_1, \omega_2, \omega_3, J, M, \lambda | \omega_1', \omega_2', \omega_3', J', M', \lambda' > =$$

$$\prod_{i=1}^{3} \delta(\omega_i - \omega_i') \, \delta_{JJ'} \, \delta_{MM'} \, \delta_{\lambda\lambda'} \qquad \text{(II. 5)}$$

With these conventions we now determine A. We will expand (II. 5) by using the completeness of the momentum states. Thus (II. 5) may be written

$$\int d\vec{p}_1 \, d\vec{p}_2 <\omega_1, \omega_2, \omega_3, J, M, \lambda \, | \, \vec{p}_1 \vec{p}_2 \vec{p}_3 >< \vec{p}_1 \vec{p}_2 \vec{p}_3 \, | \, \omega_1', \omega_2', \omega_3', J', M', \lambda' >$$

$$= \int d\vec{p}_1 \, d\vec{p}_2 \prod_{i=1}^{3} \delta\left(\omega_i - \frac{p_i^2}{2m_i}\right) \delta\left(\omega'_i - \frac{p_i^2}{2m_i}\right) |A|^2 \qquad \text{(II. 6)}$$

$$\times \mathcal{D}^{J}_{\lambda M}(\psi, \theta, \phi) \mathcal{D}^{*J'}_{\lambda' M'}(\psi, \theta, \phi)$$

This integral can easily be done if the variables of integration are changed to $p_1^2/2m_1$, $p_2^2/2m_2$, $p_3^2/2m_3$, ψ, θ, ϕ. For $\vec{p}_1$ and $\vec{p}_2$ restricted to lie in the plane defined by ψ, θ, ϕ we can write

$$d\vec{p}_1 \, d\vec{p}_2 = m_1 \, m_2 \, m_3 \, d\left(\frac{p_1^2}{2m_1}\right) d\left(\frac{p_2^2}{2m_2}\right) d\left(\frac{p_3^2}{2m_3}\right) dR \qquad \text{(II. 7)}$$

where dR is the measure of the rotation group

$$dR = \sin\theta \, d\theta \, d\psi \, d\phi \qquad \text{(II. 8)}$$

and

$$\int dR = 8\pi^2 = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \qquad \text{(II. 9)}$$

Thus (II. 6) may be simplified to

$$|A|^2 m_1 m_2 m_3 \prod_{i=1}^{3} \delta(\omega_i - \omega_i') \int dR \; \mathcal{D}^{J}_{\lambda M}(R) \, \mathcal{D}^{*J'}_{\lambda' M'}(R) \qquad \text{(II. 10)}$$

Using the orthogonality relation for the $\mathcal{D}$ functions gives us

$$|A|^2 \; m_1 m_2 m_3 \prod_{i=1}^{3} \delta(\omega_i - \omega_i') \; \delta_{JJ'} \, \delta_{MM'} \, \delta_{\lambda\lambda'} \, \frac{8\pi^2}{(2J+1)} \qquad \text{(II. 11)}$$

Comparing (II. 11) with (II. 5) gives us $|A|^2$

$$|A|^2 = \frac{2J+1}{8\pi^2 m_1 m_2 m_3} \qquad \text{(II. 12)}$$

We must now specify the completeness relation for the basis $|\omega_1, \omega_2, \omega_3, J, M, \lambda>$. The completeness relation has the form

$$I = \int d\omega_1 \, d\omega_2 \, d\omega_3 \sum_{JM\lambda} |\omega_1, \omega_2, \omega_3, J, M, \lambda><\omega_1, \omega_2, \omega_3, J, M, \lambda| \times X(\omega_1, \omega_2, \omega_3). \qquad \text{(II. 13)}$$

The undetermined factor $X(\omega_1, \omega_2, \omega_3)$ is the density of states factor. By using the normalization condition (II. 5) we may calculate X. Multiplying

(II.13) by $\langle\omega_1',\omega_2',\omega_3',\ J,\ M,\lambda|$ gives us

$$\langle\ \omega_1',\omega_2',\omega_3',J',\ M',\ \lambda'| \ = \int d\omega_1\ d\omega_2\ d\omega_3 \sum_{JM\lambda}\ \prod_{i=1}^{3} \delta(\omega_i-\omega_i')\,\delta_{JJ'}\ \delta_{MM'}\ \ \delta_{\lambda\lambda'}$$

(II.14)

$$\langle\omega_1,\omega_2,\omega_3,\ J,\ M,\lambda\ |\ X(\omega_1,\omega_2,\omega_3)$$

Thus,

$$\langle\omega_1',\omega_2',\omega_3',\ J',\ M',\ \lambda'\ \Big| = \langle\omega_1',\omega_2',\omega_3',\ J',\ M',\ \lambda'\Big|\ X(\omega_1',\omega_2',\omega_3')$$

(II.15)

and consequently $X \equiv 1$.

Now let us express Faddeev's equations in our new basis $|\omega_1,\omega_2,\omega_3,\ J,\ M,\lambda\rangle$. If no confusion results we shall abbreviate the notation for $|\omega_1,\omega_2,\omega_3,\ J,\ M,\lambda\rangle$ to $|\vec{\omega}\ J,M,\lambda\rangle$. Taking the expectation value of Faddeev's operator Eq. (I.84) between states $\langle\vec{\omega}'\ J,\ M,\ \lambda'|$ and $|\vec{\omega},\ J,\ M,\lambda\rangle$ gives

$$\langle\vec{\omega}'\ J,\ M,\ \lambda'\ |\ T^i(z)\big|\vec{\omega},\ J,\ M,\lambda\rangle\ =\ \langle\vec{\omega}'\ J,\ M,\lambda'\ \Big|\ T_i(z)\Big|\ \vec{\omega}\ J,\ M,\lambda\rangle$$

(II.16)

$$-\langle\vec{\omega}'\ J,\ M,\lambda'\ |\ \Big\{\ T_i(z)\ G_0(z)\ \Big[T^j(z)+T^k(z)\Big]\Big\}\Big|\vec{\omega}\ J,\ M,\lambda\rangle\ \ .$$

We have used the same J and M indices, in the initial and final state because J^2 and J_z both commute with the total Hamiltonian, H, and from the definition of the three-body transition operator, (I.68), it follows immediately that T(z) is diagonal in J and M. The above equation is turned into an integral equation by inserting a complete set of states between $T_i(z)$ and $G_0(z)$, and between $G_0(z)$

and $[T^j(z) + T^k(z)]$. The expectation value of $G_0(z)$ is very simple in the basis $|\vec{\omega}, J, M,\lambda>$ because all the operators J^2, J_z, J_λ commute with each other and H_0 so that

$$<\vec{\omega}', J', M', \lambda' \mid G_0(z) \mid \vec{\omega}, J, M, \lambda> = \frac{\left[\prod_{i=1}^{3} \delta(\omega_i - \omega_i')\right] \delta_{JJ'} \delta_{MM'} \delta_{\lambda\lambda'}}{\omega_1 + \omega_2 + \omega_3 - z} \tag{II.17}$$

Our integral equation will now be

$$<\vec{\omega}', J, M, \lambda' \mid T^i(z) \mid \vec{\omega}, J, M,\lambda> = <\vec{\omega}', J, M, \lambda' \mid T_i(z) \mid \vec{\omega}, J, M,\lambda>$$

$$- \sum_{\lambda''=-J}^{J} \int d\vec{\omega}'' \frac{<\vec{\omega}', J, M, \lambda \mid T_i(z) \mid \vec{\omega}'', J, M, \lambda''>}{\omega_1'' + \omega_2'' + \omega_3'' - z} \tag{II.18}$$

$$<\vec{\omega}'', J, M, \lambda'' \mid T^j(z) + T^k(z) \mid \vec{\omega}, J, M,\lambda>$$

This will be a perfectly good integral equation once we have determined the driving term and the kernel in terms of known two-body functions.

B. Three Body Coordinate Systems and Geometry.

We now want to re-express $<\vec{\omega}', J, M, \lambda' \mid T_i(z) \mid \vec{\omega}, J, M, \lambda>$ by transforming to the plane wave momentum base, where $T_i(z)$ is the solution to the two-particle Lippman-Schwinger equation and therefore a known function. In order to carry out all the integrals we shall encounter, we need to specify in detail the geometry of our three-body coordinate systems. As above the kinetic energy of the i^{th} particle is ω_i, its momentum $\vec{p}_i$. Thus

$$\vec{p}_i^{\,2} = 2m_i \omega_i \tag{II.19}$$

The relative momentum of particle i with respect to particle j is defined as $\vec{q}_{ij}$, where

$$\vec{q}_{ij} = \frac{m_j \vec{p}_i - m_i \vec{p}_j}{m_i + m_j} \tag{II.20}$$

Defining θ_{ij} as the angle between $\vec{p}_i$ and $\vec{p}_j$, this angle is determined solely as a function of $\omega_1, \omega_2, \omega_3$. Explicity we have

$$\vec{p}_k^{\,2} = (\vec{p}_i + \vec{p}_j)^2 = \vec{p}_i^{\,2} + \vec{p}_j^{\,2} + 2\,\vec{p}_i \cdot \vec{p}_j \tag{II.21}$$

$$\cos\theta_{ij} = \frac{2m_k\omega_k - 2m_i\omega_i - 2m_j\omega_j}{2\sqrt{2m_i\omega_i\, 2m_j\omega_j}}$$

$$= \frac{m_k\omega_k - m_i\omega_i - m_j\omega_j}{2\sqrt{m_i m_j\ \omega_i\ \omega_j}} \tag{II.22}$$

In addition we shall need the angle, γ_i, between $\vec{p}_i$ and $\vec{q}_{jk}$ in terms of $\omega_1, \omega_2, \omega_3$.

$$\cos\gamma_i = \frac{\vec{p}_i \cdot \vec{q}_{jk}}{p_i\, q_{jk}}$$

$$\vec{p}_i \cdot \vec{q}_{jk} = \frac{(m_j + m_k)(m_k\omega_k - m_j\omega_j) + (m_j - m_k)\, m_i\omega_i}{(m_j + m_k)} \tag{II.23}$$

For brevity we have omitted the algebra leading to Eq. (II.23). The only trick needed to derive this formula is just to replace all dot products of the type $\vec{p}_i \cdot \vec{p}_j$ by using (II.21). The square of the length of $\vec{q}_{ij}$ is also determined by

the coordinates. Once more we have, omitting simple algebra

$$q_{ij}^2 = \frac{2m_i m_j}{m_i+m_j}(\omega_1 + \omega_2 + \omega_3) - \frac{2m_i m_j (m_1 + m_2 + m_3)\omega_k}{(m_i + m_j)^2} \qquad \text{(II.24)}$$

We complete the list of the required kinematic formulae by writing out the expressions for $\sin^2\theta_{ij}$ and $\sin^2\gamma_i$. For convenience define $\lambda(\omega_1, \omega_2, \omega_3)$ to be

$$\lambda(\omega_1,\omega_2,\omega_3) \equiv m_1^2\omega_1^2 + m_2^2\omega_2^2 + m_3^2\omega_3^2 - 2m_1 m_2 \omega_1\omega_2 - 2m_1 m_3 \omega_1\omega_3$$

$$- 2m_2 m_3 \omega_2\omega_3 \qquad \text{(II.25)}$$

Then

$$\sin^2\theta_{ij} = -\frac{\lambda(\omega_1, \omega_2, \omega_3)}{4m_i m_j \omega_i\omega_j} \qquad \text{(II.26)}$$

and

$$\sin^2\gamma_i = \frac{-\lambda(\omega_1, \omega_2, \omega_3)}{\frac{4m_1 m_2 m_3\omega_i}{(m_j + m_k)}\left(\omega_k + \omega_j - \frac{m_i}{(m_j+m_k)}\omega_i\right)} \qquad \text{(II.27)}$$

The forms of $\sin^2\theta_{ij}$ and $\sin^2\gamma_i$ show that they have coincident zeros. This is just an expression of the fact that when $\vec{p}_i$ and $\vec{p}_j$ are parallel ($\sin\theta_{ij} = 0$) then $\vec{q}_{ij}$ will be in the same direction as $\vec{p}_k$ so that $\sin\gamma_i$ is zero.

C. Evaluation of Transition Amplitudes in New Basis

We now resume our evaluation of $\langle\vec{\omega}', J, M, \lambda' \mid T_i(z) \mid \vec{\omega}, J, M \lambda\rangle$. For explicitness let $i = 1$. Introducing a complete set of plane wave momentum

states leads to

$$\langle \vec{\omega}', J, M, \lambda' \mid T_1(z) \mid \vec{\omega}, J, M, \lambda \rangle \equiv \langle T_1 \rangle =$$

$$= \int d\vec{p}_2 \; d\vec{p}_3 \int d\vec{p}'_2 \; d\vec{p}'_3 \langle \vec{\omega}', J, M, \lambda' \mid \vec{p}_2 \, \vec{p}_3 \rangle$$

$$\langle \vec{p}_2 \, \vec{p}_3 \mid T_1(z) \mid \vec{p}'_2 \, \vec{p}'_3 \rangle \langle \vec{p}'_2 \, \vec{p}'_3 \mid \vec{\omega}, J, M, \lambda \rangle \qquad \text{(II.28)}$$

Changing the variables of integration according to Eq. (II.7), using (II.3), and relating T_1 to t_1 via (I.54) allows us to rewrite (II.28) as

$$\langle T_1 \rangle = (m_1 m_2 m_3)^2 \, |A|^2 \int dR \; d\left(\frac{p_1^2}{2m_1}\right) \; d\left(\frac{p_2^2}{2m_2}\right) \; d\left(\frac{p_3^2}{2m_3}\right) \int dR' \; d\left(\frac{p_1'^2}{2m_1}\right)$$

$$d\left(\frac{p_2'^2}{2m_2}\right) \; d\left(\frac{p_3'^2}{2m_3}\right) \prod_{i=1}^{3} \delta\left(\omega_i' - \frac{p_i'^2}{2m_i}\right) \prod_{j=1}^{3} \delta\left(\omega_j - \frac{p_j^2}{2m_j}\right) \qquad \text{(II.29)}$$

$$\mathcal{D}^{J}_{\lambda' M}(R) \, \mathcal{D}^{J*}_{\lambda M}(R') \; \delta(\vec{p}_1 - \vec{p}'_1) \; \langle \vec{q}_{23} \mid t_1 \left(z - \frac{p_1^2}{2n_1}\right) \mid \vec{q}'_{23} \rangle$$

The only complicated factor in this integral is $\delta(\vec{p}_1 - \vec{p}'_1)$, since the ingredient $\vec{p}_1$ and $\vec{p}_2$ depend on the Euler angles in R and R'. If we choose the body-fixed z axis the lie along the direction of the momentum of particle 1 then ψ is polar angle with respect to the space-fixed z axis and θ is the azimuthal angle.

Thus we may use the representation[26] of the delta function in spherical coordinates to get

$$\delta(\vec{p}_1 - \vec{p}_1') = \frac{1}{p_1^2} \delta(p_1 - p_1') \delta(\cos\theta - \cos\theta') \delta(\psi - \psi')$$

$$= \frac{1}{m_1 p_1} \delta(\omega_1 - \omega_1') \delta(\cos\theta - \cos\theta') \delta(\psi - \psi') \qquad \text{(II. 30)}$$

Choosing our body-fixed axis along the direction of the first particle specifies exactly the operator J_λ. We denote this choice of J_λ by J_{λ_1}. Clearly

$$J_{\lambda_1} \mathscr{D}^J_{\lambda_1 M}(R) = \lambda_1 \mathscr{D}^J_{\lambda_1 M}(R) \qquad \text{(II. 31)}$$

For this specific coordinate system we may use (II. 30) to evaluate (II. 29). We have

$$\langle \vec{\omega}', J, M, \lambda_1' \mid T_1(z) \mid \vec{\omega}, J, M, \lambda_1 \rangle =$$

$$(m_1 m_2 m_3)^2 |A|^2 \frac{\delta(\omega_1 - \omega_1')}{m_1\sqrt{2m_1\omega_1}} \int d\phi \, d\cos\theta \, d\psi \int d\phi' \, d\cos\theta' \, d\psi' \, \delta(\psi - \psi') \qquad \text{(II. 32)}$$

$$\times \delta(\cos\theta - \cos\theta') \mathscr{D}^J_{\lambda_1' M}(\phi, \theta, \psi) \mathscr{D}^{J*}_{\lambda_1 M}(\phi', \theta', \psi')$$

$$\times \langle \vec{q}_{23} \mid t_1\left(z - \frac{m_1}{n_1}\omega_1\right) \mid \vec{q}_{23}' \rangle$$

Integrating out the delta functions and using our formula (II. 12) for $|A|^2$ gives us

$$\langle T_1 \rangle = \frac{(2J+1)}{8\pi^2} \frac{m_1 m_2 m_3}{m_1\sqrt{2m_1\omega_1}} \delta(\omega_1 - \omega_1') \int d\phi \, d\phi' \, d\cos\theta \, d\psi \qquad \text{(II. 33)}$$

$$\langle \vec{q}_{23} \mid t_1\left(z - \frac{m_1}{n_1}\omega_1\right) \mid \vec{q}_{23}' \rangle \mathscr{D}^J_{\lambda_1' M}(\phi, \theta, \psi) \mathscr{D}^{J*}_{\lambda_1 M}(\phi', \theta, \psi)$$

The ϕ dependence of $\mathscr{D}^{J}_{\lambda M}$ is a simple exponential. In fact

$$\begin{aligned}\mathscr{D}^{J*}_{\lambda M}(\phi', \theta, \psi) &= e^{-i\lambda\phi'}\, d^{J}_{\lambda M}(\theta)\, e^{-iM\psi} \\ &= e^{i\lambda(\phi-\phi')}\, e^{-i\lambda\phi}\, d^{J}_{\lambda M}(\theta)\, e^{-iM\psi} \\ &= e^{i\lambda(\phi-\phi')}\mathscr{D}^{J*}_{\lambda M}(\phi,\theta,\psi)\end{aligned} \qquad \text{(II. 34)}$$

Because we choose our body-fixed axis to lie along the $\vec{p}_1$ direction, this direction is uniquely determined by θ, ψ. Furthermore the only dependence on the Euler angles $\langle \vec{q}_{23} \left| t_1 \left(z - \frac{m_1}{n_1}\omega_1\right)\right| \vec{q}'_{23}\rangle$ will have is on the difference $\phi-\phi'$. The reason for this is as we scatter $\vec{q}'_{23} \to \vec{q}_{23}$, the plane of momentum triangle described by $(\vec{p}_2\vec{p}_3)$ is just rotated by $\phi-\phi'$ about the fixed $\vec{p}_1$ into the plane of $(\vec{p}'_2\vec{p}'_3)$. Thus we can do all the integrations with the exception of one over $\phi-\phi'$. Defining $u = \phi-\phi'$ we have, using (II. 34)

$$\langle T_1\rangle = \frac{m_1m_2m_3}{m_1\sqrt{2m_1\omega_1}}\,\delta(\omega_1-\omega'_1)\left[\frac{(2J+1)}{8\pi^2}\int dR\, \mathscr{D}^{J}_{\lambda'_1 M}(R)\mathscr{D}^{J*}_{\lambda_1 M}(R)\right] \qquad \text{(II. 35)}$$

$$\times \int_0^{2\pi} du\; e^{i\lambda_1 u}\, \langle \vec{q}_{23} \left| t_1\left(z - \frac{m_1}{n_1}\right)\right| \vec{q}'_{23}\rangle$$

By the definition for orthogonality for the $\mathscr{D}$ functions the term in the square brackets is $\delta_{\lambda_1\,\lambda'_1}$. For this choice of the body-fixed axis we have the result that

$$\langle \vec{\omega}', J, M, \lambda'_1 \left| T_1(z)\right| \vec{\omega}, J, M, \lambda_1\rangle = \frac{m_2m_3}{\sqrt{2m_1\omega_1}}\,\delta(\omega_1-\omega'_1)\,\delta_{\lambda'_1\lambda_1}\int_0^{2\pi} du\, e^{i\lambda_1 u} \qquad \text{(II. 36)}$$

$$\langle \vec{q}_{23} \left| t_1\left(z - \frac{m_1}{n_1}\omega\right)\right| \vec{q}'_{23}\rangle .$$

Although we have found an explicit expression for $\langle\vec{\omega}', J, M, \lambda_1' \mid T_1(z) \mid \vec{\omega}, J, M, \lambda_1\rangle$ in terms of two-body amplitudes we have not completely solved the problem. Once we have made the choice of a particular coordinate system we must use it consistently through the problem. The method described above would not work for $\langle\vec{\omega}, J, M, \lambda_1' \mid T_2(z) \mid \vec{\omega}, J, M, \lambda_1\rangle$. However, we may perform a rotation on $|\vec{\omega}, J, M, \lambda_1\rangle$ in order to make the result valid for a more general reference frame than having the fixed-body axis lying along the $\vec{p}_1$ direction.

