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INFINITE LADDERS AND THE ALGEBRA OF INTEGRATED CURRENT COMPONENTS

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I. INTRODUCTION

It was recently emphasized[1,2] that the role played by Lie algebras in particle physics is by no means restricted to the conventional notion of an invariance group of the Hamiltonian. The most fruitful among the other proposed applications has been, so far, that of an algebra of integrated current components[3] (or more generally, an algebra of some physically meaningful transition operators) which satisfy the required commutation relations without necessarily implying that particle states fall in a simple way into the irreducible representations of the algebra. The generators of such an algebra may, in general, connect any two physical states, provided that the "old-fashioned" selection rules (isotopic spin, strangeness, etc.) are satisfied. It is, however, expected[1] that a clever choice of the algebra may lead to a situation in which the generators connect every state only to a few nearby states, while all other transitions are relatively small and may be neglected.[4] In some cases we may even find a finite set of states which are connected among themselves by the generators of the algebra, but (to a certain approximation) are not connected to any other state. Such a situation inevitably leads to the classification of these particle states in a finite, though not necessarily irreducible, representation of the algebra. The Wigner-Eckart theorem can then be used for deriving relations between matrix elements of operators whose commutation relations with the generators are known.

On the other hand, it has been suggested that a non-compact algebra may generate an infinite sequence of physical states[2,5,6] (similar to those encountered in atomic or nuclear physics), while its maximal compact subalgebra connects only states within a given "level" of the infinite sequence and may serve as an approximate symmetry of the problem. It is interesting to try to extrapolate the known spectrum of mesons or baryons and to expect many more states to be discovered. Such states, which may have higher spins, isotopic spins and strangeness, could then form infinite ladders, defined by the unitary representations of some non-compact group. We do not require or assume that the non-compact group is an approximate symmetry of the strong interactions, as this would lead to an approximate mass degeneracy of all states in the infinite ladder. We may find, however, that a certain finite set of states within the infinite ladder forms a representation of our compact algebra of currents in the sense that transitions between any two states which are, respectively, inside and outside the given set of states are extremely weak and can be neglected.[7] In such a case it may even be reasonable to assume approximate mass degeneracy within this finite set of states which, in general, might include more than one level[8] of the infinite ladder.

In this paper we discuss the possible relations between various proposed current algebras and some non-compact groups which may generate infinite ladders of particle states. In particular, the attractive possibility of identifying the compact algebra of currents with the compact version of the non-compact spectrum-generating algebra is studied. Some examples based on the SU(3), SU(4) and SU(6) current algebras and, respectively, on the SU(2,1), SL(4,R) and SL(6,R) non-compact algebras, are considered, and it is shown that the particles within the first few "levels" of the obtained infinite ladders approximately fall into finite representations of the relevant current algebras.

II. CURRENT ALGEBRAS AND NON-COMPACT GROUPS

We now consider the possible relations between the proposed non-compact algebra G, its maximal compact subalgebra K, and the compact algebra of integrated current components G'. The simplest possible relation is obtained when G' is identified with K (the maximal compact subalgebra of G), which is expected to be an approximate symmetry of the problem.[9] In this case, the integrated currents generating G' connect only states which are in the same representation of K. If the particle states can be approximately identified with the basis vectors of the representations of K, we are essentially led to a sequence of multiplets of a symmetry group K, arranged in infinite ladders of G, while no transitions between different rungs of a given ladder can be induced by the integrated currents. If we want to use such a scheme while assuming an SU(3) algebra of currents, we will obtain SL(3,C) [or possibly SU(3,1)] as the non-compact algebra. Similarly, an $SU(3) \times SU(3)$ algebra of currents will lead to SU(3,3) as the spectrum-generating algebra.

We suggest, however, a different possibility which leads to a more attractive classification of the known particles and is consistent with experimental facts such as the absence of positive strangeness baryons. It is also consistent with the usual results of SU(3), SU(6), etc., while it allows infinite multiplets of particles.

