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on

INSTABILITIES IN STORED PARTICLE BEAMS

A SUMMARY REPORT

August 1965

Edited by

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Report 1

INTRODUCTION

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A. The Summer Study

Interest in high-energy particle research with colliding beam storage rings has been increasing rapidly during the past few years. The Italian group has observed the bremsstrahlung interaction between electrons and positrons in a 130-MeV storage ring (ADA) operating at low beam intensities. The Princeton-Stanford group has now achieved a large enough interaction rate to observe wide-angle electron-electron scattering at 300 MeV (in its 500-MeV rings). Beams have been stored in 130-MeV electron-electron rings and in a 700-MeV electron-positron ring at Novosibirsk. Two electron-positron rings are nearing completion in Europe - a 450-MeV ring at Orsay, and a 1.5-GeV ring at Frascati. Proposals have been made for a 3-GeV electron-positron ring by two groups in the United States. Colliding-beam storage rings for 30-GeV protons are to be built at CERN.

The successful operation of colliding-beam storage rings requires that two high intensity beams of small cross section be stored in an intersecting geometry for periods of many minutes to hours. In the past five years several phenomena have been discovered experimentally or theoretically which can cause instabilities in intense stored beams leading either to a catastrophic loss of the beams or to a drastic reduction of the interaction rates in the beam intersection regions. Many of these phenomena were first observed and studied in the pioneering work of the Princeton-Stanford storage ring group. By early 1965, this group - aided by contributions from Courant, Ritson, Sessler, and others - had arrived at a reasonable understanding of the basic limitations of the interaction rates and had been able to select operating parameters of the rings which lead to stored currents in the colliding beams of several tens of milliamperes and beam interaction rates high enough to carry out measurements of wide-angle electron-electron scattering.

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The successful operation of the 500-MeV electron-electron rings gave reasons for believing that one could proceed with some confidence with the construction of a 3-GeV electron-positron ring, although some questions were still not completely clear. What laws governed the extrapolations to higher energy? Would new phenomena appear with high rf harmonics (and, therefore, with many circulating bunches)? Would new phenomena appear with counter-circulating beams in one ring which did not show up in the two-ring (electron-electron) configuration?

These questions were discussed and clarified somewhat at a Review Meeting held at Stanford in February 1965, attended by workers from throughout the United States, and at an international meeting held at Novosibirsk in March 1965. In view of some of the questions raised at these meetings, and in view of the imminence of a decision to proceed with the ambitious project of a 3-GeV electron-positron ring, the summer of 1965 appeared as a particularly appropriate time to bring together the interested workers for a concentrated attack on the outstanding problems. There was reason to believe that such an effort would result in significant progress - as, in fact, turned out to be the case.

In this Summary Report we have tried to bring together, very quickly, brief reports of the principle results which came forth during the Summer Study. In the next section we review the most relevant previous work. This is followed by Report 2 by A. M. Sessler, which contains a review of the main conclusions of the Study. In the rest of this Summary Report will be found short reports of the work of various participants. It is expected that more complete reports of these efforts will be published by the authors at some later time and in the normal literature. We feel, however, that this Summary Report will, in the meantime, be useful to the workers in the field.

The Storage Ring Summer Study was held at the Stanford Linear Accelerator Center from June 28 to July 30, 1965. Several participants were able to attend for only a part of the time, and some contributed reports of work carried out wholly or in part at their home institutions. The appended table contains a list of the participants in the Study.

The Study was supported by the U.S. Atomic Energy Commission. The attendance of some of the participants was sponsored financially by

by SLAC and of others, by their home laboratories, for whose cooperation we are grateful. We are also grateful for the contribution to the Study from each of the participants. Andrew Sessler made a special contribution in stimulating and, by taking part in several of the analyses, in helping to guide the course of the Study, and in helping to put together on a crash schedule this Summary Report.

List of Participants

Matthew Allen	Stanford
Fernando Amman	Frascati
Carl Barber	Stanford
Ernest Courant	Brookhaven
Gabriel Gendreau	Orsay
Bernard Gittleman	Princeton
Mervyn Hine	CERN
Eberhard Keil	CERN
Jackson Laslett	Lawrence Radiation Laboratory
M. Lee	Stanford
Philip Morton	Lawrence Radiation Laboratory
Jerry O'Neil	Princeton
Claudio Pellegrini	Frascati
John Rees	Stanford
Burton Richter	Stanford
David Ritson	Stanford
Kenneth Robinson	Cambridge Electron Accelerator
Matthew Sands	Stanford
Arnold Schoch	CERN

B. Earlier Work and Open Questions

Early in the Summer Study (on July 6-7) a General Review Meeting was held to review the current status of our knowledge on beam instabilities. This two-day meeting was attended, in addition to the Summer Study participants, by physicists from SLAC, from the Lawrence Radiation Laboratories, from MURA, and from Argonne. Reviews were given of the status of the various storage-ring projects around the world, and of past theoretical and experimental studies of instabilities in circulating

beams. What follows is an attempt to summarize the conclusions that could be drawn from these reviews.

Instabilities which may limit the intensities of accelerated or stored beams in circular accelerators have been observed and studied at many laboratories and now appear to be reasonably well understood. A summary of this subject was recently given by E. Courant.¹

It was first pointed out by Amman and Ritson² that the electromagnetic interaction between two colliding beams ("space charge effects") would limit the interaction rates which could be obtained. In the pioneering work of the Princeton-Stanford storage ring group, new and unexpected instabilities were encountered at intensities below the Amman-Ritson limit. The troublesome instabilities appear to be of two distinct type: coherent instabilities, in which all of the particles (or a major fraction of the particles) in a part of the beam execute anti-damped oscillations about the expected equilibrium positions; and incoherent instabilities, in which the dimensions of the beam are enlarged due to a growth of the oscillation amplitudes of individual particles. The incoherent instabilities limit the interaction rates because of the reduction of the current density in the beam. The coherent instabilities can cause a catastrophic loss of one or both beams; they can also limit the reaction rates by causing the beams to avoid each other or by leading to a growth of the beam size if the coherence of the oscillations is subsequently broken up - as may happen if the guide field gives a nonlinear restoring force.

Transverse coherent oscillations of single beams in circular accelerators have been observed at MURA, Brookhaven, CERN, Argonne, Stanford, and elsewhere. These instabilities are in most cases, explained quantitatively by the theory of Laslett, Neil, and Sessler³ (LNS) or by that of Hereward.⁴ Only the observations at MURA are in serious quantitative

¹E. Courant, Proceedings of the Particle Accelerator Conference, Washington, D.C., March 10-12, 1965; p. 550.

²F. Amman and D. Ritson, Proceedings of the Brookhaven International Conference on High Energy Accelerators (1961); p. 471.

³L. J. Laslett, V. K. Neil, A. M. Sessler, Rev. Sci. Instr. 32, 276 (1961).

⁴H. G. Hereward, CERN International Report MPS/Int. DL 64-8, CERN, Geneva (1964).

disagreement with the theory. In the LNS theory it is shown that the image currents in a resistive wall of the vacuum chamber produce fields which can drive the beam particles into coherent transverse oscillations of increasing amplitude. In the Hereward instability the driving forces are provided by the interaction of the beam with residual ions in the vacuum chamber. Both of these instabilities can be quenched by an electronic feedback system (Brookhaven, Argonne, Stanford) or by a non-linear focusing field which introduces a spread in the betatron frequencies of the particles in the beam.

Coherent longitudinal oscillations have also been observed at CERN, MURA, and Saclay and have been analyzed by Laslett, Neil, Nielsen, Sessler, and Symon.^{5,6,7} Coherent longitudinal oscillations of a beam bunch about its equilibrium rf phase may also arise from the interaction between the beam and an accelerating cavity. This effect has been analyzed by Robinson⁸ and can be avoided by proper tuning of the cavity.

It was first suggested by Sessler that two individually stable beams might become unstable when placed in a colliding geometry. The nonlinear electromagnetic interaction at the intersection region can combine with the driving forces from the resistive-wall effects in the individual beams to produce unstable coherent oscillations. It is likely that this effect has been observed in the Princeton-Stanford storage rings. Analyses by Ritson and Rees⁹ and by Sessler suggested that this instability could be avoided if the betatron frequencies of the two beams differ by an amount related to the shift in betatron frequency of the particles of one beam caused by forces from the other beam. The Princeton-Stanford rings are being operated with different Q's in the two rings.

⁵C. E. Nielsen and A. M. Sessler, Rev. Sci. Inst. 30, 80 (1959).

⁶C. E. Nielsen, A. M. Sessler, and K. R. Symon, Proc. International Conference on High Energy Accelerators and Instrumentation, CERN, Geneva 1959, p. 239.

⁷L. J. Laslett, V. K. Neil, and A. M. Sessler, Rev. Sci. Instr. 32, 245 (1961).

⁸K. W. Robinson, CEA Report No. 11, Cambridge Electron Accelerator, Cambridge, Massachusetts (1956).

⁹D. Ritson and J. Rees, SLAC Internal Report, Stanford Linear Accelerator Center, Stanford, California (June 1965).

When all coherent oscillations have apparently been suppressed in the Princeton-Stanford storage rings, there is still observed an incoherent increase in the vertical dimension of one of the two beams when they are brought into collision. This effect does not occur at low beam currents, but appears at currents significantly below the space-charge limit originally recognized by Amman and Ritson.¹⁰ Robinson has suggested that this incoherent beam growth may be due to the excitation of non-linear resonances by the strong nonlinearity of the space-charge fields at the interaction region. Courant¹¹ has carried out extensive numerical computations which simulate the motion of a particle moving in a linear guide field but subjected to impulses each revolution (at the interaction point) which are non-linear in both lateral coordinates of the particles (see Ref. 11). Courant finds that for beam currents well below the Amman-Ritson limit there can be a "quasi-random" growth of the transverse oscillations of the particle. The onset of this growth does not appear to be related to any simple machine resonance, but begins when one of the beams reaches a certain critical charge density. Below this critical charge density there is no discernible growth over hundreds of thousands of revolutions of the particle. These results are in reasonable agreement with the behavior of the beams in the Princeton-Stanford storage ring.

When the Summer Study was initiated there appeared to be several questions which warranted further analysis:

(1) The resistive wall instability had been analysed in detail only for azimuthally uniform beams; to what extent were these analyses applicable to beams with one, or with many, short bunches as one might have in high-energy electron storage rings? Also, to what extent are the discrepancies between the theory and the MURA observations fundamental?

(2) In what ways are the effects in an electron-positron-ring - in which both beams circulate in the same chamber - the same or different from the effects in electron-electron rings?

¹⁰F. Amman and D. Ritson, Proc. International Conf. on High Energy Accelerators, Brookhaven, 1961; p. 471.

¹¹E. Courant, BNL Internal Report AADD-69, Brookhaven National Laboratory, New York, (March 1965).

(3) Are colliding beams stable generally with respect to coherent longitudinal (phase) oscillations?

(4) Is the colliding beam incoherent stability sufficiently well understood (on the basis of the experience with the electron-electron rings and the semi-heuristic calculations of Courant and others) to warrant extrapolations to a 3-GeV machine with currents as high as one ampere?

(5) Are there any effects not thought of yet - particularly effects which might only show up at ultra-relativistic energies?

These questions were the ones faced - and in some instances answered - by the Storage Ring Summer Study.

Report 2

CONCLUSIONS OF THE STUDY

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A. Summary of the Results Obtained

The next sections consist of summary reports by various participants in the Summer Study. These reports, which are brief, superficially may appear unmotivated or to lack relation to each other and to the problems of direct concern to the construction and/or operation of storage rings. It is the purpose of this section to attempt to put the contributions into context (1) by discussing the problems which motivated the specific and detailed investigations, and (2) by describing the significance of the various contributions.

Subsequent to the General Meeting held on July 6 and 7, there was a group discussion held on July 9 concerning outstanding problems of beam instabilities in e^-e^+ storage rings. Much of the material in this section is drawn from the conversations of that day.

The extension of the theory of coherent resistive instabilities for uniform beams (LNS theory) to azimuthally-bunched beams was considered to be the most pressing problem at the session's beginning. Prior to the conference, work performed by Courant --- of a semi-heuristic nature --- indicated the results which might be expected, while work by Skrinsky (of a preliminary character at the time of the Novosibirsk Meeting, where he would only speak informally of his results) shed insight into the physical origin of Courant's conclusions.

The contributions by Morton and Sessler, Laslett and Sessler, Robinson, Courant and Sessler, and Allen, Lee and Rees are all devoted to the coherent resistive instability of a single bunched

beam. Report 3 examines, in great detail, the wake fields behind a shot of charge fired centrally down a straight pipe of circular cross section. It is shown that the fields remain, after the charge has passed, for a surprisingly long time when the walls of the pipe are not perfectly conducting. The longitudinal field falls off only algebraically in the distance behind the particle out to a critical distance which is proportional to σd^2 , where d is either the radius of the pipe or the thickness of the wall, whichever is smaller. It is these very-long-persisting asymptotic fields which are the physical basis of the resistive wall instability. The contribution by Laslett and Sessler (Report 4) gives expressions for the wake fields of transversely oscillating and longitudinally moving charges in a straight pipe of rectangular cross section. The results for the longitudinal field agree in character with that obtained by Morton and Sessler, but do not exhibit the critical distance (an unimportant feature in practice) because the wall thickness was taken as infinite and resistance put only on the top and bottom surfaces, so that corner or curvature effects were neglected. The asymptotic field (transverse magnetic) for an oscillating charge drops as the inverse square root of the distance (with no critical distance, presumably because of the approximations), which is in agreement with the previous preliminary results of Skrinsky, Courant, and Ferlenghi and Pellegrini --- all of whom make approximations which would preclude obtaining the correct "super asymptotic" behavior.

Report 5 by Robinson gives a simple physical derivation of the wake fields in the asymptotic (distances much larger than the pipe diameter or length of charge) --- but not the "super asymptotic" --- regime. This contribution is particularly important for the physical insight it supplies: One sees that in the laboratory frame the field left behind a charge simply slowly decays with time as $t^{-1/2}$ (where t is the time since the charge passed).

The paper by Courant and Sessler (Report 6) studies the dynamics of bunches in a circular accelerator which are subject to the wake fields of each other (including the self-interaction). It is shown that a single bunch, which has a slight stabilizing effect directly upon itself, upon completing a turn is subject to its own wake field which will be either stabilizing or unstabilizing depending upon the phase of its oscillation; hence the result that for Q between an integer and an integer plus one-half the motion is stable, while for Q below an integer the motion will be unstable unless adequately Landau-damped. Further results of the Courant-Sessler paper concern many bunches in an accelerator: a general formalism, and analytic results for equal-intensity bunches. In the latter case it is shown that half of the modes are unstable; consequently it is important to study the transition between stable motion (Q just above an integer) of unequal bunches, and the unstable motion of equal bunches. This is the motivation behind the computational program described in the contribution of Allen, Lee, and Rees (Report 7).