The properties of the states $|\vec{\omega}, J, M, \lambda\rangle$ under a rotation of coordinate system follow from the properties of the $\mathscr{D}$ functions and the definition of $|\vec{\omega}, J, M, \lambda\rangle$ via (II.3). We will now show how to do such a rotation. The rotation we have in mind is the rotation of one body-fixed axis into another different body-fixed axis. The spacially fixed coordinate system remains constant and thus the operators J^2 and J_z are the same and so are their eigenvalues J and M. The operator that is changed is J_λ --the projection of the angular momentum on the body-fixed axis. Let the first coordinate system have Euler angles $(\psi_1, \theta_1, \phi_1)$. If this coordinate system is transformed by a finite rotation (α, β, γ) into the coordinate system $(\psi_2, \theta_2, \phi_2)$ the group property of the $\mathscr{D}$ functions states[27]

$$\mathscr{D}^{*J}_{\lambda_2 M}(\psi_2, \theta_2, \phi_2) = \sum_{\lambda_1'' = -J}^{J} \mathscr{D}^{*J}_{\lambda_2 \lambda_1''}(\alpha, \beta, \gamma) \quad \mathscr{D}^{*J}_{\lambda_1'' M}(\psi_1, \theta_1, \phi_1) \tag{II.37}$$

Multiplying this by $A \prod_{i=1}^{3} \delta\left(\frac{p_i^2}{2m_i} - \omega_i\right)$ and using Eq. (II.3) we have at once

$$\langle\vec{p}_1 \vec{p}_2 \vec{p}_3 \mid \vec{\omega}, J, M, \lambda_2\rangle = \sum_{\lambda_1'' = -J}^{J} \mathscr{D}^{*J}_{\lambda_2 \lambda_1''}(\alpha, \beta, \gamma) \langle\vec{p}_1 \vec{p}_2 \vec{p}_3 \mid \vec{\omega}, J, M, \lambda_1''\rangle \tag{II.38}$$

Since the state $<\vec{p}_1\vec{p}_2\vec{p}_3|$ is an arbitrary state in the three-body momentum space, it follows that

$$|\vec{\omega},\ J,\ M,\lambda_2> \ = \ \sum_{\lambda_1'' = -J}^{J} \mathscr{D}^{*J}_{\lambda_2\lambda_1''}\ (\alpha,\beta,\gamma)\,|\vec{\omega},\ J,\ M,\lambda_1''> \qquad \text{(II.39)}$$

Let us specialize this somewhat by choosing $(\psi_1, \theta_1, \phi_1)$ be the coordinate system with the body-fixed z axis along $\vec{p}_1$ and with x axis lying in the plane of $\vec{p}_1,\vec{p}_2,\vec{p}_3$.[28] Suppose (ψ_2,θ_2,ϕ_2) is similarly defined except that $\vec{p}_2$ is the body-fixed axis. For θ_{12} the angle between $\vec{p}_1$ and $\vec{p}_2$, the rotation needed to transform (ψ_1,θ_1,ϕ_1) into (ψ_2,θ_2,ϕ_2) is just a rotation about the y axis by the amount θ_{12}. Thus in this case

$$\mathscr{D}^{*J}_{\lambda_2\lambda_1''}\ (\alpha,\beta,\gamma) \ = \ \mathscr{D}^{*J}_{\lambda_2\lambda_1''}\ (0,\theta_{12},\ 0) \ = \ d^{J}_{\lambda_2\lambda_1''}\ (\theta_{12}) \qquad \text{(II.40)}$$

Our transformation (II.39) then becomes

$$|\vec{\omega},\ J,\ M,\lambda_2> \ = \ \sum_{\lambda_1''}\ d^{J}_{\lambda_2\lambda_1''}\ (\theta_{12})\,|\vec{\omega},\ J,\ M,\lambda_1''> \qquad \text{(II.41)}$$

Here it should be noted that θ_{12} depends only on $\vec{\omega}$. The companion relationship of Eq. (II.41) for the ket state is just given by taking the adjoint. Because $d^{J}_{\lambda_2\lambda_1''}$ is a real function we have

$$<\vec{\omega},\ J,\ M,\lambda_2| \ = \sum_{\lambda_1''} <\vec{\omega},\ J,\ M,\lambda_1''\ |\ d^{J}_{\lambda_1''\lambda_2}\ (\theta_{12}) \qquad \text{(II.42)}$$

Now we can write out $\langle\vec{\omega}', J, M, \lambda_1' \mid T_j(z) \mid \vec{\omega}, J, M, \lambda_1\rangle$ for any $j = 1, 2, 3$.

$$\langle\vec{\omega}', J, M, \lambda_1' \mid T_j(z) \mid \vec{\omega}, J, M, \lambda_1\rangle =$$

$$= \sum_{\lambda_j'' \lambda_j'''} \langle\vec{\omega}', J, M, \lambda_1' \mid \vec{\omega}', J, M, \lambda_j''\rangle \langle\vec{\omega}', J, M, \lambda_j'' \mid T_j(z) \mid \vec{\omega}, J, M, \lambda_j'''\rangle \tag{II.43}$$

$$\langle\vec{\omega}, J, M, \lambda_j''' \mid \vec{\omega}, J, M, \lambda_1\rangle$$

Using (II.41) and (II.42) this simplifies to

$$= \sum_{\lambda_j'' \lambda_j'''} d^J_{\lambda_1' \lambda_j''}(\theta_{1j}') \; d^J_{\lambda_j''' \lambda_1}(\theta_{1i}) \langle\vec{\omega}', J, M, \lambda_j'' \mid T_j(z) \mid \vec{\omega}, J, M, \lambda_j'''\rangle \tag{II.44}$$

Further simplification results from the fact that T_j is diagonal in λ_j. Using $\theta_{j1} = -\theta_{1j}$ we find,

$$\langle\vec{\omega}', J, M, \lambda_1' \mid T_j(z) \mid \vec{\omega}, J, M, \lambda_1\rangle = \sum_{\lambda_j''} d^J_{\lambda_1' \lambda_j''}(\theta_{1j}') \; d^J_{\lambda_j'' \lambda_1}(-\theta_{1j}) \tag{II.45}$$

$$\times \langle\vec{\omega}', J, M, \lambda_j'' \mid T_j(z) \mid \vec{\omega}, J, M, \lambda_j''\rangle \; .$$

This last equation allows us to find $T_j(z)$ for all j in terms known two-body functions via (II.36).

D. Introduction of Two Body Partial Wave Expansion.

Our formula, (II.36), for $\langle\vec{\omega}', J, M, \lambda_1' \mid T_i(z) \mid \vec{\omega}, J, M, \lambda'\rangle$ still has one remaining integral over du. We will now do this integral explicitly by expanding the two-body amplitude in its partial wave form. The amplitude $\langle\vec{q}_{23} \mid t_1\left(z - \frac{m_1}{n_1}\omega_1\right) \mid \vec{q}_{23}'\rangle$

depends on only $|\vec{q}_{23}|$, $|\vec{q}'_{23}|$ and $\vec{q}_{23}\cdot\vec{q}'_{23}$. For the body-fixed z axis along the $\vec{p}_1$ direction the polar angle for $\vec{q}_{23}$ is just the angle between $\vec{p}_1 \cdot \vec{q}_{23}$ or γ_1, according to (II.23). Likewise γ'_1 is the polar angle for $\vec{q}'_{23}$. The angle between the planes of $(\vec{p}_1, \vec{q}_{23})$ and $(\vec{p}_1, \vec{q}'_{23})$ is just u. With these identifications we can write down the following well known expansion.

$$\hat{q}_{23} \cdot \hat{q}'_{23} = \cos\gamma_1 \cos\gamma'_1 + \sin\gamma_1 \sin\gamma'_1 \cos u \qquad \text{(II.46)}$$

Using the partial wave projection defined in Appendix A, Eq. (A.2) we may expand the two-body amplitude as

$$\langle \vec{q}_{23} | t_1 \left(z - \frac{m_1}{n_1}\omega_1\right) | \vec{q}'_{23} \rangle = \frac{\hbar^2}{4\pi^2\mu_{23}} \sum_{\ell} (2\ell+1)\, t'_\ell \left(q_{23}, q_{23}; z - \frac{m_1}{n_1}\omega_1\right) P_\ell (\cos\gamma_1 \cos\gamma'_1 + \sin\gamma_1 \sin\gamma'_1 \cos u) \qquad \text{(II.47)}$$

The Legendre polynomial may be now expressed in terms of $Y_{\ell m}$'s.

$$P_\ell (\cos\gamma_1 \cos\gamma'_1 + \sin\gamma_1 \sin\gamma'_1 \cos u) = \frac{4\pi}{2\ell+1} \sum_m Y^*_{\ell m}(\gamma_1, \phi_1)\, Y_{\ell m}(\gamma'_1, \phi'_1)$$

$$= \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\gamma_1, 0)\, Y_{\ell m}(\gamma'_1, 0)\, e^{-imu} \qquad \text{(II.48)}$$

Combining (II.48) together with (II.47) we see that the integral in (II.36) depends on u only through exponential factors. Specifically

$$\int_0^{2\pi} du\ e^{i\lambda_1 u} \langle \vec{q}_{23} \left| t_1 \left(z - \frac{m_1}{n_1}\ \omega_1 \right) \right| \vec{q}'_{23} \rangle$$

$$= \frac{\hbar^2}{\pi \mu_{23}} \sum_{\ell} t^1_{\ell} \left(q_{23},\ q'_{23};\ z - \frac{m_1}{n_1} \omega_1 \right) \sum_{m} Y_{\ell m}(\gamma_1, 0)\ Y_{\ell m}(\gamma'_1, 0)$$

$$\left[\int_0^{2\pi} du\ e^{i\lambda_1 u}\ e^{-imu} \right] \tag{II.49}$$

The integral term in square brackets is zero unless $m = \lambda_1$. Thus the expectation value of $T_1(z)$ can be written as

$$\langle \vec{\omega}',\ J,\ M, \lambda'_1 \left| T_1(z) \right| \vec{\omega},\ J,\ M, \lambda_1 \rangle = \frac{2(m_2+m_3)\hbar^2}{\sqrt{2m_1\omega_1}}\ \delta(\omega_1 - \omega'_1)\ \delta_{\lambda_1 \lambda'_1}$$

$$\times \sum_{\ell} Y_{\ell \lambda_1}(\gamma_1, 0)\ Y_{\ell \lambda'_1}(\gamma'_1, 0)\ t^1_{\ell}\ (q_{23},\ q'_{23},\ z - \frac{m_1}{n_1}\ \omega_1) \tag{II.50}$$

We can write the most general form of this matrix element by using Eq. (II.45)

$$\langle \vec{\omega}',\ J,\ M, \lambda'_1 \left| T_j(z) \right| \vec{\omega},\ J,\ M, \lambda_1 \rangle = \frac{2(m_i+m_k)\hbar^2}{\sqrt{2m_j\omega_j}}\ \delta(\omega_j - \omega'_j)$$

$$\left[\sum_{\lambda_j} d^J_{\lambda'_1 \lambda_j}(\theta'_{1j})\ d^J_{\lambda_j \lambda_1}(-\theta_{1j}) \sum_{\ell} Y_{\ell \lambda_j}(\gamma_j, 0)\ Y_{\ell \lambda_j}(\gamma'_j, 0) \right] \tag{II.51}$$

$$t^j_{\ell} \left(q_{ik},\ q'_{ik},\ z - \frac{m_j}{n_j}\ \omega_j \right)$$

E. Complete Angular Momentum Reduction

Placing Eq. (II.51) into Eq. (II.18) gives us the complete angular momentum reduction of the problem. The term in the square brackets contains all the three-body kinematic behavior of the problem. The dynamics enters through the two-body t matrix. Let us denote the kinematic factor $K^{J\ell}_{\lambda'_1\lambda_1}(\vec{\omega}',\vec{\omega})$ by,

$$K^{J\ell}_{\lambda'_1\lambda_1}(\gamma_j, \theta_{1j}, \gamma'_j\theta'_{1j}) = \sum_{\lambda_i=-J}^{J} Y_{\ell\lambda_j}(\gamma_j, 0)\, d^J_{\lambda_j\lambda_1}(-\theta_{1j})\, Y_{\ell\lambda_j}(\gamma'_j, 0)\, d^J_{\lambda'_1\lambda_j}(\theta'_{1j}) \tag{II.52}$$

It should be recalled that this form of the kinematic term is dependent on our choice of the body-fixed axis along the $\vec{p}_1$ direction. The only effect of some other choice would be to replace the d functions with the more general $\mathscr{D}$'s. One should note that if $|\lambda_j|$ exceeds ℓ in sum in Eq. (II.52) the term is zero since $Y_{\ell\lambda_j}$ vanishes for $|\lambda_j| > \ell$. Using our notation for the kinematic factor the matrix element for $T_i(z)$ may be written

$$\langle \vec{\omega}',\ J,\ \lambda'_1 | T_i(z) | \vec{\omega},\ J, \lambda_1 \rangle = \frac{2(m_j+m_k)\,\hbar^2}{\sqrt{2m_i\omega_i}}\,\delta(\omega'_i-\omega_i)$$

$$\sum_{\ell} K^{J\ell}_{\lambda'_1\lambda_1}(\gamma'_i, \theta'_{1i}; \gamma_i, \theta_{1i})\ t^i_\ell\left(q'_{jk},\ q_{jk};\ z-\frac{m_i}{n_i}\,\omega_i\right) \tag{II.53}$$

We have omitted the M dependence of T_i, since as can be seen from the right hand side of (II.53) there is no M dependence. This means also that the three-body transition amplitude like that of the two-body transition amplitude will not have any M dependence. This independence on M, which expresses a spherical symmetry inherent in the problem, results directly from the commutation of $\vec{J}$ with

with the exact Hamiltonian H. Thus the raising and lower operators, $J_x \pm iJ_y$, commute with H and from this it is easily seen that no observable for this system may have an M dependence.

The complete integral equation, in its angular momentum decomposed form is obtained from (II. 53) and (II. 18)

$$\langle \vec{\omega}', J, \lambda'_1 | T^i(z) | \vec{\omega}, J, \lambda_1 \rangle = \frac{2(m_j+m_k)\hbar^2}{\sqrt{2\, m_i \omega_i}}$$

$$\left\{ \delta(\omega'_i - \omega_i) \sum_\ell K^{J\ell}_{\lambda'_1 \lambda_1}(\gamma'_i, \theta'_{1i}; \gamma_i, \theta_{1i})\; t^i_\ell \left(q'_{jk}, q_{jk}; z - \frac{m_i}{n_i}\, \omega_i \right) \right. \tag{II. 54}$$

$$- \sum_{\lambda''_1 = -J}^{J} \int d\omega''_i\, d\omega''_j\, d\omega''_k \frac{\sum_\ell K^{J\ell}_{\lambda'_1 \lambda''_1}(\gamma'_i, \theta'_{1i}; \gamma''_i, \theta''_{1i})}{\omega'_i + \omega''_j + \omega''_k - z} \times \delta(\omega''_i - \omega'_i)$$

$$\left. t^i_\ell \left(q'_{jk}, q''_{jk}; z - \frac{m_i}{n_i}\, \omega''_i \right) \langle \vec{\omega}'', J, \lambda''_1 | T^j(z) + T^k(z) | \vec{\omega}, J, \lambda_1 \rangle \right\}$$

At this point let us assess what we have accomplished. The three-body partial wave form, Eq. (II. 54), of Faddeev's reduction is an integral equation in three real variables $(\omega'_1, \omega'_2, \omega'_3)$ and six parameters $(z, J, \vec{\omega}, \lambda_1)$. This is a considerable simplification with respect to the momentum version of Faddeev's equations which were 3 coupled equation in six real variables and seven continuous parameters. For each value of J we have a different integral equation and a different driving term.

An interesting physical consequence concerning the angular dependence of $\underline{T}(z)$ results from the invariant M dependence. Consider the expectation value

of T(z) in the momentum representation.

$$\langle \vec{\omega}, \psi, \theta, \phi \,|\, T(z) | \vec{\omega}', \psi', \theta', \phi' \rangle$$

$$= \sum_{JM\,\lambda\lambda'} \langle \vec{\omega}, \psi, \theta, \phi \,|\, \vec{\omega}, J, M, \lambda \rangle \langle \vec{\omega}, J, M, \lambda \,|\, T(z) | \vec{\omega}', J, M, \lambda' \rangle \qquad \text{(II. 55)}$$

$$\langle \vec{\omega}' \, J \, M \, \lambda' \,|\, \vec{\omega}' \, \psi', \theta', \phi' \rangle$$

From (II. 3) we have that

$$= \sum_{J\,\lambda\lambda'} \frac{2J+1}{8\pi^2} \left(\sum_{M} \mathcal{D}^{*J}_{\lambda M}(\psi, \theta, \phi)\, \mathcal{D}^{*J}_{M\lambda'}(-\psi', -\theta', -\phi') \right) \qquad \text{(II. 56)}$$

$$\langle \vec{\omega}\, J \lambda \,|\, T(z) | \vec{\omega}, J \, \lambda' \rangle$$

Let $R(\alpha, \beta, \gamma)$ be the rotation obtained by applying $R(-\psi', -\theta', -\phi')$ and then $R(\psi, \theta, \phi)$. The group property for rotations tells us

$$\langle \vec{\omega}, \psi, \theta, \phi \,|\, T(z) | \vec{\omega}', \psi', \theta', \phi' \rangle = \sum_{J\,\lambda\lambda'} \frac{2J+1}{8\pi^2} \mathcal{D}^{*J}_{\lambda\lambda'}(\alpha, \beta, \gamma) \qquad \text{(II. 57)}$$

$$\times \langle \vec{\omega}\, J \lambda \,|\, T(z) | \vec{\omega}' \, J \lambda' \rangle$$

Thus we have proved there are only three independent angles needed to describe the scattering not six as the left hand side of (II. 57) suggests. This is analogous to the well-known result for the two-body amplitude, $t(\vec{p}, \vec{p}', z)$, which states that t depends only on $|p|, |p'|$ and $\hat{p} \cdot \hat{p}'$.

— While we are on the subject of general properties of the transition amplitude it is convenient to study parity conservation. The argument given here is a specialized

version of a more general one given by S. M. Berman and M. Jacob.[29] The wave function can be represented in the form

$$|\vec{\omega},\ J,\ M,\ \lambda\rangle = \sqrt{\frac{2J+1}{8\pi^2}}\int \mathcal{D}^{*J}_{M\lambda}(\alpha,\beta,\gamma)\ |\vec{\omega},\alpha,\beta,\gamma\rangle\, d\alpha \sin\beta\, d\beta\, d\gamma$$

(II. 58)

$$= \sqrt{\frac{2J+1}{8\pi^2}}\int \mathcal{D}^{*J}_{M\lambda}(\alpha,\beta,\gamma)\ R(\alpha,\beta,\gamma)\ |\vec{\omega},\ 0,\ 0,\ 0\rangle\, d\alpha \sin\beta\, d\beta\, d\gamma$$

where $R(\alpha,\beta,\gamma)$ is the rotation operator[30] $e^{-\alpha J_z} e^{-i\beta J_y} e^{-i\beta J_z}$. We can easily see that (II. 58) is equivalent to our former definition of $|\vec{\omega},\ J,\ M,\ \lambda\rangle$ by multiplying from the left by $\langle\vec{\omega},\psi,\theta,\phi|$ to get

$$\langle\vec{\omega},\psi,\theta,\phi\,|\vec{\omega},\ J,\ M,\lambda\rangle = \int \mathcal{D}^{*J}_{M\lambda}(\alpha,\beta,\gamma)\ \delta(\psi-\alpha)\ \delta(\cos\theta-\cos\beta)\ \delta(\phi-\gamma)$$

$$\sqrt{\frac{2J+1}{8\pi^2}}\ d\alpha \sin\beta\ d\beta\ d\gamma \qquad \text{(II. 59)}$$

where we have used

$$\langle\vec{\omega},\psi,\theta,\phi|\vec{\omega},\alpha,\beta,\gamma\rangle = \delta(\psi-\alpha)\ \delta(\cos\theta-\cos\beta)\ \delta(\phi-\gamma)$$

Thus (II. 59) is identical with (II. 3)

Now the parity operator P commutes with the rotation operator R so we may write

$$P\,|\vec{\omega},\ J,\ M,\lambda\rangle = \sqrt{\frac{2J+1}{8\pi^2}}\int \mathcal{D}^{*J}_{M\lambda}(\alpha,\beta,\gamma)\ R(\alpha,\beta,\gamma)\ P\,|\vec{\omega},\ 0,\ 0,\ 0\rangle\, d\alpha$$

(II. 60)

$$\sin\beta\, d\beta\, d\gamma$$

If we denote by Y the reflection through the x - y plane then the parity operator may be written as

$$P = e^{i\pi J_z} Y \tag{II.61}$$

Supposing that η_1, η_2, η_3 are intrinsic parities of partieles 1, 2 and 3, we have the effect of Y on $|\vec{\omega},\ 0,\ 0,\ 0>$.

$$Y\ |\vec{\omega},\ 0,\ 0,\ 0> = \eta_1 \eta_2 \eta_3 |\vec{\omega},\ 0,\ 0,\ 0> \tag{II.62}$$

Combining (II.62), and (II.61) Eq. (II.60) may be written

$$P\ |\vec{\omega},\ J,\ M, \lambda> = \eta_1\eta_2\eta_3\sqrt{\frac{2J+1}{8\pi^2}}\int \mathcal{D}^{*J}_{M\lambda}\ (\alpha, \beta\ , \gamma)\ R(\alpha, \beta, \gamma - \pi)$$

$$|\vec{\omega},\ 0,\ 0,\ 0> d\alpha \sin\beta d\beta d\gamma \tag{II.63}$$

Now noting that

$$\mathcal{D}^{*J}_{M\lambda}(\alpha, \beta, \gamma) = (-1)^{\lambda}\ \mathcal{D}^{*J}_{M\lambda}\ (\alpha, \beta, \gamma - \pi),$$

we have

$$P\ |\vec{\omega},\ J,\ M, \lambda> = \eta_1\eta_2\eta_3(-1)^{\lambda}\ |\vec{\omega},\ J,\ M, \lambda> \tag{II.64}$$

So we have that matrix elements of three-body operators with commute with a parity conserving three-body Hamiltonian H vanish unless the initial λ_i and the final λ_f differ by an even integer. Thus roughly half (depending on whether J is even or odd) of the matrix elements $<\vec{\omega},\ J, \lambda\ |\ T(z)\ |\vec{\omega}',\ J,\ \lambda'>$ will vanish in the λ index.

CHAPTER III

SEPARABILITY IN TWO BODY AMPLITUDES

The purpose of this section is to study representations and approximations of the two-body transition amplitudes that will be useful in the three-body problem. The principal ingredient in the kernel of Faddeev's integral equations for the three-body problem is the two-body transition amplitude. The two-body transition amplitude is a function of three independent variables: the incident momenta, the final momenta and the energy of the scattering process. The transition amplitude is called on-shell if both incident and final momenta are equal to the momenta implied by the energy. If one of the two momenta are independent of energy then the amplitude is considered half-off-shell. If both are independent of energy then the amplitude is said to be full-off-shell. The form of the two-body amplitude needed in Faddeev's equation is the full-off-shell amplitude. The approximation which is nearly universally used in treating the three-body problem is to assume that the two-body transition amplitude may be factored into a product of two terms each a function of only one of the two momenta in the scattering process. This assumption of separability allows one eventually to reduce Faddeev's equations to integral equations in one variable. For most authors this separability has been a conditio sine qua non for proceeding. Generally, separability of the transition amplitude arises from assuming that two-body potential is separable. Contact with physics is made by fitting the parameters of the chosen separable potential so that the two-body scattering length and boundstate energy are correctly reproduced. Here we want to study all forms of separability for the two-body amplitudes, in particular separable approximations for the amplitude which do not necessarily follow from the restrictive assumption of a separable potential. To this end we shall develop an exact representation of the general off-shell

amplitude, that is a sum of a separable term plus a non-separable remainder which vanishes for all half-off-shell momenta. Furthermore this representation will give us a way to calculate the off-shell transition amplitude from a set of Fredholm integral equations. This last result is a significant improvement since the integral equations satisfied by the transition amplitude is the singular Lippman-Schwinger equations. Most of the ideas for the following development were first given by K. L. Kowalski and D. Feldman[31,32] and by H. P. Noyes.[33]

A. Half-Off-Shell Function $f_\ell(p,s)$

We begin our investigation by writing down the Lippman-Schwinger equation which uniquely determines the off-shell transition amplitude. In its partial wave form we have

$$t_\ell(p, q; s + i\epsilon) = v_\ell(p, q) - \frac{2}{\pi}\int_0^\infty \frac{p'^2 dp'}{p'^2 - s - i\epsilon} v_\ell(p,p')\, t_\ell(p', q; s + i\epsilon) \tag{III.1}$$

where p, q, p' are momenta, $s/(2\mu)$ is the kinetic energy in the center-of-mass coordinate system, and μ the reduced mass. The transition amplitude is $t_\ell(p,q,; s + i\epsilon)$ and v_ℓ is the partial wave projection of the potential defined in Appendix A to this chapter. The Lippman-Schwinger equation is a singular integral equation in the one variable q. Both p and s are parameters.