We first observe the following interesting property of some ladder representations of non-compact groups: In many cases (including most known physical applications such as the hydrogen atom or the harmonic oscillator) the first n "levels" of a given ladder representation of a non-compact group G form an irreducible representation of the compact version of G, i.e., the compact algebra obtained from G by appropriately changing the signs of the structure constants.[10] A "level" is defined, for this purpose, as a set of one or more[11] irreducible representations of the maximal compact subalgebra K such that:

1. Every level in the ladder contains the same number of irreducible representations of K.

2. A one-to-one correspondence exists between the representations of K in two adjacent levels. Any transition between the corresponding representations in these levels involves the operation of the "non-compact operators" of G (those generators of G which do not belong to K).

A simple example of this structure is provided by some of the ladder representations of SO(4,1). For any positive integral or half-integral value of S there exists an infinite dimensional irreducible unitary representation of SO(4,1), the first level of which consists of[12]:

$$(S,O)\ \ (S-\tfrac{1}{2},\ \tfrac{1}{2})\ (S-1,1)\ldots(\tfrac{1}{2},S-\tfrac{1}{2})\ (O,S)\ ;$$

(k,ℓ) is a representation of the maximal compact subgroup SO(4) and is characterized by the two orthogonal "spins" which form an SO(4) algebra. The next levels of the same ladder representation of SO(3,1) are:

$$(S+\tfrac{1}{2},\tfrac{1}{2})\ (S,1)\ (S-\tfrac{1}{2},\tfrac{3}{2})\ldots(1,S)\ (\tfrac{1}{2},S+\tfrac{1}{2})\ ;$$

$$(S+1,1)\ (S+\tfrac{1}{2},\tfrac{3}{2})\ (S,2)\ldots(\tfrac{3}{2},\ S+\tfrac{1}{2})(1,\ S+1);\ldots$$

For every value of n, the first n levels form an irreducible representation of SO(5) for which $\lambda = 2S$; $\mu = n-1$ in the usual (λ,μ) notation.[13] Note that each level of the ladder contains $2S+1$ irreducible representations of SO(4), every one of which is obtained by applying the four non-compact operators of SO(4,1) to the appropriate representations of SO(4) in the previous level.

Using this mathematical structure we suggest that, in some cases, the following relation may exist between the non-compact algebra G and the compact algebra of integrated currents G'. G' is the compact version of G [14] and its generators may consequently connect only pairs of states which belong either to the same level or to adjacent levels. Every state is then automatically connected only to a few nearby states, although it is not necessarily clear that an "isolated" finite set of states exists. In some cases we may find, however, that if we "cut" the infinite ladder after a finite number of levels, we obtain a set of states which form an irreducible representation of G', while transitions connecting these states to higher levels in the ladder are negligibly small. [7] It is clear that such a "miracle" can occur only as a consequence of a certain underlying dynamical structure, which we are not able to describe at present. However, any proposal of an approximate symmetry involves the assumption that a finite set of states is "isolated" by the operations of the integrated currents, and if such a situation occurs at all, it may well apply to the first n levels of some infinite ladder.

III. SL(4,R) AND THE STRONG COUPLING THEORY

We now proceed to some examples. It was recently pointed out[5,15] that the old strong coupling theory[16] leads to an infinite dimensional representation of SL(4,R) as a classification scheme for a sequence of nucleon resonances with $(I,J) = (\frac{1}{2},\frac{1}{2});\ (\frac{3}{2},\frac{3}{2});\ (\frac{5}{2},\frac{5}{2});\ldots$

The maximal compact subalgebra of SL(4,R) is SO(4) or SU(2) $\times$ SU(2) which includes the spin and isospin operators $\vec{\sigma}$ and $\vec{\tau}$. The compact "version" of SL(4,R) is, or course, SU(4).

We now assume that the following integrated current components form an SU(4) algebra:[17]

a) Space integrals of the space components of the isovector, axial vector current $\int A_i^{(1)}(\vec{x},t)d^3x\,(i=1,2,3)$

b) Space integrals of the space components of the isoscalar, axial vector current $\int A_i^{(0)}(\vec{x},t)d^3x\,(i=1,2,3)$

c) Space integrals of the space components of the isoscalar, vector current $\int V_0^{(1)}(\vec{x},t)d^3x$.