The validity of the theory of resistive wall instabilities is of vital importance since it is being employed so crucially in the design of storage rings. Consequently, although the agreement with observations on the Cosmotron, Argonne, AGS, and Stanford electron rings is good, the discrepancy with the observations at MURA must be taken seriously. This was the motivation behind the investigations of Laslett (Report 8) and of Briggs, Neil, and Sessler (Report 9). In particular it is necessary to understand a discrepancy of approximately two orders of magnitude in both U and V --- the out-of-phase and in-phase forces of the LMS theory. The work was further motivated by a desire to ascertain what values of U and V can really be expected in storage rings. Chirikov (paper presented at Novosibirsk) has already shown that laminated walls (as in a betatron) could change V by two orders of magnitude; in addition, the $(1 - \beta^2)$ cancellation in U reduces it in relativistic machines to a negligible term, but if

the cancellation is removed, U could again become very important in ultra-relativistic machines.

Let us digress to a discussion of the contributions to U . In a translationally invariant, perfectly conducting unloaded guide $U \approx (1 - \beta^2) = 1/\gamma^2$, which is the "Panofsky Theorem" of rf separators. Terms in U not varying as γ^{-2} are:

- (1) Resistive wall terms ($U \approx V$; consequently, these terms are usually small for metal walls).
- (2) Dielectric loading, which can make a large term.
- (3) Corrugations in the longitudinal direction (or pins and ferrites, etc.) which can give a large term.
- (4) Contributions from the beam variation in the longitudinal direction -- " k^2 terms of LNS" -- which are quite small.
- (5) Clearing electrodes (etc...) with impedance loading to the walls, which can give large terms.
- (6) Curvature of vacuum tank, or of particle orbit between two conducting planes, which gives contribution that is small, of the order of the " k^2 terms" (unpublished result of Laslett, Neil, and Sessler).

It is presently thought that slow electrons and ions are not sufficiently trapped in the MURA accelerator to explain the discrepancy (unpublished work by Morton and Sessler), but that dirty clearing electrodes or effect (5) above may be the explanation. (A recent private communication from R. A. Otte indicates that a modification of the termination of the clearing electrodes converts the $n = 5$ mode growth time from 100 times larger than theory to damping with approximately the same magnitude!)

With the solution of the single bunched beam coherent resistive instability problem -- or, at least, its reduction to numerical studies -- the most pressing problem to solve was the two-beam coherent instability. Preliminary work on this problem had been done for two unbunched continuously interacting beams in June 1964 by Sessler; this work was extended and amplified by Ritson and Rees

prior to the Summer Study. Report 10 by Pellegrini and Sessler consists of a straightforward extension of the work of Courant and Sessler to two interacting bunched beams. The consequences are really remarkably simple: for individual beams with the same equilibrium orbits and with equal bunches, the modes of oscillation obtained by Courant and Sessler are all unaltered except for the one mode (of each beam) having net center-of-mass motion. Various methods of handling possible instability in these two modes are discussed.

Report 11 by Robinson is addressed to coherent longitudinal motion in bunched beams. It does to the theory of Neil and Sessler what Courant and Sessler have done to the LNS theory of transverse instabilities. The conclusion is that for e^-e^+ rings the radiation damping of synchrotron motion will-- in practice -- dominate any resistive instability.

We may mention here a tentative result obtained by Amman, but not included in these reports. Amman attacked the point first raised by Robinson and Collins in the spring of 1964; namely, if storage rings are operated on a high harmonic of the circulation frequency, a particle of one beam as it passes through an interaction region will be influenced by many bunches of the other beam. This will tend to average out the nonlinear forces and perhaps reduce the incoherent beam blow-up. The conclusion of Amman -- under certain assumptions, the limiting nature of which is still moot -- is that, in fact, there is not gain from such an averaging effect in the quantity of direct interest, namely the luminosity.

The two contributions by the CERN people are of more immediate interest to proton storage rings than e^-e^+ storage rings. In particular, Hine is concerned with the possible long-term instability of a particle subject to periodic nonlinear forces such as one particle experiences from intersecting the other beam. He reports the results of numerical computations which follow a particle through a large number of revolutions. For e^-e^+ storage rings

the radiative damping makes a slow growth (such as might show up in these calculations) of no significance. The contribution of Keil concerns itself with short-term growth; this subject was first explored in mid-1964 by Sessler and Courant. Subsequent studies have been made by Gendreau; Keil's studies are the most exhaustive to date. The resulting requirement on beam intensity and shape -- as reflected in the one parameter ΔQ which gives the frequency shift of a single particle by an intense beam -- is that ΔQ must be less than 0.05 to avoid blow-up in proton storage rings.

B. The Future: Areas of Confidence and Uncertainty; Subjects for Further Study

Twice during the Summer Study general discussions were held, in which all participants contributed, on the subjects of what problems remained to be studied and what implications could be drawn concerning present or future storage rings from the progress made to date. These meetings, on July 9 and 27, form the basis for the material in this section, but it is realized that the comments contained here are one man's opinion; each reader will want to form his own opinions.

The discussion is limited, as was the whole Summer Study, to problems associated with beam instabilities. First, a survey was made of subjects requiring further study; this was followed by a discussion of the relative importance of the problems, which then leads into more general comments concerning areas of confidence and areas of uncertainty.

Theoretical problems needing further work can be simply listed with a few explanatory comments:

1. Coherent instability in a single unbunched beam:
 - (a) The influence of various wall materials and types to ascertain the influence on the $(1 - \beta^2)$ cancellations in U.
 - (b) Influence of variously terminated clearing electrodes, primarily to attempt to remove the discrepancy between theory and the observations at MURA.

- (c) Studies of nonlinear phenomena and the effect of ions, primarily to ascertain the contributions to the Landau damping.
2. Coherent instability in a single bunched beam:
- (a) Are the very high modes corresponding to internal motion of a bunch stable? One expects that the rf mixing will strongly damp such modes, but a quantitative study is lacking.
 - (b) Computation studies to learn how the equal bunch approximation goes over into independent bunch motion. This is of interest to explain observations at Cornell, ZGS, AGS, and CERN. It is also important, when coupled with 1(a), to the behavior to be expected in storage rings.
 - (c) Careful evaluation of the fields due to one bunch:
 - (i) Asymptotically, to see if vacuum tank dimensions and wall thickness enter in the transverse fields as they do in the one simple case of longitudinal field so far tested.
 - (ii) Asymptotically, but considering the finite circumference of the accelerator so that sums are retained and not replaced by integrals. Since the asymptotic behavior is dominated by a small range of wave numbers, this could easily make a sizeable numerical difference.
 - (iii) At all distances, in order to ascertain the effect on nearby bunches and of a bunch on itself numerically more accurately.
3. Coherent instabilities involving two beams:
- (a) Extend the analysis of Pellegrini and Sessler for continuously interacting bunched beams to discretely interacting continuous beams -- of interest for application to proton storage rings.

- (b) Numerical studies to learn how to handle the unstable modes by Q-splitting, or Landau damping, or unequal bunches. (The analog of 2(b) for two beams.)
- (c) Employ the Italian studies (private communication from F. Amman) of incoherent-incoherent beam interaction to obtain beam shape and hence make accurate estimates of the cross-beam contribution to U (the term in $1/z$ of LNS) and to the Landau damping (the cross-beam ΔS of LNS and Ritson and Rees).

4. Incoherent two-beam effects:

Further study of the Robinson-Collins proposal to employ many bunches to average away some nonlinear effects; in particular, to see the sensitivity of the cancellation to geometry and variation in intensity from bunch to bunch.

5. Longitudinal coherent motion:

Further study -- for one and two beams -- of the coherent motion of bunches with respect to each other, including the interaction with the rf system (although it does not now appear that there is likely to be a problem).

Experimental studies are, of course, very much limited by the availability of suitable accelerators for beam behavior studies. It is clear that any further information from the CERN P.S., the AGS, and the ZGS would be most valuable, but extensive experimental studies would seem to be unlikely. On the other hand, continued studies may be expected at MURA, on the CERN electron model, and on the Stanford storage rings. The sorts of studies required are clear and will not be detailed here, other than to comment upon the special requirement at MURA to modify the clearing electrodes in regard to their surface (make it clean and good conducting), extent (presently as much as 20% of the circumference is not cleared, much of it where magnetic fields might be trapping ions and electrons), and electrical properties (the present results are clearly sensitive to the mode of termination).

Where then, do we stand at the present time? Despite the formidable array of theoretical problems, I think that there is a large area of phenomena which we can feel confident that we both understand and can control. This assumes that the theory is correct; further studies on coherent phenomena in the Stanford rings will be vital in reinforcing (or destroying!) this assumption, which is the only reasonable basis upon which to rest our plans.

In particular, incoherent beam-beam effects are adequately understood to allow a design of storage rings which (awaiting the study of (4), above) at worst is overly conservative. Also, it is probable that coherent longitudinal motion is no source of difficulty. That leaves coherent transverse motion; and although the studies outlined above as (1), (2), and (3) will make quantitative differences -- and thus introduce an area of uncertainty which is reflected in a conservative design -- the theory suggests a number of ways of designing storage rings so as to be assured of successful operation:

1. Single beam coherent instabilities can be controlled, as in the present Stanford rings or at Cornell, by Landau damping introduced via octopoles. It can also be controlled, as at the Cosmotron or ZGS, by feedback. Theoretically, by loading or modifying a smooth vacuum tank or by having unequal bunches, and choosing Q above an integer, the instability should be suppressed. This remains to be confirmed by experiment.
2. Two-beam coherent instabilities can be handled by Q -splitting, as at the Stanford electron rings; or by feedback on the one mode with center-of-mass motion in the equal bunch case (Landau damping is not adequate for this one mode, in most designs); or by unequal loading and choice of Q -values (in loaded structures) so as to damp all modes; or by choice of Q -values to damp the two coupled modes and Landau damping to handle the modes which are unstable (equal bunch case) but unaffected by beam-beam effects. And there are, clearly, other possibilities also.

Thus, at the end of this Summer Study, we would appear to have gained considerable insight into beam instabilities in storage rings, enough insight to successfully circumvent all presently known difficulties.

Report 3

THE LONGITUDINAL WAKE FIELDS OF A
PULSE OF CHARGE MOVING DOWN A STRAIGHT PIPE
OF CIRCULAR CROSS SECTION WITH FINITE CONDUCTIVITY

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A. Introduction

It has been shown that, for the case of unbunched beams, the finite conductivity of the walls of an accelerator can lead to unstable coherent oscillations of the beam.^{1,2} The question of stability then arises for a bunched beam in which the distance between bunches is large compared to the radius of the vacuum pipe. If the electric and magnetic fields fall off fast enough, the motion of separate bunches would be independent of each other, and one might expect to stabilize the coherent beam oscillations by bunching the beam longitudinally. In fact, for the case of infinite conductivity the fields fall off exponentially in a distance of the order of the pipe radius, which is typically small compared to the distance between bunches.

It is the purpose of this paper to compute the falloff of the fields at large distances from a bunch of charge in the case of finite wall conductivity. In all cases it has been assumed that the point of observation is at a distance which is large compared with the pipe radius and the bunch length, and that the conductivity of the vacuum walls is such that the displacement current in the wall can be neglected compared to the conduction current.

B. Field Solution

We present here the solution for the electric and magnetic fields produced by a pulse of charge traveling, in the z direction with velocity βc , inside an infinitely long straight circular pipe of conductivity σ . The inner and outer radii of the pipe will be designated by b and d ,

respectively. The pulse of charge will have a constant radial density inside of radius a so that the charge density can be written as:

$$\rho(r, z) = e n(r) f(z - vt) \quad (1)$$

with

$$n(r) = \begin{cases} n_0 & r < a \\ 0 & r > a \end{cases}$$

Fourier transformation will be used in solving for the fields; the convention that a curl above a quantity designates the transform of the quantity will be adopted, for example:

$$f(z - vt) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ik(z-vt)} dk \quad (2)$$

The transformed Maxwell equations are employed to find general solutions for the four regions $r < a$, $a < r < b$, $b < r < d$, and $r > d$. The boundary conditions that \tilde{E}_z and \tilde{B}_θ be continuous then determine the solution; for the special case that $r = a = b$, we obtain:

$$\tilde{E}_z = \left\{ \frac{4\pi e n_0 \tilde{f}}{ik} \frac{q\beta^2 \gamma^2}{\alpha} J_1(qb) \right\} \times \left\{ \frac{\alpha [N_0(\alpha b) J_1(\alpha d) - J_0(\alpha b) N_1(\alpha d)] H_0^{(1)}(qd) + q\beta^2 \gamma^2 [N_0(\alpha b) J_0(\alpha d) - N_0(\alpha d) J_0(\alpha b)] H_1^{(1)}(qd)}{\alpha [N_1(\alpha b) J_1(\alpha d) - J_1(\alpha b) N_1(\alpha d)] H_0^{(1)}(qd) + q\beta^2 \gamma^2 [N_1(\alpha b) J_0(\alpha d) - N_0(\alpha d) J_1(\alpha b)] H_1^{(1)}(qd)} \right\} \quad (3)$$

with

$$q^2 = -\frac{k^2}{\gamma^2}, \quad \alpha^2 = iRk \quad \text{and} \quad R = \frac{4\pi\beta\sigma}{c}$$

The choice of sign for q and α is such that the imaginary part of q and α is always positive

C. Asymptotic Limit

It will be assumed that the outer pipe radius d , and the length of the bunch are both much smaller than $|s|$, where $s \equiv (z - vt)$ and (z, t) is the point at which the field is evaluated. (The bunch of charge has trajectory $z = vt$.) In this case we need only retain in \tilde{E}_z the dominant terms in the small quantities qb and qd . We will be interested in two cases for the relative values of pipe inner and outer radii: Case 1, where the wall thickness of the pipe is large compared to the inner radius; and Case 2, where the radius of the pipe is larger than the pipe thickness.

For Case 1 there are three regions of interest: $|s| \ll Rb^2$; $Rb^2 \ll |s|$; and $|s| \gg Rd^2$. The relevant approximations for \tilde{E}_z are: $\alpha b \gg 1$; $\alpha d \gg 1 \gg \alpha b$; and $\alpha d \ll 1$, respectively, so that if N is the number of particles in the pulse,

$$\tilde{E}_z = i \frac{eN \beta^2}{\pi b} \left(\frac{|k|}{2R}\right)^{\frac{1}{2}} \left[i + \text{sign}(k) \right] \quad |s| \ll Rb^2 \quad (4)$$

$$\tilde{E}_z = -i \frac{eN \beta^2}{\pi} k \left[\ln\left(i|k|^{\frac{1}{2}} + |k|^{\frac{1}{2}} \text{sign}(k)\right) \right] \quad Rb^2 \ll |s| \ll Rd^2 \quad (5)$$

$$\tilde{E}_z = \frac{eN \beta^2}{\pi R(d^2 - b^2)} \quad Rd^2 \ll |s| \quad (6)$$

We use the fact that the Fourier inversion integral depends upon the singularities of k to obtain³

$$E_z(z, t) = \frac{eN \beta^2}{\sqrt{\pi R} b} \frac{S(z, t)}{|s|^{\frac{3}{2}}} \quad |s| \ll Rb^2 \quad (7)$$

$$E_z(z, t) = eN \beta^2 \frac{S(z, t)}{|s|^2} \quad Rb^2 \ll |s| \ll Rd^2 \quad (8)$$

$$E_z(z, t) = \frac{eN \beta^2}{\pi R(d^2 - b^2)} \delta(s) \quad Rd^2 \ll |s| \quad (9)$$

where $S(z,t) = \begin{cases} 1 & \text{for } (s) < 0 \\ 0 & \text{for } (s) > 0 \end{cases}$, and $\delta(s)$ is a Dirac δ function.