We now want to introduce the half-off-shell extension function, $f_\ell(p,s)$, which is defined by

$$f_\ell(p,s) \equiv \frac{t_\ell(p,k; s + i\epsilon)}{t_\ell(k,k; s + i\epsilon)} \quad , \quad p \geq 0\,, \quad k \geq 0, \quad s \in [-\infty, +\infty]\,. \tag{III.2}$$

Here the momentum k is the on-shell momentum associated with the energy $s/2\mu$, for $s \geq 0$, $k = +\sqrt{s}$. We want to use this definition for f_ℓ for negative s as well

as positive. In order to do this we must define the relationship between k and s when s is negative. This definition is somewhat arbitrary since there are no physical momenta associated with negative energies. S-matrix theory chooses $k = +\sqrt{s}$ for all real s, and has as a consequence that $t_\ell(k, k; s + i\epsilon)$ possesses all the simple analyticity properties customarily associated with the on-shell amplitude in cut complex s plane. However, at this stage we wish to keep free the choice of the relationship between k and s for $s < 0$. Thus we only assume that $|k^2| = |s|$ for all s. Note that the meaning of the phase of the on-shell amplitude is dependent on the functional relation between k and s.

Our next step is to relate the half-off-shell extension function f_ℓ to the on-shell amplitude and to find the integral equation that f_ℓ satisfies. In order to do this we substitute equation (III. 2) into the half-off-shell version of the Lippman-Schwinger equation, and find

$$f_\ell(p,s)\, t_\ell(k, k; s + i\epsilon) = v_\ell(p, k) - t_\ell(k, k; s + i\epsilon)\, \frac{2}{\pi} \int_0^\infty \frac{q^2 dq}{q^2 - s - i\epsilon}\, v_\ell(p,q)\, f_\ell(q,s) \tag{III. 3}$$

Since by definition $f_\ell(k,s) = 1$, it follows immediately that

$$t_\ell(k, k; s + i\epsilon) = \frac{v_\ell(k,k)}{1 + \frac{2}{\pi}\int_0^\infty \frac{q^2 dq}{q^2 - s - i\epsilon}\, v_\ell(k,q)\, f_\ell(q,s)} \tag{III. 4}$$

This equation shows us how to recover the on-shell amplitude t_ℓ when the half-off-shell extension function f_ℓ is known. We will now get a simple integral equation for

f_ℓ if we eliminate $t_\ell(k, k; s + i\epsilon)$ from Eq. (III. 3) by using Eq. (III. 4)

$$f_\ell(p,s) = v_\ell(p,k)\left\{\frac{1 + \frac{2}{\pi}\int_0^\infty \frac{q^2 dq}{q^2 - s - i\epsilon}\, v_\ell(k,q)\, f_\ell(q,s)}{v_\ell(k,k)}\right\} - \frac{2}{\pi}\int_0^\infty \frac{q^2 dq}{q^2 - s - i\epsilon}\, v_\ell(p,q)\, f_\ell(q,s)$$

Rearranging the terms gives us the desired integral equation.

$$f_\ell(p,s) = \frac{v_\ell(p,k)}{v_\ell(k,k)} + \int_0^\infty \Lambda_\ell(p,q;s)\, f_\ell(q,s)\, dq \qquad \text{(III. 5)}$$

where the kernel Λ_ℓ is

$$\Lambda_\ell(p,q;s) = \frac{2}{\pi}\,\frac{q^2}{q^2 - s - i\epsilon}\left[\frac{v_\ell(p,k)\, v_\ell(k,q)}{v_\ell(k,k)} - v_\ell(p,q)\right], \quad p, q \in [0,\infty] \qquad \text{(III. 6)}$$

Eq. (III. 5) is Fredholm and has none of the singularity difficulties associated with the partial wave Lippman-Schwinger equation for $s > 0$. For positive s the denominator has a zero when $q^2 = s$, but when $k^2 = s$ the portion of the numerator in square brackets also vanishes. Consequently Λ_ℓ is a continuous function in the neighborhood of $q^2 \approx s$ and the $i\epsilon$ may be dropped. Since the $i\epsilon$ was the only complex factor in Λ_ℓ, this shows that Λ_ℓ and f_ℓ are real. Once we have solved Eq. (III. 5) for f_ℓ we can use Eq. (III. 4) to find the on-shell amplitudes and phase shifts. The immediate utility of f_ℓ and the integral equation it satisfies is that it provides convenient direct way to obtain the half-off-shell and on-shell solutions to the Lippman-Schwinger equation for the scattering amplitudes.

B. Derivation of Kowalski-Noyes Representation

We now want to carry the analysis further by finding a representation of the full-off-shell amplitude $t_\ell (p, q; s + i\epsilon)$ in terms of f_ℓ. The resolvent kernel for Λ_ℓ is defined to be the solution $R_\ell (p, q; s)$ of the integral equation

$$R_\ell (p, q; s) + \Lambda_\ell (p, q; s) = \int_0^\infty R_\ell (p, x; s)\, \Lambda_\ell (x, q; s)\, dx = \int_0^\infty \Lambda_\ell (p, x; s)\, R_\ell (x, q; s)\, dx \qquad \text{(III. 7)}$$

The most useful property of the resolvent is that it allows one to write the solution $k(p, s)$ of a Fredholm integral equation as the integrated product of an inhomogeneous term $g(p, s)$ times the resolvent function; that is, given

$$h(p, s) = g(p, s) + \int_0^\infty \Lambda (p, q; s)\, h (q, s)\, dq \qquad \text{(III. 8)}$$

then

$$h (p, s) = g(p, s) - \int_0^\infty R (p, q; s)\, g (g, s)\, dq$$

If we restrict the Lippman-Schwinger equation to its half-off-shell form, multiply by $v_\ell (p, k)\, v_\ell (k, k)^{-1}$ and use the definition for f_ℓ given by Eq. (III. 2), the following relation is obtained

$$0 = \frac{v_\ell (p, k)}{v_\ell (k, k)} f_\ell (q, s)\, t_\ell (k, k; s + i\epsilon) - \frac{v_\ell (k, q)\, v_\ell (p, k)}{v_\ell (k, k)} + \frac{2}{\pi} \int_0^\infty \frac{x^2 dx}{x^2 - s - i\epsilon} \frac{v_\ell (p, k)\, v_\ell (k, x)}{v_\ell (k, k)}\, t_\ell (x, q; s + i\epsilon) \qquad \text{(III. 9)}$$

Adding Eq. (III.9) to the full-off-shell Lippman-Schwinger equation (Eq. (III.1)) gives

$$t_\ell(p,q;s+i\epsilon) = \frac{v_\ell(p,k)}{v_\ell(k,k)} f_\ell(q,s)\, t_\ell(k,k;s+i\epsilon) + \left[v_\ell(p,q) - \frac{v_\ell(k,q)\, v_\ell(p,k)}{v_\ell(k,k)} \right] \tag{III.10}$$

$$+ \frac{2}{\pi} \int_0^\infty \frac{x^2 dx}{x^2 - s - i\epsilon} \left[\frac{v_\ell(k,q)\, v_\ell(p,x)}{v_\ell(k,k)} - v_\ell(p,x) \right] t_\ell(x,q;s+i\epsilon)$$

Substituting our definition of Λ_ℓ into the above equation reduces the result to

$$t_\ell(p,q;s+i\epsilon) = \left\{ \frac{v_\ell(p,k)}{v_\ell(k,k)} f_\ell(q,s)\, t_\ell(k,k;s+i\epsilon) - \frac{\pi}{2} \frac{q^2-s}{q^2} \Lambda_\ell(p,q;s) \right\} \tag{III.11}$$

$$+ \int_0^\infty dx\, \Lambda_\ell(p,x;s)\, t_\ell(x,q;s+i\epsilon)$$

We now use the resolvent property given in Eq. (III.8) to rewrite the solution $t_\ell(p,q;s+i\epsilon)$ of the integral equation (III.11). Here the driving term is the entire expression within the curly brackets. The result reads

$$t_\ell(p,q;s+i\epsilon) = f_\ell(q,s)\, t_\ell(k,k;s+i\epsilon) \left[\frac{v_\ell(p,k)}{v_\ell(k,k)} - \int_0^\infty dp'\, R_\ell(p,p';s) \frac{v_\ell(p',k)}{v_\ell(k,k)} \right] \tag{III.12}$$

$$- \frac{\pi}{2} \frac{q^2-s}{q^2} \left[\Lambda_\ell(p,q;s) - \int_0^\infty dp'\, R_\ell(p,p';s)\, \Lambda_\ell(p',q;s) \right]$$

Now using the resolvent property once more this simplifies to

$$t_\ell(p,q;s+i\epsilon) = f_\ell(q,s)\, t_\ell(k,k;s+i\epsilon)\, f_\ell(p,s) + \frac{\pi}{2} \frac{q^2-s}{q^2} R_\ell(p,q;s). \tag{III.13}$$

This is the general off-shell representation of Kowalski and Noyes that we have set out to derive. The equality is valid for all p, q and s, and the first term is separable in p and q. The non-separable term is completely determined by the resolvent of the kernel $\Lambda_\ell(p, q; s)$. From the definition of $\Lambda_\ell(p, q; s)$ it follows that for the half-off-shell case, e.g., either $p^2 = s$ or $q^2 = s$ then $\Lambda_\ell(p, q; s)$ is zero and so consequently is $R_\ell(p, q; s)$. This result is even more immediately obtained from the definition of $f_\ell(p, s)$ in Eq. (III.2). Since the non-separable term vanishes for on-shell and half-on-shell values of the momenta p and q it is not unreasonable to hope the separable first term will be a good approximation to $t_\ell(p, q; s + i\epsilon)$. In any case the representation gives us a convenient method for finding the full-off-shell t matrix in terms of functions $f_\ell(q, s)$ and $R_\ell(p, q; s)$ both of which satisfy a Fredholm integral equation.

Let us now study the approximation to $t_\ell(p, q; s + i\epsilon)$ obtained when the non-separable term, $\frac{\pi}{2} \frac{q^2 - s}{s} R_\ell(p, q; s)$ of Eq. (III.12) is neglected. First, we will describe the basic qualitative properties of the exact amplitude and see which of these properties are preserved in the approximations we make. These basic properties are:

(a) Symmetry in the variables p, q, e.g., $t_\ell(p, q; z) = t_\ell(q, p; z)$. This follows from time reversal invariance, which implies that $v(\vec{p}, \vec{q})$ is symmetric.

(b) $t^\ell(p, q; z)$ has simple analyticity in the variable z. In the left half complex energy (z) plane t_ℓ is meromorphic with all of its poles lying on the negative real z axis. The position of each pole is a bound state (or states) energy and the residue of the t_ℓ matrix factors into a separable product of terms, each of which is related to the bound state wave function. Note that left hand cut associated with the transition amplitude comes about by setting $p = \sqrt{z}$, $q = \sqrt{z}$ for negative z. In Faddeev's treatment of the three-body problem only real positive

momenta occur so that successful approximate forms of the two-body transition amplitude need not reproduce the exact amplitude for imaginary values of the momenta.

(c) $t_\ell(p,q;z)$ has the unitarity property. For all positive values of z, t_ℓ has a discontinuity across real axis given by

$$\text{Im } t_\ell(p,q;z) = \sqrt{z}\; t_\ell(p,\sqrt{z};z)\; \overline{t_\ell(q,\sqrt{z};z)} \quad . \tag{III.14}$$

This equation given in Appendix A, implies that the on-shell normalization in terms of the phase shift $\delta_\ell(k)$ is

$$t_\ell(k,k;k^2+i\epsilon) = \frac{1}{k}\, e^{i\delta_\ell(k)} \sin\delta_\ell(k) \tag{III.15}$$

(d) Asymptotic behavior. In the limit as $|z|\to\infty$ the Born term dominates.

$$\lim_{|z|\to\infty} t_\ell(p,q;z) = v_\ell(p,q) \tag{III.16}$$

In searching for approximate off-shell transition forms it seems reasonable to expect that all or most of these basic properties be satisfied.

C. Low Equation

At this point it is instructive to present the Low equation in terms of the f_ℓ functions. The Low equation gives us another representation for T matrix and if we re-write it in terms of the f_ℓ functions we may compare the two representations in detail. The general operator form of the Low equation is

$$T(z) = V + V\,\frac{1}{z-H}\,V \tag{III.17}$$

where $H = H_0 + V$. Let the complete set of states associated with H be written as

$$1 = \sum_{\alpha} |\psi_\alpha><\psi_\alpha| + \int d\vec{k} \, |\psi_{\vec{k}}><\psi_{\vec{k}}| \qquad \text{(III.18)}$$

If we now pre and post multiply Eq. (III.17) by the plane wave states $<\vec{p}|$ and $|\vec{q}>$ respectively and use Eq. (III.18) to expand $<\vec{p}| \, V \frac{1}{z - H} V |\vec{q}>$ we get

$$<\vec{p}|T(z)|\vec{q}> = <\vec{p}|V|\vec{q}> + \int d\vec{k} \, \frac{<\vec{p}|V|\psi_{\vec{k}}><\psi_{\vec{k}}|V|\vec{q}>}{z - \frac{k^2\hbar^2}{2\mu}} + \sum_{\alpha} \frac{<\vec{p}|V|\psi_\alpha><\psi_\alpha|V|\vec{q}>}{z + \frac{\alpha^2\hbar^2}{2\mu}} \qquad \text{(III.19)}$$

Identifying $<\vec{p}|V|\psi_{\vec{k}}>$ as the half-off-shell transition amplitude $<\vec{p}|T\ (k^2)|\vec{k}>$ we get the customary form of the Low equation.

$$<\vec{p}|T(z)|\vec{q}> = <\vec{p}|V|\vec{q}> + \int d\vec{k} \, \frac{<\vec{p}|T(k^2)|\vec{k}>\overline{<\vec{q}|T(k^2)|\vec{k}>}}{z - \frac{k^2\hbar^2}{2\mu}} + \sum_{\alpha} \frac{<\vec{p}|V|\psi_\alpha><\psi_\alpha|V|\vec{q}>}{z + \frac{\alpha^2\hbar^2}{2\mu}} \qquad \text{(III.20)}$$

By doing the standard partial wave reduction we obtain

$$t_\ell(p, q; z) = v_\ell(p, q) + \frac{2}{\pi} \int_0^\infty \frac{t_\ell(p, k; k^2)\,\overline{t_\ell(q, k; k^2)}}{\frac{2\mu z}{\hbar^2} - k^2} k^2 dk + \frac{2}{\pi} \sum_{\alpha_\ell} \frac{\phi_\alpha^\ell(p)\,\overline{\phi_\alpha^\ell(q)}}{\frac{2\mu z}{\hbar^2} + k_{\alpha_\ell}^2} \qquad \text{(III.21)}$$

where $\phi_\alpha^\ell(p) = \int_0 V_\ell(p, p')\,\psi_\alpha^\ell(p')\,p'^2\,dp'$, and ψ_α^ℓ is the radial part of the bound state wave function. The analyticity of $t_\ell(p, q; z)$ in z for fixed p, q can

be obtained directly from Eq. (III.21). To do this we just take derivative d/dz of both sides of Eq. (III.21) and note that derivative operation and the integration may be interchanged, provided there is no singularity in the denominator of the integrand. Thus $t_\ell(p,q;z)$ is analytic except for the explicitly displayed bound state poles and the right hand unitarity cut. Furthermore it is obvious from Eq. (III.21) that $t_\ell(p,q;z) \longrightarrow v_\ell(p,q)$ as $|z| \longrightarrow \infty$. An additional fact emerging from Low equation is that if we know $t_\ell(p,k;k^2)$ and $v_\ell(p,q)$ we can construct the full-off-shell amplitude $t_\ell(p,q,z)$. Thus all possible physical information is contained in the half-off-shell amplitudes. We now want to see what form Eq. (III.21) takes when expressed in terms of $f_\ell(p,k)$ and the phase shifts. Using Eq. (III.2) and (III.15) we have

$$t_\ell(p,q;s) = v_\ell(p,q) + \frac{2}{\pi}\int_0^\infty \frac{\sin^2\delta_\ell(k)\, f_\ell(p,k^2)\, f_\ell(q,k^2)\, dk}{s + i\epsilon - k^2} + \frac{2}{\pi}\sum_\alpha \frac{\phi_\alpha^\ell(p)\,\overline{\phi_\alpha^\ell(q)}}{s + \alpha_\ell^2} \qquad \text{(III.22)}$$

where $s = \frac{2\mu z}{\hbar^2}$. Letting $s = q^2$ gives

$$\frac{e^{i\delta_\ell(q)}\sin\delta_\ell(q)}{q} f_\ell(p,q^2) = v_\ell(p,q) + \frac{2}{\pi}\int_0^\infty \frac{\sin^2\delta_\ell(k)\, f_\ell(p,k^2)\, f_\ell(q,k^2)\, dk}{q^2 - k^2 + i\epsilon} + \frac{2}{\pi}\sum_{\alpha_\ell} \frac{\phi_\alpha^\ell(p)\,\overline{\phi_\alpha^\ell(q)}}{q^2 + \alpha_\ell^2} \qquad \text{(III.23)}$$

As noted above this equation can be used to find $v_\ell(p,q)$. If the phase shifts are assumed known from experiment then $f_\ell(p,k)$ represents our ignorance of the

off-shell behavior of t_ℓ, or equivalently our ignorance of $v_\ell(p,q)$. Taking the imaginary part of Eq. (III.22) will just yield the unitarity relation. The final term, the term containing the bound state poles, of Eq. (III.23) may be re-expressed by using the partial wave momentum space version of the Schroedinger equation for the bound state wave function. In our notation this equation reads

$$(p^2 + \alpha_\ell^2)\ \psi_{\alpha_\ell}(p) = -\frac{2}{\pi}\int_0^\infty q^2\, dq\ v_\ell(p,q)\ \psi_{\alpha_\ell}(q) = -\frac{2}{\pi}\ \phi_{\alpha_\ell}(p) \qquad \text{(III.24)}$$

for a bound state of energy $-\dfrac{\hbar^2 \alpha_\ell^2}{2\mu}$. Consequently

$$\frac{2}{\pi}\sum_{\alpha_\ell}\frac{\phi_\alpha^\ell(p)\ \phi_\alpha^\ell(q)}{s+\alpha_\ell^2} = \frac{\pi}{2}\sum_{\alpha_\ell}\frac{(p^2+\alpha_\ell^2)\ (q^2+\alpha_\ell^2)\ \psi_{\alpha_\ell}(p)\ \psi_{\alpha_\ell}(q)}{s+\alpha_\ell^2}\,. \qquad \text{(III.25)}$$

D. Optimal Separable Potentials

Now let us return to the problem of finding the best separable approximation suggested by the representation of $t_\ell(p,q;\ s+i\epsilon)$ given in Eq. (III.13). The representation as derived above did not specify the functional relationship between the energy s and the on-shell momenta k. The on-shell amplitude used in S-matrix theory is $t_\ell\ (\sqrt{s},\ \sqrt{s};\ s+i\epsilon)$. Here relationship between k and s is $k = \sqrt{s}$. As a consequence of this choice $t_\ell(\sqrt{s},\ \sqrt{s};\ s+i\epsilon)$ is analytic functions of s for $\text{Im}\ s \neq 0$ and as a consequence of this analyticity we may use the Cauchy formulae to derive the N/D formulation. Thus $k = \sqrt{s}$ seems the most natural choice for the definition of the on-shell momenta. However, we shall see that this functional relation will not lead to a useful separable approximation. Setting $k = \sqrt{s}$ the separable approximation obtained from Eq. (III.13) is

$$t_\ell(p,q;\ s+i\epsilon) \cong f_\ell(q,s)\, t_\ell(\sqrt{s},\ \sqrt{s};\ s+i\epsilon)\, f_\ell(p,s) \qquad \text{(III.26)}$$

The domain over which t_ℓ is integrated in Faddeev's equation is all positive p, q and $s\epsilon\ (z, -\infty)$, where z is the energy of the three-body system. Thus Eq. (III. 26) must be examined for negative s. Substituting the definition of f_ℓ gives at once

$$t_\ell(p, q;\ s+i\epsilon\ ) \cong \frac{t_\ell(p, \sqrt{s}\ ;\ s+i\epsilon)\ t_\ell(q, \sqrt{s};\ s+i\epsilon)}{t_\ell(\sqrt{s},\ \sqrt{s};\ s+i\epsilon)} \qquad \text{(III. 27)}$$

Since $t_\ell(\sqrt{s},\ \sqrt{s};\ s+i\epsilon)$ has an interaction cut for s sufficiently negative, the right hand number of Eq. (III. 27) will have this cut. However, for $p \geq 0$, $q \geq 0$ $t_\ell(p, q;\ s+i\epsilon)$ has only the bound state poles and is real for $s < 0$. Thus from the beginning of the interaction cut to $-\infty$ Eq. (III. 27) will not be a good approximation, and furthermore the approximation does not share the analyticity properties of $t_\ell(p, q;\ s+i\epsilon)$ in the s variable.

Since the prescription $k = \sqrt{s}$ has failed to yield a useful approximation, we will try $k = \sqrt{|s|}$. Now Eq. (III. 27) will take the form

$$t_\ell(p, q;\ s+i\epsilon) \cong \frac{t_\ell(p, \sqrt{|s|};\ s+i\epsilon)\ t_\ell(q, \sqrt{|s|}\ ;\ s+i\epsilon\ )}{t_\ell(\sqrt{|s|}, \sqrt{|s|}\ ;\ s+i\epsilon)} \qquad \text{(III. 27')}$$

Since the momentum arguments of $t_\ell(\sqrt{|s|}, \sqrt{|s|}\ ;\ s+i\epsilon)$ are positive, this function will not have the interaction cut. The objections which applied to Eq. (III. 27) do not apply to (III. 27'). There has been some objection[34] to $k = \sqrt{|s|}$ on the grounds that this is not an analytic relation and as a result the right hand term of Eq. (III. 27') will not be analytic at the point $s = 0$. It is however continuous at this point, and separately analytic for $s > 0$ and $s < 0$.

In order to further evaluate this approximation we need to know more about the behavior of $t_\ell(p, q; s+i\epsilon)$ and $f_\ell(p, s)$ in the bound state region. We first evaluate $f_\ell(p, s)$ as $s \to -\alpha_\ell^2$, where $(2\mu)^{-1}\alpha_\ell^2$ is any bound state energy in the ℓ^{th} partial wave channel. For $s \simeq -\alpha_\ell^2$ the pole term coming from the Low equation is for the half-off-shell amplitude

$$t_\ell(p, k; s) = \frac{\pi}{2} \frac{(p^2+\alpha_\ell^2)(k^2+\alpha_\ell^2)\psi\alpha_\ell(p)\ \psi\alpha_\ell(k)}{s+\alpha_\ell^2} + \text{non-pole terms} \qquad \text{(III. 28)}$$

where, as above $k = \sqrt{|s|}$. Alternately from the definition of f_ℓ we have

$$\begin{aligned} t_\ell(p, k; s) &= f_\ell(p, s)\ t_\ell(k, k; s) \\ &= f_\ell(p, s) \left\{ \frac{\pi}{2} \frac{(k^2+\alpha_\ell^2)(k^2+\alpha_\ell^2)\psi\alpha_\ell(k)\psi\alpha_\ell(k)}{s+\alpha_\ell^2} \right. \\ &\qquad \left. + \text{non-pole terms} \right\} \end{aligned} \qquad \text{(III. 29)}$$

Equating the coefficients of the pole terms in Eq. (III. 28) and (III. 29) gives us the relation

$$f_\ell(p, s) \cong \frac{(p^2+\alpha_\ell^2)\psi\alpha_\ell(p)}{(k^2+\alpha_\ell^2)\psi\alpha_\ell(k)} \qquad \text{(III. 30)}$$

This relation becomes exact as $s \to -\alpha_\ell^2$. Thus each time the energy $(2\mu)^{-1}s$ becomes equal a bound state energy the f_ℓ is for that energy proportional to the bound state wave function. Returning to the separable approximation we see that in the neighborhood of a bound state we can write

$$f_\ell(p, s)\ t_\ell(k, k; s)\ f_\ell(q, s) = \frac{\pi}{2} \frac{(p^2+\alpha_\ell^2)(q^2+\alpha_\ell^2)\psi\alpha_\ell(p)\psi\alpha_\ell(q)}{s+\alpha_\ell^2} \qquad \text{(III. 31)}$$

This tells us that the separable term in the representation given by Eq. (III. 13) contains all of the bound state pole terms in the amplitude $t_\ell(p, q; s+i\epsilon)$.