It follows immediately that these integrals connect only adjacent levels in any given SL(4,R) ladder representation. Moreover, if we assume that the transitions among the four spin-isospin states of the nucleon are much stronger than the $N \longleftrightarrow N^*$ transitions, we are led to the classification of the nucleon in a 4-dimensional representation of SU(4),[18] consistent with the classification of Wigner's supermultiplet theory[19] as applied to problems in which no N^* is expected to be involved. If, however, dynamics tells us that the transitions to the $N^*(1238)$ are important (and as far as we can tell this is really the case), the nucleon states are no more "isolated" and the N^* and the nucleon will be in the same SU(4) multiplet. Note that the assumption that all sum rules based on evaluating the commutation relations of integrated currents between <u>nucleon</u> states are saturated by the N and N^* immediate states is not sufficient for specifying the SU(4) representation of the nucleon. Consequently, this assumption does not lead to unique values for quantities such as the axial-vector coupling constant or the magnetic moments ratio for the nucleon. The assumption is consistent with any classification which is based on grouping the first n levels ($n \geq 2$) of the SL(4,R) ladder into an $\binom{n+3}{3}$-dimensional irreducible representation of SU(4). Each value of n will lead to a different value for G_A/G_V and $\mu(n)/\mu(p)$. We obtain:

$$\frac{G_A}{G_V} = \frac{2n+1}{3}$$

$$\frac{\mu(n)}{\mu(p)} = -\frac{2n^2-2}{2n^2+1}$$

If we now want to suggest that the nucleon and $N^*(1238)$ form a $\underline{20}$-representation of SU(4) (the classification implied by the usual SU(6) assignment), we must assume that transitions between the $(\frac{3}{2},\frac{3}{2})$ state and the $(\frac{5}{2},\frac{5}{2})$ state can be neglected and that sum rules which are evaluated between <u>N^* states</u> are saturated in all cases by the N and N^* intermediate states. This assumption sets $n = 2$ and leads to the usual results of $G_A/G_V = \frac{5}{3}$ and $\mu(n)/\mu(p) = -\frac{2}{3}$. It is, again, clear that only the dynamics is capable of dictating where we should "cut" the infinite ladder and in this case it looks as if the second stage is the right place to do so, although even this is, at best, only a crude approximation.

With some slight modifications we may carry this SL(4,R) analysis to the strange baryons or to other nucleon resonances. In the case of baryons with non-zero strangeness we must introduce a "strange spin"[20] which is carried with every unit of strangeness [similar to the λ-quark spin[21] in SU(6)] . The SL(4,R) ladder for $S = -1$ baryons will then be (0,0); (1,1); (2,2);... The first level describes an $I = 0$ state with a zero "non-strange" spin which, together with the strange spin, forms an $I = 0$, $J = \frac{1}{2}$, $S = -1$ particle: $\Lambda(1115)$. The second level includes the $\Sigma(1190)$ and the $Y_1^*(1385)$, which have $I = 1$, "strange spin" = $\frac{1}{2}$ and "non-strange spin" = 1. They differ in the total value of J. In the Σ case, the two spins are coupled to $J = \frac{1}{2}$ while for Y_1^* they are coupled to $J = \frac{3}{2}$. The next states in this $S = -1$ ladder will have $I = 2$ and $J^P = \frac{3^+}{2}; \frac{5^+}{2}$. We can assign the Λ to a scalar representation of the SU(4) algebra (and assume that $\Lambda - \Sigma$ and $\Lambda - Y_1^*$ transitions are negligible) or classify Λ, Σ and Y_1^* into a <u>10</u> representation. This last possibility coincides with the usual SU(6) classification.

Note that the existence of SL(4,R) ladders does not necessarily depend on the validity of the strong coupling theory which is usually used, in this connection, only as an example of a dynamical structure which may lead to a non-compact spectrum-generating algebra.