The functions $S(z,t)$ and $\delta(s)$ appear here only because we have ignored all fields with fall-off distances of the order of the length of the bunch or the radius of the pipe.

For Case 2, we have regions of interest where $|s| \ll R(d-b)^2$; $R(d-b)^2 \ll |s| \ll Rd^2$; and $|s| \gg Rd^2$. With the appropriate approximations for αb and $\alpha(d-b)$ we have:

$$\tilde{E}_z = i \frac{eN \beta^2}{\pi b} \left(\frac{|k|}{2R}\right)^{\frac{1}{2}} [i + \text{sign}(k)] \quad |s| \ll R(d-b)^2 \quad (10)$$

$$\tilde{E}_z = \frac{eN \beta^2}{\pi R d(d-b)} \quad R(d-b)^2 \ll |s| \ll Rd^2 \quad (11)$$

$$\tilde{E}_z = \frac{eN \beta^2}{\pi R (d^2 - b^2)} \quad Rd^2 \ll |s| \quad (12)$$

Upon inverting \tilde{E}_z to find E_z , we obtain

$$E_z(z,t) = \frac{eN \beta^2}{\sqrt{\pi R} b} \frac{S(z,t)}{|s|^{\frac{3}{2}}} \quad |s| \ll R(d-b)^2 \quad (13)$$

which is the same expression that we have for the thick wall case except that now the cutoff distance is $R|d-b|^2$. Since the value of R for metal walls is of the order of 10^7 cm^{-1} , we see that the falloff distance of the fields behind a pulse of charge is sufficiently large for one bunch of particles to affect the motion of a later bunch. Also, since pulses in an accelerator return to their previous position, it is possible for a bunch to leave behind fields that it will subsequently encounter.

The expression for $E_z(z,t)$ with $|s| \ll Rb^2$ or $R(d-b)^2$ can be obtained⁴ from Ref. 1, for the case of a circular pipe, by letting $\omega = kv$ in all expressions and noting that boundary condition is valid only for values of $k > 0$ and must be modified for $k < 0$. One

readily obtains

$$\tilde{E}_z(k) = - \left[1 - i \operatorname{sign}(k) \right] \left(\frac{|k| \beta c}{8\pi\alpha} \right)^{\frac{1}{2}} B_\phi$$

from which $E_z(s)$ may be computed. The result is identical to Eqs. (7) and (13) of this paper. The approximations of Ref. 1, for the circular pipe case, are such that it is not possible to obtain the cutoff distances that are obtained here.

The fields presented in Ref. 1 for the rectangular case are exact, and would lead (following the procedure of the previous paragraph) to fields which do not exhibit a cutoff. This may be traced to the use of infinitely thick walls, and to the absence of curvature (or corner) effects arising from the fact that only the top and bottom walls were taken to have finite conductivity.

REFERENCES

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2. L.J. Laslett, V.K. Neil, and A.M. Sessler, Rev. Sci. Instr. 36, 436 (1965).
3. M.J. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge Press, Cambridge, Massachusetts (1958).
4. This was first observed by E.D. Courant (private communication).

Report 4

THE ASYMPTOTIC PERTURBATION FIELDS OF A LONGITUDINALLY BUNCHED BEAM WITHIN A RECTANGULAR PIPE WITH RESISTIVE WALLS

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A. Introduction

In connection with an examination of the stability of completely bunched beams, it is of importance to recognize the presence of a continuous spectrum of spatial frequencies, κ , in the longitudinal charge distribution and to examine specifically the remote (asymptotic) fields that could affect neighboring bunches (or affect the same bunch on a subsequent revolution within a cyclic accelerator). Previous unpublished notes¹ have presented the fields of a bunched beam oscillating transversely within a straight rectangular pipe with resistance in its upper and lower walls. The formulas for the field components were explicitly given, however, for bunching represented by a density distribution with a single Fourier component

$$\sigma(x, z, t) = \left[\sum_n \sigma_n \sin \frac{n\pi}{w} x \right] \cos \kappa (\beta_p ct - z),$$

or, in complex notation,

$$\sigma(x, z, t) = \left[\sum_n \sigma_n \sin \frac{n\pi}{w} x \right] e^{j\kappa (\beta_p ct - z)}.$$

It is of interest to synthesize from these results the dominant remote fields of a completely bunched beam.²

¹L. Jackson Laslett, "Electromagnetic Fields in a Rectangular Pipe due to a Modulated Beam which is Displaced Transversely," unpublished notes (Lawrence Radiation Laboratory, 13 November 1963). In the present work we set the bunching factor, B , equal to zero.

²Although we here derive the remote fields for a bunched beam by an asymptotic synthesis of Fourier components, an alternative procedure, such as that proposed by K. W. Robinson (see Report 5), might be followed to obtain such fields directly.

After indicating the notation to be employed, we summarize in the present paper the formulas for the electric and magnetic fields of interest¹ and then exhibit the corresponding asymptotic expressions for the remote fields. These latter fields have been employed elsewhere³ in analyzing the coherent resistive instability of bunched beams.

B. Notation

The pipe is taken to have cross-sectional dimensions $w \times h$, with the side walls at $x = 0, w$ and the other boundaries at $y = \pm h/2$. The "surface impedance" (ratio of tangential E and H fields) is characterized by $(1+j)\mathcal{R}$,⁴ in which (in unrationalized Gaussian units) the dimensionless quantity \mathcal{R} is given in terms of the resistivity (ρ) of the (thick) boundary material and the angular frequency (ω) of the electromagnetic waves as $\mathcal{R} = \sqrt{|\omega|\rho/8\pi}$.

The surface density of charge in the beam will be represented by

$$\sigma(x, z, t) = \left[\sum_n \sigma_n \sin \frac{n\pi}{w} x \right] \int_{-\infty}^{\infty} \tilde{f}(\kappa) e^{j\kappa(\beta_p ct - z)} d\kappa$$

and the corresponding current density (e.s.u./cm) in the longitudinal z-direction is $\beta_p c$ times $\sigma(x, z, t)$. It is convenient to express the asymptotic fields in terms of a charge density per unit width of the beam, formed by integrating $\sigma(x, z, t)$ through the bunch:

$$\Lambda(x) \equiv \int_{-\infty}^{\infty} \sigma(x, z, t) dz = \sum_n \Lambda_n \sin \frac{n\pi}{w} x,$$

where

$$\Lambda_n = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \tilde{f}(\kappa) e^{j\kappa s} d\kappa = 2\pi \tilde{f}(0) \sigma_n.$$

³See Reports 6 and 10.

⁴With the time dependence expressed as $e^{+j\omega t}$, a factor $1 + j$ denotes a phase advance and here serves to represent a partially inductive impedance (electric field leading the magnetic field by 45 degrees, as is characteristic of the skin effect). For a time dependence $e^{-i\omega t}$, j may be replaced by $-i$.

The beam is considered either to be circulating in the median plane ($y=0$) or to be undergoing a small transverse oscillation

$$y = \xi e^{jk(\beta_w ct - z)}$$

of specified k and β_w . Although local perturbation fields require recognition of a (non-resistive) " $-2/\tau$ term" that applies to a beam of finite thickness τ in the y direction, this feature may be seen not to affect the remote fields to be obtained in the present report.

We employ, for brevity, the following auxiliary quantities:

$$\beta_s = \frac{k\beta_w + \kappa\beta_p}{k + \kappa} \qquad \beta_d = \frac{k\beta_w - \kappa\beta_p}{k - \kappa}$$

$$\theta_w = k(\beta_w ct - z)$$

$$\theta_p = \kappa(\beta_p ct - z)$$

$$\theta_s = (k+\kappa)(\beta_s ct - z) = \theta_w + \theta_p$$

$$\theta_d = (k-\kappa)(\beta_d ct - z) = \theta_w - \theta_p$$

$$\mu = \frac{n\pi}{w} \qquad \mu_w^2 = \mu^2 + k^2(1-\beta_w^2) \qquad \mu_p^2 = \mu^2 + \kappa^2(1-\beta_p^2)$$

$$\mu_s^2 = \mu^2 + (k+\kappa)^2(1-\beta_s^2) \qquad \mu_d^2 = \mu^2 + (k-\kappa)^2(1-\beta_d^2) ,$$

in which the index n and the functional dependence on κ are to be implicitly understood. The angular frequencies $k\beta_w c$, $\kappa\beta_p c$, $(k+\kappa)\beta_s c$, and $(k-\kappa)\beta_d c$ will lead respectively to the four resistance parameters \mathcal{R}_w , \mathcal{R}_p , \mathcal{R}_s , and \mathcal{R}_d .

C. The Perturbation Fields

We list here the continuous field components in the median plane ($y=0$) that are of interest for a perturbation analysis, together with the values near the axis of certain related field components that may be of interest in the interpretation of the results. Summation over the index n and integration over κ is to be implicitly understood in these expressions. The upper sign given in the complex coefficient of \mathcal{R} applies in each case to a positive frequency in θ , as derived in the 13 November 1963 notes (Ref. 1); the lower sign (obtained by

replacing \mathcal{R} by $(1-j/1+j)\mathcal{R}$ is to be applied when these results are extended to negative frequencies.

1. Longitudinal Electric Field and Related Field Components for a Bunched Beam in the Median Plane

For a bunched beam moving in the median plane, the longitudinal electric field is, with the summation over n omitted for brevity,

$$E_z = 2\pi\sigma_n \left[j \frac{\kappa}{\mu_p} (1-\beta_p^2) \tanh \mu_p \frac{h}{2} - (1\pm j)\beta_p \mathcal{R}_p \operatorname{sech}^2 \mu_p \frac{h}{2} \right] \cosh \mu_p y \sin \mu x e^{j\theta_p};$$

also

$$E_x = -2\pi\sigma_n \frac{\mu}{\mu_p} \tanh \mu_p \frac{h}{2} \cosh \mu_p y \cos \mu x e^{j\theta_p}$$

$$E_y = 2\pi\sigma_n \left[-\tanh \mu_p \frac{h}{2} + (\pm 1 - j) \frac{\beta_p \kappa}{\mu_p} \mathcal{R}_p \operatorname{sech}^2 \mu_p \frac{h}{2} \right] \sinh \mu_p y \sin \mu x e^{j\theta_p}.$$

Correspondingly,

$$H_x = 2\pi\sigma_n \left[\beta_p \tanh \mu_p \frac{h}{2} + (\pm 1 - j) \frac{\mu_p^2 - \kappa^2}{\mu_p \kappa} \mathcal{R}_p \operatorname{sech}^2 \mu_p \frac{h}{2} \right] \sinh \mu_p y \sin \mu x e^{j\theta_p}$$

$$H_y = -2\pi\sigma_n \left[\frac{\mu}{\mu_p} \beta_p \tanh \mu_p \frac{h}{2} + (\pm 1 - j) \frac{\mu}{\kappa} \mathcal{R}_p \operatorname{sech}^2 \mu_p \frac{h}{2} \right] \cosh \mu_p y \cos \mu x e^{j\theta_p}$$

$$H_z = 2\pi\sigma_n \left[(1\pm j) \frac{\mu}{\mu_p} \mathcal{R}_p \operatorname{sech}^2 \mu_p \frac{h}{2} \right] \sinh \mu_p y \cos \mu x e^{j\theta_p}.$$

These field components, as listed, may be seen to satisfy the homogeneous Maxwell's equations for vacuum.

2. Fields Associated with Transverse Oscillations of a Bunched Beam

The field components E_y and H_x are of greatest significance for the transverse stability of a beam undergoing transverse oscillations. The relevant (ξ -dependent) terms for these field components are:

$$\begin{aligned}
E_y = 2\pi\sigma_n \xi & \left\{ \left[\frac{\mu_p^2 + k \left\{ k(1 - \beta_{pw}) + \kappa \left[2 - \beta_p(\beta_p + \beta_w) \right] \right\}}{2\mu_s} \coth \mu_s \frac{h}{2} \right. \right. \\
& \left. \left. + (k+\kappa) \beta_s \frac{\mu^2 + (k+\kappa)^2(1-\beta_{ps})}{2\mu_s^2} (\pm 1 - j) \mathcal{L}_s \operatorname{csch}^2 \mu_s \frac{h}{2} \right] \cosh \mu_s y e^{j\theta_s} \right. \\
& \left. + \left[\frac{\mu_p^2 + k \left\{ k(1 - \beta_{pw}) - \kappa \left[2 - \beta_p(\beta_p + \beta_w) \right] \right\}}{2\mu_d} \coth \mu_d \frac{h}{2} \right. \right. \\
& \left. \left. + (k-\kappa) \beta_d \frac{\mu^2 + (k-\kappa)^2(1-\beta_{pd})}{2\mu_d^2} (\pm 1 - j) \mathcal{L}_d \operatorname{csch}^2 \mu_d \frac{h}{2} \right] \cosh \mu_d y e^{j\theta_d} \right\} \sin \mu x,
\end{aligned}$$

$$H_x = 2\pi\sigma_n \xi \left\{ \begin{aligned} & \left[\begin{aligned} & - \frac{\mu_s^2 \beta_p + k(k+\kappa)(\beta_w - \beta_p)}{2\mu_s} \coth \mu_s \frac{h}{2} \\ & + \frac{\mu_s^2 \beta_p [\mu_s^2 - (k+\kappa)^2] - k(k+\kappa)^3 \beta_s^2 (\beta_w - \beta_p)}{2\mu_s^2 (k+\kappa) \beta_s} (\pm 1 - j) \mathcal{R}_s \operatorname{csch}^2 \mu_s \frac{h}{2} \end{aligned} \right] \cosh \mu_s y e^{j\theta_s} \\ & + \left[\begin{aligned} & - \frac{\mu_d^2 \beta_p + k(k-\kappa)(\beta_w - \beta_p)}{2\mu_d} \coth \mu_d \frac{h}{2} \\ & + \frac{\mu_d^2 \beta_p [\mu_d^2 - (k-\kappa)^2] - k(k-\kappa)^3 \beta_d^2 (\beta_w - \beta_p)}{2\mu_d^2 (k-\kappa) \beta_d} (\pm 1 - j) \mathcal{R}_s \operatorname{csch}^2 \mu_d \frac{h}{2} \end{aligned} \right] \cosh \mu_d y e^{j\theta_d} \end{aligned} \right\} \sin \mu x .$$

D. The Wake Fields

The general nature of the resistance-dependent terms in the fields of Section C may be examined in the asymptotic limit of $|z - \beta_p ct|$ large⁵ if we presume that $\tilde{f}(\kappa)$ has the character required to construct a particle bunch localized in the immediate vicinity of $z = \beta_p ct$. We introduce, for dimensional convenience, a distance l_p given by $\beta_p c\rho/8\pi$. The symbol $S(z,t)$ is defined as unity for $\beta_p ct - z > 0$, and vanishes for $\beta_p ct - z < 0$ save for higher-order terms in $1/(\beta_p ct - z)$.