Now let us summarize the important features which we have derived for the Kowalski-Noyes representation given in Eq. (III. 13). First, the decomposition into a separable and a non-separable piece is exact for all s, p, q. If either $p^2 = s$ or $q^2 = s$ then the non-separable term identically vanishes. Since $R_\ell(p, q; s)$ is real all the imaginary part of the amplitude is contained in the separable term. The on-shell amplitude and equivalently the phase shift is given exactly by the separable term. Also it follows immediately from definition (Eq. (III. 2)) of the half-off-shell extension function f_ℓ that the separable term will satisfy half-off-shell unitarity. However, the approximation will not satisfy full-off-shell unitarity. The non-separable term in the Kowalski-Noyes representation is purely real. Neglecting it will destroy the relation between real and imaginary parts of the amplitude that unitarity requires. The separable term contains all the bound state singularities and has the correct residue. If we choose $k = \sqrt{|s|}$ then there will not be any non-physical cuts in the separable term for negative s. However, this choice of $k = \sqrt{|s|}$ means that the separable term will not be analytic at $s = 0$ where as the full amplitude is analytic. An additional difficulty with the separable term is that it may have poles which do not occur in the full amplitude. Suppose there is a zero in phase shift at $k = k_0$, e.g., that $\delta(k_0) = 0$. The on-shell t matrix will behave as

$$t_\ell(k, k; +k^2) \simeq (k - k_0)^\beta t'_\ell(k_0) \ , \ k \simeq k_0 \qquad \text{(III. 32)}$$

where $\beta > 0$ and $t'_\ell(k_0)$ is finite. By using the definition of $f_\ell(p, k^2)$ we have

$$f_\ell(p, k) \simeq (k - k_0)^{-\beta} f'_0(p)$$

where $f'_0(p)$ is finite. The full separable term will have a dependence of the form

$$(k-k_0)^{-\beta} f'_0(p) (k - k_0)^{\beta} t'_\ell(k_0) (k - k_0)^{-\beta} f'_0(q)$$

which diverges as $(k - k_0)^{-\beta}$. Generally the $t_\ell(p,q;k_0^2)$ will be finite. Thus for each zero in the phase shift the separable term will have a pole which is not present in the exact amplitude. Clearly for the representation to remain valid there must also be a pole in the non-separable term which cancels that of the separable term. It is this feature of non-physical poles in the separable term which is the severest limitation of the approximation. One would not expect the separable approximation to work for three-body energies that are greater than the energy for which the first zero in the phase shift occurs. For a system like the N-P system you could not use the approximation in a three-body calculation for scattering amplitudes close to or above 250 MeV. However, for lower energies the approximation should still be valid.

E. Alternate Approaches to Separability

Having evaluated the merits of the separable approximation arising from the Kowalski-Noyes representation, it is now useful to examine different approaches to separability that also exist in the literature. The method most commonly used to acquire a separable t-matrix is to assume a potential form that is separable.[35] This is the approach used by all authors working with Faddeev's equations who carry out numerical calculations of three-body amplitudes. The relationship between the separable potential approach and the method described above is established by noting that the kernel Λ_ℓ given in Eq. (III. 6) vanishes identically if the potential $v_\ell(p,k)$ is separable. Thus R_ℓ is always zero and the first term in the Kowalski-Noyes representation is

exact even off-shell. For separable potentials the integral equation for f_ℓ, Eq. (III.5), reduces to

$$f_\ell(p, s) = \frac{v_\ell(p, k)}{v_\ell(k, k)} \qquad \text{(III. 33)}$$

The advantages of the separable potential are as follows: The separability allows one to solve the Lippman-Schwinger equation analytically. The resulting two-body t matrix is completely separable and obeys full-off-shell unitarity. The disadvantages are that a single separable potential can support only one bound state regardless of the size of the coupling constant. When the separable potential is transformed to coordinate space by the appropriate Fourier transform the resulting potential is non-local. This non-locality is probably the most serious physical draw-back to the separable potential.

An attempt to extend the flexibility of the separable potential has been made by T. Mongan.[36] He observes that we can simply generalize the Lippman-Schwinger equation by replacing the energy independent potential, $v_\ell(p, q)$ with an energy dependent potential $v_\ell(p, q; s)$. The additional freedom that the energy dependence allows would then permit more accurate fits of scattering data. However, this prescription runs into the following difficulty. The full Hamiltonian is the sum of the unperturbed Hamiltonian, H_0, and the potential. Thus if the potential has an energy dependence then the full Hamiltonian has an energy dependence. An immediate consequence of this is that eigenstates of different energies are not orthogonal. Thus the completeness and orthogonality properties of the set of all wave functions are destroyed. One should not take the above argument as a restriction against velocity dependent or $\vec{\ell}\cdot\vec{s}$ dependent potentials. Although these potentials are sometimes referred to as energy dependent this is somewhat of a misnomer since these operators really act on the momentum

variables of the wave function. An apparent energy dependence is only acquired when one goes on-shell with the momentum variables.

Another approach to separable approximations has been given by Lovelace.[16] It is instructive to compare his results with those resulting from the Kowalski-Noyes representation. Instead of looking at the t matrix Lovelace considers the K matrix. The K matrix satisfies the principal value form of the Lippman-Schwinger equation (III. 1) obtained when the $i\epsilon$ is dropped and the integral becomes a principal value integral. Let $A_\ell(p, q;k^2)$ be the amplitude which is the solution of the K-matrix equation. The relation between A_ℓ and t_ℓ derived by Lovelace is

$$t_\ell(p, q;k^2+i\epsilon) = A_\ell(p, q;k^2) - \frac{2\pi^2 i k A_\ell(p, k;k^2)A_\ell(k, q;k^2)}{1 + 2\pi^2 i k A_\ell(k, k;k^2)} \qquad \text{(III. 34)}$$

The t_ℓ which Lovelace uses is $4\pi^2$ times the one we have used. However, this different normalization will not affect the equations which follow. The completely on-shell form of Eq. (III. 34) simplifies to

$$t_\ell(k, k;k^2+i\epsilon) = \frac{A_\ell(k, k;k^2)}{1+2\pi^2 i A_\ell(k, k;k^2)} \quad . \qquad \text{(III. 35)}$$

If we set $q = k$ in Eq. (III. 34) and use Eq. (III. 35) we obtain after a few algebraic manipulations

$$\frac{A_\ell(p, k;k^2)}{A_\ell(k, k;k^2)} = \frac{t_\ell(p, k;k^2)}{t_\ell(k, k;k^2)} \equiv f_\ell(p, k) \qquad \text{(III. 36)}$$

Equation (III. 36) shows that the f_ℓ we defined earlier is also the ratio of the half-off-shell K-matrix amplitude to the on-shell K-matrix amplitude.

The on-shell K-matrix is related to the phase shift by

$$A_\ell(k, k; k^2) = -\frac{1}{2\pi^2} \; k \cot \; \delta_\ell \qquad \text{(III. 37)}$$

Thus A_ℓ will have a pole for each resonant energy $\left(\delta_\ell(k_r) = \pi/2\right)$. This is in contrast to the resonant behavior of t_ℓ, where the pole corresponding to a resonance lies in some neighboring non-physical sheet. A_ℓ will, of course, have all the bound state poles of t_ℓ since for negative energies A_ℓ and t_ℓ are identical. The operator represented by the kernel of the K matrix equation is compact and operator analytic in the energy variable. Consequently the residue of each pole is of finite rank, or equivalently a finite sum of separable products. If there is only one bound state or resonance at the pole position, k_r, then

$$A_\ell(p, q; k^2) \cong \frac{g_\ell(p)\, g_\ell(q)}{k_r^2 - k^2} \;, \quad k \cong k_r \qquad \text{(III. 38)}$$

Combining Eq. (III. 38) and Eq. (III. 36) allows us to relate our f_ℓ to Lovelace's form factor g_ℓ. We have

$$f_\ell(p, k) \cong \frac{g_\ell(p)}{g_\ell(k_r)} \;, \quad k \cong k_r \qquad \text{(III. 39)}$$

where the relation is exact for $k = k_r$. It is interesting to see what our separable approximation becomes when expressed in terms of g_ℓ via Eq. (III. 39). By using Eq. (III. 35) and Eq. (III. 38) we can approximate the on-shell t matrix by

$$t_\ell(k, k; k^2 + i\epsilon) \cong \frac{g_\ell(k)^2}{(k_r^2 - k^2) + 2\pi i k g(k)^2} \qquad \text{(III. 40)}$$

Now with the aid of Eq. (III. 39) we can write our separable approximation as

$$f_\ell(p,k)\, t_\ell(k,k;k^2+i\epsilon)\, f_\ell(q,k) \simeq \frac{g_\ell(p)\, g_\ell(q)}{(k_r^2-k^2)+2\pi i k g(k)^2} \qquad \text{(III. 41)}$$

where we have equality for $k = k_r$. The right hand side of Eq. (III. 41) is just the separable approximation that Lovelace derives by combining Eq. (III. 38) and Eq. (III. 34). Consequently, we may obtain Lovelace's separable approximation from the Kowalski-Noyes representation just by setting $k = k_r$ in the half-off-shell functions f_ℓ . In addition these results show that the separable term of the Kowalski-Noyes representation contains all of the resonance pole contributions to the amplitude.

The underlying idea of the approximations advocated by Lovelace is one of pole dominance. Basically we are hoping that if we have an expression which is accurate in the region where the amplitude has a pole then the three-body results obtained when this approximate expression is used in lieu of the exact amplitude will differ by a small amount. There are good reasons to be skeptical of this pole dominant idea even before carrying out detailed numerical comparisons. The range of two-body energies for which the amplitude t_ℓ is needed in solving Faddeev's equation varies from the energy of the three-body system to $-\infty$. For problems where the three-body scattering energy is considerably below that of the nearest two-body resonance the ingredient two-body amplitude is required only in the region where it is small and distant from any pole. Therefore constructing an approximation which is optimized at the pole will not necessarily yield an accurate approximation in the region of energies used in a three-body calculation.

CHAPTER IV

REDUCTION OF FADDEEV-OMNES INTEGRAL EQUATION TO TWO VARIABLES

In this chapter, we continue our analysis of Faddeev's equations in their angular momentum form. The final result is to reduce Faddeev's equations to an infinite set of coupled integral equations in two continuous variables. The final equations will become a finite set if there are only a finite number of partial waves contributing to the ingredient two-body amplitudes. First of all, we change our notation so that it is more in accord with our previously published work.[37] Let us re-express the two-body amplitude T_i in the three-body Hilbert space. From Eq. (II.51) we have

$$\langle \vec{\omega}' J \lambda' \left| T_i \right| \vec{\omega} J \lambda \rangle = \frac{m_k + m_j}{2\pi p_i} \delta(\omega_i - \omega_i') \hbar^2 \sum_{\ell} \sum_{m} (2\ell + 1) \cdot$$

$$\frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(\cos\gamma_i) P_\ell^m(\cos\gamma_i') d^J_{\lambda' m}(+\alpha_i') d^J_{m\lambda}(-\alpha_i) \qquad \text{(IV.1)}$$

$$t_\ell^{(i)}\left(E' - r_i \omega_i' , E - r_i \omega_i ; z - r_i \omega_i\right) .$$

Here we have used

$$Y_{\ell m}(\theta, \phi) = (-1)^m \left[\frac{(2\ell+1)(\ell - m)!}{4\pi (\ell + m)!}\right]^{1/2} P_\ell^m(\cos\theta) e^{i m \phi} \qquad \text{(IV.2)}$$

to replace the $Y_{\ell\lambda}(\gamma, 0)$ with $P_\ell^\lambda(\cos\gamma)$. Furthermore, we do not assume that the body-fixed axis lies along the direction of particle 1, but only that it lies in the plane of the three particles. The angle α_i is the angle between $\vec{p}_i$, measured counter-clockwise, and the body fixed axis $\vec{\alpha}$. For example, in Eq. (II.51) α_i is $\alpha_i = \theta_{1i} = -\theta_{i1}$. In writing Eq. (IV.1), we have written the transition

amplitude as $t^{(i)}\left(q'^2_{jk}/2\mu_{jk},\ q^2_{jk}/2\mu_{jk};\ z - r_i\,\omega_i\right)$ instead of $t^{(i)}\left(q'_{jk},\ q_{jk},\ z - r_i\,\omega_i\right)$. Then the arguments of $t^{(i)}$ have been written in terms of the $\vec{\omega}$ via Eq. (II.24). For simplicity, we have defined $E = \omega_1 + \omega_2 + \omega_3$, which is the total intermediate energy. Lastly, r_i is a dimensionless ratio of masses given by

$$r_i = \frac{m_i}{n_i} = \frac{m_1 + m_2 + m_3}{m_j + m_k} \quad . \tag{IV.3}$$

A. Expansion of $T_i(z)$ in Terms of Two Variable Functions

Our two-body partial wave transition amplitude is that defined in Chapter III with on-shell normalization $e^{i\delta_\ell}\sin\delta_\ell/k$. The critical observation which allows this derivation to proceed is that $t^{(i)}_\ell$ depends only on $\vec{\omega}'$ through the combination $E' - r_i\,\omega'_i$. It is this fact, not the presence of the delta function in Eq. (IV.1), which allows us to get a set of integral equations in two variables. What the delta function does, however, is that it allows us to write the resulting kernel in algebraic form.

Now let us substitute Eq. (IV.1) into Faddeev's equation and factor out all the common terms we can. This gives us

$$\begin{aligned}\langle\vec{\omega}'J\lambda'\left|T^i(z)\right|\vec{\omega},J,\lambda\rangle = \frac{m_j+m_k}{2\pi p'_i}\sum_{\ell=0}^{\infty}\sum_{m=-J}^{J}(2\ell+1)\,\frac{(\ell-m)!}{(\ell+m)!}\,P_\ell^m(\cos\gamma'_i)\,d^J_{\lambda m}(+\alpha'_i)\\ \Bigg\{\delta(\omega_i-\omega'_i)P_\ell^m(\cos\gamma_i)\,d^J_{m\lambda}(-\alpha_i)\,t^{(i)}_\ell(q_{jk},q'_{jk},\ z-r_i\,\omega_i)\\ -\sum_{\lambda''=-J}^{J}\int d\vec{\omega}''\,P_\ell^m(\cos\gamma''_i)\,d^J_{m\lambda''}(-\alpha''_i)\,t^{(i)}_\ell(q''_{jk},q'_{jk},z-r_i\omega''_i)\\ \times\,\delta(\omega''_i-\omega'_i)\,\frac{\langle\vec{\omega}''J\lambda''\left|T^j(z)+T^k(z)\right|\vec{\omega}J\lambda\rangle}{E''-z}\Bigg\}\end{aligned} \tag{IV.4}$$

In Eq. (IV.4), the behavior in $\vec{\omega}'$ of the bracketed expression comes only from the term t_ℓ. As noted before, this term depends just on E' and the independent particle energy ω_i. Thus we have justified the following expansion for $T^i(z)$.

$$\langle \vec{\omega}' J \lambda' \left| T_i(z) \right| \vec{\omega} J \lambda \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-J}^{J} (2\ell+1) \frac{(\ell-m)!}{(\ell+m)!} \, d^J_{\lambda' m}(+\alpha_i') \, P_\ell^m(\cos \gamma_i') \tag{IV.5}$$

$$F_{\ell m}^{P(i)}(E', e_i')$$

where $e_i' = \omega_i'$. We note that using this expansion forces us to use a different coordinate system for each different value of i. We can get an integral equation for the $F_{\ell m}$ just by substituting Eq. (IV.5) into Eq. (IV.4). Equating terms with coefficients $(2\ell + 1)\frac{(\ell - m)!}{(\ell + m)!} \; P_\ell^m(\cos \gamma_i) \, d^J_{\lambda' m'}(\alpha_i')$ gives us

$$F_{\ell m}^{P(i)}(E', e_i') = \frac{(m_i + m_k)}{2\pi p_i'} \left\{ \delta(\omega_i - \omega_i') \, P_\ell^m(\cos \gamma_i) d^J_{m\lambda}(-\alpha_i) \, t_\ell^{(i)}(E' - r_i\omega_i', E - r_i\omega_i; z - r_i\omega_i) \right.$$

$$- \int d\vec{\omega}'' \sum_{\lambda''=-J}^{J} \sum_{\ell''=0}^{\infty} \sum_{m''=-J}^{J} \sum_{s=j,k} \frac{t_\ell^{(i)}\left(E' - r_i e_i', \, E'' - r_i e_i'', \, z - r_i e_i''\right)}{E'' - z} \tag{IV.6}$$

$$\delta(\omega_i'' - \omega_i') \; (2\ell'' + 1)\frac{(\ell'' - m'')!}{(\ell'' + m'')!} \; P_\ell^m(\cos \gamma_i'') \, P_{\ell''}^{m''}(\cos \gamma_s'')$$

$$\left. d^J_{m\lambda''}(-\alpha_i'') \, d^J_{\lambda'' m''}(+\alpha_s'') \quad F_{\ell'' m''}^{P(s)}(E'', e_s'') \right\}$$

This is our basic result and from now on we will just simplify it as much as we can. The λ'' sum can be done by using the addition formulas for d functions,

which read

$$\sum_{\lambda''=-J}^{J} d^J_{m\lambda''}(-\alpha_i'')\, d^J_{\lambda''m''}(+\alpha_s'') = d^J_{mm''}(\alpha_s'' - \alpha_i'')$$

$$= d^J_{mm''}(-\theta''_{si}) = d^J_{mm''}(\theta''_{is}) \qquad \text{(IV.7)}$$

The last pair of equalities follows from simple angular geometry, and has the interesting consequence that the body-fixed axis coordinates α_i drop out of the kernel. They still, of course, remain in the inhomogeneous term.

B. Simplification of the Kernel

From the integral term in Eq. (IV.6), we see that we must change the variables in integration so that they are compatible with E'', e_s''. We note that the third independent variable orthogonal to E, e_j is $\cos\gamma_j$, where $\cos\gamma_j$ is the angle between $\vec{q}_{jk}$ and $\vec{p}_i$, given in Eq. (II.23). Explicitly, the transformation from coordinates $\vec{\omega}$ to $E, e_j, \cos\gamma_j$ is given by

$$\omega_i = \frac{m_k}{m_i+m_k}\left\{\left[\frac{m_j m_i}{m_k(m_i+m_k)} - r_j\right] e_j + E + 2\left(\frac{m_j m_i e_j}{m_k(m_k+m_i)}\right)^{\frac{1}{2}} (E - r_j e_j)^{\frac{1}{2}} \cos\gamma_j\right\}$$

$$\omega_j = e_j \qquad \text{(IV.8)}$$

$$\omega_k = \frac{m_i}{m_i+m_k}\left\{\left[\frac{m_k m_j}{m_i(m_i+m_k)} - r_j\right] e_j + E - 2\left(\frac{m_j m_k e_j}{m_i(m_k+m_i)}\right)^{\frac{1}{2}} (E - r_j e_j)^{\frac{1}{2}} \cos\gamma_j\right\}$$

Cyclic permutation will give us any of the other transformations that we need. The range of e_j is still from 0 to ∞ if $r_j e_j \leq E \leq \infty$, while $\cos\gamma_j$ can vary from -1 to +1 independent of the energies. The Jacobian for this variable

transformation is

$$\frac{\partial(\omega_i, \omega_k)}{\partial(E, \cos\gamma_j)} = \frac{\sqrt{2m_j e_j}\,\sqrt{2\mu_{ik}}}{m_i + m_k}\,\sqrt{E - r_k\omega_k} \tag{IV.9}$$

Introducing the new coordinate systems into Eq. (IV.6) and using Eq. (IV.7) allows us to write

$$F^{P(i)}_{\ell m}(E', e'_i) = \frac{(m_j + m_k)\hbar^2}{2\pi\sqrt{2m_i e'_i}} \left\{ \delta(\omega_i - e'_i)\, P^m_\ell(\cos\gamma_i)\, d^J_{\lambda m}(-\alpha_i) \right.$$

$$t^{(i)}_\ell(E - r_i e_i, E' - r_i e'_i, z - r_i e_i) - \sum_{s=j,k} \int_0^\infty de''_s \int_{r_s e''_s}^\infty dE''$$

(IV.10)

$$\left. \frac{t^{(i)}_\ell(E' - r_i e'_i, E'' - r_i e'_i; z - r_i e'_i)}{E'' - z} \sum_{\ell''m''} K^{(i,s)}_{\ell m;\ell''m''}(E', e'_i; E'', e''_s)\, F^{P(s)}_{\ell''m''}(E'', e''_s) \right\}$$

where

$$K^{(i,s)}_{\ell m;\ell''m''}(E', e'_i; E'', e''_s) = \frac{2m_s e''_s}{m_s + m_i}\,\sqrt{2\mu_{is'}}\,\sqrt{E'' - \gamma_s e''_s} \tag{IV.11}$$

$$\times \int_{-1}^{+1} d\cos\gamma''_s\, \delta\left(e'_i - \omega''_i(E'', e''_s, \cos\gamma''_s)\right) (2\ell'' + 1)\frac{(\ell'' - m'')!}{(\ell'' + m'')!}$$

$$\times P^m_\ell\left(\cos\gamma_i(E'', e''_s, \cos\gamma''_s)\right) P^{m''}_{\ell''}(\cos\gamma''_s)\, d^J_{mm''}(\theta''_{is})$$

Here $s' = k$ if $s = j$, or $s' = j$ if $s = k$.

We may think of $K^{(i,s)}_{\ell m;\ell'' m''}$ as just the kinematic part of the kernel. Doing the integration over the delta function gives

$$K^{(i,s)}_{\ell m;\ell'' m''}(E', e'_i; E'', e''_s) = (2\ell''+1)\frac{(\ell''-m'')!}{(\ell''+m'')!} P^m_\ell\left(\cos\gamma_i(E'', e''_s, \cos\Gamma''_s)\right)$$

$$P^{m''}_{\ell''}(\cos\Gamma''_s)\, d^J_{mm''}(\theta''_{is}) \left\{\Theta\left[E'' - r_s e''_s - \left(\sqrt{\frac{m_{s'}\, m_i}{m_{s'}}}\, e'_i - \sqrt{\frac{m_i m_s}{m_{s'}(m_{s'}+m_i)}}\, e''_s\right)^2\right]\right. \tag{IV.12}$$

$$\left. - \Theta\left[E'' - r_s e''_s - \left(\sqrt{\frac{m_{s'}+m_i}{m_{s'}}}\, e'_i + \sqrt{\frac{m_i m_s e''_s}{m_{s'}(m_s+m_i)}}\right)^2\right]\right\}$$

where $\cos\ \Gamma_s$ is the value of $\cos\gamma''_s$ determined by the delta function in Eq. (IV.11) and is given by

$$\cos\ \Gamma_s(e'_i, E'', e''_s) = \pm \frac{\frac{m_{s'}+m_i}{m_{s'}}\, e'_i - (E'' - r_s e''_s) - \frac{m_i m_s e''_s}{m_{s'}(m_{s'}+m_i)}}{2\left(\frac{m_i m_s e''_s}{m_{s'}(m_{s'}+m_i)}\right)^{1/2}(E'' - r_s e'')^{1/2}} \tag{IV.13}$$

where the + applies for s = j and the - for s = k.