IV. SU(2,1) AS AN "INTERNAL" NON-COMPACT ALGEBRA

A more interesting example is provided by the groups SU(2,1) and SU(3). The baryons can be assigned to infinite ladder representations of the non-compact SU(2,1)[6] whose maximal compact subalgebra is the isospin-strangeness U(2). If we consider only those SU(3,1) representations which have a bounded spectrum of S-values, we find that these can be characterized[6,22] by the maximal (or minimal) weight (I,S) where I and S are, respectively, the total isotopic spin and strangeness of a U(2) multiplet. The first level of such a representation includes the following U(2) multiplets:

$$(I,S)(I - \frac{1}{2}, S - 1)(I - 1, S-2)\ldots(O,S - 2I) ;$$

The next levels are:

$$(I+\tfrac{1}{2},\ S-1)(I,\ S-2)(I-\tfrac{1}{2},\ S-3)\ldots(\tfrac{1}{2},\ S-2I-1)\ ;$$

$$(I+1,\ S-2)(I+\tfrac{1}{2},\ S-3)(I,\ S-4)\ldots(1,\ S-2I-2);$$

$$(I+\tfrac{3}{2},\ S-3)(I+1,\ S-4)(I+\tfrac{1}{2},\ S-5)\ldots(\tfrac{3}{2},\ S-2I-3);\ \ldots$$

For every value of n, the first n levels form an irreducible representation of the compact SU(3), provided that the isosinglet operator of SU(3) is properly defined. In the usual (λ,μ) notation, the SU(3) representation constructed by the first n levels of an SU(2,1) ladder is $(2I,\ n-1)$ where I is the maximal isotopic spin of the first level.

The $J^P = \frac{1}{2}^+$ baryons will clearly fit into the following representation:

$$(\tfrac{1}{2},\ 0)(0,-1);\ (1,-1)(\tfrac{1}{2},-2);\ (\tfrac{3}{2},-2)(1,-3);\ \ldots$$

The first level includes the nucleon and Λ; the second level Σ and Ξ. In the third level, yet to be experimentally discovered, $\Xi^*(I=\frac{3}{2})$ and $\Omega^*(I=1)$. If we now assume an SU(3) algebra of integrated currents, including the space integrals of the time components of the eight vector currents, we find again that all allowed transitions in the ladder occur either within a given level or between adjacent levels. Moreover, dynamics may tell us that the first level is "isolated," forming a Sakata-type $\underline{3}$ representation or, preferably, that the first two levels are "isolated," thus constructing the usual SU(3) octet. The inclusion of the next two hypothetical Ξ^* and Ω^* states might lead to a $\underline{15}$ representation of SU(3). However, we have at least two reasons to believe that the best approximation is to "cut" the ladder after the second level. These are the success of the octet assignments for the baryons, with respect to various electromagnetic and weak phenomena, and the apparently large mass difference between states in the second and third levels.[23] The known $J^P = \frac{3}{2}^+$ resonances fit into the first level of another SU(2,1) infinite ladder:

$$(\tfrac{3}{2},\ 0)(1,-1)(\tfrac{1}{2},-2)(0,-3);\ \ (2,-1)(\tfrac{3}{2},-2)(1,-3)(\tfrac{1}{2},-4);\ldots$$

The first level is an SU(3) decuplet. The first two levels may form a $\underline{24}$ representation, including $Y_2^* \equiv^* (I = \frac{3}{2})$, $\Omega^*(I = 1)$ and $X(S = -4, I = \frac{1}{2})$.

The mesons can be assigned to an unbounded representation which allows an ordinary octet at its center,[6] while higher states include $S = \pm 2, \pm 3, \ldots$ $I = 1, \frac{3}{2}, 2 \ldots$ mesons.

Higher baryon resonances fit nicely into nucleon-type representations. In both the $J^P = \frac{3^-}{2}$ and the $J^P = \frac{5^+}{2}$ cases the only well established states have the internal quantum numbers of N and Λ. For $J^P = \frac{3^-}{2}$ we find N*(1510) and $Y_0^{**}(1520)$ in the first level of a nucleon-type representation, predicting a Y_1^* and a Ξ^* with $J^P = \frac{3^-}{2}$ in the next level. For $J^P = \frac{5^+}{2}$, N*(1688) and $Y_0^*(1815)$ may be assigned to the first level of another nucleon-type representation.