1. Longitudinal Electric Field and Related Field Components for a Bunched Beam in the Median Plane

Of the resistance-dependent fields given in Section C.1, that which arises from E_z dominates at large distances:

$$E_z \approx \sqrt{2\pi} \Lambda_n l_p^{\frac{1}{2}} \beta_p \frac{\text{sech}^2 \mu \frac{h}{2} \cosh \mu y \sin \mu x}{(\beta_p ct - z)^{3/2}} S(z,t) .$$

The component E_y falls off more rapidly, to exhibit the asymptotic form (consistent with $\text{div } \vec{E} = 0$):

$$E_y \approx -3\sqrt{\pi/2} \Lambda_n l_p^{\frac{1}{2}} \beta_p \frac{\text{sech}^2 \mu \frac{h}{2} \sinh \mu y \sin \mu x}{\mu (\beta_p ct - z)^{5/2}} S(z,t) .$$

The corresponding dominant magnetic-field components fall off more slowly, to assume the form (consistent with $\text{div } \vec{H} = 0$):

$$H_x \approx 2\sqrt{2\pi} \Lambda_n l_p^{\frac{1}{2}} \mu \frac{\text{sech}^2 \mu \frac{h}{2} \sinh \mu y \sin \mu x}{(\beta_p ct - z)^{1/2}} S(z,t) .$$

$$H_y \approx -2\sqrt{2\pi} \Lambda_n l_p^{\frac{1}{2}} \mu \frac{\text{sech}^2 \mu \frac{h}{2} \cosh \mu y \cos \mu x}{(\beta_p ct - z)^{1/2}} S(z,t) .$$

⁵Cf. M.J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions*, (Cambridge Press, Cambridge, Massachusetts, 1958).

It is seen that the \vec{E} field may be regarded as induced by the changing magnetic field, as required by the equation $\text{curl } \vec{E} = - (1/c)\partial\vec{H}/\partial t$, in that $\partial E_z/\partial y = -(1/c)\partial H_x/\partial t$ and $\partial E_z/\partial x = (1/c)\partial H_y/\partial t$.

2. Fields Associated with Transverse Oscillations of a Bunched Beam

From the expressions given in Section C.2 for the field components E_y and H_x , it is evident that the greatest contribution of these in the asymptotic limit comes from the resistance-dependent terms in H_x . The essential limiting process for such terms is, in fact, analogous to that applicable to H_x and to H_y in Section C.1, with $k \beta_w/\beta_p \pm \kappa$ now playing the role of κ .⁶ One thus obtains the asymptotic form (for k small in comparison to the reciprocal of the bunch length):

$$H_x \approx 2\sqrt{2\pi} \Lambda_n \ell_p^{\frac{1}{2}} \mu^2 \frac{\text{csch}^2 \mu \frac{h}{2} \cosh \mu y \sin \mu x}{(\beta_p ct - z)^{\frac{1}{2}}} S(z, t) \xi e^{-jk \frac{\beta_p - \beta_w}{\beta_p} z}$$

The "phase factor," $\exp[-jk(\beta_p - \beta_w/\beta_p)z]$, may be understood by noting that if we increase z by Δz and advance t by $\Delta t = \Delta z/\beta_p c$ (thus keeping the bunch the same distance ahead of the observation point), the situation remains essentially as before, save that the phase of the transverse wave has changed by $k(\beta_w c \Delta t - \Delta z) = -k(\beta_p - \beta_w/\beta_p)\Delta z$. Specifically, the magnitude of the resistance-dependent perturbation field at the observation point is directly proportional to the transverse displacement of the bunch at the time it passed that point.

As in Section D.1, there is a similar asymptotic field component

$$H_y \approx -2\sqrt{2\pi} \Lambda_n \ell_p^{\frac{1}{2}} \mu^2 \frac{\text{csch}^2 \mu \frac{h}{2} \sinh \mu y \cos \mu x}{(\beta_p ct - z)^{\frac{1}{2}}} S(z, t) \xi e^{-jk \frac{\beta_p - \beta_w}{\beta_p} z}$$

⁶ $(k+\kappa)\beta_s = k\beta_w + \kappa\beta_p = \beta_p(k\beta_w/\beta_p + \kappa)$, $\mathcal{R}_s = \sqrt{|(k+\kappa)\beta_s c| \rho / 8\pi} = \sqrt{|k\beta_w/\beta_p + \kappa|} \ell_p^{\frac{1}{2}}$, and similarly for $(k-\kappa)\beta_d$ and \mathcal{R}_d (with the sign of κ reversed). It also is convenient to write

$$e^{j\theta_s} = e^{j(k\beta_w/\beta_p + \kappa)(\beta_p ct - z)} \cdot e^{-jk \frac{\beta_p - \beta_w}{\beta_p} z},$$

and $e^{j\theta_d}$ similarly with the opposite sign for κ .

and an associated $\xi \cdot \mathcal{R}$ -dependent E_z that is proportional to $1/(\beta_p ct - z)^{3/2}$ and is related to \vec{H} through the law of induction,

$$E_z \approx \sqrt{2\pi} \Lambda \ell_p^{1/2} \mu \beta_p \frac{\text{csch}^2 \mu \frac{h}{2} \sinh \mu y \sin \mu x}{(\beta_p ct - z)^{3/2}} S(z, t) \xi e^{-jk \frac{\beta_p - \beta_w}{\beta_p} z}.$$

The component E_y falls off even more rapidly - proportionally to $1/(\beta_p ct - z)^{5/2}$ - so the effect of H_x will dominate in determining the effective $E_y + \beta_p \times H_x$ field in the wake zone.

Report 5

THE WAKE FIELDS OF A CHARGED PARTICLE TRAVELING IN A VACUUM CHAMBER WITH RESISTIVE WALLS

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A. Wall Currents

The wake fields associated with a charge moving in a vacuum chamber can be obtained readily from the wake currents; i.e., the currents left in the resistive walls of the vacuum chamber after the passage of the charged particle. We may compute these currents easily if we know the current generated in the wall by the application of a changing magnetic field. The later problem is solved by employing Maxwell's equations in the conductor, where:

$$\frac{\partial B_x}{\partial y} \approx \mu i_z ,$$

$$\frac{\partial E_z}{\partial y} = - \frac{\partial B_x}{\partial t} ,$$

$$i_z = \sigma E_z ,$$

are appropriate to a conducting media for $y \geq 0$, and a time-varying magnetic field $B_x(0,t)$ applied at the surface $y = 0$ and directed along the surface (x-direction). Eliminating B_x and E_z , one obtains

$$\frac{\partial^2 i_z}{\partial y^2} = - \mu \sigma \frac{\partial i_z}{\partial t} .$$

The solution of this equation for a specified applied field $B_x(0,t)$ is

$$i_z(y,t) = \frac{\sigma^{1/2}}{\pi^{1/2} \mu^{1/2}} \int_{-\infty}^t \frac{\partial B_x(0,t')}{\partial t'} \frac{\exp \left[-\frac{\mu\sigma}{4} \frac{y^2}{(t-t')^2} \right]}{(t-t')^{1/2}} dt'$$

B. Longitudinal Field

The longitudinal field due to the passage of a charge q down the center of a vacuum chamber, having a circular cross section with inner radius b and an infinitely thick wall, is obtained directly from the wall current:

$$E_z(t) = \frac{i_z(0,t)}{\sigma} = \frac{1}{\pi^{1/2} \sigma^{1/2} \mu^{1/2}} \int_{-\infty}^z \frac{\partial B_x(0,t')}{\partial t'} \frac{dt'}{(t-t')^{1/2}}$$

In this expression the curvature of the vacuum chamber wall has been neglected, as has the small variation of E_z with radial position for $r \leq b$. If we are interested in the fields at a time after particle passage which is large compared with the duration of the direct fields produced by the particle, then we may expand the t -dependent term in E_z to obtain

$$E_z(t) \approx \frac{1}{\pi^{1/2} \sigma^{1/2} \mu^{1/2}} \left[\frac{1}{t^{1/2}} \int_{-\infty}^{\infty} \frac{\partial B_x(0,t')}{\partial t'} dt' + \frac{1}{2t^{3/2}} \int_{-\infty}^{\infty} \frac{\partial B_x(0,t')}{\partial t'} t' dt' \right]$$

The first term vanishes; after integration by parts and employing the relations $dt' = dz/\beta c$ and $B_x = \beta E_n/c$, one obtains

$$\begin{aligned} E_z(t) &= \frac{1}{2\pi^{1/2} \sigma^{1/2} \mu^{1/2} t^{3/2} c^2} \int_{-\infty}^{\infty} E_n dz \\ &= \frac{q}{4\pi^{3/2} \epsilon_0 \mu^{1/2} c^2 \sigma^{1/2} b t^{3/2}} \end{aligned}$$

C. Transverse Magnetic Field

The transverse magnetic wake field of a slowly oscillating particle of charge q , having amplitude of oscillation ξ , may be obtained by computing the field produced by a traveling dipole of charges $\pm q$ with displacements $\pm \xi/2$. The geometry is indicated in Fig. 1. The wake field B_x will be generated by dipole currents in the walls:

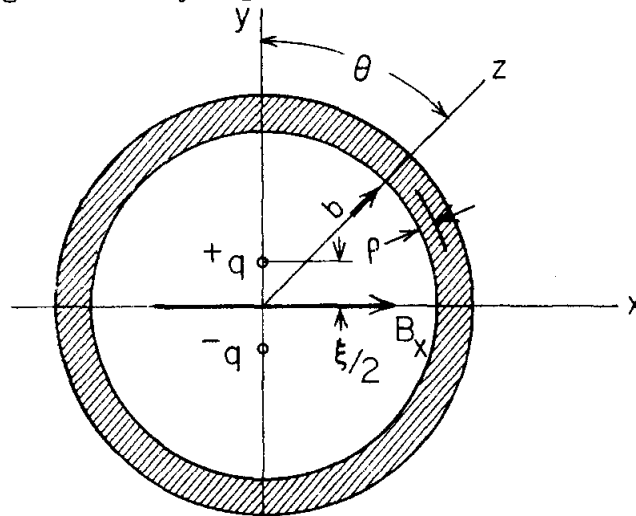


Fig. 1--Geometry for transverse magnetic wake field calculation.

$$I_{z\rho} = \int_0^{\infty} i_z(\rho, t) \rho d\rho = \frac{\sigma^{1/2}}{\pi^{1/2} \mu^{1/2}} \int_0^{\infty} \int_{-\infty}^t \frac{\partial B_x(0, t')}{\partial t'} \times \frac{\exp\left[-\frac{\mu \sigma \rho^2}{4(t-t')}\right]}{(t-t')^{1/2}} \rho dt' d\rho$$

Integrating over ρ we obtain:

$$I_{z\rho} = \frac{2}{\pi^{1/2} \sigma^{1/2} \mu^{3/2}} \int_{-\infty}^t \frac{\partial B_x(0, t')}{\partial t'} (t-t')^{1/2} dt'$$

A series expansion of the factor $(t-t')^{1/2}$ now yields, from the second term, after integration by parts

$$I_{z\rho} = \frac{1}{\pi^{1/2} \sigma^{1/2} \mu^{3/2} t^{1/2}} \int_{-\infty}^{\infty} B_x(0, t') dt'$$

The integral involving B_x may be expressed in terms of an integral involving E_n :

$$\int_{-\infty}^{\infty} B_x(0, t') dt' = \frac{1}{c^2} \int_{-\infty}^{\infty} E_n dz$$

which may be evaluated for a charge q whose longitudinal distribution is such that the density varies only slightly in a distance b . In this two-dimensional electrostatic approximation,

$$\int_{-\infty}^{\infty} E_n dz = \frac{q \xi \cos \theta}{\pi \epsilon_0 b^2}$$

which, when employed in $I_{z\rho}$ and combined with the formula

$$B_x = \int_0^{2\pi} \frac{\mu I_{z\rho}}{2\pi b^2} \cos \theta b d\theta ,$$

yields for the transverse magnetic wake field:

$$B_x = \frac{q \xi}{2\pi^{3/2} \mu^{1/2} \epsilon_0 c^2 b^3 \sigma^{1/2} t^{1/2}}$$

Report 6

TRANSVERSE COHERENT INSTABILITY OF BUNCHED BEAMS

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Laslett, Neil, and Sessler¹ have studied the instability of transverse oscillations that arises by virtue of the interaction of a beam circulating in a metallic vacuum chamber with the field induced by the beam in the walls of the chamber. They show that instability may be caused by the finite resistivity of the vacuum chamber walls. Their treatment is confined to a continuous beam, of azimuthally constant density and dimensions, oscillating coherently in such a mode that its transverse electric dipole moment is of the form

$$P_y(\theta, t) \equiv \int y \rho(r, \theta, z, t) dr dz = P_n e^{i(n\theta - \omega t)} \quad (1)$$

where ρ is the density of the beam, and we use cylindrical coordinates r, θ, z ; y is the direction of transverse oscillation [$y = z$ for vertical oscillations; $y = r - R$ ($R =$ radius of orbit) for horizontal oscillations].

The mean force field acting on the beam is found to be of the form

$$F_y = E_y - \beta B_x = P_n \left[U + W \sqrt{\frac{1}{\omega}} \right] e^{i(n\theta - \omega t)} \quad (2)$$

¹L. J. Laslett, V. K. Neil, and A. M. Sessler, Rev. Sci. Inst. 36, 436 (1965)

where U and W depend on the geometry of the beam and the vacuum chamber. For a circular beam (radius a) in a circular vacuum chamber (radius b) they obtain, approximately,

$$U = -\frac{2}{\gamma^2} \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \quad (3)$$

$$W = \frac{4c\beta^2}{b^3} (4\pi\sigma)^{-\frac{1}{2}}$$

where σ is the conductivity of the wall material, expressed in Gaussian units (dimension t^{-1}). The expressions (3) are valid if $\sigma \gg \omega$, $\sigma \gg c^2/d^2\omega$ (d = thickness of vacuum chamber wall \gg skin depth), $R/n \gg b$ (wave length of oscillation \gg transverse dimension of chamber). For other geometries the expressions for U and W are different, but subject to the above conditions, they still possess the following characteristics:

U and W are independent of mode number n

U has the factor $1/\gamma^2$; W does not

U is sensitive to the beam dimensions; W is not

W is proportional to $\sigma^{-\frac{1}{2}}$

The resistive (W) term in Eq. (2) arises from the skin effect in the chamber wall. The derivation of this effect shows that the sign of the square root must be chosen, regardless of the sign of ω , such that $\sqrt{i/\omega}$ has a positive real part, corresponding to an attenuated wave in the metal.