The argument of the d function θ''_{is} must be determined in terms of E'', e''_s. This can be done by taking the ratio of $\sin^2\theta_{is}$ and $\sin^2\gamma_s$. From Eq. (II.26) and Eq. (II.27), we have

$$\frac{\sin^2\theta_{is}}{\sin^2\gamma_s} = \frac{\frac{4m_1m_2m_3}{(m_i+m_{s'})}\omega_s\left(\omega_i+\omega_s+\omega_{s'} - \left(1+\frac{m_s}{m_{s'}+m_k}\right)\omega_s\right)}{4\, m_i m_s \omega_i \omega_s}$$

$$= \frac{m_{s'}}{m_i + m_{s'}}\ \frac{(E - r_s\omega_s)}{\omega_i} \tag{IV.14}$$

Thus the argument to be used in $d^J_{mm''}(\theta''_{is})$ is determined by

$$\sin^2\theta''_{is} = \frac{m_{s'}}{m_i+m_{s'}} \sin^2\Gamma_s(e'_i, E'', e''_s) \frac{(E''-r_s e''_s)}{\omega_i(E'', e''_s, \Gamma_s)} \qquad \text{(IV.15)}$$

Eqs. (IV.10), (IV.12), (IV.13) and (IV.15) represent our reduction to a coupled set of integral equations in two continuous variables. If a finite number of partial waves dominate the two-body transition amplitude (and this is frequently the case) then the infinite sum over ℓ and ℓ'' will become finite. Thus our formalism is particularly suited to bound state and low energy scattering where the s-wave is dominant or in energy regions where there are resonances in some partial wave channel. In fact,as we will later note, for three-body bound states involving n-p like forces the correction to assuming that the s-wave dominates is about one percent.

C. Three Identical Boson Problems

The reduction to a set of integral equations in two variables marks the end of our study of the general three-body formalism. From now on we shall restrict ourselves to the study of one special three-body problem, that of three identical bosons. We shall make one assumption, namely that the two-body amplitude is dominated by the ℓ=0 partial wave. The reasons for making these assumptions is to simplify the problem enough so that we can obtain numerical solutions. It is our goal to obtain solutions with no further approximations. In particular, we do not want to assume, as is conventional, that the two-body amplitude is separable. Part of our motive in doing this work is to obtain an evaluation of the separable approximation in three-body problems, and in order to do this we need more general solutions.

First of all, we specialize our results to three identical particles acting only through two-body s-wave interactions. We will consider only the J=0

state. One might think that the J=0 state is implied by the assumption of s-waves in all the two-body interactions. This is not the case, since one cannot add the angular momenta of the different two-body systems because the coordinate systems are different. As Eq. (IV.6) shows, it is possible to have an amplitude with $J > 0$ when only $t_0^{(i)}(E' - r_i e'_i, E'' - r_i e''_i, z - r_i e''_i)$ are present. We note that kinematic part of the kernel $K^{(i,s)}_{\ell m;\ell'' m''}$ becomes just the difference of two theta functions when $\ell = m = \ell'' = m'' = 0$.

Now that we have restricted the scope of our problem such that we can hope to obtain solutions,we want to write our equations in a form that will be easily adaptable to numerical computation. Furthermore, since we are interested in the separable approximation we want our general equation to be as close as possible to the equations valid when separability is imposed. We will now reformulate Eq. (IV.10) so that these conditions are met. First we transform the theta functions in the kinematic kernel (IV.12) into lower and upper bounds on the E'' integration. Using the fact, that for identical particles the masses will all be the same, the condition for the upper bound on the E'' integration is from (IV.12)

$$E'' \leq \frac{3}{2} e''_s - \left(\sqrt{2e'_i} + \sqrt{\frac{e''_s}{2}}\right)^2$$

$$E'' \leq 2(e''_s + e'_i + \sqrt{e'_i e''_s}) \qquad \text{(IV.16)}$$

Similarly the condition for the lower bound on E'' is

$$E'' \geq 2(e''_s + e'_i - \sqrt{e'_i e''_s}) \qquad \text{(IV.17)}$$

For the case of J=0, ℓ=0, the expansion (IV. 5) is trivial since

$$d^{o}_{00}(\alpha'_i) = 1 = P^{o}_{0}(\cos\gamma'_i) \quad .$$

Thus the full three-body transition amplitude is

$$\langle \vec{\omega}' \, 00 \,|T(z)|\, \vec{\omega} \, 00 \rangle = \sum_{i=1}^{3} F^{P(i)}_{00}(E', e'_i) \qquad \text{(IV. 18)}$$

Since the three particles are assumed identical, the full transition amplitude will be symmetric under the interchange of any two particles. Therefore, the three F_{00} functions will be the same. Dropping the two zero subscripts to simplify the notations, we have

$$F^{P(1)}(x,y) = F^{P(2)}(x,y) = F^{P(3)}(x,y) \qquad \text{(IV. 19)}$$

For this special case Faddeev's equation, (IV. 10), reduces to

$$F(E', e'_i) = \frac{m}{\pi\sqrt{2me'_i}} \left\{ \delta(e_i - e'_i)\, t\left(E - \frac{3}{2}e_i,\, E' - \frac{3}{2}e'_i\,;\, z - \frac{3}{2}e_i\right) \right.$$

$$\left. - \sum_{s=j,k} \int_0^\infty de''_s \int_{2(e''_s + e'_i - \sqrt{e'_i e''_s})}^{2(e''_s + e'_i + \sqrt{e'_i e''_s})} dE'' \, \frac{t\left(E' - \frac{3}{2}e'_i,\, E'' - \frac{3}{2}e'_i;\, z - \frac{3}{2}e'_i\right)}{E'' - z} \right.$$

$$\left. F(E'',\, e''_s) \right\} \qquad \text{(IV. 20)}$$

Since the F has no s dependence except through the argument e''_s, and the kernel has no s dependence we may drop the variable subscripts in (IV. 20). Already we have dropped the superscript on t, which are all identical. Our

equation becomes then

$$F(E',e') = \frac{m}{\pi\sqrt{2me_i}} \left\{ \delta(e_i - e_i')\, t\left(E - \frac{3}{2}e_i,\ E' - \frac{3}{2}e_i';\ z - \frac{3}{2}e_i'\right) \right.$$

$$\left. - 2\int_0^\infty de'' \int_{2(e''+e'-e'e'')}^{2(e''+e'+e'e'')} \frac{t\left(E' - \frac{3}{2}e',\ E'' - \frac{3}{2}e';\ z - \frac{3}{2}e'\right) F(E'',e'')}{E'' - z} \right\} \qquad \text{(IV. 21)}$$

The above equation is still not suited for a numerical calculation, even when $z < 0$. If we iterate (IV. 21) once we will have an integrand of the type

$$\int_0 \frac{de''}{\sqrt{e''}}$$

This, of course, an integrable singularity, but not an easy thing to do numerically. The problem vanishes if we change to momentum coordinates. Therefore, let us introduce the following change of variables.

$$e' = \frac{q'^2}{2m} \quad , \qquad de' = \frac{q'}{m}\, dq' \quad , \qquad \sqrt{2me'} = q' \qquad \text{(IV. 22)}$$

With this coordinate transformation Faddeev's equation now reads

$$F(E',q') = \frac{m}{\pi q'} \left\{ t\left(E' - \frac{3}{4}\frac{q'^2}{m},\ E - \frac{3}{4}\frac{q'^2}{m};\ z - \frac{3}{4}\frac{q'^2}{m}\right)\delta\left(\frac{q'^2}{2m} - e\right) \right.$$

$$\left. + 2\int_0^\infty \frac{q''dq''}{m} \int_{\frac{1}{m}(q'^2+q''^2-q'q'')}^{\frac{1}{m}(q'^2+q''^2+q'q'')} dE'' \frac{t\left(E' - \frac{3}{4}q'^2,\ E'' - \frac{3}{4}\frac{q'^2}{m};\ z - \frac{3}{4}\frac{q'^2}{m}\right)}{z - E''} F(E'',q'') \right\} \qquad \text{(IV. 23)}$$

The units will be simpler and the notation more symmetric if we now transform $E'' \rightarrow \frac{k''^2}{m}$. Dropping the ubiquitous factor of m in the first two arguments of t, we have

$$F(k'^2, q') = \frac{m}{\pi q'} \, t\left(k'^2 - \frac{3}{4} q'^2, k^2 - \frac{3}{4} q'^2; z - \frac{3}{4} \frac{q'^2}{m}\right) \delta\left(\frac{q'^2}{2m} - e\right)$$

$$+ \frac{4}{\pi} \int_0^\infty q''^2 dq'' \left[\int_{q'^2+q''^2-q'q''}^{q'^2+q''^2+q'q''} dk''^2 \; \frac{1}{2q'q''} \right.$$

$$\left. \frac{t\left(k'^2 - \frac{3}{4} q'^2, k''^2 - \frac{3}{4} q'^2; z - \frac{3}{4} \frac{q'^2}{m}\right)}{mz - k''^2} \; F(k''^2, q'') \right] \qquad \text{(IV. 24)}$$

This equation has been checked against a similar one given by Wong and Zambotti. After making some slight changes in notation we found that the two equations are exactly the same. This verifies that our angular momentum reduction and simplification to two variables is correct for the J=0, ℓ=0 case we are now considering.

D. Inclusion of Kowalski-Noyes Representation

We will not modify Eq. (IV. 24) so that the effect of a separable two-body transition amplitude is apparent. We assume, as is described in Chapter III, that the two-body transition amplitude separates, or approximately does, in the following form:

$$t\left(k'^2 - \frac{3}{4} q'^2, k''^2 - \frac{3}{4} q'^2; z - \frac{3}{4} \frac{q'^2}{m}\right)$$

$$\cong f\left(\sqrt{k'^2 - \frac{3}{4} q'^2}\,; \; z - \frac{3}{4} \frac{q'^2}{m}\right) \; t\left(z - \frac{3}{4} \frac{q'^2}{m}\right) \; f\left(\sqrt{k''^2 - \frac{3}{4} q'^2}\,; \; z - \frac{3}{4} \frac{q'^2}{m}\right) \qquad \text{(IV. 25)}$$

where $t\left(z - \frac{3}{4}\frac{q'^2}{m}\right)$ is the on-shell t matrix.

Let us define a new amplitude $H(k'^2, q')$ given by

$$F(k'^2, q') = f\left(k'^2 - \frac{3}{4}q'^2,\ z - \frac{3}{4}\frac{q'^2}{m}\right)\ H\ (k'^2, q') \tag{IV. 26}$$

The virtue of this substitution, as we will later shall see, is that when t is separable then $H(k'^2, q')$ is constant in the variable k'^2. The equation for H is

$$f\left(k'^2 - \frac{3}{4}q'^2,\ z - \frac{3}{4}\frac{q'^2}{m}\right)\ H\ (k'^2, q') = I$$

$$+ \frac{4}{\pi}\int_0^\infty q''^2 dq'' \left[\frac{1}{2q'q''}\int_{q'^2+q''^2-q'q''}^{q'^2+q''^2+q'q''} dk''^2\ \frac{1}{mz - k''^2}\right. \tag{IV. 27}$$

$$\left. t\left(k'^2 - \frac{3}{4}q'^2, k''^2 - \frac{3}{4}q'^2; z - \frac{3q'^2}{4m}\right)\ f\left(k''^2 - \frac{3}{4}q''^2,\ z - \frac{3q'^2}{4m}\right)\ H\ (k''^2, q'')\right] .$$

Here I represents the inhomogeneous term. The square roots of the momentum arguments of f are dropped for simplicity. It should be clear that Eq. (IV. 27) is still general and remains valid whether or not t is separable.

Now let us assume that Eq. (IV. 25) is valid, or at least that replacing the kernel by the separable term of the t matrix is a good approximation. Substituting Eq. (IV. 25) into (IV. 27) we note the entire dependence involving $k'^2 - \frac{3}{4}q'^2$ is carried by an f. Thus the f's cancel on both the right and left.

We are left with

$$H(k'^2, q') = \frac{m}{\pi q'} \delta\left(\frac{q'^2}{2m} - e\right) f\left(k^2 - \frac{3}{4} q'^2,\ z - \frac{3}{4}\frac{q'^2}{m}\right) t\left(z - \frac{3}{4}\frac{q'^2}{m}\right)$$

$$+ \frac{4}{\pi} t\left(z - \frac{3}{4}\frac{q'^2}{m}\right) \int_0^\infty q''^2 dq'' \left[\frac{1}{2q'q''} \int_{q'^2+q''^2-q'q''}^{q'^2+q''^2+q'q''} \right.$$

$$\left. dk''^2 \frac{f\left(k''^2 - \frac{3}{4} q'^2; z - \frac{3q'^2}{4m}\right) f\left(k''^2 - \frac{3}{4} q''^2, z - \frac{3q''^2}{4m}\right)}{mz - k''^2} \right] H(k''^2, q'')$$

(IV. 28)

In examing Eq. (IV. 28) we see that neither the driving term nor the kernel has a k' dependence. Thus we have proved that when t is separable, H is constant in its k' dependence. The one variable integral equation satisfied by $H(k'^2, q') \equiv H(q')$ is

$$H(q') = \frac{m}{\pi q'} \delta\left(\frac{q'^2}{2m} - e\right) f\left(k^2 - \frac{3}{4} q'^2, z - \frac{3}{4}\frac{q'^2}{m}\right) t\left(z - \frac{3}{4}\frac{q'^2}{m}\right)$$

(IV. 29)

$$+ \frac{4}{\pi} t\left(z - \frac{3}{4}\frac{q'^2}{m}\right) \int_0^\infty q''^2 dq'' M(q', q'', z) H(q'')$$

where the kernel M is just

$$M(q', q'', z) = \frac{1}{2q'q''} \int_{q'^2+q''^2-q'q''}^{q'^2+q''^2+q'q''} dk''^2 \frac{f\left(k''^2 - \frac{3}{4} q'^2; z - \frac{3q'^2}{4m}\right) f\left(k''^2 - \frac{3}{4} q''^2, z - \frac{3q''^2}{4m}\right)}{mz - k''^2}$$

(IV. 30)

The simplicity of Eq. (IV. 29) relative to the two variable comparison equation (IV. 27) shows how effective the separable assumption is in simplifying the problem. The one variable equation for $H(q')$ can be solved in a few seconds on computer, whereas the two-variable equation requires two orders of magnitude more time. Furthermore, the variable limit of integration in the interior integral makes the two variable integral equation numerically difficult to handle. For these reasons no one has been able, until now, to obtain direct solutions for (IV. 27).

In the appendix to this section, Appendix B, we present the formula for $M(q', q'', z)$ which results when simple separable potentials are used.

CHAPTER V

NUMERICAL METHODS

We will now describe the numerical techniques used to solve our one- and two-variable integral equations. The difficult equation is, of course, the two-variable form, Eq. (IV.27), of Faddeev's equations for three identical particles. The kernel of this equation is itself a function which must be obtained by solving an integral equation. So let us turn first to the study of the easier one-variable integral equations.

A. Quadrature Rules

Our basic objective is to describe the techniques which allow us to approximate an integral equation by a finite linear system which may be easily solved with a computer. Suppose the unknown function, f(x), satisfies the equation

$$f(x) = g(x) + \int_a^b K(x,y)\, f(y)\, dy, \qquad a \le x \le b \tag{V.1}$$

Here it is assumed that both g(x) and K(x,y) are known functions. This discussion will also assume that a unique solution, f(x), does exist. This question of the existence of a solution can usually be examined a priori either with Fredholm theory or by showing that the kernel K represents a compact operator. The first step in transforming Eq. (V.1) into a finite matrix problem is to replace the integral with a sum.

Rules which approximate integrals by finite sums are called quadrature rules. The general form of these rules follows. Let the set,

$$\mathcal{D}(N) = \left\{ y_i : y_i \in (a,b), \quad i \le N \right\}$$

be a discrete set of N points on the interval (a,b) of integration. It is useful to write the integrand as a product of two factors w(x), h(x). All the singularities

and rapidly varying portions of the integrand are placed in $w(x)$. On the other hand it is intended that $h(x)$ be a smooth, singularity-free function with a rapidly converging Taylor series. With this notation we may now write down the general form of a quadrature rule.

$$\int_a^b h(y)\, w(y)\, dy = \sum_{i=1}^{N} h(y_i)\, w_i + \mathcal{E}_N(h) \qquad (V.2)$$

The factor $w(y)$ is usually called the weighting function, w_i are the weights, and $\mathcal{E}_N(h)$ is the error associated with this quadrature rule. The weights $\{w_i\}$ are a set of N numbers, determined by requiring that $\mathcal{E}_N(h)$ be small. Specifically, the weights may be calculated by requiring that $\mathcal{E}_N(y^k) \equiv 0$ for $k = 0, 1, \ldots, N-1$. That is, we calculate

$$I_k \equiv \int_a^b y^k\, w(y)\, dy$$

either analytically or by some numerical scheme for each integer value of k and then solve the linear $N \times N$ matrix problem given by

$$I_k = y_1^k w_1 + y_2^k w_2 + \ldots + y_N^k w_N, \quad k = 0, \ldots, N-1 \qquad (V.3)$$

for the w_i.

Such a quadrature rule has a simple relation to the Taylor series expansion of $h(y)$. Let the series be written

$$h(y) = \sum_{n=0}^{\infty} a_n y^n \qquad y \in (a,b)$$

$$= \sum_{n=0}^{N-1} a_n y^n + R_N(y) \qquad (V.4)$$

where $R_N(y)$ is the remainder term. By definition of the w_i the error term is

just

$$\mathcal{E}_N(h) = \int_a^b R_N(y)\, w(y)\, dy \qquad (V.5)$$

Consequently the first N terms in the power series for h(y) are integrated exactly by our quadrature rule, Eq. (V.2).

A quadrature rule is considered acceptable if $|\mathcal{E}_N(\tilde{k})| < \delta$, where δ is some small positive number. For a given δ we just increase N until the above inequality is satisfied. For successful integration rules this value of N should be less than 10 or 20.

Now let us convert Eq. (V.1) into a matrix equation by using a quadrature rule to replace the integral term. Thus, we have

$$f(x) = g(x) + \sum_{i=1}^{N} K(x,y_i)\, f(y_i)\, w_i + \mathcal{E}_N\big(K(x,\,.)f(.)\big) \qquad (V.6)$$

This equation is valid for all $x \in (a,b)$. If we consider only the x given by the N point subset $\mathcal{D}(N)$ of (a,b) then Eq. (V.6) becomes N equations for N unknowns $f(y_i)$. Assuming that the quadrature rule is accurate means $\mathcal{E}_N\big(K(x,\,.)f(.)\big) < \delta$ where δ is small enough so that $\mathcal{E}_N\big(K(x,\,.)\,f(.)\big)$ may be dropped from the right hand side of Eq. (V.6). The resulting $N \times N$ matrix equation is then

$$\tilde{f}(x_i) = g(x_i) + \sum_{j=1}^{N} K(x_i, y_j)\ \tilde{f}(y_j)\, w_j \qquad (V.7)$$

We write a tilde over f to indicate that this equation is different from Eq. (V.6) because of the neglect of $\mathcal{E}_N\big(K(x,\,.)\,f(.)\big)$. The linear system represented by Eq. (V.7) can easily be solved on a computer if N is not too large. Since the computational time required to solve this system is proportional to N^3 it is important to find the smallest N consistent with not introducing intolerable errors.

B. Error Analysis

The function $\mathcal{E}_N\left(K(x_i, .)\, f(.)\right)$ is unattainable unless $f(x)$ is exactly known; however, it is hopefully small when our integration rules are adequate. We now want to explore the consequence of neglecting $\mathcal{E}_N\left(K(x_i, .)\, f(.)\right)$. As above let $\tilde{f}(x_i)$ be the solution of the resulting matrix equation, and let us ask how close $\tilde{f}(x_i)$ is to $f(x)$, the exact solution. The equation satisfied by $f(x_i)$ is

$$f(x_i) = g(x_i) + \sum_{j=1}^{N} K(x_i, y_j)\, f(y_j)\, w_j + \mathcal{E}_N\left(K(x_i, .)\, f(.)\right) . \tag{V.8}$$

Subtracting Eqs. (V.6) and (V.7) and defining the error in $\tilde{f}$ to be

$$E(x_i) = f(x_i) - \tilde{f}(x_i) ,$$

then $E(x_i)$ satisfies the integral equation

$$E(x_i) = \mathcal{E}_N\left(K(x_i, .)\, f(.)\right) + \sum_{j=1}^{N} K(x_i, y_j)\, E(y_j)\, w_j \quad . \tag{V.9}$$

Thus the error is determined by

$$E(x_i) = \sum_{j=1}^{N} \left[1 - Kw\right]^{-1}_{ij}\ \mathcal{E}_N\left(K(x_j, .)\, f(.)\right) \tag{V.10}$$

where $\left[1 - Kw\right]^{-1}_{ij}$ is the inverse of the matrix determined by $\delta_{ij} - K(x_i, y_i) w_j$. Hence, if $\left|\mathcal{E}_N\left(K(x_i .)\, f(.)\right)\right| \ll \left|g(x_i)\right|$ we can expect a small percentage error in replacing $f(x_i)$ by $\tilde{f}(x_i)$.

Once we have obtained $f(x_i)$ it is easy to find f for all x by using Eq. (V.6) and neglecting the error term. This method of interpolating f is generally much more accurate than linear or Langrangian schemes. One note of caution should be made about neglecting $\mathcal{E}_N\left(K(x, .)\, f(.)\right)$. It often is the case that $\mathcal{E}_N\left(K(x, .) f(.)\right)$

is a cyclic function of x with minima at x_i. So when the integral equation is used to interpolate $f(x)$ the error may be considerably larger for a general point x than that for the points in $\mathscr{D}(N)$. In spite of this, the integral equation is its own best interpolater.

C. Treatment of Lippman-Schwinger Equation

If our integral equation is the two-body Lippman -Schwinger equation of the separable one-variable three-body equation then the domain is the entire positive real axis. Consequently we need quadrature rules like Eq. (V.2) but valid for the infinite interval $[0,\infty)$. We obtain such rules by conformally mapping $[0,\infty)$ into a finite interval, say $[0,1)$, and then using Gaussian formulas for the finite interval. Specifically, suppose we want to integrate h on $[0,\infty)$, according to the quadrature rule

$$\int_0^\infty h(x)\,dx \cong \sum_{i=1}^{N} w_i\, h(x_i) \quad . \tag{V.11}$$

The problem is to determine w_i and x_i so that the approximation is the best we can obtain for N points.