Note that all these assignments allow us to predict more and more states with decreasing strangeness without requiring that positive strangeness baryon resonances exist. The nonexistence of such states is, so far, one of the most striking regularities of the baryon spectrum. In ordinary SU(3), any representation higher than $\underline{8}$ or $\underline{10}$ will predict such $S > 0$ states and every $I = \frac{5}{2}$, $I = 2$ or $S = -4$ baryon resonance will immediately lead[24] to such a prediction. On the other hand, the SU(2,1) classification is consistent with the absence of $S > 0$ baryons, although such states are by no means forbidden.

V. AN SU(6)-TYPE LADDER

We may incorporate both the SU(2,1) and the SL(4,R) infinite ladders into an SU(6)-type scheme in the following simple way: We assign the baryonic states into infinite ladders of SL(6,R) while assuming an SU(6) algebra of currents. The integrated currents included in this SU(6) algebra are the space components of the octet and singlet axial vector currents and the time components of the octet vector currents.[25,1]

The maximal compact subalgebra of SL(6,R) is SO(6) (isomorphic to the SU(4) algebra) and the baryon ladder representation is constructed from the following SO(6) representations:[26] $\underline{6}$; $\underline{50}$; $\underline{196}$; $\underline{540}$, ... The first level accommodates the six spin states of p, n and Λ and it is easy to verify that the states of the $\underline{50}$ representation of SO(6) have the correct quantum numbers of Σ, Ξ, N*, Y_1^*, Ξ^* and Ω. If we "cut" the ladder after the first level, assuming that transitions such as $N \longleftrightarrow N^*$ or $\Lambda \longleftrightarrow \Sigma$ can be neglected, we again get a Sakata-type assignment of p, n, Λ into a $\underline{6}$ of SU(6). If, however, dynamics tells us that the first two levels are "isolated" from the rest of the ladder (i.e., neglecting $N^* \longleftrightarrow N^{**} (\frac{5}{2}, \frac{5}{2})$ and similar transitions) we have the usual $\underline{56}$ of SU(6) as the basic baryon multiplet. The third level of the SL(6,R) ladder includes the following 196 spin states: Ξ^* and Ω_3^*, with $J^P = \frac{1^+}{2}$; four $J^P = \frac{3^+}{2}$ states which formed the second level of the SU(2,1) ladder of the decuplet: Y_2^*, Ξ^*, Ω^*, and an $S = -4$, $I = \frac{1}{2}$ state; and six $J^P = \frac{5^+}{2}$ resonances, including the $(\frac{5}{2}, \frac{5}{2})$ state from the SL(4,R) ladder of the nucleon.

We can apply similar methods to the full SU(3) $\times$ SU(3) or SU(6)$\times$ SU(6) algebra of currents (including both vector and axial vector currents). The corresponding non-compact groups are, respectively, SL(3,C) and SL(6,C). In the SL(6,C) case, for example, the baryons may be classified into a $\underline{56}$; $\underline{700}$; $\underline{4536}$;... ladder and the mesons are in $\underline{1}$; $\underline{35}$; $\underline{405}$; $\underline{2695}$; ... The corresponding SU(6)$\times$ SU(6) representations obtained by "cutting" the ladders after the first, second or third level will then be: for baryons - $(\underline{56}, \underline{1})$; $(\underline{126}, \overline{\underline{6}})$; $(\underline{252}, \overline{\underline{21}})$... and for mesons - (1,1); $(6, \overline{6})$; $(21, \overline{21})$; $(56, \overline{56})$; ... These should not be confused with the similar sequences suggested by Dothan, Gell-Mann and Ne'eman[2] as the components of U(6,6) ladders.