If we consider a single particle (called the r^{th} particle) circulating with angular velocity Ω oscillating with angular frequency $v\Omega$, and an amplitude ζ , we have

$$P(\theta, t) = \zeta_r e^{i(\varphi_r + v\Omega t)} \delta_p(\theta - \theta_r - \Omega t) \quad (4)$$

where δ_p is the periodic delta function, and φ_r and θ_r are the transverse phase and azimuthal location of the particle at time $t=0$. Writing δ_p as a Fourier series

$$\delta_p(\theta - \theta_r - \Omega t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik(\theta - \theta_r - \Omega t)} \quad (5)$$

we obtain

$$P(\theta, t) = \frac{\zeta_r}{2\pi} e^{i\varphi_r} \sum_k e^{i[k(\theta - \theta_r) - (k-v)\Omega t]} \quad (6)$$

which is a sum of terms of the form Eq. (1). The deflecting field is, therefore, by Eq. (2),

$$F = \frac{\zeta_r}{2\pi} e^{i\varphi_r} \sum_k \left[U + W \sqrt{\frac{i}{(k-v)\Omega}} \right] e^{i[k(\theta - \theta_r) - (k-v)\Omega t]} \quad (7)$$

$$= U P(\theta, t) + \frac{W}{\sqrt{\Omega}} \frac{\zeta_r}{2\pi} e^{i\varphi_r} \sum_k \sqrt{\frac{i}{k-v}} e^{i[k(\theta - \theta_r) - (k-v)\Omega t]} \quad (8)$$

Thus the nonresistive part of the field is -- in this approximation -- an exact copy of the dipole moment itself. The resistive part may be written in the form

$$\frac{W}{\sqrt{\Omega}} \frac{\zeta_r}{2\pi} e^{i(\varphi_r + v\Omega t)} G(\alpha, v) \quad (9)$$

with

$$\alpha = \theta_r + \Omega t - \theta$$

and the function

$$G(\alpha, v) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{i}{k-v}} e^{-ik\alpha} \quad (10)$$

will be shown in the Appendix to be equal to

$$G(\alpha, v) = 2 \int_0^{\infty} \frac{e^{-\alpha(y+iv)}}{1-e^{-2\pi(y+iv)}} \frac{dy}{\sqrt{y}} \quad (11)$$

for $0 < \alpha \leq 2\pi$, (and periodic in α for other values of α).

Now consider many particles. The force on a particular particle, say the s^{th} particle, is Eq (8) evaluated at $\theta = \Omega t + \theta_s$, plus the restoring force from the external focusing field. Thus its equation of transverse motion is

$$M\gamma (\ddot{y}_s + \Omega^2 v_s^2 y_s) = eUP(\theta_s + \Omega t, t) + \frac{eW}{2\pi\sqrt{\Omega}} \sum_r \zeta_r e^{i(\varphi_r + v\Omega t)} G(\theta_r - \theta_s, v)$$

where we have written the U term in terms of the dipole moment of all particles. Since it is assumed that all particles move with $y = \zeta_s e^{i(\varphi_s + v\Omega t)}$ we may rewrite this as

$$M\gamma\Omega^2 (v_s^2 - v^2) \zeta_s e^{i(\varphi_s + v\Omega t)} = eUP(\theta_s + \Omega t, t) + \frac{eW}{2\pi v\Omega} \sum_r \zeta_r e^{i(\varphi_r + v\Omega t)} G(\theta_r - \theta_s, v) \quad (12)$$

Here $v_s \Omega$ is the frequency of free oscillations of the s^{th} particle, while $v\Omega$ is the frequency of the normal mode we want to investigate. There will, in general, be a spread in v_s because of spreads in energy, synchrotron oscillation amplitude, and nonlinearity in the focusing field.

Now assume that the particles are bunched tightly into several bunches. Within the m^{th} bunch there are N_m particles, with an assumed distribution in $\theta, \zeta, \varphi, v_s$

$$\psi(\theta, \zeta, \varphi, v_s) = \frac{N_m}{\alpha} D(\zeta, \varphi) f(v_s) \quad (13)$$

over an azimuthal range of width α and zero elsewhere; D and f are normalized to unity. The dipole moment of the bunch is

$$Q_m = N_m \int D(\zeta, \varphi) \zeta e^{i\varphi} d\zeta d\varphi ;$$

the dipole moment per unit length is Q_m/α .

We now multiply Eq. (12) by ψ as given by Eq. (13), divide by $v_s^2 - v^2$, and integrate over the m^{th} bunch. At the same time, we replace the summation over r in Eq. (12) by integration over all bunches.

We get

$$M\gamma\Omega^2 Q_m = \int \frac{f(v_s)}{v_s^2 - v^2} dv_s \left[\frac{eU}{\alpha} N_m Q_m + \frac{eW N_m}{2\pi\alpha^2\sqrt{\Omega}} \sum_n Q_n \int_0^\alpha d\theta \int_0^\alpha d\theta' G(\theta' - \theta, v) \right] \quad (14)$$

where the sum is taken over all bunches.

For all bunches $n \neq m$ the argument of G in the double integral can be taken as constant, equal to $\theta_n - \theta_m$, the azimuthal distance between the m^{th} and n^{th} bunch. For $n = m$ we note, from the Appendix, for small α :

$$G(\alpha, v) \approx 2\sqrt{\frac{\pi}{\alpha}} + G(2\pi, v) \quad 0 < \alpha \ll 2\pi$$

$$G(-\alpha, v) \approx G(2\pi, v)$$

so that the double integral for $n = m$, because of

$$\alpha^2 G(2\pi, v) + \frac{8}{3} \sqrt{\pi} \alpha^{3/2}$$

and Eq. (14) becomes

$$M\gamma\Omega^2 Q_m = eN_m \int \frac{f(v_s) dv_s}{v_s^2 - v^2} \left[\left(\frac{U}{\alpha} + \frac{4W}{3\sqrt{\pi\Omega\alpha}} \right) Q_m + \frac{W}{\sqrt{\Omega}} Q_m G(2\pi, v) + W \sum_{n \neq m} Q_n G(\theta_n - \theta_m, v) \right] \quad (15)$$

We must now solve for v ; if the solution has a negative imaginary part, the oscillations will grow exponentially; if it has a positive

imaginary part, they will damp. We assume that $f(v_s)$ is different from zero only in the vicinity of $v_s = v_0$ (or $-v_0$), and we replace v by v_0 in the argument of G . Let us write

$$\lambda = \frac{M\gamma\Omega^2}{e \int \frac{f(v_s)dv_s}{v_s^2 - v^2}} \quad (16)$$

Then Eq. (15) can be written in the form

$$(N_m U' - \lambda)Q_m + N_m W' \sum_n Q_n G_{nm} = 0 \quad (17)$$

where

$$U' = \frac{U}{\alpha} + \frac{4W}{3\sqrt{\pi\Omega\alpha}} ; \quad W' = \frac{W}{\sqrt{\Omega}} , \quad (18)$$

and

$$G_{nm} = G(\theta_n - \theta_m, v) ; \quad G_{mm} = G(2\pi, v).$$

Equation (17) is in the standard form of an eigenvalue problem: We must find λ such that the determinant vanishes. Evidently for a delta function distribution v^2 can have a negative imaginary part (instability) only if λ has a positive imaginary part.

In the case of a finite distribution it can be seen, just as in the case treated by LNS, that λ may also be allowed to have a positive imaginary part provided the real part is large enough; however, if $\text{Im } \lambda < 0$, instability certainly does not arise.

LNS have also shown that in most cases (especially when γ is not very large) the nondissipative parts of the field are large compared

to the dissipative ones, i.e.,

$$|U'| \gg |W' G_{nm}| \quad (19)$$

In that case, and if the bunch strengths N_m are substantially different, we may treat W' as a perturbation compared to U' . Then, by standard perturbation theory, the eigenvalues depend to first-order on the diagonal perturbations $W' G_{mm}$, but only to second-order on the off-diagonal perturbations $W' G_{mn}$, and the eigenvalues are simply

$$\lambda = N_m [U' + W' G(2\pi, \nu)]. \quad (20)$$

In this case -- which is equivalent to the case of a single bunch -- the imaginary part of λ is simply $N_m W'$ times the imaginary part of $G(2\pi, \nu)$.

Now from Eq. (11) we can see that the imaginary part of the integrand, for $\alpha = 2\pi$, is just

$$\text{Im} \frac{z}{1-z} = \text{Im} \frac{z(1-\bar{z})}{|1-z|^2} = \frac{\text{Im} z}{|1-z|^2} = \frac{-e^{-2\pi y} \sin(2\pi \nu)}{|1-z|^2}$$

where $z = e^{-2\pi(y+iv)}$. Therefore, instability can occur ($\text{Im} \lambda > 0$) only if $\sin(2\pi \nu) < 0$, i.e., if ν lies between an integer and the next lower half-integer. This is true in the case of a single bunch; for multiple bunches it is still true only if $|U'| \gg |W' G_{nm}|$, and if the bunches are substantially different in intensity (numerical calculations indicate that when $U' = 10^3 W'$ and there are twelve bunches, their intensities must vary by at least a few percent).

In the case of many bunches, they may well be nearly enough alike to vitiate the above condition. For the case of equal N_m 's and equally spaced bunches, it turns out that Eq. (17) can be solved explicitly.

We write Eq. (17) in the form

$$\begin{aligned}
 (A_0 - \lambda) Q_0 + A_1 Q_1 + \dots + A_{M-1} Q_{M-1} &= 0 \\
 A_{M-1} Q_0 + (A_0 - \lambda) Q_1 + \dots + A_{M-2} Q_{M-1} &= 0 \\
 A_{M-2} Q_0 + A_{M-1} Q_1 + \dots + A_{M-3} Q_{M-1} &= 0 \\
 A_1 Q_0 + A_2 Q_1 + \dots + A_{M-1} Q_{M-2} + (A_0 - \lambda) Q_{M-1} &= 0.
 \end{aligned} \tag{21}$$

Let $a = e^{-2\pi \frac{m}{M} i}$ be an M^{th} root of unity. Then a solution is

$$Q_0 = 1, Q_1 = a, Q_2 = a^2, \dots, Q_{M-1} = a^{M-1} \tag{22}$$

for any m from 0 to $M-1$, and the eigenvalue is

$$\lambda_m = \sum_{r=0}^{M-1} A_r a^r \tag{23}$$

as is easily verified. Now

$$A_0 = N [U' + W' G(2\pi, v)]$$

$$A_r = NW' G\left(\frac{2\pi r}{M}, v\right) \quad r \geq 1$$

and

$$\lambda_n = NU' + NW' \sum_{r=1}^M e^{-\frac{2\pi m r}{M} i} G\left(\frac{2\pi r}{M}, v\right) \tag{24}$$

In the Appendix it is shown that

$$\sum_{r=1}^M e^{-\frac{2\pi m r}{M} i} G\left(\frac{2\pi r}{M}, \nu\right) = \sqrt{M} G\left(2\pi, \frac{\nu+m}{M}\right). \quad (25)$$

By the theorem on the sign of $G(2\pi, \nu)$ the imaginary part is positive when $(\nu+m)/M$ lies between an integer and the next lower half-integer and negative in the other half-interval. Therefore, if M is even, half the eigenvalues have positive and half have negative imaginary parts; if M is odd, one more has a positive imaginary part than a negative one (or vice versa). The only case where there is no eigenvalue with a positive imaginary part is when there is only one bunch and ν lies in the proper range.

When the bunches are not evenly spaced, no explicit solution has been found. It might be expected that, when they are all confined to one vicinity, the stability condition might be similar to that for a single bunch. Numerical work at Brookhaven has shown this expectation to be false: with two bunches very near to each other, there was always one stable and one unstable mode. The conditions for stability are just too stringent. We also believe that if twisting within a bunch is allowed, there will always be unstable modes. Of course, Landau damping can, in practice, kill these instabilities -- as can electronic feedback circuits.

One more remark: When $|U'| \gg |W' G_{nm}|$, LNS have shown that the threshold particle intensity is nearly proportional to U' and almost independent of W' (just as long as the sign of G is right). From Eq. (15) it is seen that this means that the thresholds depend on the tightness of bunching (i.e., on α), while the growth rates (depending on G) are independent of α .

APPENDIX

The function

$$G(\theta, \nu) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{i}{k-\nu}} e^{-ik\theta} \quad (A1)$$

is evaluated as follows:

$$G = \int \sqrt{\frac{i}{k-\nu}} \sum_{n=-\infty}^{\infty} \delta(k-n) e^{-ik\theta} dk \quad (A2)$$

The path of integration is above singularity $k=\nu$ so as to make $\text{Re} \sqrt{i/(k-\nu)}$ positive.

$$\sum_{n=-\infty}^{\infty} \delta(k-n) = \sum_{s=-\infty}^{\infty} e^{-2\pi s k i} \quad (A3)$$

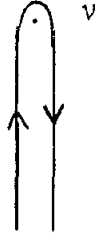
$$G = \sqrt{i} \sum_s \int_{\nu} e^{-ik(\theta+2\pi s)} \frac{dk}{\sqrt{k-\nu}} \quad (A4)$$

Let

$$0 < \theta \leq 2\pi$$

Then, for $s < 0$, integral contour may be closed in upper half plane, where there are no irregularities: integral = 0. Only $s \geq 0$ contributes.

Change path of integration to



in k-plane, change to variable

$$y = i(k-v) :$$

$$G = 2 \sum_s \int_0^{\infty} e^{-(y+iv)(\theta+2\pi s)} \frac{dy}{\sqrt{y}} \quad (A5)$$

$$= 2\sqrt{\pi} \sum_s \frac{e^{-iv(\theta+2\pi s)}}{\sqrt{\theta + 2\pi s}} \quad (A6)$$

$$= 2 \int_0^{\infty} \frac{e^{-\theta(y+iv)}}{1-e^{-2\pi(y+iv)}} \frac{dy}{\sqrt{y}} \quad (A7)$$

For numerical calculations it is convenient to write

$$G = 2 \left(\sum_{s=0}^{K-1} + \sum_{s=K}^{\infty} \right)$$

and to use the form (A6) for the first sum, (A7) for the second:

$$G = 2\sqrt{\pi} \sum_{s=0}^{K-1} \frac{e^{-iv(\theta+2\pi s)}}{\sqrt{\theta + 2\pi s}} + 2 \int_0^{\infty} \frac{e^{-(\theta+2\pi K)(y+iv)}}{1-e^{-2\pi(y+iv)}} \frac{dy}{\sqrt{y}} \quad (A8)$$

The second term has the asymptotic form

$$2\sqrt{\pi} \frac{e^{-iv(\theta+2\pi K)}}{\sqrt{\theta+2\pi K}} \frac{1}{1-e^{-2\pi iv}} \left(1 + \frac{e^{-2\pi vi}}{2(1-e^{-2\pi vi})} (2\pi K + \theta) + \dots \right) .$$

Addition theorem (25) :

From (A6)

$$\begin{aligned} & \sum_{r=1}^M e^{-\frac{2\pi m r}{M} i} G\left(\frac{2\pi r}{M}, v\right) \\ &= 2\sqrt{\pi} \sum_{s=0}^{\infty} \sum_{r=1}^M \frac{\exp\left[-2\pi i \left(\frac{m+v}{M} r + vs\right)\right]}{\sqrt{\frac{2\pi}{M} (Ms+r)}} \\ &= \sqrt{2M} \sum_{r,s} \frac{e^{-2\pi i \left(\frac{m+v}{M}\right) (Ms+r)}}{\sqrt{(Ms+r)}} \end{aligned}$$

(Let $Ms+r = t$)

$$= 2\sqrt{\pi M} \sum_{t=1}^{\infty} \frac{e^{-2\pi i \left(\frac{m+v}{M}\right) t}}{\sqrt{2\pi t}} \equiv \sqrt{M} G\left(2\pi, \frac{m+v}{M}\right) .$$

Report 7

THE TRANSVERSE STABILITY OF ULTRA-RELATIVISTIC
MULTIPLE-BUNCHED BEAMS

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A. Computer Program at SLAC For Bunched Beams

Courant and Sessler have applied the existing LNS theory for a continuous beam to a rigid bunched beam of many bunches. They have shown in Report 6 that, for a beam with equal bunch length and the same distribution in amplitude and frequency in each bunch, the stability of the beam can be investigated by finding the roots of the following equation.

$$\begin{vmatrix}
 N_1 A_0 - \lambda & N_1 A_1 & \dots & N_1 A_{M-1} \\
 N_2 A_{M-1} & N_2 A_0 - \lambda & \dots & \\
 \vdots & & & \\
 N_M A_1 & N_M A_2 & \dots & N_M A_0 - \lambda
 \end{vmatrix} = 0$$

where

$$A_0 = \frac{U}{\alpha} + \frac{4W}{3\sqrt{\pi\Omega\alpha}} + \frac{W}{\sqrt{\Omega}} G(2\pi\nu)$$

$$A_r = \frac{W}{\sqrt{\Omega}} G\left(\frac{2\pi r}{M}, \nu\right), \quad r \geq 1$$

$$U = \frac{U_{LNS}}{S}$$

$$W = \frac{\sqrt{2\omega}}{S} V_{LNS}$$

$$S = \frac{\rho_0 e n a^2}{2v\Omega \alpha m}$$

$$\omega = (n - v) \Omega$$

Ω = angular velocity of the particle

ρ_0 = average charge density per unit length

L = average circumferential length of the machine

α = length of a bunch in radians

a = radius of a bunch

N_j = number of particles in the j th bunch

M = total number of bunches

with $G(\theta, v)$, the bunch function, given by:

$$G(\theta, v) = \sum_{k=0}^{\infty} \sqrt{\frac{2}{k + \frac{\theta}{2\pi}}} e^{-i(2\pi k + \theta)v}$$

which takes into account the wake fields of all bunches. The stability of a beam can be characterized in terms of the normal mode frequencies, v , whose values are related to the λ 's through a dispersion relationship:

$$\frac{1}{\lambda_i} = \frac{1 \cdot e}{m\gamma \Omega^2} \int \frac{f(v_s) dv_s}{v_s^2 - v_i^2}$$

with some suitable distribution function of the natural frequency of the particles of the bunch. A beam is stable if $\text{Im}(v_i) > 0$ for all i .