First we consider the map $x \to y \in [0,1]$ given by

$$y = \frac{x}{1+x} \quad , \quad x = \frac{y}{1-y} \quad . \tag{V.12}$$

The Jacobian is

$$\frac{dx}{dy} = \frac{1}{(1-y)^2} = (1+x)^2 \geq 0. \tag{V.13}$$

Thus in the y coordinates our integral becomes

$$\int_0^\infty h(x)\,dx = \int_0^1 h(x(y))\,(1-y)^{-2}\,dy \tag{V.14}$$

Now Gauss' method gives us an easy way to do the integral on the right. This

method is constructed so that

$$\int_0^1 F(y)\,dy = \sum_{i=1}^{N} F(y_i)\,\tilde{w}_i \tag{V.15}$$

is exact for F in the class of polynomials of order $2N-1$ or less. Thus the $2N$ free constants, $\{y_1, y_2, \ldots, y_N, \tilde{w}_1, \ldots, \tilde{w}_N\}$, are chosen so that for $F \equiv \{1, y, y^2, \ldots, y^{2N-1}\}$, then Eq. (V.14) is exact. These values of abscissa y_i and weights w_i are tabulated[24] for many values of $N \leq 100$. By using the Gaussian abscissa and weights appropriate to Eq. (V.11) we obtain an integration rule for $h(x)$,

$$\int_0^\infty h(x)\,dx = \sum_{i=1}^{N} h(x_i)(1+x_i)^2\,\tilde{w}_i \tag{V.16}$$

where the x_i are determined by Eq. (V.12) from y_i. The weights for the quadrature rule, Eq. (V.11), are obviously,

$$w_i = (1+x_i)^2\,\tilde{w}_i \tag{V.17}$$

By using these techniques it is easy to turn the one-variable Lippman-Schwinger equation into a finite matrix equation. A typical example is the Lippman-Schwinger equation resulting from a Yukawa potential. In the boundstate region we can obtain $t(p, q, z)$ with a percentage error characteristically smaller than .1% if N is chosen to be 10. This is illustrative of the rapid convergence of the Gaussian quadrature rules. When the potential and therefore the kernel oscillate in sign, as in the case of a superposition of Yukawa potentials, then two to three times this number of points may be needed.

D. Convergence of Separable Three-Body Problem

Now let us examine the convergence of our quadrature rules for the one-variable separable three-body problem described by Eq. (IV.29). We

present the results in tabular form. The label Yamaguchi means that kernel M is derived from the Yamaguchi potential fitted to low energy N-P triplet scattering data. The label Yukawa means that M is derived from the separable f's coming from a Yukawa potential. The quadrature order is N

TABLE 1

N	YAMAGUCHI	YUKAWA
5	- 4.54	- 4.93
6	- 0.74	- 3.19
7	- 0.11	- 0.63
8	- 0.005	- 0.14
10	- 0.002	- - -

The percentage errors quoted represent the error in determining the three-body boundstate energy. This table shows us we can do fairly well with just 7 points. This observation becomes crucial when we attempt to do the two-variable integral equation.

At this juncture we should mention an alternate method of obtaining finite matrix equations which represent integral equations. If we know of a complete orthogonal basis we may expand the inhomogeneous term, the solution, and the kernel in terms of this basis. The integral equation is thereby transformed into a linear matrix equation for the coefficients of the expansion. Usually infinite matrix equations result and these are made finite by truncating the expansion for f(x) at some finite number of terms. Whether or not expanding in a set of basis functions is competitive in terms of computational ease and efficiency depends many factors: whether or not such a basis exists, how easily the basis functions are to obtain, if the expansion is rapidly convergent, whether or not the integrals determining the entries of the matrix kernel are easily done. As far as this author is aware there are no basis expansions for the two-body Lippman-Schwinger equation that compete with the 10-point Gaussian quadrature method. Wong and Zambotti did

expand the numerator function, $N(p, q, t)$ terms of powers of

$$\left(\frac{p^2}{p^2 + \mu^2}\right) \quad \text{and} \quad \left(\frac{q^2}{p^2 + \mu^2}\right) .$$

They claimed five terms in the series are sufficient for making accurate three-body calculations. This seems too optimistic a statement. We have done the same problem as they report in their paper but find the three-body boundstate 20% more tightly bound than they report. Thus the question of the relative merits of discretizing the functions or expanding them in a series of function remains open. Our method of discretizing the variable dependence is successful for solving the three identical particle problem but is not powerful enough yet to solve a more realistic physical problem such as N-D scattering.

In closing these comments on numerical methods for one-variable integral equations we should point out the following. Gauss' method as used in Eqs. (V.12) through (V.17) are equivalent to expanding $h\Big(x(y)\Big)(1 - y)^{-2}$ in polynomials in y on $[0,1]$. Thus for an N^{th} order quadrature rule we have the expansion

$$h\Big(x(y)\Big)(1 - y)^{-2} = \sum_{i=1}^{2N-1} a_i y^i , \qquad y \in [0,1] \tag{V.18}$$

Re-expressing this in terms of x gives

$$h(x) = \frac{1}{(1 + x)^2} \sum_{i=0}^{2N-1} a_i \left(\frac{x}{1 + x}\right)^i \tag{V.19}$$

If x is thought of as p^2 then this expansion is kindred to that of Wong and Zambotti's. We see that choosing $N = 10$ is equivalent to keeping 20 terms in the expansion, Eq. (V.17).

E. Discretization of the Two-Variable Integral Equation

We now turn our attention to transforming our two-variable integral equation into a matrix equation. The basic form of our equation is

$$H(x,y) = g(x,y) + \int_0^\infty dx' \int_{x'^2+x''^2-x'x''}^{x'^2+x''^2+x'x''} dy' \, K(x,y;x',y';z) H(x',y') , \qquad (V.20)$$

We can discretize the x variable by the methods described above. However, the y variable presents quite a problem. The domain of the y' integration is dependent on both x and x'.

Let us proceed by first discretizing the x dependence by a quadrature rule of the type Eq. (V.11). We have

$$H(x_i,y) = g(x_i,y) + \sum_{j=1}^{N_x} w_j \int_{x_i^2+x_j^2-x_ix_j}^{x_i^2+x_j^2+x_ix_j} dy' \, K(x_i,y,x_j,y';z) \, H(x_j,y) \qquad (V.21)$$

Suppose we have some discretization of y, say $\mathcal{D}_y(N_y) = \{ y_i : i = 1, \ldots, N_y \}$. For N_y a reasonably small number (20 or less) the average number of points inside the integration interval $(x_i^2 + x_j^2 - x_ix_j, \; x_i^2 + x_ix_j)$ will be a fraction of N_y and may frequently be zero. How can we have an accurate integration rule if we have no points at all in the interval over which the integration is performed? The answer is that if the function to be integrated may be expanded in polynomials on some larger domain than the domain of integration we can construct Gaussian-like quadrature rules which use points outside the domain of integration. Interestingly enough, Maxwell[38] found when studying quadrature rules for triple integrals over a cubic domain that he was forced to choose values of the integration variables outside the limit of integration.

Our procedure for obtaining integration rules for the y variable is as follows. We first assume some discretization of the y variable $\mathcal{D}_y(N_y)$. For each i j pair we construct a separate quadrature rule. We seek weights such that

$$\frac{1}{2x_i x_j} \int_{x_i^2+x_j^2-x_i x_j}^{x_i^2+x_j^2+x_i x_j} dy'\, S_{ij}(y',z)\, F(y') = \sum_{\ell=1}^{N_y} W(ij,z)_\ell\, F(y_\ell) \qquad (V.22)$$

where $S_{ij}(y', z)$ will contain all the singular structure in the kernel.

Although $S_{ij}(y', z)$ is singular, the singularities always lie outside the region of integration. Thus in extending the domain of integration outside the upper and lower limits we must be sure never to extend it so far as to include the singularities present in S. Basically, expanding the domain of integration assumes that Taylor series expansions of the integrand converge everywhere in the enlarged domain. Clearly when singularities of the integrand are encountered then the Taylor series will fail to converge. In practice it is found that using points very distant from the region of the upper and lower limits does not significantly improve the accuracy of the quadrature rule.

Define the points in the enlarged integration domain to be the set

$$\{\ell_{ij}, \ell_{ij+1}, \ldots, \ell_{ij+n_{ij}}\}.$$

The n_{ij} non-zero weights in our quadrature rule are determined by

$$\int_{x_i^2+x_j^2-x_i x_j}^{x_i^2+x_j^2+x_i x_j} dy'\, S_{ij}(y',z)\, y'^r = \sum_{\ell=\ell_{ij}}^{\ell_{ij}+n_{ij}} W(ij,z)_\ell\, (y_\ell)^r \qquad (V.23)$$

The n_{ij} integrals on the left are found by numerical integration. The resulting linear system is solved for the n_{ij} unknown weights

$$\{W(ij,z)_{\ell_{ij}}, W(ij,z)_{\ell_{ij+1}}, \ldots, W(ij,z)_{\ell_{ij+n_{ij}}}\}$$

By definition if $F(y')$ is a polynomial of order $n_{ij} - 1$ or less then the quadrature rule is exact. Typically n_{ij} varies between 3 and 6, so we are not taking into account very many terms in the Taylor series for $f(y')$. It is the core size of the computer that forces us to work with such a small number of points for the y' integration. The computer used, the Burroughs B5500, has available about 10,000 words of fast memory in which we can store the matrix which results for our discretization. From Table 1 we see that we can choose $N_x = 7$ for the outer x' integration and expect about 1/2% error to arise from this quadrature. Since the dimension of the final matrix is $N_x \times N_y$ we can have $N_y = 15$. With this arrangement there are generally one or two points in each y' integration interval. Using enlarged domains the average number of points for a y' quadrature rule is brought up to 4 or 5.

In principal we could use all N_y points for each ij integration that do not lie beyond the nearest singularities in the integrand. This is not practicable for the following reason. If we use many points outside the region of integration in our integration rule, then the resulting weights become very large and oscillatory, so that even the integral for $F(y') = 1$ is a sum of terms whose first 8 or 9 digits all cancel out when summed. The finite word length of the computer, here roughly 11 decimal digits, soon destroys the accuracy of the method. The 3 to 6 points used represents, then, a compromise which avoids large oscillatory weights yet provides enough points for an accurate quadrature rule.

F. Accuracy of Three-Body Method

A convenient check on the accuracy of this method is to use our quadrature rules to find the separable kernel $M(q_i, q_j, z)$ by doing the intergral in Eq. (IV.30). If the ingredient f functions are derived from a Yukawa interaction then we find that all 49 matrix elements have errors less than 2% and all but three have errors smaller than 1%. The maximum absolute error is less than 10^{-3}.

For f functions that arise from Yamaguchi's or Bander's potential the accuracy is somewhat better. These results suggest the accuracy of the y' integration is sufficient. We expect, but can not prove, that the accuracy of the zero of the Fredholm determinant will be better than 2% because errors of different sign tend to cancel and 70% of the matrix elements $M(q_i, q_j; z)$ are determined to better than .1%. These error estimates will generalize to our two-variable integral Eq. (IV.27) provided that $H(k''^2, q'')$ has only a mild dependence on k''^2. We defined H with this goal in mind, and it turns out that when the eigenfunctions of Eq. (IV.27) are calculated that the k''^2 dependence of H is very nearly constant. An additional numerical check on the y' integration scheme was to add one additional integration point to N_y. The additional point was placed in the intervals with the largest errors. If one of the matrix elements has a large error then the resulting energy eigenvalue should change when the extra point is added. No significant change(greater than .1%) was observed. Ideally, one would like to double the number of points as a check. The fixed core size of the computer however makes this impossible. This could be done by rewriting all the programs in order to use the larger core sizes of more modern computers now available.

The most striking and convincing check on our accuracy comes from our attempt to repeat the results of Wong and Zambotti. Using a method completely different from ours, they give the boundstate energy for three identical particles interacting via a Yukawa potential, $\lambda (e^{-r}/r)$. For $\lambda = -1.8$ we can read off from their curve of boundstate energies versus the Yukawa coupling constant, λ, the corresponding energy eigenvalue, $s = Zm \cong -.25\ F^{-2}$. Our calculation for the same potential gives $-.293$. This 16% discrepancy is considerably more than allowed by estimated error ($\sim$ 1%) in our method and the 2% accuracy Wong and Zambotti quote. Recently Wong[39] has redone the calculation by a method different

from his previous calculations.[12] He expands the Lippman-Schwinger kernel, $t(p, q; z)$ in terms of the separable Sturmian functions. His new result for the above problem, with no quoted error estimates, is now $-.295$. This agrees very well with our result and the discrepancy of .7% is in accord with what our error analysis leads to expect. He also finds for this case that including the higher angular momentum states increases the binding energy by less than 1%, which justifies our only formal approximation of including only relative s-waves between the interacting pairs.

Thus we conclude that our numerical method is adequate to solve three-body problems resulting from smooth non-oscillating potentials like the one term Yukawa interaction. We expect our results to have about 1% error.

CHAPTER VI

RESULTS AND CONCLUSIONS

In this section we report on the results of our numerical calculation of three-body boundstate solutions. As described in Chapter V we are able to reduce the three-body problem to a finite, solvable matrix problem if we are dealing with the interaction of three identical spinless particles. Since our formulation is valid for any potential form our solutions are not limited to those arising from separable potentials. In the following we describe the boundstate spectra for two local potentials which have been extensively studied.

A. Three-Body Boundstate Spectra

We first present our results for a Yukawa potential. In accordance with the two-body angular momentum decomposition given in Appendix A, we can write the s-wave projection of the Yukawa potential in momentum space as

$$v_0(p, q) = \frac{\lambda}{2pq} Q_0 \left(\frac{\mu^2 + p^2 + q^2}{2pq} \right) \qquad \text{(VI.1)}$$

Using this potential in Eq. (IV.28) we have calculated the boundstate spectra as a function of λ for a fixed value of μ. The value of μ is chosen from a fit of μ and λ to low energy N-P triplet scattering data given by Preston.[40] This value of μ is .633 F^{-1}. When the coupling constant, λ, equals 1.58 we have a good fit of the Yukawa potential to N-P triplet force. Figure #1 presents the boundstate spectrum associated with the two-body system. Here the scattering length is plotted against the coupling constant λ. Each time the scattering length becomes infinite we have the introduction of one more two-body boundstate in the spectrum--thus at $\lambda \simeq -1$ the first boundstate emerges while at $\lambda \simeq -3.8$ the second boundstate emerges. The value of the coupling constant appropriate for deuteron is noted by a dashed curve.

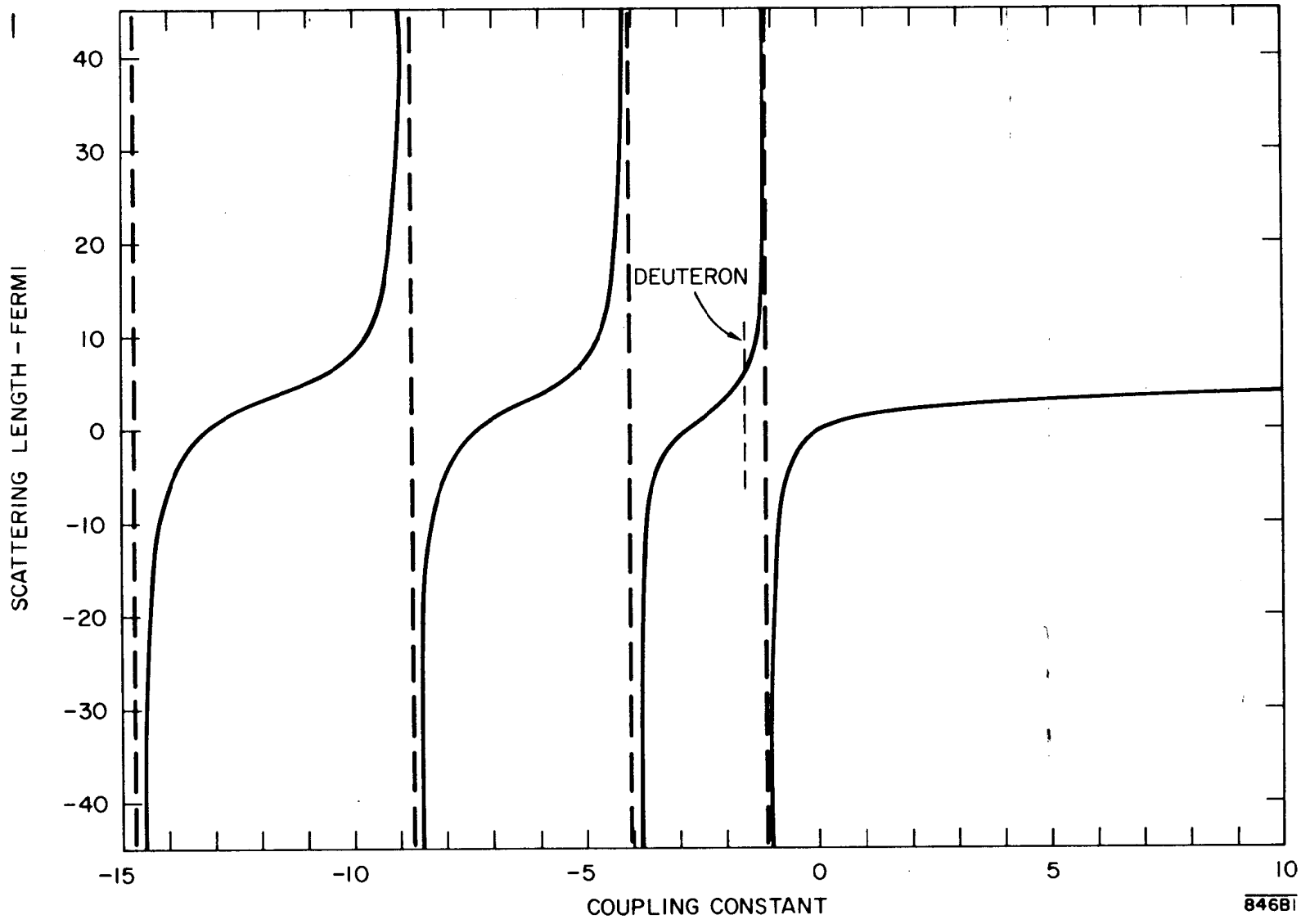

FIG. 1--S-wave scattering length of a two-body system interacting via a local Yukawa potential as a function of the coupling constant.

Figure # 2 shows the corresponding three and two-body boundstate energies as function of the coupling constant λ. The ground state energy of the two-body system when plotted against λ appears indistinguishable from a straight line. The linear behavior stems from the 1/r portion of the Yukawa potential. If we assume Coulomb wave functions as a first approximation and use perturbation theory to calculate the shift in energy corresponding to a small change in λ for the Yukawa potential, then the linear behavior noted above appears. The position of the two-body boundstate energy marks the beginning of the continuum in three particle system. Above this energy there can not be any three-body boundstates. Figure # 2 shows us that the three-body system is bound for a smaller value of the coupling constant that is the two-body system. These curves also show us that when the two-body boundstate emerges it is accompanied by a three-body state at a slightly lower energy. The behavior of all of these states is relatively linear until we get near the value of λ at which a new three-body boundstate is found. Here, interestingly, the trajectory of the first excited state rapidly changes and continues along a linear extension of the first boundstate trajectory. From this point on the ground state of the three-body system binds roughly seven times more rapidly than before the appearance of the second excited state. Furthermore, the second excited states seem to continue along an extension of the first excited state. All of this structure occurs for values of λ that are less than twice the physical N-P value.

B. Boundstate Wave Functions

In order to shed more light on this behavior we have calculated the boundstate eigenfunctions for various values of λ. The physical function the eigenfunctions represent is $H(k^2, q)$ described in Chapter IV. The reader will recall that H is defined such that if a separable term of the two-body t-matrix dominates the

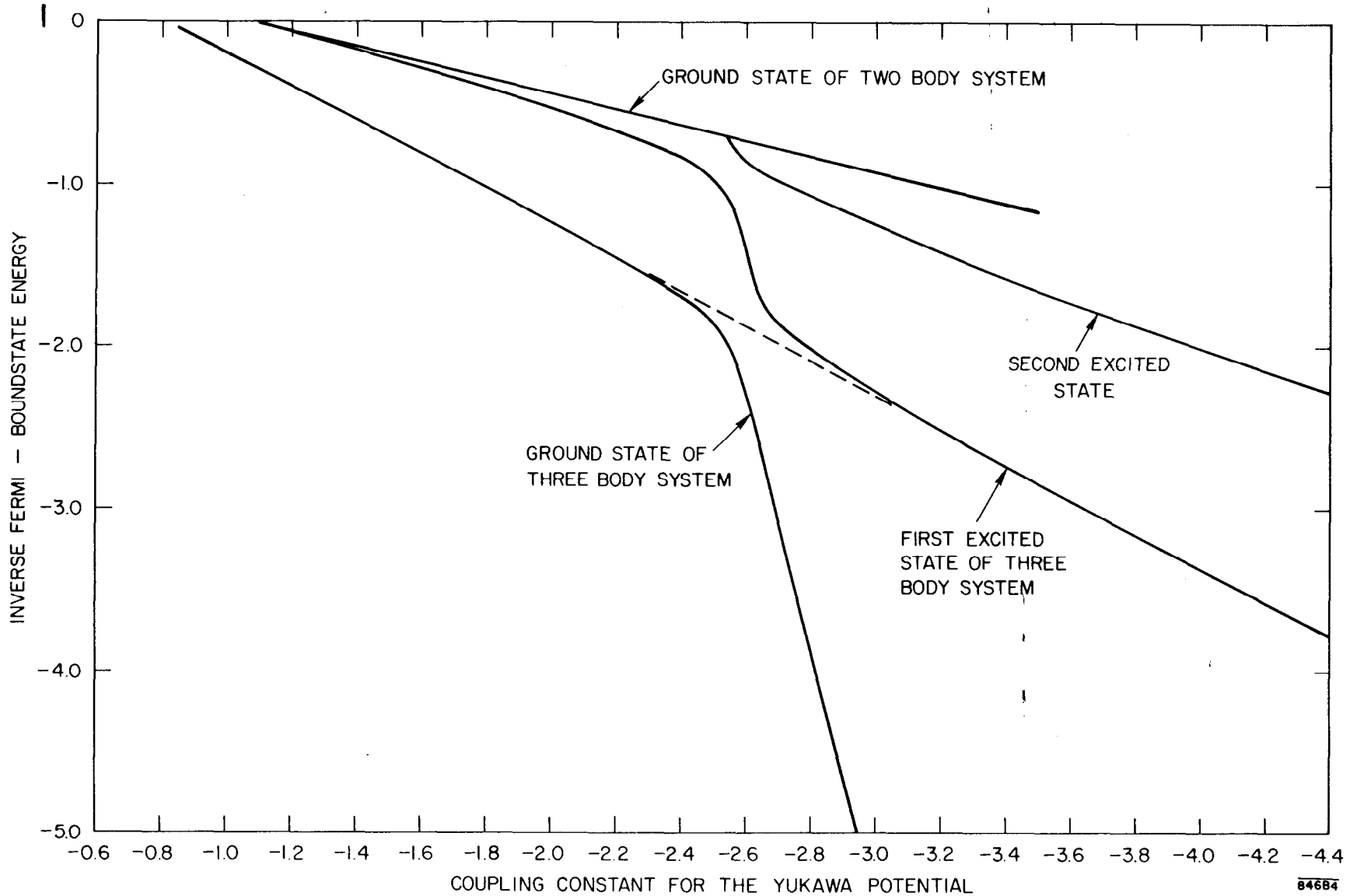

FIG. 2--Binding energies of the J = 0 states of two and three spinless particles interacting via the S-wave part of a local Yukawa potential as a function of the coupling constant.

interaction then H will be nearly constant in the first variable k^2. The eigenfunction is transformed into a wavefunction just by multiplying $f(k^2 - \frac{3}{4}q^2, z - \frac{3}{4}\frac{q^2}{m})$ as in Eq. (IV.26). We have calculated the wavefunctions for four values of λ: - 1.4, - 2.4, - 2.6 and - 2.8.