VI. SOME FINAL COMMENTS

The assumption that the integrated currents satisfy the exact commutation relations of a given algebra is clearly much weaker, and probably closer to reality, than the assignment of physical states to the irreducible representations of the algebra. However, we cannot ignore many successful results which have been obtained by classifying particles into the representations of SU(3) or SU(6), and we should not abandon this way of describing various processes. Although we cannot explain why the same assumption of an approximate symmetry sometimes leads to accurate predictions while it fails completely in other cases, we feel that the classification of particles into finite multiplets of the algebra of currents can provide, in many cases, a reasonable first approximation to the physical situation.

On the other hand, an infinite spectrum of states as described by non-compact groups appears natural both from the point of view of the analogy to atomic and nuclear systems and from the apparent absence of an upper bound for the masses of resonant states. Knowing that the classification of an infinite number of particles in one representation of a non-compact group cannot lead, in any meaningful way, to an approximate symmetry based on this group, we are led to speculations on the possible relation between the algebra of currents (or the finite multiplets to which it may approximately lead) and the non-compact algebra.

The possibility pointed out in this paper may provide a good example of such a relation, leading to a scheme in which both the approximate symmetry and the algebra of currents are built into the structure of the non-compact spectrum-generating algebra. It remains to be seen, however, if the next states to be discovered will fit into the next levels of the proposed infinite ladders.

It should also be pointed out that the only way of constructing a fully covariant "vertex symmetry" (i.e., a symmetry of the interaction part of the Lagrangian) which includes the Lorentz group as a subgroup in a non-trivial way, must be based on classifying the physical states according to infinite dimensional representations, <u>without</u> necessarily assigning an infinite number of physical states to every representation. The infinite SO(4,1) ladders discussed as a mathematical example in Section II (and footnote 13) may be a suitable starting point for such a theory, avoiding the unitarity difficulties and the limitations on the number of derivatives[27] which are present in symmetry schemes such as the U(2,2) subgroup of U(6,6).[28]

FOOTNOTES AND REFERENCES

1. R. F. Dashen and M. Gell-Mann, Phys. Letters 17, 142 (1965).
2. Y. Dothan, M. Gell-Mann and Y. Ne'eman, Phys. Letters 17, 148 (1965).
3. M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
4. It was pointed out by W. I. Weisberger (private communication) that it is sufficient to demand that the sum rules derived by evaluating the commutators of two operators between one-particle states are saturated by a few intermediate one-particle states. This assumption is weaker than the one made in the text and does not lead to the difficulties discussed by S. Coleman (CERN preprint, October 1965) for the case of an $SU(6)_W$ algebra of currents. The final sum rules which are obtained are not changed if we use this weaker assumption. The author is indebted to S. Coleman for a helpful discussion concerning this point.
5. Y. Dothan and Y. Ne'eman, paper presented at the Athens (Ohio) Conference, 1965.
6. C. Fronsdal, Proc. Roy. Soc. (London) A288, 98 (1965).
7. The weaker assumption that the sum rules are saturated by states which belong to the considered finite set leads, again, to the same results and conclusions.
8. The precise definition of a "level" is given in Sec. II.
9. This is the case for some of the examples discussed in Refs. 2 and 5.
10. This property was independently noticed and discussed by N. Mukunda, L. O'Raifertaigh and E.C.G. Sudarshan, Phys. Rev. Letters 15, 1041 (1965) and by R. Herman, lecture notes, Seminar on High-Energy Physics, Trieste, 1965. Mukunda et al. discuss the hydrogen atom, the rigid rotator and the harmonic oscillator, while Herman discusses various mathematical aspects of this relation. It is not known whether there is a simple necessary and sufficient criterion for this relation to occur.
11. The applications discussed in Ref. 10 always involve only one irreducible representation of K for each level. The generalization to the case of any finite number of representations is mathematically trivial and crucial to our examples and discussion.
12. L. H. Thomas, Ann. Math. 42, 113 (1940); T. D. Newton, Ann. Math. 51, 730 (1950).