In particular, $f(v_s) = \delta(v_s - v_i)$ for the case of a rigid bunch. Then

$$\int \frac{f(v_s) dv_s}{v_s^2 - v_i^2} = -\frac{1}{2v_i} \int \frac{\delta(v_s - v_i)}{v_s - v_i} dv_s = -\frac{1}{4v_i^2}$$

and

$$\lambda_i = - \left(\frac{4M\gamma\Omega^2}{1 \cdot e} \right) v_i^2 .$$

Thus, the growth rate is given by:

$$\text{Im} (v_i) \Omega = - \frac{e}{8M\gamma\omega_i} \text{Im} \lambda_i \quad (\text{sec}^{-1})$$

where

$$\omega_i = \text{Re} (v_i) \Omega > 0.$$

For protons of a few GeV where the coherent frequency shift is large compared to the resistive-wall growth rates: $U \gg V$, Courant has investigated the effects of variations in the bunch populations. By suitable changes in N_i it is possible to cause the corresponding coherent frequency shift changes to be large relative to the changes in growth rates. This results in all coherent modes obeying the single-bunch stability criterion; namely, stability if ν lies between an integer and the next higher half-integer.

To solve these coherent normal mode problems, a computer program has been brought into operation at SLAC for finding the eigenvalues and eigenvectors of general complex matrices. Its first use has been for computing the normal modes of bunched single-beam motion for a beam with equal number of particles in each bunch.

In the electron-positron storage rings now planned or under construction, the coherent frequency shift may be much smaller than the growth rate because of cancellation of electric and magnetic forces. It appears that the technique of varying bunch population may not work to achieve stability. The behavior of the modes may be more like that of the case of identical bunch populations, where about half the modes are unstable. The computations are being carried out to investigate whether this is indeed the case.

Report 8

THE INFLUENCE OF DIELECTRIC MEDIA ON THE ELECTROMAGNETIC FIELDS OF A COASTING BEAM IN A STRAIGHT RECTANGULAR PIPE

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A. Introduction

In an earlier treatment¹ of transverse instabilities of intense coasting beams, use was made of electromagnetic fields generated by the perturbed beam in an evacuated pipe with resistive walls. Terms in the effective $\vec{E} + \vec{\beta}_p \times \vec{H}$ field that remain in the limit that the wavelength of the perturbation becomes infinite ($k \rightarrow 0$) exhibited, as expected, a strong $1 - \beta_p^2$ cancellation between the effects of the in-phase components of the electric and magnetic fields in the neighborhood of the beam. Because of this strong cancellation, the real (resistance-independent) term U , which entered in the dispersion analysis of Ref. 1, appeared smaller than might be required to account for some of the empirical observations of transverse instability.

As a result of discussions during the SLAC Storage Ring Summer Study, concerning observations of pressure-dependent instabilities and the possible presence of high-permittivity dielectric layers (oxide of titanium?) on the inner surfaces of devices evacuated with titanium pumps,² Dr. Sessler suggested the relevance of reevaluating the perturbation fields for situations in which a dielectric medium occupies a strip centered about the

¹L.J. Laslett, V.K. Neil, and A.M. Sessler, Rev. Sci. Instr. 36, 436 (1965). In deriving the electromagnetic fields in a pipe of rectangular cross section, resistance was assumed to be present only in the top and bottom wall surfaces. Unrationalized Gaussian units are used throughout.

²An alternative possibility, not examined in the present work, would involve the presence of a thin, spongy (and hence, poorly conducting) metallic layer on the chamber walls.

median plane (as might simulate a plasma layer in the region transversed by the beam) or in which the material is located in layers adjacent to boundary surfaces of the pipe (insulating oxide layer on metallic surfaces above and below the beam). It would be expected that the presence of such dielectric media would reduce the strong $1 - \beta_p^2$ cancellation otherwise present in certain prominent terms in U and, if a loss factor is introduced through the mechanism of an imaginary term in the permittivity, could also contribute to the magnitude of V .

Since the effects to be sought are primarily those that affect a change in the long-wavelength (k -independent) terms, the effect of wall resistance is considered to be ignorable in the present context, and it is permissible, if desired, to evaluate the fields in the quasi-static limit ($k \rightarrow 0$, so that the perturbation fields have essentially an electrostatic and magnetostatic transverse distribution, albeit with ac boundary conditions applied to the latter). In the work reported here we present (i) the effective perturbation field in the case that the entire pipe is filled with a homogeneous dielectric medium (with wall resistance ignored, but without making the long-wavelength approximation) and (ii) the result for two separately-homogeneous media symmetrically disposed about the median plane (for the limit $k \rightarrow 0$). By setting the permittivity of one or the other of these two layers equal to unity in this last result, one obtains expressions applicable to a centered plasma layer or to the case of a dielectric coating on the upper and lower boundary surfaces.

B. Notation

We consider the pipe to be of height h and width w . The particle beam (speed $\beta_p c$) is taken to have a small vertical thickness τ , with a uniform (constant) particle density transversely within this interval, and its center is assumed to oscillate with the waveform

$$\begin{aligned}
 y_C &= \xi \exp [-i(\omega t - kz)] \\
 &= \xi \exp [-ik(\beta_w ct - z)].
 \end{aligned}$$

The surface density of charge in the other transverse direction is represented by

$$\sigma(x) = \frac{2\lambda}{w} \sum_n g_n \cos \frac{n\pi}{w} x,$$

with the coordinate x measured from the left-hand wall. We introduce the notation

$$\mu_n = \frac{n\pi}{w}$$

and also, in the first example, require the quantity

$$\mu_n'' = \sqrt{\mu_n^2 + k^2 (1 - \epsilon\beta_w^2)}.$$

C. Results

1. Dielectric Material Throughout the Pipe

We consider here the case in which material of dielectric constant ϵ fills the pipe. The effective perturbation field, evaluated in the median plane ($y = 0$) for use in a dispersion analysis, is found to be

$$\langle \epsilon_y + \beta_p H_x \rangle = \frac{4\pi\lambda}{w} \sum_n g_n^2 \left\{ -\frac{2}{\tau} \left(\frac{1}{\epsilon} - \beta_p^2 \right) + \left[\mu_n'' \left(\frac{1}{\epsilon} - \beta_p^2 \right) + \frac{k^2}{\mu_n''} (\beta_w - \beta_p)^2 \right] \coth \mu_n'' \frac{h}{2} \right\} \xi \exp[-i(\omega t - kz)]. \quad (1)$$

In the limit that k becomes zero, this clearly reduces to

$$\langle \epsilon_y + \beta_p H_x \rangle = \frac{4\pi\lambda}{w} \left(\frac{1}{\epsilon} - \beta_p^2 \right) \sum_n g_n^2 \left(-\frac{2}{\tau} + \mu_n \coth \mu \frac{h}{2} \right) \xi \exp[-i(\omega t - kz)]. \quad (3)$$

It is noted that the first of these results agrees with Eq. (2.17b) of Ref. 1, if one sets $\mathcal{R} = 0$ in the latter equation, save that the factor $1/\gamma^2 = 1 - \beta_p^2$ in the k -independent terms has now been replaced by $\frac{1}{\epsilon} - \beta_p^2$.

(3) For brevity, the subscript on μ_n is omitted in the arguments of the hyperbolic functions in this equation and in those that follow.

2. Dielectric Strip (ϵ_B , of Thickness Δ) Centered About the Median Plane and Remainder of Pipe Filled With ϵ_W

We consider here two dielectric media, in layers symmetrically disposed about the median plane:

$$\epsilon_B \text{ for } |y| < \frac{\Delta}{2} \quad \text{and} \quad \epsilon_W \text{ for } \frac{\Delta}{2} < |y| < \frac{h}{2}.$$

In this case the effective perturbation field is found to be, in the limit that k becomes zero,

$$\langle \vec{E}_y + \beta_p \vec{H}_x \rangle = \frac{4\pi\lambda}{v} \sum_n \epsilon_n^2 \left\{ -\frac{2}{\pi} \left(\frac{1}{\epsilon_B} - \beta_p^2 \right) + \mu_n \left[\frac{1}{\epsilon_B} \frac{\epsilon_W \cosh \mu \frac{\Delta}{2} \cosh \mu \frac{h-\Delta}{2} + \epsilon_B \sinh \mu \frac{\Delta}{2} \sinh \mu \frac{h-\Delta}{2}}{\epsilon_W \sinh \mu \frac{\Delta}{2} \cosh \mu \frac{h-\Delta}{2} + \epsilon_B \cosh \mu \frac{\Delta}{2} \sinh \mu \frac{h-\Delta}{2}} - \beta_p^2 \coth \mu \frac{h}{2} \right] \right\} \times \exp[-i(\omega t - kz)] \quad (2)$$

It is seen that, if $\epsilon_W = \epsilon_B$ or if $\Delta = h$, this result becomes identical with the simple equation presented in Section B.1 for the $k \rightarrow 0$ limit (with $\epsilon = \epsilon_B$). We now indicate certain specialized forms of the result just presented.

a. $\epsilon_W = 1$:

$$\langle \vec{E}_y + \beta_p \vec{H}_x \rangle = \frac{4\pi\lambda}{v} \sum_n \epsilon_n^2 \left\{ -\frac{2}{\pi} \left(\frac{1}{\epsilon_B} - \beta_p^2 \right) + \mu_n \left[\frac{1}{\epsilon_B} \frac{\cosh \mu \frac{\Delta}{2} + (\epsilon_B - 1) \sinh \mu \frac{\Delta}{2} \sinh \mu \frac{h-\Delta}{2}}{\sinh \mu \frac{\Delta}{2} + (\epsilon_B - 1) \cosh \mu \frac{\Delta}{2} \sinh \mu \frac{h-\Delta}{2}} - \beta_p^2 \coth \mu \frac{h}{2} \right] \right\} \times \exp[-i(\omega t - kz)]. \quad (3)$$

(1) For $\epsilon_B - 1 \ll 1$,

$$\langle \vec{E}_y + \beta_p \vec{H}_x \rangle = \frac{4\pi\lambda}{v} \sum_n \epsilon_n^2 \left\{ -\frac{2}{\pi} \left(\frac{1}{\epsilon_B} - \beta_p^2 \right) + \mu_n \left[\frac{1}{\epsilon_B} - \beta_p^2 - \frac{\epsilon_B - 1 \sinh \mu (h-\Delta)}{\epsilon_B \sinh \mu h} \right] \coth \mu \frac{h}{2} \right\} \times \exp[-i(\omega t - kz)]. \quad (4)$$

If Δ is small also,

$$\begin{aligned} \langle \bar{E}_y + \beta_p \bar{E}_x \rangle &= \frac{h\omega}{4\pi} \sum_u \epsilon_u^2 \left\{ -\frac{2}{v} \left(\frac{1}{\epsilon_B} - \beta_p^2 \right) + \mu_n \left[\frac{1}{\epsilon_B} - \beta_p^2 - \frac{\epsilon_B - 1}{\epsilon_B} (1 - \mu_n \Delta \coth \mu h) \right] \coth \mu \frac{h}{2} \right\} \epsilon \exp [-i(\omega t - kz)] \\ &= \frac{h\omega}{4\pi} \sum_u \epsilon_u^2 \left\{ -\frac{2}{v} \left(\frac{1}{\epsilon_B} - \beta_p^2 \right) + \mu_n \left[\frac{1}{\epsilon_B} - \beta_p^2 + \mu_n (\epsilon_B - 1) \Delta \coth \mu h \right] \coth \mu \frac{h}{2} \right\} \epsilon \exp [-i(\omega t - kz)], \end{aligned} \quad (5)$$

this latter form being identical through first-order terms in $(\epsilon_B - 1)$ to that just preceding.

(2) For Δ small,

$$\langle \bar{E}_y + \beta_p \bar{E}_x \rangle = \frac{h\omega}{4\pi} \sum_u \epsilon_u^2 \left\{ -\frac{2}{v} \left(\frac{1}{\epsilon_B} - \beta_p^2 \right) + \mu_n \left[\frac{1}{\epsilon_B} \left[1 + (\epsilon_B - 1) \frac{\mu_n \Delta}{2} (\tanh \mu \frac{h}{2} - \frac{1}{\epsilon_B} \coth \mu \frac{h}{2}) \right] - \beta_p^2 \right] \coth \mu \frac{h}{2} \right\} \epsilon \exp [-i(\omega t - kz)]; \quad (6)$$

If $\epsilon_B - 1$ is small also,

$$\langle \bar{E}_y + \beta_p \bar{E}_x \rangle = \frac{h\omega}{4\pi} \sum_u \epsilon_u^2 \left\{ -\frac{2}{v} \left(\frac{1}{\epsilon_B} - \beta_p^2 \right) + \mu_n \left[\frac{1}{\epsilon_B} - \beta_p^2 + \mu_n (\epsilon_B - 1) \Delta \coth \mu h \right] \coth \mu \frac{h}{2} \right\} \epsilon \exp [-i(\omega t - kz)] \quad (7)$$

when terms of second and higher order in $\epsilon_B - 1$ are ignored.

b. $\epsilon_B = 1$

$$\begin{aligned} \langle \bar{E}_y - \beta_p \bar{E}_x \rangle &= \frac{h\omega}{4\pi} \sum_u \epsilon_u^2 \left\{ -\frac{2}{v} (1 - \beta_p^2) + \mu_n \left[\frac{\epsilon_y \cosh \mu \frac{h}{2} \cosh \mu \frac{h}{2} + \sinh \mu \frac{h}{2} \sinh \mu \frac{h}{2}}{\epsilon_y \sinh \mu \frac{h}{2} \cosh \mu \frac{h}{2} + \cosh \mu \frac{h}{2} \sinh \mu \frac{h}{2}} - \beta_p^2 \coth \mu \frac{h}{2} \right] \right\} \epsilon \exp [-i(\omega t - kz)] \\ &= \frac{h\omega}{4\pi} \sum_u \epsilon_u^2 \left\{ -\frac{2}{v} (1 - \beta_p^2) + \mu_n \left[\frac{\cosh \mu \frac{h}{2} + (\epsilon_y - 1) \cosh \mu \frac{h}{2} \cosh \mu \frac{h}{2}}{\sinh \mu \frac{h}{2} + (\epsilon_y - 1) \cosh \mu \frac{h}{2} \sinh \mu \frac{h}{2}} - \beta_p^2 \coth \mu \frac{h}{2} \right] \right\} \epsilon \exp [-i(\omega t - kz)]; \quad (8) \end{aligned}$$

where $T = \frac{h - \Delta}{2}$ denotes the thickness of the individual dielectric layers at the top and bottom surfaces of the rectangular pipe. [It is noted that if $\epsilon_W \rightarrow \infty$, the first term in the square bracket approaches just $\coth \mu$ (reduced semi-aperture) as it should.