For all values of λ the ground state has no nodes. The first excited state has one and the second excited state has two. The eigenfunction for the weakest value of the coupling constant, - 1.4, is notably constant in the total energy variable k^2. For portions of the wave function associated with small values of q, the independent particle energy, the variation of $H(k^2, q)$ over the whole range of k^2 is never more than 30%. For large values of q, where the wave is less than one one-hundredth of its maximum, the variation of $H(k^2, q)$ is still less than a factor of 3. This relatively constant behavior in the k^2 variable is evidence that separability is working rather well for $\lambda = -1.4$.

Now let us see what the wave functions tell us about the behavior of the boundstate spectra. Let us first examine the boundstate. Here, as is generally the case, all the significant variation of the wave function occurs in the individual particle momenta q. Thus we will describe the wave functions as if they were a function only of q. These wave functions are presented in Figs. 3, 4, 5. Examining Fig. 3, which represents the ground state wave function, we see that as λ increases the high momentum portion of the wave function, H(p, k), grows at the expense of the low portion: between - 2.6 and - 2.8 the growth is very fast, and for - 2.8 the wave function is almost constant except for the last two high momentum points. Clearly the Fourier transform in coordinate space will look like a delta function. This suggests the interpretation that for $\lambda < -2.6$ the ground state wave function has collapsed. As a consequence the wave function sees only the 1/r singularity in the Yukawa potential and its dynamical behavior

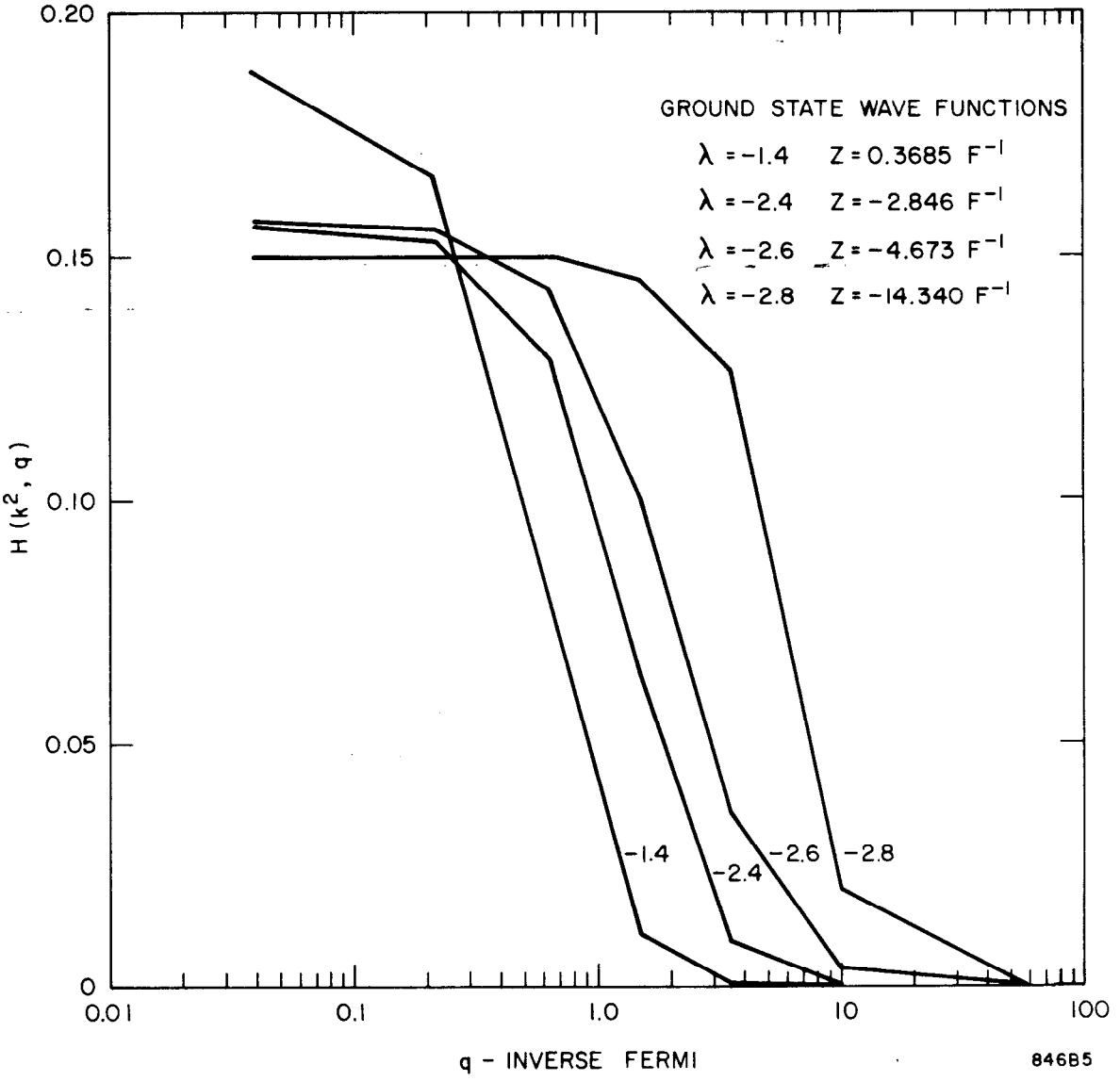

FIG. 3--Ground state wave functions of the three-boson system in momentum space for the Yukawa interaction.

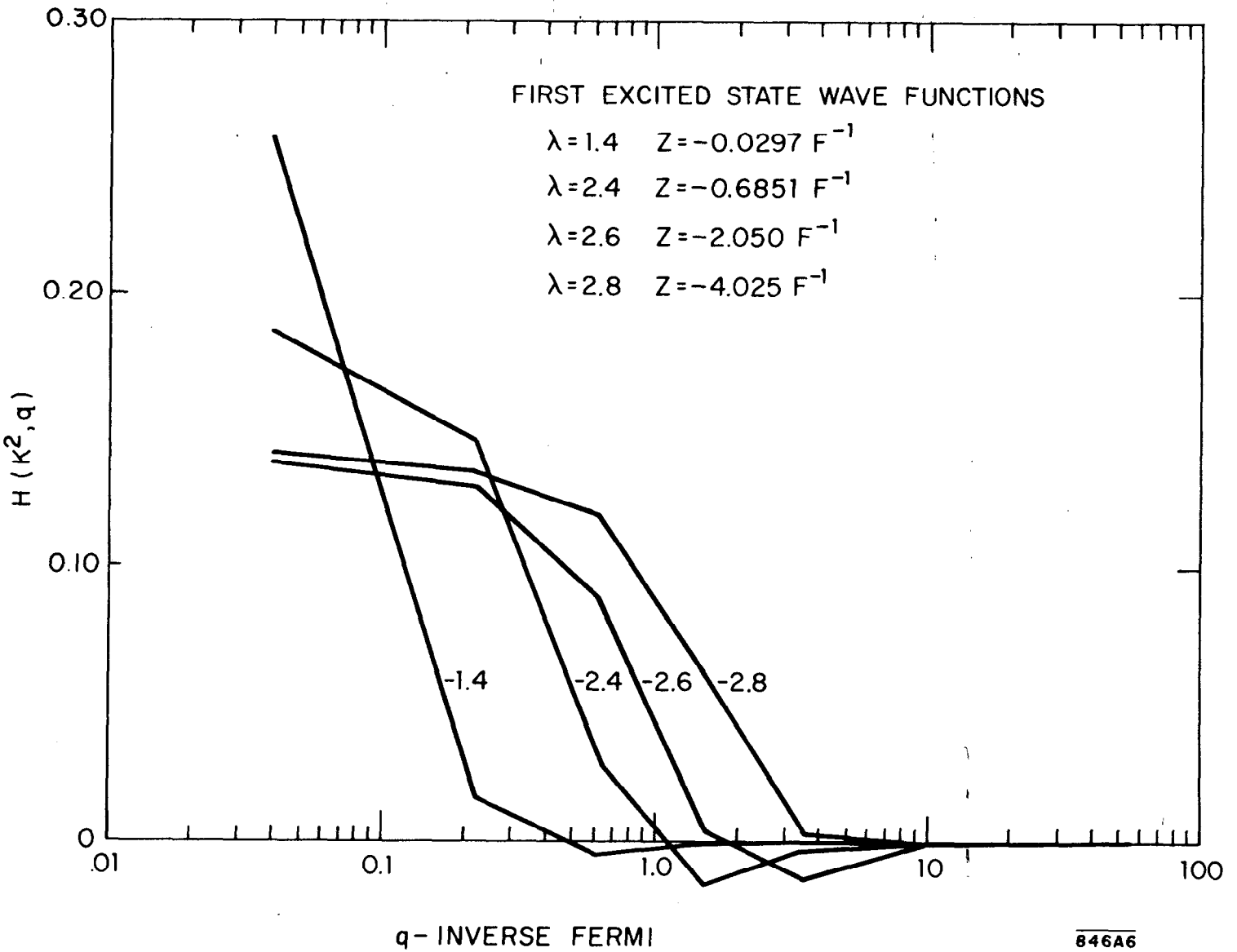

FIG. 4--First excited state wave functions of the three-boson system in momentum space for the Yukawa interaction.

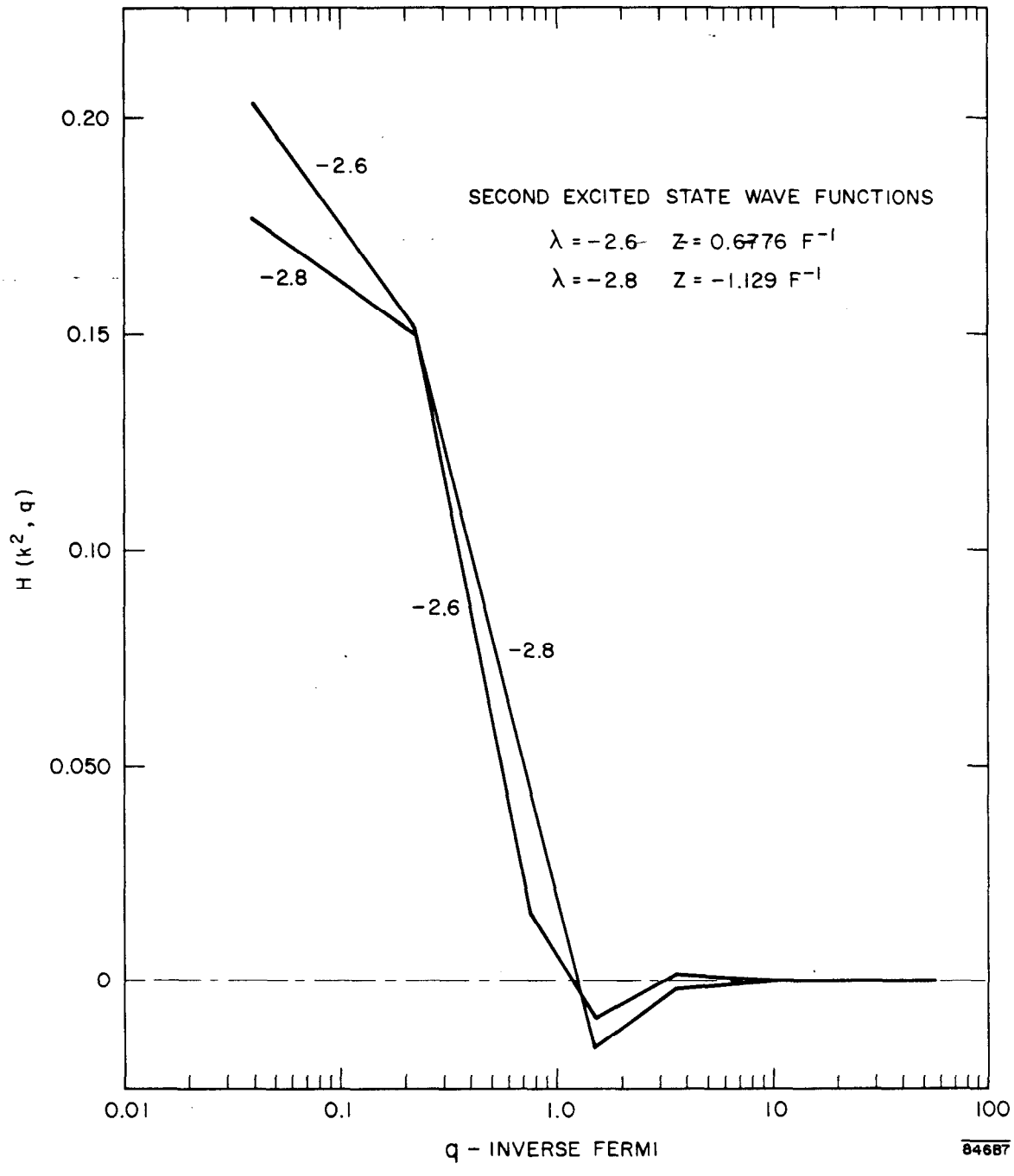

FIG. 5--Second excited state wave functions of the three-boson system in momentum space for the Yukawa interaction.

as a function of λ is governed only by this singularity. Presumably before the discontinuity at $\lambda = -2.6$ the spatially more extended wave functions are governed by exponential structure of the potential. This explains then the different rates of boundstate binding before and after $\lambda = -2.6$.

Now we want to examine why the first excited state for $\lambda < -2.6$ binds along a linear extension of the ground state trajectory. If we look at Fig. 4, which shows the first excited state for the same four values of λ as before, we see that the zero in the wave function for $\lambda = -2.8$ has moved out to $\sim 10\ F^{-1}$. The interior portion of the wave function has almost exactly the shape of the ground state wave function for $\lambda = -2.4$. Thus the straight line along which the ground state and then later the first excited state lie is related to a simple wave function shape. This shape is just that of a momentum space wave function which is large only for small momenta -- any oscillation occurs only at large ($\geq 10\ F^{-1}$) values of momenta where the wave function is small.

This same story repeats itself for the nearly linear trajectory along which the first and second excited states lie. By comparing the second excited state at $\lambda = -2.8$ shown on Fig. 5 with the first excited state at $\lambda = -2.4$ we notice that except for the tail of the wave function they have the same shape.

Finally Figs. 6 and 7 show the boundstate spectra for the two- and three-body systems which arise when the two-body interaction is the s-wave portion of an exponential potential. Figure 7 shows the three boundstates observed above in the exponential system. Here, however, the behavior of the boundstate trajectories do not show any rapid variations. Presumably this is because of the absence of the singular 1/r portion of the Yukawa potential, and gives a quantitative example of the general argument originally given by Thomas[41] for the finite range of nuclear forces.

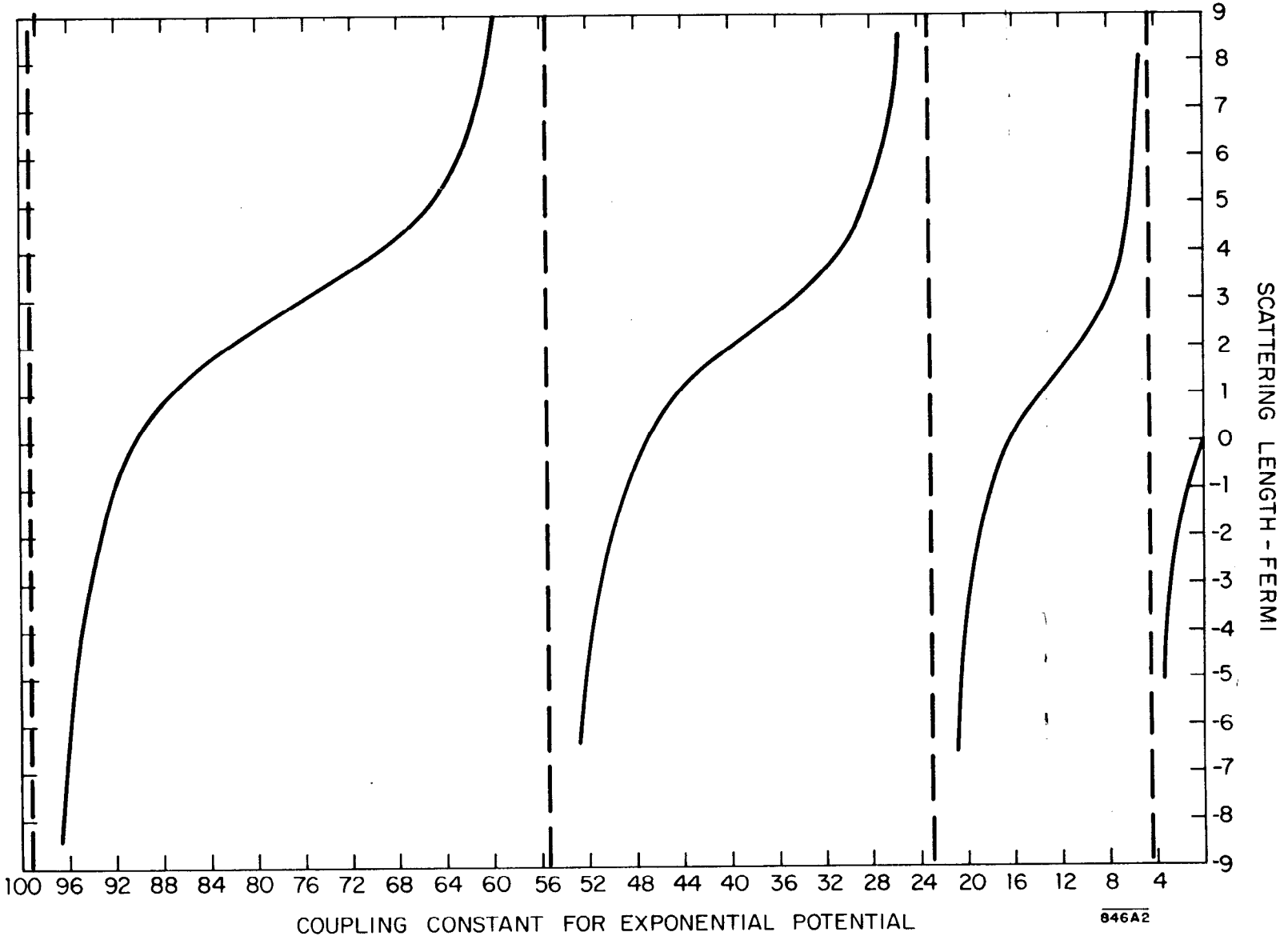

FIG. 6--S-wave scattering length of a two-body system interacting via a local exponential potential as a function of the coupling constant.

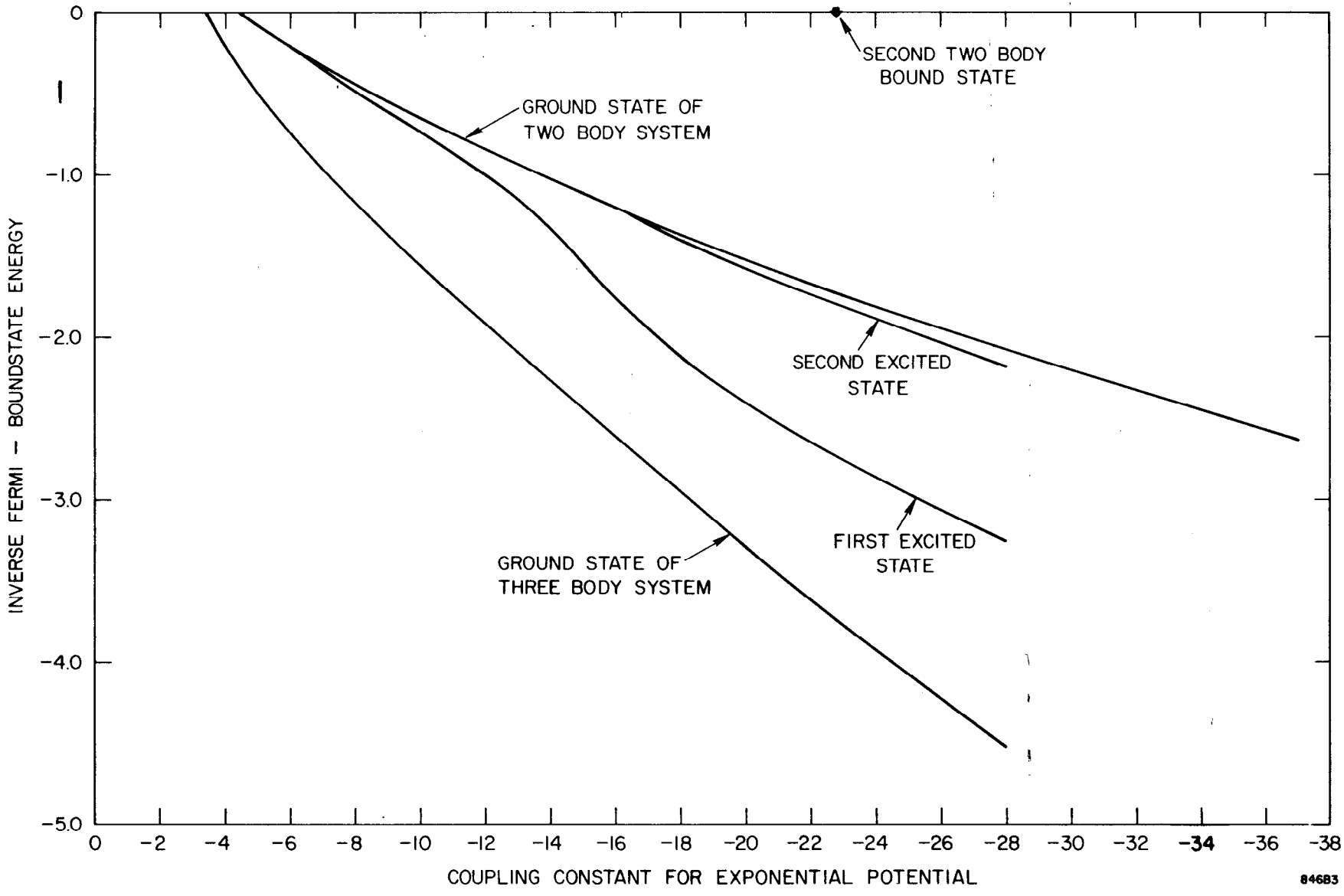

FIG. 7--Binding energies of the J = 0 states of two and three spinless particles interacting via the S-wave part of a local exponential potential as a function of the coupling constant.

C. Validity of Separable Approaches

We now turn to our results which concern the validity of separable approximations. These results are summarized in Figs. 8 and 9. We test the separable approximation by asking it to reproduce the results of the Yukawa potential. This is accomplished by calculating the two-body boundstate energy and the scattering length for the Yukawa potential. We then fit the two parameters of the Yamaguchi potential so that the same two-body boundstate and scattering length is reproduced. The resulting separable potential is then used in a calculation of the three-body ground state energy. If the separable potential is a good approximation to the Yukawa potential then the three-body boundstate energies will be similar. Figure 8 shows the corresponding boundstate trajectories for the Yamaguchi potential and the Yukawa potential. The conclusion is for potential strengths less than the triplet N-P force the Yamaguchi potential gives the boundstate energy correct to about 10% or better. For potential strength greater than - 1.6 the Yamaguchi interaction rapidly diverges from that of the exact Yukawa solution. At $\lambda \cong - 2.3$ the Yamaguchi binding energy becomes infinite -- this value of the coupling constant is only 40% greater than the value for the N-P interaction. This tells us that we can expect the separable potential to give accurate results when the two-body force is roughly equal to or weaker than the N-P triplet interaction.