13. Note that the SO(4) representations, included in the SO(5) representation which is constructed from the first n levels of an SO(4, 1) ladder starting with (S, O), correspond to those representations of the homogeneous Lorentz group which describe a free field of spin S together with its first $n + S - 1$ derivatives (for integral S) or $n + S - \frac{3}{2}$ derivatives (for half integral S). This is discussed in detail by H. Harari, Phys. Rev. Letters 14, 1100 (1965).

14. Y. Dothan and Y. Ne'eman, Ref. 5, discuss a U(12) algebra of currents and a U(6, 6) spectrum-generating non-compact algebra. In this case G' is the compact form of G. However, in their treatment, the approximate symmetry is $U(6) \times U(6)$, and the particles are expected to fall in a simple way into its irreducible representations, although not necessarily into the irreducible representations of U(12). As far as the approximate symmetry is concerned, their example fits into our first case in which the maximal compact subalgebra K defines the symmetry of our problem.

15. T. Cook, C. J. Goebel and B. Sakita, Phys. Rev. Letters 15, 35 (1965).

16. For a detailed list of references, including recent generalizations and modifications based on bootstrap mechanisms, see Ref. 5.

17. This is a subalgebra of the algebra suggested by B. W. Lee, Ref. 25, and R. F. Dashen and M. Gell-Mann, Ref. 1. An SU(4) treatment was given by C. Ryan, Rochester preprint, UR-875-89, University of Rochester, Rochester, New York, (June 1965).

18. The $\underline{\bar{4}}$-representation is equally possible, from the mathematical point of view. We prefer the $\underline{4}$ representation as this leads in the next stage to the $\underline{20}$ which gives the correct value for $\mu(n)/\mu(p)$ and the best value of G_A/G_V. This point was discussed by C. Ryan, Ref. 17.

19. E. P. Wigner, Phys. Rev. 51, 106 (1937).

20. M.A.B. Beg and V. Singh, Phys. Rev. Letters 13, 418 (1964).

21. H. J. Lipkin, Phys. Rev. Letters 13, 590 (1964).

22. L. C. Biedenharn, J. Nuyts and N. Straumann, CERN preprint TH-555, CERN, Geneva, Switzerland (1965).

23. The problem of finding mass formulae for the infinite ladders may turn out to be very complicated. If the dynamical situation is such that transitions between the first two levels of a given ladder are enhanced while the

third level does not contribute, we may expect the mass differences to be larger between the second and third levels than between the first and the second. Consequently, a simple Gell-Mann-Okubo mass formula is not acceptable, as this would lead to an equal spacing rule of an infinite number of U(2) multiplets such as $(\frac{1}{2},0)(1,-1)(\frac{3}{2},-2)(2,-3)\ldots$ Such a naive formula would lead to $m(\Xi^*_3) = 1450$ MeV; $m(\Omega^*_1) = 1520$ MeV. The success of the octet assignments for the first two levels suggests, however, that states in the third level may have much higher masses, possibly in the 2-BeV region.

24. An $I = \frac{5}{2}$ $N\pi\pi$ resonance was recently reported by two groups: G. Goldhaber, Proceedings of the Second Coral Gables Conference, January 1965, and G. Alexander et al. "Further Evidence for Possible $I = \frac{5}{2}$ N^* Resonance at 1580 MeV," Weizmann Institute preprint, Weizmann Institute, Rehovoth, Israel (June 1965). Such a resonance can be accommodated only in the 35 or 64 of SU(3), predicting an $S = 1$, $I = 2$ $K\pi N$ resonance with a small mass. See also H. Harari and H. J. Lipkin, Phys. Rev. Letters 13, 345 (1965).

25. B. W. Lee, Phys. Rev. Letters 14, 676 (1965).

26. The SU(4) Young tableaux corresponding to these representations are (1,1,0); (3,3,0); (5,5,0);... The non-compact generators of SL(6,R) transform like the (2,2,0) 20-dimensional representation of SO(6) (or SU(4)).

27. H. Harari, Ref. 13.

28. A. Salam, R. Delbourgo and J. Strathelee, Proc. Roy. Soc. (London) A284, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. Letters 14, 404 (1965); M.A.B. Beg and A. Pais, Phys. Rev. Letters 14, 269 (1965).