(1) For T small ($\mu T \ll 1$),

$$\begin{aligned} \langle \bar{z}_y - \beta_p^2 x \rangle &= \frac{h-\lambda}{\omega} \sum_n \epsilon_n^2 \left\{ -\frac{2}{\tau} (1 - \beta_p^2) + \mu_n \left[1 - \beta_p^2 + 2\mu_n \frac{\epsilon_n - 1}{\epsilon_n} \tau (\coth \mu \frac{h}{2} - \tanh \mu \frac{h}{2}) \right] \coth \mu \frac{h}{2} \right\} \exp[-i(\omega t - kz)] \\ &= \frac{h-\lambda}{\omega} \sum_n \epsilon_n^2 \left\{ -\frac{2}{\tau} (1 - \beta_p^2) + \mu_n \left[1 - \beta_p^2 + 2\mu_n \frac{\epsilon_n - 1}{\epsilon_n} \tau \coth \mu h \right] \coth \mu \frac{h}{2} \right\} \exp[-i(\omega t - kz)]. \end{aligned} \quad (9)$$

D. Conclusion

It is seen from the results presented that a dielectric medium can influence materially the effective perturbation field experienced by a transversely oscillating beam. Specifically, the image terms in the quasi-static effective field no longer contain precisely the $1 - \beta_p^2$ cancellation factor, and, if the medium fills the region traversed by the beam, this cancellation is also modified in the direct $-2/\tau$ term. The formulas that have been presented here also permit evaluation of the contribution that a small but non-vanishing loss angle will make to the imaginary part of the effective field. For this purpose, loss in the dielectric material, if characterized by a conductivity σ , may be represented by a complex dielectric constant

$$\begin{aligned} \underline{\epsilon} &= \epsilon_{\text{Real}} + \frac{4\pi i \sigma}{\omega} \\ &= \epsilon_{\text{Real}} + \frac{4\pi i \sigma}{k \beta_W c}. \end{aligned}$$

In particular, with this substitution, the factor $\frac{1}{\epsilon} - \beta_p^2$ becomes

$$\frac{1}{\epsilon_{\text{Real}}} - \beta_p^2 - i \frac{4\pi\sigma}{\epsilon_{\text{Real}}^2 \omega} ,$$

if the phase angle $4\pi\sigma/\epsilon_{\text{Real}}\omega$ is small ("relaxation time," $\epsilon_{\text{Real}}/4\pi\sigma$, small compared to $1/\omega$), and the factor $\frac{1}{\epsilon^2} - \beta_p^2$ similarly becomes

$$\frac{1}{\epsilon_{\text{Real}}^2} - \beta_p^2 - i \frac{8\pi\sigma}{\epsilon_{\text{Real}}^3 \omega} .$$

The quantities U and V employed in the dispersion analysis of Ref. 1 are given by

$$U + iV = \frac{1}{2\nu\omega_0} \frac{e}{m} \frac{\langle E_y + \beta_p H_x \rangle}{\xi \exp[-i(\omega t - kz)]}$$

where ν is the number of betatron oscillations per circumference, $\omega_0 = \beta_p c/R$ (R being the orbit radius), $\lambda = N_e/2\pi R$, $m = \gamma m_0$ denotes the (relativistic) mass of the particle, and e is the particle charge.

Report 9

MEDIA EFFECTS ON THE ELECTROMAGNETIC FIELDS
OF A COASTING BEAM IN A STRAIGHT PIPE
OF CIRCULAR CROSS SECTION

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The fields of an oscillating beam of particles are calculated exactly as in LNS,¹ Section IIB, for a pipe of circular cross section containing a beam of radius a , and dielectric media characterized by ϵ , between b and d . The walls of the pipe (at $r = d$) are taken to be perfectly conducting, with the result:

$$B_y = \frac{-2\pi \beta \lambda \xi}{\pi a^2} \left[1 - \left(\frac{a}{d}\right)^2 \right]$$

$$E_x = \frac{-4\pi \left(\frac{\lambda \xi}{\pi a^2}\right) \left(\frac{a}{b}\right)^2 \left[\left(\frac{d}{b}\right)^2 - 1\right]}{\left[(1 + \epsilon)\left(\frac{d}{b}\right)^2 + \epsilon - 1\right]} + \frac{2\pi \lambda \xi}{\pi a^2} \left[\left(\frac{a}{b}\right)^2 - 1\right]$$

where the notation is that of Ref. 1.

Letting $\epsilon' = \epsilon - 1$, this yields $F = e(E_x - \beta B_y)$ as

$$F = \frac{-2\pi \lambda e \xi}{\pi a^2 \gamma^2} \left[1 - \frac{a^2}{d^2} \right] + \frac{\lambda \pi \xi}{\pi a^2} \frac{\epsilon' (1 + b^2/d^2) (d^2 - b^2) a^2/d^2}{b^2 \left[1 + \epsilon'/2 (1 + b^2/d^2) \right]}$$

where the second term clearly vanishes when $\epsilon \rightarrow 1$ and the first term is as in LNS.

¹L.S. Laslett, V.K. Neil, and A.M. Sessler, Rev. Sci. Instr. 36, 436 (1965).

Making the replacement $\epsilon \rightarrow \epsilon = \frac{4\pi\sigma i}{\omega}$, and assuming $\frac{4\pi\sigma}{\omega} \ll 1$, we can obtain convenient formulas for U and V, namely:

$$U = \frac{-e^2 N}{2\pi Q \omega_0 \gamma R a^2 m_0} \left\{ \left(1 - \frac{a^2}{d^2}\right) \frac{1}{\gamma^2} - \frac{\epsilon' a^2}{2 d^2} \left(1 + \frac{b^2}{d^2}\right) \left(\frac{d^2 - b^2}{b^2}\right) \times \left[1 + \frac{\epsilon'}{2} \left(1 + \frac{b^2}{d^2}\right)\right]^{-1} \right\};$$

and

$$V \approx \frac{e^2 N \sigma}{Q(n - Q) \omega_0^2 \gamma R m_0 a^2} \left(1 + \frac{b^2}{d^2}\right) \left(\frac{d^2 - b^2}{b^2}\right) \left[1 + \frac{\epsilon'}{2} \left(1 + \frac{b^2}{d^2}\right)\right]^{-2}.$$

Report 10

THEORY OF COHERENT BUNCH MOTION IN ELECTRON-POSITRON STORAGE RINGS

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A. Introduction

A general theory is developed to describe the coherent motion of bunches of particles in electron-positron storage rings. The bunches are taken to be internally rigid, and to interact with each other through direct fields as well as through fields associated with resistive vacuum tank walls; thus each bunch is affected by its own local fields (including images), its own wake field as encountered on subsequent traversals, the wave fields of other bunches in the same beam, and by the bunches of the other beam through direct fields encountered at crossing as well as wake fields. It is assumed that the storage ring is so designed and operated that each bunch of one beam encounters — under similar conditions — all bunches of the other beam; in fact, one bunch collides with a given bunch of the other beam twice per revolution.

B. Basic Equations

We consider coherent motion only in the vertical direction: Let $z_i^\pm(t)$ be the vertical coordinate of the i -th bunch of the plus or minus beam, let N^\pm be the number of particles in one bunch of the plus or minus beam (we assume that all the bunches of one beam are of equal intensity, but that the e^+ current can be different from the e^- current),¹ and let N_b be the number of bunches in each beam.

¹This assumption, as well as the assumption of the previous paragraph concerning operating conditions, can easily be either changed (or removed) to develop a more general theory. The present theory appears general enough to describe all presently contemplated designs.

The equation of motion for the ℓ -th bunch may be written in the form:

$$\ddot{z}_\ell^\pm(t) + \omega_\pm^2 z_\ell^\pm(t) = J_1 + J_2, \quad (1)$$

where J_1 represents the influence of bunches in the same beam, J_2 represents the influence of bunches in the other beam, and ω_\pm is the betatron oscillation frequency ($Q_\pm \omega_0$) for a particle in the beam.

The terms J_1 and J_2 are

$$J_1 = N^+ \sum_i \sum_n z_i^\pm \left(t \mp \frac{s_i^\pm - s_\ell^\pm}{v} - nT \right) \int d\lambda \tilde{F}(\lambda) e^{i\lambda \left(\pm \frac{s_i^\pm - s_\ell^\pm}{v} + nT \right)} \quad (2)$$

$$J_2 = N^+ \sum_i \sum_n z_i^\mp \left(t \mp \frac{s_\ell^\pm - s_i^\mp}{v} - nT \right) \int d\lambda \tilde{G}(\lambda) e^{i\lambda \left(\pm \frac{s_\ell^\pm - s_i^\mp}{v} + nT \right)},$$

where the longitudinal motion of the ℓ -th bunch has been described by

$$s_i^\pm = \pm vt + \sigma_i^\pm \quad (3)$$

The phases σ_i^\pm characterize the spacing of bunches. The summation over n is a sum over contributions from previous turns; T is the period of the motion and $T = 2\pi/\omega_0 = 2\pi R/v$. From the analysis of fields² we know that the functions $\tilde{G}(\lambda)$ and $\tilde{F}(\lambda)$ are analytic in the complex λ -plane, except for a branch point at the origin and a consequent cut which we take along the negative imaginary axis. We shall not need to specify \tilde{G} and \tilde{F} in any further detail in order to arrive at the results of this paper; subsequent numerical studies, for a particular ring, will in general require more detailed information about \tilde{F} and \tilde{G} .

²See Reports 3, 4, and 5.

C. Solution of the Equations of Motion

The solution of Eq. (1) is - in view of the periodic nature of the equations - of the form

$$z_{\ell}^{\pm}(t) = \xi_{\ell}^{\pm} \sum_{m=-\infty}^{\infty} A_m^{\pm} e^{it(2m\omega_0 - \Omega)}, \quad (4)$$

where the coefficients A_m^{\pm} and the common frequency Ω are yet to be determined. Inserting this form into J_1 , and employing the analytic properties of $\tilde{F}(\lambda)$, it is easy to show that only n larger than the smallest integer n_0 such that

$$n_0 > \mp \frac{\sigma_i^{\pm} - \sigma_{\ell}^{\pm}}{L} \quad (5)$$

contribute to the sum. The sum over n is trivial; after this the integral over λ can be performed, for $\text{Im}\Omega > 0$, by closing the contour in the upper half plane and picking up contributions at the poles $\lambda_k = \Omega + k\omega_0$ associated with integer values of k . One is left with the summations over m and k , namely:

$$J_1 = \omega_0 \sum_n \sum_k A_m^{\pm} e^{i(2m\omega_0 - \Omega)t} e^{\pm i(2m\omega_0 + k\omega_0) \left(\frac{\sigma_i^{\pm} - \sigma_{\ell}^{\pm}}{v} \right)} \tilde{F}(\Omega + k\omega_0) \quad (6)$$

The same result may also be obtained for the case $\text{Im}\Omega < 0$; while the term J_2 can - by similar manipulations - be brought into the form

$$J_2 = \omega_0 \sum_m \sum_k A_{-k-m}^{\mp} e^{-it(\Omega - 2m\omega_0)} e^{\pm i \left(\frac{\sigma_{\ell}^{\pm} - \sigma_i^{\mp}}{v} \right) (2m+k)\omega_0} \tilde{G}(\Omega + k\omega_0). \quad (7)$$

Introducing the notation

$$D_m^{\pm} = \omega_{\pm}^2 - (2m\omega_0 - \Omega)^2, \quad (8)$$

and changing dummy summation variables, we arrive at the form

$$\begin{aligned} \xi_l^\pm A_m^\pm D_m^\pm = N^\pm \omega_0 \sum_i \sum_k \xi_i^\pm A_m^\pm e^{+i(2m+k)\omega_0} \left(\frac{\sigma_i^\pm - \sigma_l^\pm}{v} \right) \tilde{F}(\Omega+k\omega_0) \\ + \omega_0 N^\mp \sum_i \xi_i^\pm A_{k+m}^\mp e^{\mp ik\omega_0} \left(\frac{\sigma_l^\pm - \sigma_i^\mp}{v} \right) \tilde{G}(\Omega-2m\omega_0-k\omega_0). \end{aligned} \quad (9)$$

The self-field terms are, of course, only a small influence on the motion of bunches. When this small effect turns stable motion into unstable motion it is of great concern, but in terms of the analysis the right-hand side of Eq. (9) is a small perturbation, so that to zero-order

$$\begin{aligned} A_m^\pm = \sigma_{m,0} \\ \Omega^2 = \omega_\pm^2. \end{aligned} \quad \begin{array}{l} \text{(zero-order)} \\ \end{array} \quad (10)$$

We now solve Eq. (9) through second-order by inserting the zero-order solution on the right-hand side and solving for $A_m^\pm(m \neq 0)$ to first-order. Evaluating Eq. (10) for $m = 0$, and inserting the first-order solution on the right yields the set of equations:

$$\begin{aligned} \xi_l^\pm D_0^\pm = N^\pm \omega_0 \sum_i \xi_i^\pm P_{i-l}^\pm + N^\mp \omega_0 \sum_i \xi_i^\mp \tilde{G}(\Omega) \\ + \omega_0^2 (N^\mp)^2 \sum_i \sum_j \xi_j^\pm T_{l-i}^\pm P_{j-i}^\pm \quad \text{(second-order)} \\ + \omega_0^2 N^- N^+ N_b \sum_j \xi_j^\pm R_{j-l}^\pm, \end{aligned} \quad (11)$$

where the coefficients P, R, T are defined by:

$$P_{i-l}^{\pm} = \sum_k e^{\mp i k \omega_0} \left(\frac{\sigma_i^{\pm} - \sigma_l^{\pm}}{v} \right) \tilde{F}(\Omega + k \omega_0),$$

$$R_{i-l}^{\pm} = \sum_{k \neq 0} e^{\mp i k \omega_0} \left(\frac{\sigma_l^{\pm} - \sigma_i^{\pm}}{v} \right) \frac{\tilde{G}(\Omega - k \omega_0)^2}{D_k^{\mp}}$$

$$T_{l-i}^{\mp} = \sum_{k \neq 0} e^{\mp i k \omega_0} \left(\frac{\sigma_l^{\pm} - \sigma_i^{\mp}}{v} \right) \frac{\tilde{G}(\Omega - k \omega_0)}{D_k^{\mp}}, \quad (12)$$

and depend only on the difference $i - l$.