Further, as Fig. 8 shows, we have repeated this comparison for Bander's separable potential. Basically this potential is just the square root of the Yamaguchi potential. This means that potential will fall off more slowly in momentum space than the Yamaguchi. However, this makes only a slight difference to the three-body binding energy. Thus all the conclusions given above about the Yamaguchi separable potential remain the same for Bander's potential.

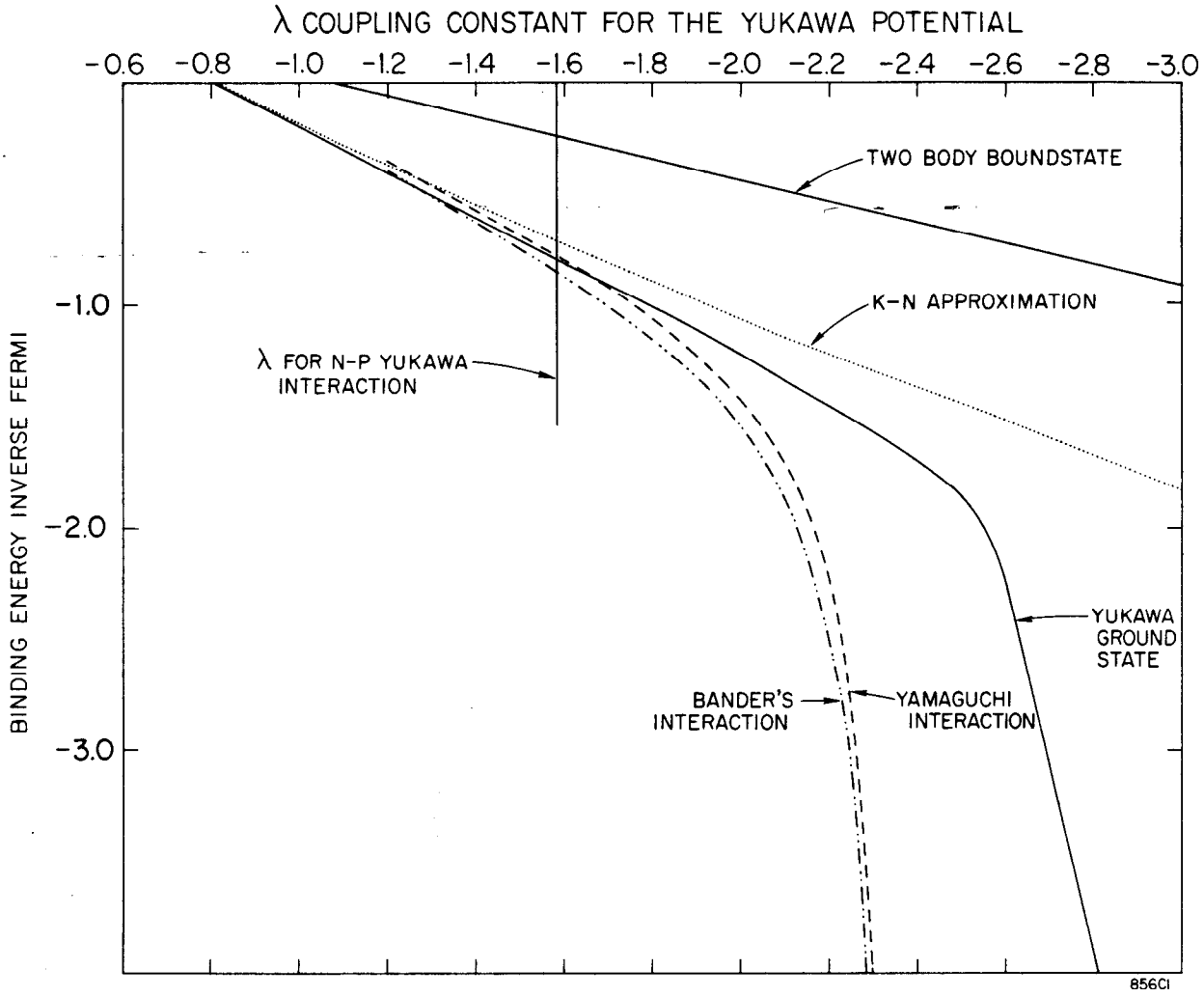

FIG. 8--Comparison of the binding energy of the ground state of three bosons as computed from the Yukawa interaction with various separable approximations fitted to the same two-body binding energy and scattering length.

As discussed in Chapter III we have another separable approximation given by the Kowalski-Noyes representation. The resultant binding energies are also presented on Fig. 8. This approximation does not diverge for finite coupling constants but is unable to reproduce the rapid change in the binding rate that occurs at $\lambda \approx -2.6$. Here too the approximation is very good for weak forces and gradually deteriorates as the coupling constant increases.

An additional measurement of the validity of the separable potential is represented in Fig. 9. Here the exact solution, which we try to have the separable potentials reproduce, is that arising from the exponential interaction. We see that even for this non-singular potential the conclusions are identical to those already presented for the Yukawa potential.

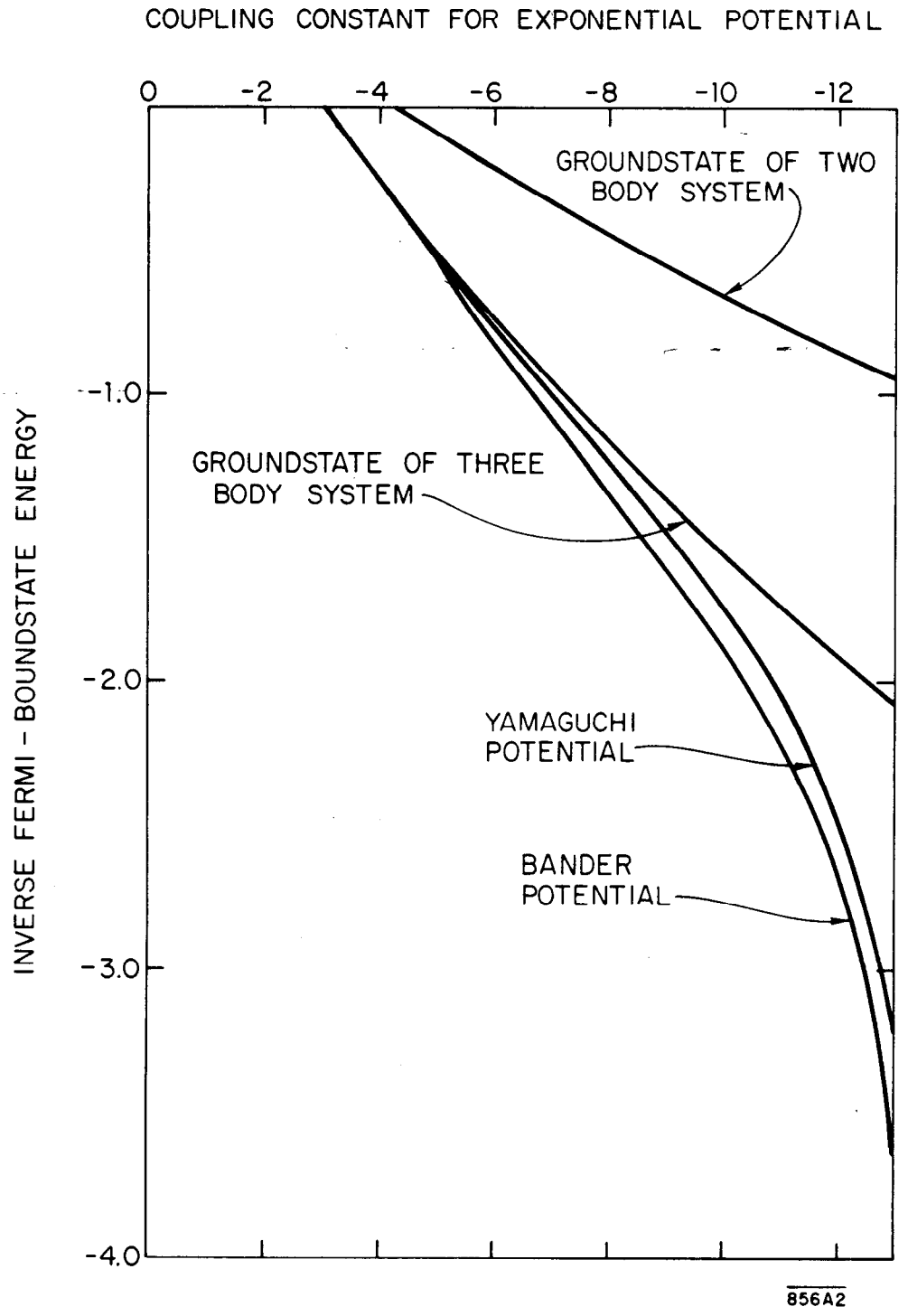

FIG. 9--Comparison of the binding energy of the ground state of three bosons as computed from the exponential interaction with various separable approximations fitted to the same two-body binding energy and scattering length.

APPENDIX A

In this appendix we write out explicitly the partial wave projections we have used to reduce our momentum space equations of motion to that of a single variable radial form. The projections define the normalization constants of the resulting equations as well as defining the units. In the last portion of the appendix we give explicit formulas for transition amplitudes resulting from several standard separable potentials.

Let O be an operator in the two-body Hilbert space, then our definition of the partial wave projection is

$$\langle \vec{p} \,|O|\, \vec{q} \rangle = \frac{\hbar^2}{\pi \mu} \sum_{\ell\ell' mm'} O_{\ell\ell' mm'}(p,q)\; Y_{\ell M}(\hat{p})\; Y^*_{\ell' m'}(\hat{q}) \;, \tag{A1}$$

where $\hat{}$ denotes the direction of the momentum vector and μ is the reduced mass. If $[O,\vec{L}] = 0$ then

$$O_{\ell\ell' mm'}(p,q) = O_\ell(p,q)\, \delta_{\ell\ell'}\, \delta_{mm'}$$

and the above equation simplifies to

$$\begin{aligned} \langle \vec{p} \,|O|\, \vec{q} \rangle &= \frac{\hbar^2}{\pi \mu} \sum_\ell O_\ell(p,q) \sum_{m=-\ell}^{m=\ell} Y_{\ell m}(\hat{p})\; Y_{\ell m}(\hat{q}) \\ &= \frac{\hbar^2}{4\pi^2 \mu} \sum_\ell (2\ell+1)\, O_\ell(p,q)\, P_\ell(\hat{p}\cdot\hat{q}) \;. \end{aligned} \tag{A2}$$

Throughout we will use the definition of $Y_{\ell m}$ given by Edmonds.

By using the partial wave projection given in (A2) we obtain the various partial wave equations given in the text. Here we will only go through the derivation for the Schoedinger equation. Suppose ψ_E is the eigenstate of the Hamiltonian H

with an eigenvalue E. If K is the kinetic energy operator then we have

$$H\psi_E = (K + V)\psi_E = E\psi_E \ . \qquad (A3)$$

Taking expectation values between plane wave momentum states for the kinetic term gives us

$$\int dq \langle \vec{p}|K|\vec{q}\rangle \langle \vec{q}|\psi_E\rangle = \int d\vec{q}\left(\frac{p^2\hbar^2}{2\mu}\right)\langle \vec{p}|\vec{q}\rangle \langle \vec{q}|\psi_E\rangle$$

$$= \frac{p^2\hbar^2}{2\mu} \langle \vec{p}|\psi_E\rangle \qquad (A4)$$

where $\langle \vec{p}|\vec{q}\rangle$ has been delta function normalized. The momentum space representation of Eq. (A 3) is

$$\frac{p^2\hbar^2}{2\mu} \langle \vec{p}|\psi_E\rangle + \int d\vec{p} \langle \vec{p}|V|\vec{q}\rangle \langle \vec{q}|\psi_E\rangle = E\langle \vec{p}|\psi_E\rangle \qquad (A5)$$

Since we assume that $[H,\vec{L}] = 0$, we can write for the wave function

$$\langle \vec{p}|\psi_E\rangle = \sum_{\ell m} Y_{\ell m}(\hat{p})\,\psi_{E,\ell}(p) \ . \qquad (A6)$$

Using this and our partial wave projection Eq. (A 2) we can write the potential term as

$$\int q^2\, dq\, d\Omega_{\hat{q}} \left\{\frac{\hbar^2}{\pi\mu}\sum_{\ell} v_\ell(p,q)\sum_{m} Y_{\ell m}(\hat{p})\, Y^*_{\ell m}(\hat{q})\sum_{m'} Y_{\ell' m'}(\hat{q}')\psi_{E,\ell'}(q)\right\}$$

$$= \frac{\hbar^2}{\pi\mu}\sum_{\ell} Y_{\ell m}(\hat{p})\int_0^\infty q^2\, dq^2\, v_\ell(p,q)\psi_{E,\ell}(q) \ . \qquad (A7)$$

Here we have used the orthonormality of the $Y_{\ell m}$'s when integrated over $d\Omega_{\hat{q}}$. By equating the coefficients of $Y_{\ell m}(\hat{p})$ we obtain the partial wave form of the Schroedinger equation that we seek

$$p^2\psi_{E,\ell}(p) + \frac{2}{\pi}\int_0^\infty q^2\, dq\ v_\ell(p,q)\,\psi_{E,\ell}(q) = \frac{2\mu E}{\hbar^2}\,\psi_{E,\ell}(p) \ . \qquad (A8)$$

Here p has dimension of inverse length and v_ℓ has dimensions of inverse length squared. Dimensional consistency is maintained by treating $\int q^2\, dq$ as dimensionless. Analogous reductions are valid for the Lippman-Schwinger equation and the Low equation. These are given in Chapter III as Eqs. (III. 1) and (III. 21), respectively.

The particular partial wave projection given by Eq. (A 1) was chosen to give a simple form to the on-shell unitarity relation. We can quickly obtain this relation by taking the imaginary part of the on-shell version of the Low equation. From Eq. (III. 21) we have

$$\mathcal{I}m\, t_\ell(k,k;k^2+i\epsilon) = \frac{2}{\pi}\,\mathcal{I}m \int_0^\infty \frac{t_\ell(k,k';k'^2)\,\overline{t_\ell(k,k';k'^2)}}{k^2-k'^2+i\epsilon}\, k^2 dk\,. \tag{A 9}$$

The integral in Eq. (A 9) may be done if we use the identity

$$\frac{1}{x+i\epsilon}\, dx = -i\pi\,\delta(x)\,dx + P\,\frac{1}{x}\, dx\,. \tag{A 10}$$

The principle value part is real and drops out leaving

$$\frac{2}{\pi}\,\mathcal{I}m\, -i\pi \int_0^\infty k'^2\, dk'\, \frac{\delta(k-k')\left|t_\ell(k,k';k'^2+i\epsilon)\right|^2}{-(k+k')}\,. \tag{A 11}$$

Thus the on-shell unitarity relation reads

$$\mathcal{I}m\, t_\ell(k,k;k^2+i\epsilon) = +k\left|t_\ell(k,k';k'^2+i\epsilon)\right|^2\,. \tag{A 12}$$

By substitution it is trivially verified that the phaseshift representation of the on-shell amplitude,

$$t_\ell(k,k;k^2+i\epsilon) = \frac{e^{i\delta_\ell(k)}}{k}\,\sin\delta_\ell(k)\,, \tag{A 13}$$

satisfies Eq. (A 12).

We now want to derive the formulae for transition amplitudes resulting from a Yamaguchi potential. The Yamaguchi potential is defined as

$$v(\vec{p},\vec{q}) = \frac{\lambda}{\left(\beta^2 + p^2\right)\left(\beta^2 + q^2\right)} \tag{A 14}$$

It should be noted that this separable form for the potential has no dependence on $\hat{p}\cdot\hat{q}$. Thus the potential has only an s-wave part.

Using Eq. (III. 33) we can write the half-off-shell extension function as

$$f_0(p,s) = \frac{v(p,k)}{v(k,k)} = \frac{\left(k^2 + \beta^2\right)}{\left(p^2 + \beta^2\right)} \quad . \tag{A 15}$$

The on-shell amplitude is given by Eq. (III.4). To get an analytic expression for Eq. (III.4) we need to do the integral

$$\frac{2}{\pi}\int_0^\infty \frac{q^2\,dq}{q^2 - s}\, v(k,q)\, f_0(q,k) = \frac{2\lambda}{\pi}\int_0^\infty \frac{q^2\,dq}{\left(q^2 + \beta^2\right)^2\left(q^2 - s\right)} \quad . \tag{A 16}$$

We will need the on-shell amplitude only for $s < 0$. The integral is easily done by partial fractions and the result is

$$\int_0^\infty \frac{q^2\,dq}{\left(q^2 + \beta^2\right)^2\left(q^2 - s\right)} = \frac{\pi}{4\beta\left(\sqrt{-s} + \beta\right)} \quad . \tag{A 17}$$

For, $k^2 = |-s|$ and $s \in [-\infty, 0]$ the on-shell t matrix is

$$t_0(k,k;s) = \frac{\lambda}{\left(\beta^2 + k^2\right)^2\left(1 + \dfrac{\lambda}{2\beta\left(\sqrt{-s} + \beta\right)^2}\right)} \tag{A 18}$$

The off-shell t matrix is obtained by multiplying Eq. (A 18) by $f_o(p,k^2)\, f_o(q,k^2)$. Thus

$$t_0(p,q;s) = \frac{\lambda}{\left(p^2 + \beta^2\right)\left(q^2 + \beta^2\right)\left(1 + \dfrac{\lambda}{2\beta\left(\sqrt{-s} + \beta\right)^2}\right)} \tag{A 19}$$

If α denotes the position of a boundstate pole then the boundstate condition is

$$\lambda = -2\beta\,(\alpha+\beta)^2 < 0\ . \tag{A 20}$$

For reference we present the formulas for scattering length and effective range which may be obtained from Eq. (A 9)

$$\frac{1}{a} = \frac{\beta}{2}\left(1+\frac{2\beta^2}{\lambda}\right), \tag{A 21}$$

and

$$r_0 = \frac{1}{\beta}\left(1-\frac{4\beta^2}{\lambda}\right). \tag{A 22}$$

The other separable potential we will need is the one used by Bander. Here the separable potential is defined as

$$v(q,p) = \frac{\lambda}{\sqrt{q^2+\beta^2}\ \sqrt{p^2+\beta^2}} \tag{A 23}$$

In this case

$$f(p,k^2) = \frac{\sqrt{k^2+\beta^2}}{\sqrt{p^2+\beta^2}} \tag{A 24}$$

The integral in the denominator of Eq. (III. 4) is

$$\int_0^\infty \frac{q^2\,dq}{\left(q^2-s\right)\left(q^2+\beta^2\right)} = \frac{\pi}{2}\,\frac{1}{\left(\beta+\sqrt{-s}\right)}, \qquad s<0\ . \tag{A 25}$$

The on-shell scattering amplitude for Bander's potential is therefore

$$t_0(k,k;s) = \frac{\lambda}{\left(k^2+\beta^2\right)\left(1+\dfrac{\lambda}{\beta+\sqrt{-s}}\right)}, \qquad s<0 \tag{A 26}$$

The off-shell form is

$$t_0(p,q;0) = \frac{\lambda}{\sqrt{q^2+\beta^2}\ \sqrt{p^2+\beta^2}\left(1+\dfrac{\lambda}{\beta+\sqrt{-s}}\right)}\ . \tag{A 27}$$

From Eq. (A 26) we see that the boundstate condition is

$$\lambda = -(\beta + \sqrt{-s}) < 0 \tag{A 28}$$

Formulas for the scattering length and effective range are:

$$a = \left(\beta + \frac{\beta^2}{\lambda}\right)^{-1} \tag{A 29}$$

$$r_0 = -\frac{2}{\lambda} \tag{A 30}$$

APPENDIX B

In this appendix we present algebraic formulae for the kernel $M(q',q'',z)$ occurring in the one variable separable version of the three-body problem.

1. Kernel for the Separable Yamaguchi Potential

The function for the Yamaguchi potential is given by Eq. (A 15) in Appendix A. Using Eq. (A 15) and Eq. (IV. 30), $M(q',q'',z)$ is defined to be

$$M(q',q'',z) = \frac{1}{2q'q''} \int_{q'^2+q''^2-q'q''}^{q'^2+q''^2+q'q''} dk''^2 \frac{\left(\left|z - \frac{3}{4}q'^2\right| + \beta^2\right)\left(\left|z - \frac{3}{4}q''^2\right| + \beta^2\right)}{\left(k''^2 - \frac{3}{4}q'^2 + \beta^2\right)\left(k''^2 - \frac{3}{4}q''^2 + \beta^2\right)\left(z - k''^2\right)} \tag{B 1}$$

For

$$k''^2 \in \left(q'^2 + q''^2 + q'q'',\ q'^2 + q''^2 - q'q''\right)$$

thus

$$\left(k''^2 - \frac{3}{4}q'^2 + \beta^2\right) > 0 \qquad \text{and} \qquad \left(k''^2 - \frac{3}{4}q''^2 + \beta^2\right) > 0 .$$

For boundstate energies, $z < 0$, we have that $M(q',q'',z)$ is negative definite. Furthermore, it is obviously symmetric in q' and q'' which is a general property of M. Defining the following intermediate quantities, the function M may be written

$$M(q',q'',z) = \frac{MF}{D(C2 - z)} \left[S(C1, C2) - S(C1, z)\right] \tag{B 2}$$

where

$$S(x,y) = \begin{cases} \dfrac{1}{(x-y)} \ln \dfrac{(B-x)(A-y)}{(A-x)(B-y)}, & x \neq y \\ \dfrac{D}{(A-x)(B-y)}, & x = y \end{cases} \tag{B 3}$$

and

$$A = q'^2 + q''^2 - q'q'' , \quad B = q'^2 + q''^2 + q'q'' , \quad D = B - A$$

$$C1 = \frac{3}{4}q'^2 - \beta^2 , \quad C2 = \frac{3}{4}q''^2 - \beta^2 ,$$

$$MF = \left(\left|z - \frac{3}{4}q'^2\right| + \beta^2\right)\left(\left|z - \frac{3}{4}q''^2\right| + \beta^2\right) \tag{B4}$$

II. Kernel for the Separable Bander Potential

The f function for Bander's potential is given by Eq. (A 24). For this case $M(q',q'',z)$ becomes

$$M(q',q'',z) = \frac{1}{2q'q''}\int_{q'^2+q''^2+q'q''}^{q'^2+q''^2+q'q''} dk''^2 \frac{\left[\left(\left|z-\frac{3}{4}q'^2\right|+\beta^2\right)\left(\left|z-\frac{3}{4}q''^2\right|+\beta^2\right)\right]^{1/2}}{\left(z-k''^2\right)\left[\left(k''^2-\frac{3}{4}q'^2\mp\beta^2\right)\left(k''^2-\frac{3}{4}q''^2+\beta^2\right)\right]^{1/2}} \tag{B 5}$$

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