Because of the cyclic nature of Eq. (11), it is easy to obtain the complete set of eigenmodes (with associated eigenvalues) to the system of equations. If we have no coupling between the beams ($\tilde{G} \equiv 0$), the normal modes of the problem are³

$$\xi_l^{\pm}(s) = e^{\frac{i 2 \pi l s}{N_b}}, \quad s = 1, \dots, N_b, \quad (13)$$

where the index s labels the various modes. We find solutions to Eq. (11) by expanding in the normal modes of the uncoupled problem; namely, by letting

$$\xi_l^{\pm} = \sum_s B_s^{\pm} e^{\frac{i 2 \pi l s}{N_b}}. \quad (14)$$

With some manipulation, and observing that only the $s = 0$ mode has net center-of-mass motion - which concept is equivalent to the formula

$$\sum_j e^{\frac{i 2 \pi s j}{N_b}} = N_b \delta_{s,0} \quad (15)$$

³This is established in Report 6.

we can obtain a coupled set of equations for the B_s^\pm . It suffices, now, to neglect second-order contributions in \tilde{F} and \tilde{G} to the eigenvalue Ω , since the radiation damping will in all $e^- - e^+$ rings easily dominate such terms. If, furthermore, the rings are designed⁴ so that D_k^\pm is not small of order \tilde{F} for $k \neq 0$, then the equations for B_s^\pm take the form

$$\sum_s B_s^\pm e^{\frac{i2\pi s l}{N_b}} \left\{ D_0^\pm - \omega_0 N^\pm \sum_k e^{\frac{i2\pi s k}{N_b}} P_k^\pm \right\} = \omega_0 N_b N^\mp \tilde{G}(\Omega) B_0^\mp \quad (16)$$

The solutions to Eq. (16) now follow immediately. There are $2(N_b-1)$ solutions (identified by mode number n) of the form

$$B_s^\pm = \delta_{s,n}, \quad n = 1, \dots, N_b-1 \quad (17)$$

each having an eigenvalue $\Omega(n)$ obtained by solving the corresponding equation:

$$D_0^\pm \equiv \omega_\pm^2 - \Omega^2(n) = \omega_0 N^\pm \sum_k e^{\frac{i2\pi n k}{N_b}} P_k^\pm, \quad n = 1, \dots, N_b-1 \quad (18)$$

The remaining two modes of the complete problem result from coupling among the $s = 0$ modes of the uncoupled problem; the eigenvalues are given by the two solutions of the determinantal equation:

$$\begin{vmatrix} D_0^+ - \omega_0 N^+ \sum_k P_k^+ & - \omega_0 N^- \tilde{G}(\Omega) \\ - \omega_0 N^+ \tilde{G}(\Omega) & D_0^- - \omega_0 N^- \sum_k P_k^- \end{vmatrix} = 0 \quad (19)$$

⁴This criterion, for small \tilde{F} and \tilde{G} , simply means [see Eq. (8)] that Q_\pm must be taken non-integral, a condition easily met in practice.

D. Discussion of the Solution

1. It has been shown that to first-order in \tilde{F} and \tilde{G} the two beams couple only through the $s = 0$ normal modes of the uncoupled problem. Thus the eigenvalues for the modes $n = 1, \dots, N_b - 1$ are exactly those of the uncoupled problem: If these modes are stable for single-beam operation (for example, by means of feedback or sufficient Landau damping), they will remain stable under two-beam operation.

2. We have obtained an explicit form for the modified $s = 0$ eigenvalues; namely, the solution of Eq. (19), which may be written in the form:

$$2\Omega^2 = \omega_+^2 + \omega_-^2 - (\alpha^+ + \alpha^-) \pm \left\{ [(\omega_-^2 - \alpha^-) - (\omega_+^2 - \alpha^+)]^2 + 4\beta^+ \beta^- \right\}^{\frac{1}{2}} \quad (20)$$

where

$$\alpha^\pm = \omega_0 N^\pm \sum_k P_k^\pm, \quad (21)$$

$$\beta^\pm = \omega_0 N^\pm \tilde{G}(\Omega) .$$

Case 2a: $(\omega_-^2 - \omega_+^2)^2 \gg 4 \beta^+ \beta^-$. Here we only have a second-order contribution to Ω , and consequent stable operation in view of the radiation damping. This criterion on Q-splitting must be evaluated numerically for any particular design. It is surely overly conservative, as we have ignored Landau damping, which is a stabilizing influence.

Case 2b: $(\omega_-^2 - \omega_+^2)^2 \ll 4 \beta^+ \beta^-$. In this case we obtain first-order terms in Ω , which is a potentially serious situation. We must be sure that the $\text{Im } \Omega < 0$, for stability; fortunately, this criterion can be easily satisfied in practice. For example, if $N^+ = N^-$, then $\alpha^+ = \alpha^- \equiv \alpha$ and $\beta^+ = \beta^- \equiv \beta$, and if $\omega_+ = \omega_- = Q \omega_0$, then Eq. (20) has the two solutions

$$\Omega = Q \omega_0 \left[1 - \frac{\alpha \pm \beta}{Q^2 \omega_0^2} \right] . \quad (22)$$

Stability will be attained if α dominates β ; namely, if the single beam is made strongly stable. This could be accomplished by feedback, or by choice of Q value in accord with the rules of Report 6.⁵ It is probably not possible to satisfy this condition if the single beam is intrinsically unstable but stabilized by Landau damping, because the two-beam interaction will produce a large frequency shift and cause the single beam to become unstable. Such considerations -- involving Landau damping -- are not yet incorporated into the theory, although such an extension is expected to be very straightforward. Clearly, Case 2b is the one requiring most detailed examination and probably, in practice, numerical studies.

3. Finally, it should be noted that analysis of some storage rings might require extensions or modifications of the theory outlined here. For example, if the number of bunches were small and the vacuum tank loaded, then it might be advantageous to operate with the number of particles in different bunches unequal and Q just above an integer. We believe such extensions of the theory to be very easy.

⁵Although α and β are both first-order terms, one can expect α to be larger than β since F is evaluated near its peak (for some value of k in the sum) while G is evaluated at the large frequency $\Omega \approx Q \omega_0$.

Report 11

LONGITUDINAL RESISTIVE INSTABILITY FOR AN AZIMUTHALLY BUNCHED BEAM

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The residual electromagnetic fields associated with the passage of a bunch of particles down a resistive vacuum chamber may affect subsequent bunches in such a way as to produce an unstable mode of oscillation of the bunches. This coherent synchrotron instability growth rate is evaluated by first obtaining the variation in the electric field associated with the variation in the relative phase position of a bunch. One finds

$$\delta E = \frac{3}{2} E_0 \frac{\delta t}{t} = \frac{3}{2} E_0 \frac{\delta \varphi}{2\pi}$$

where

$$E_0 = \frac{q}{4\pi^{3/2} \epsilon_0 \mu^{1/2} c^2 \sigma^{1/2} b t^{3/2}}$$

where q is the charge of a bunch and b is the radius of the vacuum chamber whose walls have conductivity σ . If the phase shift of the coherent synchrotron oscillations between adjacent bunches is θ , and the maximum amplitude is $\delta\varphi$, then the residual electric field acting on a bunch to antidamp the oscillation is given by

$$\delta E = \frac{3}{2} E_0 \frac{\delta\varphi}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^{3/2}} .$$

The relationship between the maximum energy and phase variation of a synchrotron oscillation is given by

$$\frac{\delta U}{U_0} = \frac{\delta \phi}{\phi_0}$$

from which it now follows that the antidamping rate is

$$\frac{1}{\tau} = \frac{1}{2} \frac{\beta c e \delta E}{\delta U} = \frac{3 \beta c e E_0 \phi_0}{8 \pi U_0} \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^{3/2}} .$$

As a numerical example we take the parameters of the 3-GeV electron-positron storage ring, with a current of one ampere. Thus $\theta = \pi/4$, $q = 2 \times 10^{-9}$ coulomb, $\sigma = 10^6$ mho/meter, $b = 2 \times 10^{-2}$ meter, $t = 2 \times 10^{-10}$ sec, and $\phi_0 \approx 6 \times 10^2$, from which one finds $E_0 = 5.6 \times 10^{-3}$ volt/meter and $1/\tau = 4.3 \times 10^{-2} \text{ sec}^{-1}$. This is a very slow growth rate; the radiation damping rate is about $3 \times 10^2 \text{ sec}^{-1}$. If the energy is reduced, the radiation damping rate becomes equal to the instability growth rate at an energy of approximately 300 MeV.

Report 12

INCOHERENT BEAM BLOW-UP IN COLLIDING BEAMS

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Computations using Courant's program¹ were performed for choices of parameters particularly interesting for the CERN intersecting storage rings (ISR).² The strength of the interaction is defined by the interaction parameter δ , which is related in first approximation to Δv_v , the linear v shift in the vertical plane by

$$\Delta v_v = - \frac{\delta}{4\pi} . \quad (1)$$

According to the Amman-Ritson theory,³ the linear stability limits are, for $v = 1.09$,

$$- 6.88 < \delta < 0.58 . \quad (2)$$

This v value was chosen because it is approximately the number of betatron wavelengths between two interaction regions of the ISR. Because v is so near to 1, the linear stability limit and the Courant limit $\Delta v \approx 0.1$ are very close together.

The absence of radiation damping in proton storage rings requires that much smaller limits on the rate of amplitude growth be imposed. We have followed the trajectories of particles for 10^6 revolutions,

¹E. D. Courant, BNL Report AADD-69, Brookhaven National Laboratory, New York (March 1965).

²CERN Internal Report AR/Int. SG/64-9, CERN, Geneva Switzerland (1964).

³F. Amman and D. Ritson, Proc. Brookhaven Conf. on High-Energy Accelerators, Brookhaven National Laboratory, New York (1961).

starting at an amplitude of two half beam heights and for various values of δ . The trajectories show an irregular amplitude increase of up to about 50% for $0.2 \leq \delta \leq 0.9$ and no such increase for smaller values of δ . This observation is confirmed by studying the amplitude increase for a large number of particles with different initial conditions as a function of δ . The amplitudes grow for values of δ greater than about 0.19.

Most of the amplitude growth can be attributed to coupling between horizontal and vertical betatron motions provided by the interactions, as indicated by two observations:

- (1) If the coupling is suppressed, most of the amplitude growth disappears.
- (2) The build-up depends on the difference of the horizontal and vertical ν values.

The behavior of the beam is much more simply interpreted if one considers particles inside the beam, e.g., with an initial amplitude of $1/2$ the beam half-height, although the finite horizontal width of the beam and/or the nonlinear distribution of particles in a gaussian beam still provide a nonlinear force. From a study of the maximum amplitude as a function of δ , one can conclude that the particle motion is stable for all δ 's inside the linear stability limits, Eq. (2), provided that the effective ν values (including the ν shifts due to the interaction forces) are sufficiently different. If the ν values are equal or nearly equal, a nonlinear coupling resonance occurs. Then the slow beating of the betatron oscillations can produce excursions outside the beam where the nonlinearity can shift the phase of the oscillations. In the cases investigated ($\delta \approx 0.1$, horizontal amplitude \approx two half beam heights, beam half width \approx five half beam heights) the width of the resonance is always much smaller than the ν shift given by Eq. (1). Because $\Delta\nu_h \ll \Delta\nu_v$, making two linear machine ν 's equal is always a safe choice under these conditions.

For initial amplitudes equal to the half beam height (peak amplitude at the edge of the beam), roughly the same things happen as inside the beam.

In the model for the beam-beam interaction used in the computations there is no mechanism which drives the particles from inside the beam to the outside atmosphere inside the linear stability limits unless the ν values are in resonance, a condition which can easily be avoided. The observed growth outside the beam is less than 5×10^{-3} in 10^6 interactions for $\delta < 0.15$. Assuming an amplitude increase proportional to the square root of time, this corresponds to about 25% blow-up in approximately one hour. Since the actual interaction parameter in the ISR is about an order of magnitude smaller than the threshold given above, it is very likely that the orbits are stable against the beam-beam interactions investigated here even for fairly long periods of time.

Report 13

ONE-DIMENSIONAL COMPUTATIONS
OF PARTICLE TRAJECTORIES IN COLLIDING BEAMS

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The incoherent instabilities in electron storage rings, which have been studied on computers by Courant and Keil, seem due in great part to the two-dimensional nature of the perturbing kick which each beam gives to the particles in the other when head-on collisions occur in the interaction region. In the CERN ISR the beams will intersect in the horizontal plane at an angle of 15° . In this case all the force components apart from that normal to the beam plane (i.e., the vertical kick) cancel in the approximation that the interaction length is very short. The vertical kick on a proton in beam 1 is then proportional only to the charge in beam 2 between the orbit of that proton and the median plane of beam 2.

To take advantage of these simplifying factors, a Fortran IV computing program called RINGS has been written at CERN for use on the CERN CDC-6600 computer. This program allows computation of trajectories of protons in a linear ring with appropriate kicks from a prescribed charge distribution in the other beam in the interaction region. The program was designed to run rapidly, because the object of the studies was to find under what conditions a slow "stochastic" build-up might occur which could have escaped notice in calculations on electron rings, in which radiation damping can be assumed to take care of any very slowly growing instability. About 10^6 resolutions per minute can be computed on the CDC-6600.

The first results of this program have been:

- (a) Surveys of phase trajectories, for a wide range of ν values and oscillation amplitudes, of single protons crossing an intense second beam with either a parabolic or rectangular current distribution

in the vertical direction.

(b) Computations of the long-term stability of an ensemble of particles initially lying close to one of the invariant curves found in (a). The particular curves were chosen because they appeared to be close to separatrices, or might otherwise be potentially unstable.

The kick size used (i.e., the strength of the perturbing beam) was larger by at least a factor of 10 than what will occur in the ISR. For particles outside the beam, the surveys in (a) showed up many cases of the "islands" or "strings of pearls" which have been found in earlier work. The long runs in (b), in which up to 5×10^6 particles times resolutions were computed, showed no significant or systematic build-up of amplitudes of oscillation, beyond the limits of about one part per thousand, which were set by the method of summarizing the (much more accurately calculated) oscillation amplitudes of the individual particles in the ensembles considered.

Since any "stochastic" build-up must be much slower for the smaller kick sizes to be expected in the actual ISR, there seems reasonable confirmation already from these preliminary results that proton lifetimes of hundreds of seconds at least can be expected. This is in agreement with recent developments in the analytic theory reported by Schoch.

However, Schoch's theory indicates that there may be different behavior if the spatial distribution of the perturbing kick does not have forward and backward symmetry in the machine, or, for example, if two different kicks were made at different places on its circumference. This case, which might (but will not necessarily) occur in the ISR, can be studied by a small modification of the program, and will be taken up in the near future.